1

MEASUREMENT

1.1 WHAT IS PHYSICS

Physics is branch of science, which deals with the study of the phenomena of nature. The word 'physics' is derived from a Greek word meaning nature. The word 'Science' comes from the Latin word scientia, which means 'to know'.

Man has been observing various natural phenomena from time immemorial. He has always been curious about nature and the world around him. The motion of the moon and other heavenly bodies in the sky has aroused owe and amazement in him. The regular repetition of sunrise and sunset and the seasons of the year have fascinated him. Man has observed these and other natural phenomena and responded to them in an orderly manner. The experiences gained over a period of time were transmitted from generation to generation and began to be termed as 'knowledge'. Each generation added new fact to the knowledge obtained from the previous generation. The systematized knowledge thus gained was termed 'Science'.

Every bit of knowledge is not called science. Only such knowledge as is collected by what is called the scientific method. The scientific method is the basis of scientific development. The method involves four steps:

1. Observation of the relevant facts,
2. Proposal of a hypothesis or a theory based on these observation,
3. Testing of the proposed theory to see if its consequences or predictions are actually observed in practice, and
4. Modification of the theory, if necessary.

1.2 SCOPE AND EXCITEMENT IN PHYSICS

The various sciences may be divided into two broad classes, physical and biological. Physical sciences deal with nonliving matter and biological sciences with living matter. About a hundred years ago, it was possible for one man to master the knowledge of both sciences and many outstanding workers in physical sciences were also competent doctors and biologists. There was no clear-cut division between the several branches of physical sciences, as we know them today. In fact, all were included in the term natural philosophy. Aristotle, Archimedes and even Galileo and Newton called themselves natural philosophers. But today the situation is different. The tremendous upsurge of scientific activity and the accumulation of knowledge in the last century have forced scientists to narrow down their field of activity. Not even a genius could hope to keep up with the developments in the various branches of physical sciences as we know them today, namely, physics, chemistry, astronomy, biology, medicine, geology, engineering etc.

Those concerned mainly with the application of science to the betterment of human life and environment are called engineers. The invention of the steam engine and the electric motor gave rise to engineering. Physicists and chemists are concerned more with the basic aspects of nonliving matter chemistry deals primarily with molecular changes and the rearrangement of the atoms that form molecules. Physics deals with the phenomena of the non-living world such as mechanics (motion), heat, sound, electricity, magnetism and light. The division of the subject of physics into these and other branches is largely a matter of convenience. These branches are inter-related, as you will discover in the course of your study of the subject. Physics may be defined as that branch of knowledge that deals with the phenomena of non-living matter.

In physics, we deal with many physical phenomena and experiences. Merely reading about the experiences and observation of others is not enough. If students are to understand and enjoy physics, they must have same of these experiences themselves. These experiences are not only exciting but also very educative. The swinging hanging lamp in a church led Galileo to a method of measuring time. The fall of an apple and the motion of the moon led Newton to his famous law of gravitation. The rattling (or dancing) of the lid of a kettle led to the invention of the steam engine. The flowing of a flute causes vibrations that produce sound. The light from stars tells us something about stars and their evolution. The study of electricity helps us to design motors and dynamos. The study of semiconductors helps us to design radios, televisions, calculators and even computers.

1.3 MEASUREMENT

Observation can be subjective or objective. An observation that varies from individual to individual is subjective. For example, different individuals observing the same thing, e.g. a painting or a flower, feel differently. Physics does not deal with such subjective observations. Physics is a science of objective observation, an observation that is the same for all individuals. An individual observer through his sense of touch or sight, but these senses is not always reliable. To illustrate the inaccuracy of our sense of touch, we consider three pans containing cold, warm and hot water. If you put your finger first in cold water and then in warm water, your sense of touch will tell you that it is hot. But if you put your finger first in hot water and then in warm water, your sense tells you it is cold. This clearly suggests the necessity of making a measurement to arrive at the truth. It is necessary to measure the degree of hotness of water in each pan. In other words, it is not enough to describe a phenomenon in a general and qualitative way. A number must be tied to it. Thus, physics is a science of measurement.

Lord Kelvin, a leading physicist of the 19th Century, once said: “When you can measure what you are talking about and express it in numbers, you know something about it; but when you cannot, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely in your thoughts advanced to the stage of a science”.

1.4 PHYSICAL QUANTITIES

Physical quantities are often divided into fundamental quantities and derived quantities. Derived quantities are those whose defining operations are based on other physical quantities. Fundamental quantities are not defined in terms of other physical quantities. The number of quantities regarded as fundamental is the minimum number needed to give a consistent and unambiguous description of all the quantities of physics. All Physical quantities occurring in mechanics can be expressed in the units of 'length', 'mass' and 'time'. The units of these three quantities are independent of one another and no one can be changed or related to any other unit. These quantities are called 'fundamental quantities' and their units are called 'fundamental units'.

In the same way, to fix the units of physical quantities occurring in electromagnetism, thermodynamics and optics, electric current, temperature and luminous intensity are taken as fundamental quantities and their units as 'fundamental units'.

The system of units based on the units of the seven fundamental quantities (length, mass, time, electric current, temperature, luminous intensity and amount of substance) is called International system of units. It is abbreviated as SI from the French name Le Systeme International d'units. It is based on the following seven fundamental (or Basic) and two supplementary Units:

	Basic physical Quantity	*Name of the Unit*	*Symbol*
1.	length	meter	m
2.	mass	kilogram	kg
3.	time	second	s
4.	Electric Current	ampere	A
5.	Temperature	Kelvin	K
6.	Luminous intensity	candela	Cd
7.	Amount of substance	mole	mol

	Supplementary physical Quantity	*Name of the Unit*	*Symbol*
1.	Plane angle	radian	rad
2.	Solid angle	steradian	Sr

1.5 FUNDAMENTAL UNITS OF S.I. SYSTEM

Basic and Supplementary Units of SI System

The seven fundamental and two supplementary units of SI system are defined as follows:

1. **Metre:** On atomic standard meter is defined as to be equal to 1,650,763.73 wavelengths in vacuum of the radiation emitted due to transition between the levels $2p^{10}$ and $5d^5$ of the isotope of krypton having mass number 86. Krypton-86 emits light of several different wavelengths. The light emitted by Krypton-86 due to transition between the levels $2p^{10}$ $5d^5$ in orange red in colour and has wavelength 6057.8021 Å or 6.0578021×10^{-7} m. The number of these wavelengths in 1 m can be counted by using an optical interferometer which comes out to be 1,650,763.3.

2. **Kilogram:** There is no definition of unit kilogram on atomic standards. Therefore in SI system, kilogram is the mass of a platinum-iridium cylinder kept in the International Bureau of weights and measures at Paris. In practice, the mass of 1 litre of water at 4°C is 1 kilogram.

3. **Second:** Unit second can again be defined on atomic standards. One second is defined to be equal to the duration of 9,192,631,770 vibrations corresponding to the transition between two hyperfine levels of Caesium –133 atom in the ground state.

4. **Kelvin:** It was adopted as the unit of temperature. The fraction 1/273.16 of the thermodyamics temperature of triple point of water is called 1K.

5. **Ampere:** It was adopted as the unit of current. It is defined as the current generating a force 2×10^{-7} Newton per metre between two straight parallel conductors of infinite length and negligible circular cross-section, when placed at a distance of one meter in vacuum.

6. **Candela:** It was adopted as the unit of luminous intensity. One candela is the luminous intensity in perpendicular direction of a surface of 1/600,000 meter^2 of a black body at a temperature of freezing platinum (2046.64 Kelvin) and under a pressure of 101,325 N/m^2. Candela was redefined in 1979 as below:

   It is the luminous intensity in a given direction due to a source which emits monochromatic radiation of frequency 540×10^{12} Hz and of which the radiant intensity in that direction is 1/683 watt per steradian.

7. **Mole:** It was adopted as the unit of amount of substance. The amount of a substance that contains as many elementary entities (molecules or atoms if the substance is monoatomic) as there are number of atoms in 0.012 kg of carbon-12 is called a mole. This number (number of atoms in 0.012 kg of carbon –12) is called Avogadro constant and its best value available is 6.022045×10^{23}.

8. **Radian:** It was adopted as the unit of plane angle. It is the plane angle between the two radii of a circle, which cut off from the circumference, an arc equal to the length of the radius.

   Plane angle in Radian = length of arc/radius

9. **Steradian:** It was adopted as the unit of solid angle with its apex at the centre of a sphere that cuts out an area on the surface of the sphere equal to the area of the square, whose sides are equal to the radius of the sphere.

   Solid angle in steradian = area cut out from the surface of sphere/radius^2

1.6 REFERENCE FRAMES

The same physical quantity may have different values if it is measured by observers who are moving with respect to each other.

The velocity of a train has one value if measured by an observer on the ground, a different value if measured from a speeding car, and the value zero if measured by an observer sitting in the train itself. None of these values has any fundamental advantage over any other; each is equally ‘correct’ from the point of view of the observer making the measurement.

In general, the measured value of any physical quantity depends on the reference frame of the observer who is making the measurement. To specify a physical quantity, each observer may choose a zero of the time scale, an origin in space and an appropriate coordinate system. We shall refer to these collectively as a frame of reference. Since the space of our experience has three dimensions, we must in general specify three coordinates to fix uniquely the position of an object. The Cartesian coordinates x, y, z are commonly used in mechanics. Thus, the position and time of any event may be specified with respect to the frame of reference by three Cartesian coordinates x, y, z and the time t.

1.7 INERTIAL AND NON-INERTIAL FRAMES

A system relative to which the motion of any object is described is called a frame of reference. The motion of a body has no meaning unless it is described with respect to some well defined system. There are generally two types of reference systems:

1. The frames with respect to which unaccelerated body is unaccelerated. This also includes the state of rest.
2. The frames with respect to which an unaccelerated body is accelerated.

The frames with respect to which an unaccelerated body is unaccelerated, *i.e.*, is at rest or moving with constant linear velocity are called inertial frames *i.e.*, unaccelerated frames are inertial frames.

Let us consider any coordinate system relative to which body in motion has coordinates (x, y, z). If the body, is not acted upon by any external force, then

$$m\frac{d^2x}{dt^2} = 0, \qquad m\frac{d^2y}{dt^2} = 0, \qquad m\frac{d^2z}{dt^2} = 0$$

Hence,

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = 0, \qquad \frac{d^2z}{dt^2} = 0$$

Which gives

$$\frac{dx}{dt} = u_x = \text{constant}, \quad \frac{dy}{dt} = u_y = \text{constant}, \quad \frac{dz}{dt} = u_z = \text{constant}$$

Where u_x, u_y and u_z are the components of velocity in x, y and z directions respectively. From above equations we see that the components of velocity are constant, *i.e.*, we say that without application of an external force, a body in motion continues its motion with uniform velocity in a straight line, which is Newton's first law.

Hence we may say 'An inertial frame is one in which law of inertia or Newton's first law invalid'.

The frame with respect to which an unaccelerated body is accelerated are called non-inertial frames, *i.e.*, accelerated frames are called non-inertial frames.

A frame of reference moving with constant velocity relative to an inertial frame is also inertial. Since acceleration of the body in both the frames is zero, the velocity of the body is different but uniform.

Experiments suggest that a frame of reference fixed in stars is an inertial frame. A coordinate system fixed in earth is not an inertial frame, since earth rotates about its axis and also about the sun.

Galilean Transformation

Galilean transformations are used to transform the coordinates of a particle from one inertial frame to another. They relate the observations of position and time made by two of observers, located in two different inertial frames.

Let us consider two inertial frames S and S′. S being at rest and S′ moving with a constant velocity v relative to S. The positions of two observers O and O′ observing an event at any point P coincide with the origin of the two frames S and S′. Then the problem is to transform the data of the event recorded in the first frame to those recorded in the second.

Case I: When the second frame moves relative to first along positive direction of x-axis.

Let the origins O and O′ of two frames S and S′ coincide initially. Let the event happening at P be denoted by (x, y, z, t) in frame S and by $(x', y', z'$ and $t')$ in frames S′, if we count time from the instant when O and O′ momentarily coincide, then after a time t, the frame S′ is separated from frame S by a distance vt in the direction of x-axis as shown in figure 1. Then the observations of two observers O and O′ of the same event happening at P may be seen to be related by the following equations:

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \quad \ldots(1)$$

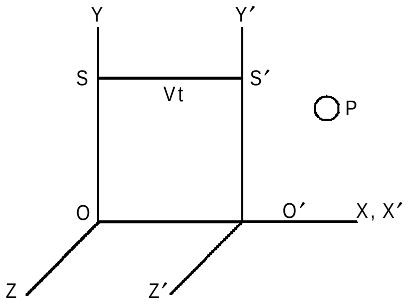


Fig. 1

These equations are called Galilean transformation equations and relate to observations of position and time made by two sets of observers, located in two different inertial frames.

Case II: When the second frame is moving along a straight line relative to first along any direction as shown in Fig. 2.

Let the second frame S′ be moving relative to first frame S with a velocity v such that

$$\vec{v} = \hat{i}v_x + \hat{j}v_y + \hat{k}v_z$$

Where v_x, v_y, and v_z are the components of v along x, y and z axes respectively.

Let (x, y, z, t) and $(x', y', z', t)'$ be the coordinates of an event happening at P at any instant as observed by the two observers situated at O and O′ of frames S and S′ respectively. If the origins of two systems coincide initially, then after a time t the frame S′ is separated from frame S by a distance v_xt, v_yt and v_zt along x, y and z axes respectively. Then referring to figure 2, we have

$$\left.\begin{aligned} x' &= x - v_xt \\ y' &= y - v_yt \\ z' &= z - v_zt \\ t' &= t \end{aligned}\right\} \quad \ldots(2)$$

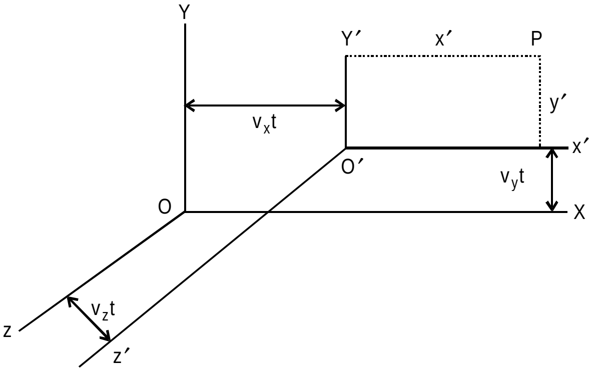


Fig. 2

These are the Galilean transformations, relating to the observations of position and time made by two observers in two different inertial frames.

Case III: When the second frame has uniform angular velocity relative to first. Let us consider two frames S and S′, the latter moving with uniform angular velocity ω relative to S about z-axis. Let the origins and axes of two frames coincide initially, *i.e.*, at $t = t' = 0$. In Fig. 3, z-axis is taken perpendicular to the plane of the paper. After time t, x' and y' axes are rotated by an angle ωt relative to x and y-axes respectively as shown in Fig. 3.

The observations (x', y', z', t') taken by observer in S′ are related to those (x, y, z, t) taken by observer in S of the same event of P by the equations.

x' = Component of x along x' + component of y along x' + component of z along x'

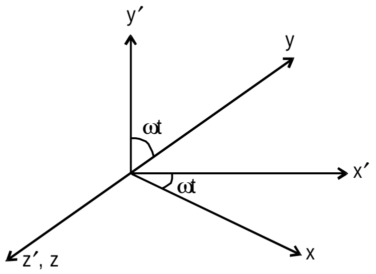


Fig. 3

$$= x \cos \omega t + y \sin \omega t + z \cos 90°$$

or $$x' = x \cos \omega t + y \sin \omega t \quad ...(3)$$

also y' = Component of x along y' + component of y along y' + component of z along y'

$$= x \cos (90° + \omega t) + y \cos \omega t + z \cos 90°$$

or $$y' = -x \sin \omega t + y \cos \omega t \quad ...(4)$$

and z' = Component of x along z' + component of y along z' + component of z along z'

$$= x \cos 90° + y \cos 90° + z \cos 0°$$

or $$z' = z \quad ...(5)$$

and $$t' = t \qquad ...(6)$$

Equations (3), (4), (5) and (6) are called time dependent Galilean transformations since they are time dependent and were obtained by Galileo.

Galilean Transformation of the Velocity of a Particle: Let us consider two inertial frames S and S′, the frame S′ moving with velocity v relative to S, which is given by

$$\vec{v} = \hat{i}v_x + \hat{j}v_y + \hat{k}v_z$$

Let r and r' be the position vectors of any particle at time t as observed by observer in frame S and S′ respectively. Then from Galilean transformations, we have

$$r' = r - vt$$

and $$t' = t \qquad ...(1)$$

Differentiating eq. (1), keeping v constant, we get

$$dr' = dr - vdt$$

and $$dt' = dt \qquad ...(2)$$

$$\therefore \qquad \frac{dr'}{dt'} = \frac{dr}{dt'} - v\frac{dt}{dt'} = \frac{dr}{dt} - v \qquad ...(3)$$

because since $\frac{dr'}{dt'} = u'$ = velocity of a particle relative to frame S and $\frac{dr}{dt} = u$ = velocity of particle relative to frame S′

Hence, Eq. (3) gives

$$u' = u - v \qquad ...(4)$$

Which shows that the velocities measured by the observers in the two frames of reference are not the same. Alternatively, we can say that velocity of a body is not invariant under Galilean transformations.

The inverse transformation from S′ to S, is obviously given by

$$u = u' + v \qquad ...(5)$$

Relation (4) or (5) are known as Galilean law of addition of velocities.

Invariance of Newton's Second Law

Newton's second law of motion, which also includes the first law, is the real law of motion. To test it let a force $\vec{F}$ be acting on a mass 'm' in frame S, then since force is the rate of change of momentum, we have

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt'} = m\vec{a}'$$

Where the mass has been assumed to be independent of velocity and $\vec{a}$ is the acceleration produced in the mass

Similarly, in frame S′, if F′ be the force acting on m, then

$$\vec{F}' = \frac{d}{dt'}(m\vec{v}') = m\frac{d\vec{v}'}{dt'} = m\vec{a}'$$

where $\vec{a}'$, is the acceleration produced in mass m in frame S′. But since in inertial frames,

$$\vec{a} = \vec{a}'$$

We have $$\vec{F} = \vec{F}'$$

This shows that force and hence Newton's second law of motion is invariant under Galilean transformation.

1.8 SCALARS AND VECTORS

The physical quantities are of two types: Scalars and Vectors.

Scalar Quantities: The quantities which have only magnitude and no direction, are called 'Scalar Quantities', e.g., mass, distance, time, speed, volume, density, work, charge, electric current, potential, frequency etc.

A scalar quantity can be completely defined by a number and a unit. For example, the 'mass' of the truck is 200 Kg, the distance of my college is 5 km from my residence. In all these statements we have given complete information about the quantity. The summation, subtraction, multiplication and division of scalar quantities can be done by ordinary algebra.

Vector Quantities

There are certain physical quantities whose complete description not only requires their magnitude (*i.e.*, a numerical value with appropriate unit) but also their direction in space e.g., velocity of a train. The magnitude of velocity is represented by a number such as 100 Km/hour. This tells how fast the train is moving. But the description of velocity is complete only when we specify the direction of velocity also. We can represent the magnitude of velocity and the tip of the arrow represents its direction.

If a particle is subjected to two velocities simultaneously its resultant velocity is different from the two velocities and is obtained by using a special rule. Suppose a particle is moving inside a long tube with a speed of 6 m/sec and the tube itself is moving in the room at a speed of 8 m/sec along a direction perpendicular to its length. Figure 1 represents the position of the tube and the particle at initial instant and after a time interval of 1 sec. Geometrical analysis gives the result that particle has moved a distance of 10 m in a direction θ = 53° from the tube. Hence, the resultant velocity of the particle is 10 m/sec along this direction.

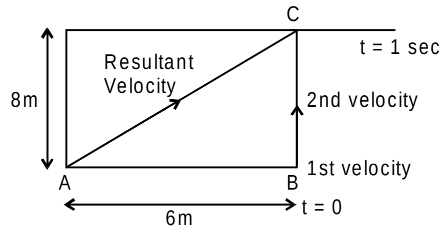


Fig. 4

In figure 4, line AB represents the first velocity with point B as the head. Then we draw another line BC representing the second velocity with its tail coinciding with the head of the first line. Thus the line AC with C as head and A as the tail represents the resultant velocity. The resultant may also be called as the sum of the two velocities. We have added two velocities AB and BC and have obtained the sum AC. This rule of addition is called as the triangle law of addition.

Thus, the physical quantities which have magnitude and direction and which can be added according to the triangle rule, are called Vector quantities.

Different Types of Vectors

1. **Like Vectors:** Two vectors are said to be like vectors if they have same direction, but different magnitude. Fig. 5 show two vectors $\vec{A}$ and $\vec{B}$ which have different magnitude but are paralled to each other.

Fig. 5

2. **Equal Vectors:** Two vectors are said to be equal, if they have the same magnitude and direction. Fig. 6 shows two vectors $\vec{A}$ and $\vec{B}$ having the same magnitude and same direction and therefore, $\vec{A} = \vec{B}$

Fig. 6

For two vectors to be equal, it does not matter, whether the two vectors have their tails at the same point or not. If the scales selected for both the vectors is the same, they are represented by two equal and parallel lines.

3. **Unlike Vectors:** The vectors having opposite direction and different magnitude, are called unlike vectors. Fig. 7 shows two such vectors $\vec{A}$ and $\vec{B}$ which have different magnitudes and are antiparallel to each other.

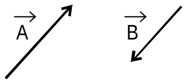


Fig. 7

4. **Opposite Vectors:** The vectors having same magnitude but opposite direction, are known as opposite vectors. Fig. 8 shows two such vectors $\vec{A}$ and $\vec{B}$ having the same magnitude and opposite direction and therefore,

$$\vec{A} = -\vec{B}$$

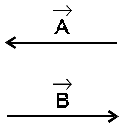


Fig. 8

5. **Unit Vectors:** A vector divided by its magnitude is called a unit vector along the direction of the vector. Obviously, the unit vector has unit magnitude and direction is the same as that of the given vector.

A unit vector in the direction of same vector $\vec{A}$ is written as $\hat{A}$ and is read as 'A cap' or 'A caret' or 'A hat'. Therefore, by definition or $$\hat{A} = \frac{\vec{A}}{A} \quad \text{or} \quad \vec{A} = \hat{A}A$$

Thus, any vector can be expressed as magnitude times the unit vector along its own direction, unit vectors along x, y and z axes are represented by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively.

6. **Co-initial Vectors:** Vectors are said to be co-initial, if they have a common initial point. In Fig. 9, $\vec{A}$ and $\vec{B}$ starting from the same point O as their origin are called Co-initial vectors.

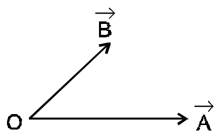
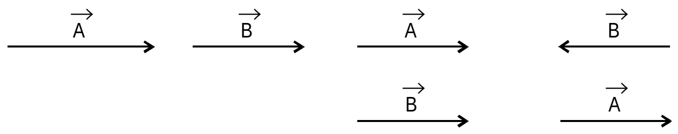


Fig. 9

$\vec{B}$ $\vec{A}$

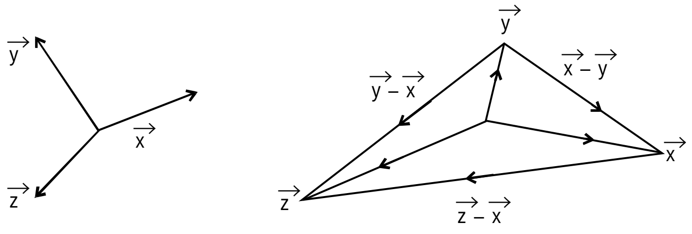
7. **Co-linear Vectors:** Two vectors having equal or unequal magnitude, which either act along the same line. Fig. 10(a) or along the parallel lines in the same direction Fig. 1-(b) or along the parallel lines in opposite direction. Fig. 10(c), are called co-linear vectors. Like, unlike, equal and opposite vectors are collinear.



(a) (b) (c)

Fig. 10

8. **Coplanar Vectors:** Vectors lying in the same plane are called coplanar vectors. Fig. 11(a) shows three vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$ along mutually $\perp$ axis x, y and z respectively. These vectors are non coplanar but the vectors $\vec{x}-\vec{y}$, $\vec{y}-\vec{z}$ and $\vec{z}-\vec{x}$ are coplanar Fig. 11(b).



(a) (b)

Fig. 11

9. **Null Vector:** It is defined as a vector having zero magnitude. It has a direction, which is indeterminate as its magnitude is zero.

1.9 ADDITION OF VECTORS

Since vectors have both magnitude and direction, they cannot be added by ordinary algebra. In Fig. 12 is shown the method of addition of two vectors $\vec{A}$ and $\vec{B}$. Fig. 12(a). For this, we first draw vector $\vec{A}$. Then starting from the arrow-head of $\vec{A}$ we draw the vector $\vec{B}$ Fig. 12(*b*). Finally, we draw a vector $\vec{R}$ starting from the initial point of $\vec{A}$ and ending at the arrow-head of $\vec{B}$. Vector $\vec{R}$ would be the sum of $\vec{A}$ and $\vec{B}$.

$$\vec{R} = \vec{A} + \vec{B}$$

The magnitude of $\vec{A}+\vec{B}$ can be determined by measuring the length of $\vec{R}$ and the direction can be expressed by measuring the angle between $\vec{R}$ and $\vec{A}$ (or $\vec{B}$).

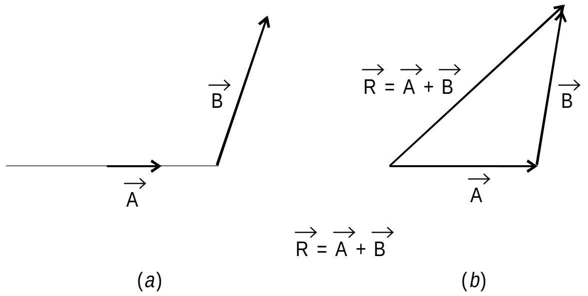


Fig. 12

Geometrical Method of Vector Addition

There are three laws of Vectors addition, namely triangle, parallelogram and polygon laws of vector addition. These laws can be used to add two or more vectors having an inclination with each other.

(***i***) ***Triangle law of Vector addition:*** According to this law if two vectors are represented both in magnitude and direction by the two sides of a triangle taken in the same order, then their resultant is represented totally (both in magnitude and direction) by the third side of the triangle taken in the opposite order.

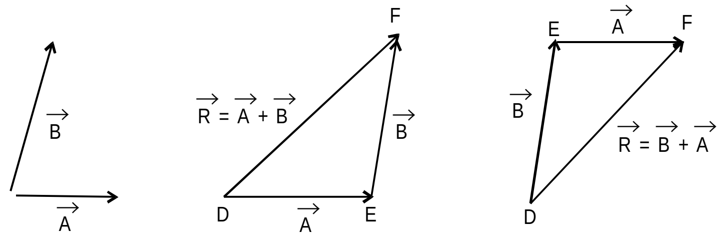


Fig. 13

Figure 13 shows two vectors $\vec{A}$ and $\vec{B}$. In order to find the sum of these vectors by using triangle law of vector addition, draw vector $\vec{DE} = \vec{A}$. Then move vector $\vec{B}$ parallel to itself until its tail coincides with the tip of vector $\vec{A}$. Show the arrow head of $\vec{B}$ by point F. Then vector DF = $\vec{R}$ drawn from the tail of vector $\vec{A}$ to the tip of vector $\vec{B}$ is the sum or resultant of vector $\vec{A}$ and $\vec{B}$. Thus

$$\vec{R} = \vec{A} + \vec{B}$$

(*ii*) ***Vector addition in Commutative:*** From Figure 13, it is clear that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. This property of the vector addition according to which the vector addition is independent of the order in which the vectors are added is called cumulative property of vector addition. A physical quantity having both magnitude and direction is not a vector, if it does not obey commutative law.

(*iii*) ***Parallelogram law of Vector addition:*** According to this law if two vectors acting simultaneously at a point can be represented both in magnitude and direction by two adjacent sides of a parallelogram, the resultant is represented completely (both in magnitude and direction) by the diagonal of the parallelogram passing through that point.

Suppose we have to find the resultant of two vector $\vec{A}$ and $\vec{B}$ as shown in Fig. 14. In order to find the resultant, draw the vector $\overrightarrow{DE} = \vec{A}$. Then move vector $\vec{B}$ parallel to itself, till its tail coincides with the tail of vector $\vec{A}$. If we represent it arrow head by point G, then vector $\overrightarrow{DG}$ represents vector $\vec{B}$. Then complete the parallelogram.

Now $\overrightarrow{EF} = \overrightarrow{DG} = \vec{B}$

and $\overrightarrow{GF} = \overrightarrow{DE} = \vec{A}$

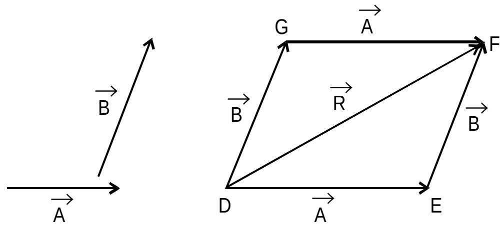


Fig. 14

It is clear from figure that diagonal $\overrightarrow{DF}$ of the parallelogram represents the resultant of sum of vectors $\vec{A}$ and $\vec{B}$.

(*iv*) ***Polygon Law of Vector Addition:*** This law helps us to obtain the resultant of more than two vectors and is just the extension of the triangle law of vector addition.

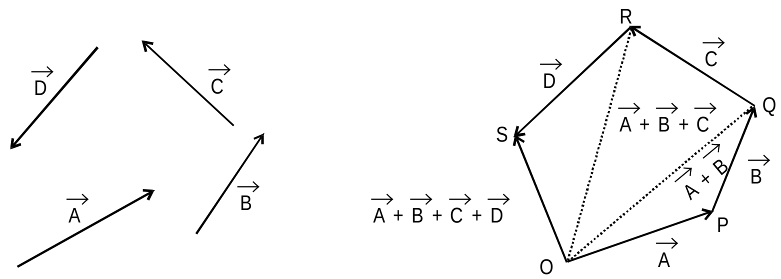


Fig. 15

According to this law if a number of vectors are represented in magnitude and direction by the sides of a polygon taken in same order, then their resultant is represented in magnitude and direction by the closing side of polygon taken in the opposite order. Figure 15 represents the addition of four vectors $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$ by this method.

Analytical Method of Vector Addition

(*i*) ***Triangle law of vector addition:*** Suppose two vectors $\vec{A}$ and $\vec{B}$ represent both in magnitude and direction the sides $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ of the triangle PQR taken in same order. Then according to triangle law of vector addition, the resultants $\vec{R}$ is represented by the closing side PR taken in the opposite order.

Magnitude of the resultant $\vec{R}$: Draw a perpendicular RS from the point R, on the side PQ which meets the line PQ at point S when produced forward. Then, from the ΔPSR, we get

$$(PR)^2 = (PS)^2 + (SR)^2 = (PQ + QS)^2 + (SR)^2$$

$$= (PQ)^2 + (QS)^2 + 2\ PQ.QS + (SR)^2$$

But $$(QS)^2 + (SR)^2 = (QR)^2$$

$\therefore$ $$(PR)^2 = (PQ)^2 + (QR)^2 + 2PQ.QS \qquad ...(i)$$

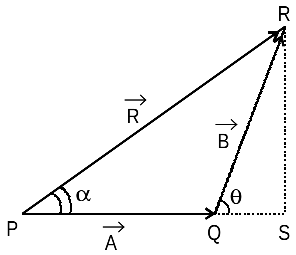


Fig. 16

From right angled triangle QRS, we get

$$\cos\ \theta = QS/QR$$

or $$QS = QR\ \cos/QR\ \cos = \theta$$

Therefore, eq. (*i*) becomes

$$(PR)^2 = (PQ)^2 + (QR)^2 + 2\ PQ.\ QR.\ \cos\ \theta$$

Now, $$PR = R,\ PQ = A,\ QR = B$$

$\therefore$ $$R^2 = A^2 + B^2 + 2\ AB\ \cos\ \theta$$

or $$R = \sqrt{[A^2 + B^2 + 2AB\cos\theta]}$$

Direction of the resultant $\vec{R}$: Suppose the resultant $\vec{R}$ makes an angle α with the direction of $\vec{A}$ then from right angled triangle PRS, we get from Fig. 16

$$\tan\ \alpha = \frac{RS}{PS} = \frac{RS}{PQ + QS}$$

Now $$PQ = A,\ QS = B\cos\ \theta,\ RS = B\sin\ \theta$$

Hence, $\tan\alpha = \dfrac{B\sin\theta}{A + B\cos\theta}$

(*ii*) ***Parallelogram Law of Vector Addition:*** Suppose two vectors $\vec{A}$ and $\vec{B}$ inclined to each other at an angle θ be represented in magnitude and direction both by the concurrent sides $\overrightarrow{PQ}$ and $\overrightarrow{PT}$ of the parallelogram. PQUT as shown in figure 17. Then according to parallelogram law, resultant of $\vec{A}$ and $\vec{B}$ is represented both in magnitude and direction by the diagonal $\overrightarrow{PU}$ of the parallelogram.

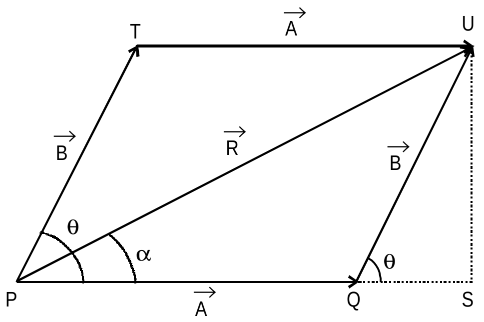


Fig. 17

Magnitude of the resultant $\vec{R}$: Drop a perpendicular from the point U on the line PQ which meets the line PQ at some point S.

From right handed triangle PSU, we get

$$PU^2 = PS^2 + SU^2$$
$$= (PQ + QS)^2 + SU^2$$
$$= (PQ)^2 + (QS)^2 + 2\ PQ.QS + (SU)^2$$

But $(QS)^2 + (SU)^2 = (QU)^2$

∴ $(PU)^2 = (PQ)^2 + (QU)^2 + 2\ PQ.QS$

Now $PU = R,\ PQ = A$ and $QU = B$

Further $QS = B\cos\theta$

Hence, $R^2 = A^2 + B^2 + 2\ AB\cos\theta$

or $R = \sqrt{[A^2 + B^2 + 2AB\cos\theta]}$

Direction of the resultant $\vec{R}$: Suppose the resultant vector $\vec{R}$ makes an angle α with vector $\vec{A}$.

Then from right angled triangle PUS, we get

$$\tan\alpha = \frac{US}{PS} = \frac{US}{PQ + QS} = \frac{B\sin\theta}{A + B\cos\theta}$$

1.10 RESOLUTION OF VECTORS

The process of splitting up a vector into two or more vectors is known as resolution of a vector. The vectors into which a given vector is split are called component vectors. The resolution of a vector into two mutually perpendicular vectors is called the rectangular resolution of vector in a plane or two dimensions.

Figure 18 shows a vector $\overrightarrow{OR} = \vec{r}$ in the X-Y plane drawn from the origin O. Let the vector makes an angle α with the *x*-axis and β with the *y*-axis. This vector is to be resolved into two component vectors along two mutually perpendicular unit vectors $\hat{i}$ and $\hat{j}$ respectively, where $\hat{i}$ and $\hat{j}$ are the unit vectors along *x*-axis and *y*-axis respectively as shown in figure 15. From point R, drop perpendiculars RP and RQ on *x* and *y*-axis respectively. The length OP is called the projection of $\overrightarrow{OR}$ on *x*-axis while length OQ is the projection of $\overrightarrow{OR}$ on *y*-axis. According to parallelogram law of vector addition

$$\vec{r} = \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{OQ}$$

Thus we have resolved the vector $\vec{r}$ into two parts, one along OX and the other along OY, the magnitude of the part along OX is OP = $r_x = r\cos\alpha$ and the magnitude of the part along OY is OQ = $r_y = r\cos\beta$ *i.e.*, in terms of unit vector $\hat{i}$ and $\hat{j}$, we can write

$$\overrightarrow{OP} = \hat{i}\,\vec{r}\cos\alpha = \hat{i}\,r_x$$

and

$$\overrightarrow{OQ} = \hat{j}\,\vec{r}\sin\alpha = \hat{j}\,r_y$$

$$= \hat{j}\,\vec{r}\cos\beta = \hat{j}\,r_y$$

Thus

$$\vec{r} = \hat{i}\,\vec{r}\cos\alpha + \hat{j}\,r\cos\beta$$

$$= \hat{i}\,r_x + \hat{j}\,r_y$$

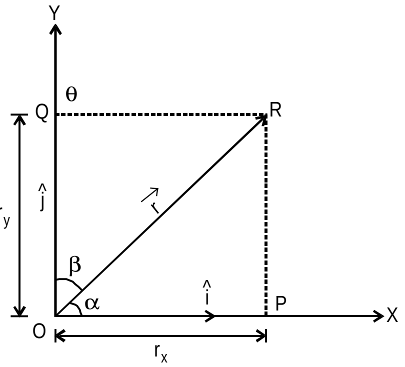


Fig. 18

If the vector $\vec{r}$ is not in the X-Y plane, it may have non zero projection along *x, y* and *z*-axes and we can resolve it into three components *i.e.*, along the *x, y* and *z*-axes. If α, β and γ be the angles made by the vector $\vec{r}$ with respect to *x, y* and *z*-axes respectively, then we can write

$$\vec{r} = \hat{i}\,r\cos\alpha + \hat{j}\,r\cos\beta + \hat{k}\,r\cos\gamma$$

$$\vec{r} = \hat{i}\,r_x + \hat{j}\,r_y + \hat{k}\,r_z$$

Where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along *x, y* and *z* axes respectively, the magnitude ($r\cos\alpha$) is called the component of $\vec{r}$ along *x*-axis, $r\cos\beta$ is called the component along *y*-axis and $r\cos\gamma$ is called the component along *z*-axis.

Above equation also shows that any vector in three-dimensional can be expressed as a linear combination of the three unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

1.11 MULTIPLICATION OF VECTORS

There are three kinds of multiplication operations for vectors

i. multiplication of a vector by a scalar,

ii. multiplication of two vectors in such a way as to yield a scalar, and

iii. multiplication of two vectors in such a way as to yield another vector.

There are still other possibilities, but we shall not consider them here.

The multiplication of a vector by a scalar has a simple meaning. The product of a scalar k and a vector $\vec{a}$, written $k\vec{a}$, is defined to be a new vector whose magnitude is k times the magnitude of $\vec{a}$. The new vector has the same direction as $\vec{a}$ if k is positive and the opposite direction if k is negative. To divide a vector by a scalar we simply multiply the vector by the reciprocal of the scalar.

Scalar Product (Dot Product): The scalar product of two vectors is defined as a scalar quantity having magnitude equal to the product of the magnitude of two vectors and the cosine of the smaller angle between them. Mathematically, if θ is the angle between vectors $\vec{A}$ and $\vec{B}$ then

$$\vec{A}.\vec{B} = AB\cos\theta$$

Above equation can also be expressed as below

$$\vec{A}.\vec{B} = A(B\cos\theta) = B(A\cos\theta)$$

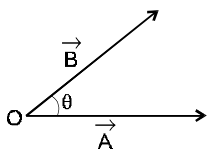


Fig. 19

where B cos θ is the magnitude of component of $\vec{B}$ along the direction of vector $\vec{A}$ and A cos θ is the magnitude of component of $\vec{A}$ along the direction of vector $\vec{B}$. Therefore, the dot product of two vectors can also be interpreted as the product of the magnitude of one vector and the magnitude of the component of other vector along the direction of first vector.

Dot product of two vectors can be positive or zero or negative depending upon θ is less than 90° or equal to 90° or $90° < \theta < 180°$.

Properties

1. Dot product of two vectors is always commutative *i.e.*,

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

$$\vec{A}.\vec{B} = AB\cos\theta$$

where θ is the angle between $\vec{A}$ and $\vec{B}$ measured in anti clock wise direction and $\vec{B}.\vec{A} = BA\cos(-\theta) = AB\cos\theta$

Thus $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

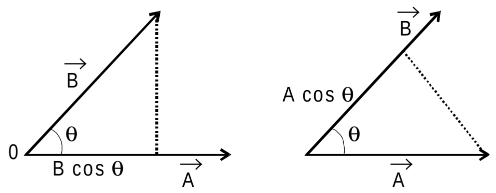


Fig. 20

2. The dot product of a vector with itself gives square of its magnitude *i.e.*,

$$\vec{A}.\vec{A} = A.A \cos 0° = A^2$$

3. The dot product of two mutually perpendicular vectors is zero *i.e.*, if two vectors $\vec{A}$ and $\vec{B}$ are perpendicular then

$$\vec{A}.\vec{B} = AB \cos 90° = 0$$

4. The dot product obeys the distributive law *i.e.*,

$$\vec{A}.(\vec{B} + \vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$$

5. Two vectors are collinear, if their dot product is numerically equal to product of their magnitudes *i.e.*,

When θ = 0° or 180°

$$\left|\vec{A}.\vec{B}\right| = AB$$

For example $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \vec{k}.\vec{k} = 1$ (as θ = 0°)

$$\hat{i}.\hat{j} = \hat{j}.\vec{k} = \vec{k}.\hat{i} = 0 \quad (\text{as } \theta = 90°)\cdot$$

6. Dot product of two vectors in term of their rectangular components in three dimensions.

$$\vec{A}.\vec{B} = (\hat{i}\ A_x + \hat{j}\ A_y + \hat{k}\ A_z).(\hat{i}\ B_x + \hat{j}\ B_y + \hat{k}\ B_z) = A_xB_x + A_yB_y + A_zB_z$$

Examples of some physical quantities which can be expressed as scalar product of two vectors:

(*a*) Work (W) is defined as the scalar product of force $(\vec{F})$ and the displacement $(\vec{S})$ *i.e.*,

$$W = \vec{F}.\vec{S}$$

(*b*) Power (P) is defined as the scalar product of force $(\vec{F})$ and the velocity $(\vec{v})$ *i.e.*,

$$P = \vec{F}.\vec{v}$$

(*c*) Magnetic flux (ϕ) linked with a surface is defined as the dot product of magnetic induction ($\vec{B}$) and area vector ($\vec{A}$) *i.e.*,

$$\phi = \vec{B}.\vec{A}$$

Vector Product (Cross Product): The vector product of two vectors is defined as a vector having magnitude equal to the product of the magnitudes of two vectors with the sine of angle between them and direction perpendicular to the plane containing the two vectors in accordance with right handed screw rule or right hand thumb rule.

If θ is the angle between vectors $\vec{A}$ and $\vec{B}$, then

$$\vec{A} \times \vec{B} = AB \sin \theta \ \hat{n}$$

The direction of vector $\vec{A} \times \vec{B}$ is the same as that of unit vector $\hat{n}$. It is decided by any of the following two rules:

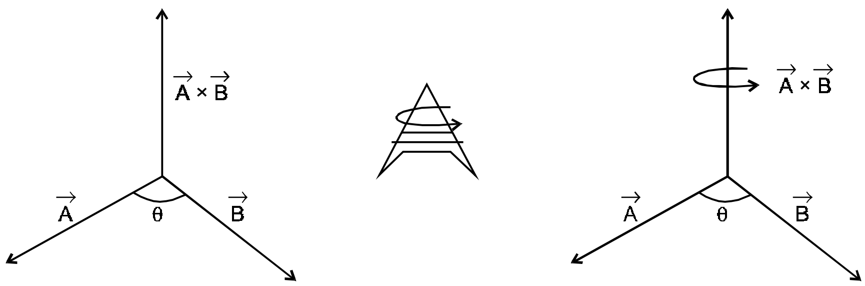


Fig. 21

Right handed screw rule: Rotate a right handed screw from vector $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the direction of motion of screw gives the direction of vector $\vec{A} \times \vec{B}$.

Right hand thumb rule: Bend the finger of the right hand in such a way that they point in the direction of rotation from vector $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the thumb points in the direction of vector $\vec{A} \times \vec{B}$.

The cross product of two vectors in term of their rectangular components is

$$\begin{aligned}\vec{A} \times \vec{B} &= (\hat{i}\, A_x + \hat{j}\, A_y + \hat{k}\, A_z) \times (\hat{i}\, B_x + \hat{j}\, B_y + \hat{k}\, B_z)\\ &= (A_y B_z - A_z B_y)\ \hat{i} + (A_z B_x - A_x B_z)\ \hat{j}\\ &\quad + (A_x B_y - A_y B_x)\ \hat{k}\\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}\end{aligned}$$

Properties of Vector Product

1. The cross product of the two vectors does not obey commutative law.

$$\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$$

i.e., $$\vec{A} \times \vec{B} \neq (\vec{B} \times \vec{A})$$

2. The cross product follows the distribution law *i.e.*

$$\vec{A} \times (\vec{B} - \vec{C}) = \vec{A} \times \vec{B} - \vec{A} \times \vec{C}$$

3. The cross product of a vector with itself is a NULL vector *i.e.*,

$$\vec{A} \times \vec{A} = (A) \,.\, (A) \sin 0° \; \hat{n} = 0$$

4. The cross product of two vectors represents the area of the parallelogram formed by them.

From figure 22, show a parallelogram PQRS whose adjacent sides PQ and PS are represented by vectors $\vec{A}$ and $\vec{B}$ respectively

Now, area of parallelogram $= QP \times SM$

$= AB \sin \theta$

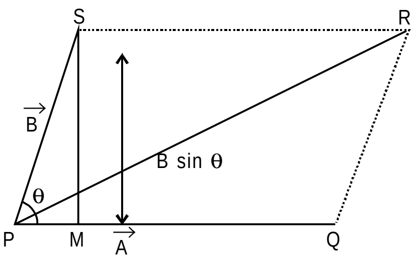


Fig. 22

Because, the magnitude of vector $\vec{A} \times \vec{B}$ is AB sin θ, hence cross product of two vectors represents the area of parallelogram formed by it. It is worth noting that area vectors $\vec{A} \times \vec{B}$ acts along the perpendicular to the plane of two vectors $\vec{A}$ and $\vec{B}$.

5. The Cross product of unit vectors are

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = (1)\,(1) \sin 0° \; \hat{n} = 0$$

$$\hat{i} \times \hat{j} = (1)\,(1) \sin 90° \; \hat{k} = \hat{k}$$

Where $\hat{k}$ is a unit vector perpendicular to the plane of $\hat{i}$ and $\hat{j}$ in a direction in which a right hand screw will advance, when rotated from $\hat{i}$ to $\hat{j}$.

Also $$-\hat{j} \times \hat{i} = (1)\,(1) \sin 90° \; (-\hat{k}) = \hat{k}$$

Similarly, $$\hat{j} \times \hat{k} = -\hat{k} \times \hat{j} = \hat{i}$$

and $$\hat{k} \times \hat{i} = -\hat{i} \times \hat{k} = \hat{j}$$

Examples of some physical quantities which can be expressed as cross product of two vectors:

(*a*) The instantaneous velocity $(\vec{v})$ of a particle is equal to the cross product of its angular velocity $(\vec{\omega})$ and the position vector $(\vec{r})$ *i.e.*,

$$\vec{v} = \vec{\omega} \times \vec{r}$$

(*b*) The tangential acceleration $(\vec{a_t})$ of a particle is equal to cross product of its angular acceleration $(\vec{\alpha})$ and the position vector $(\vec{r})$ *i.e.*,

$$\vec{a_t} = \vec{\alpha} \times \vec{r}$$

(*c*) The centripetal acceleration $(\vec{a_c})$ of a particle is equal to the cross product of its angular velocity and the linear velocity $(\vec{v})$ *i.e.*,

$$\vec{a_c} = \vec{\omega} \times \vec{v}$$

(*d*) The force $\vec{F}$ on a charge q moving inside magnetic field is equal to charge times the cross product of its velocity $(\vec{v})$ and magnetic induction $(\vec{B})$ *i.e.*,

$$\vec{F} = q(\vec{v} \times \vec{B})$$

(*e*) The torque $(\vec{\tau})$ of a force $(\vec{F})$ is equal to cross product of the position vector $\vec{r}$ and the force $(\vec{F})$ applied *i.e.*,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

(*f*) The angular momentum $(\vec{L})$ is equal to cross product of position vector $(\vec{r})$ and linear momentum $(\vec{P})$ of the particle *i.e.*,

$$\vec{L} = \vec{r} \times \vec{P}$$

1.12 VECTORS AND THE LAWS OF PHYSICS

Vectors turn out to be very useful in physics. It will be helpful to look a little , more deeply into why this is true. Suppose that we have three vectors $\vec{a}, \vec{b}$ and $\vec{r}$ which have components a_x, a_y, a_z: b_x, b_y, b_z and r_x, r_y, r_z respectively in a particular coordinate system *xyz* of our reference frame. Let us suppose further that the three vectors are related so that

$$\vec{r} = \vec{a} + \vec{b} \qquad ...(1)$$

By a simple extension of above equation

$$r_x = a_x + b_x$$
$$r_y = a_y + b_y \quad ...(2)$$
and
$$r_z = a_z + b_z$$

Now consider another coordinate system $x'\ y'\ z'$ which has these properties

1. Its origin does not coincide with the origin of the first, or *xyz*, system and
2. Its three axes are not parallel to the corresponding axes in the first system.

In other words, the second set of coordinates has been both translated and rotated with respect to the first.

The components of the vectors $\vec{a}$, $\vec{b}$ and $\vec{r}$ in the new system would all prove, in general, to be different, we may represent them by $a_{x'}$, $a_{y'}$, $a_{z'}$, $b_{x'}$, $b_{y'}$, $b_{z'}$ and $r_{x'}$, $r_{y'}$, $r_{z'}$ respectively. These new components would be found, however, to be related in that.

$$r_{x'} = a_{x'} + b_{x'}$$
$$r_{y'} = a_{y'} + b_{y'} \text{ and} \quad ...(3)$$
$$r_{z'} = a_{z'} + b_{z'}$$

That is, in the new system we would find once again that

$$\vec{r} = \vec{a} + \vec{b}$$

In more formal language: relations among vectors, of which Eq.(1) is only one example, are invariant (that is, are unchanged) with respect to translation or rotation of the coordinates. Now it is a fact of experience that the experiments on which the laws of physics are based and indeed the laws of physics themselves are similarly unchanged in form when we rotate or translate the reference system. Thus the language of vectors is an ideal one in which to express physical laws. If we can express a law in vector form the invariance of the law for translation and rotation of the coordinate system is assured by this purely geometrical property of vectors.

1.13 SPEED AND VELOCITY

The speed of a moving object is the rate at which it covers distance. It is important to distinguish between average speed and instantaneous speed. The average speed v of something that travels the distance S in the time interval t is

$$v = \frac{s}{t}$$

Average speed = distance traveled/time interval

Thus a car that has gone 150 km in 5 h had a average speed of

$$v = \frac{s}{t} = (150/5) \text{ km/h} = 30 \text{ km/h}$$

The average speed of the car is only part of the story of its journey, however, because knowing v does not tell us whether the car had the same speed for the entire 5 h or sometimes went faster than 30 km/h and sometimes slower.

Even though the car's speed is changing, at every moment it has a certain definite value (which is what is indicated by its speedometer). To find this instantaneous speed v at a particular time t, we draw a straight-line tangent to the distance- time curve at that value of t. The length of the line does not matter. Then we determine v from the tangent line from the formula

$$v = \Delta s/\Delta t$$

Where Δs is the distance interval between the ends of the tangent and Δt is the time interval between them. (Δ is the Greek capital letter delta). The instantaneous speed of the car at $t = 40$ s is from figure

Total distance, (m)	0	100	200	300	400	500
Elapsed time, (S)	0	28	40	49	57	63

$$V = \Delta s/\Delta t = 100 \text{ m}/10 \text{ s} = 10 \text{ m/s}$$

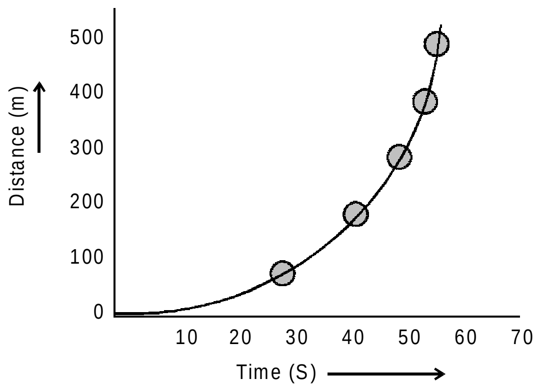


Fig. 23

When the instantaneous speed of an object does not change, it is moving at constant speed.

The speed of a moving object is a scalar quantity. Its unit is meter/second and its dimension are (LT^{-1}).

Velocity

The object's velocity, however, includes the direction in which it is moving and is a vector quantity. If the object undergoes a displacement $\vec{s}$ in the time interval, its average velocity $\vec{v}$ during this interval is

$$\vec{v} = s/t$$

Instantaneous velocity v is the value of $\Delta s/\Delta t$ at a particular moment, and for straight-line motion is found by the same procedure as that used for instantaneous speed v but with the direction specified as well.

1.14 ACCELERATION

If the velocity of a moving object is changing, then its motion is called 'accelerated motion'. The change in velocity may be in magnitude (speed), or in direction or in both. If the object is moving along a straight line, then only the magnitude of velocity (speed) changes.

The time-rate of change of velocity of an object is called the 'acceleration' of that object. That is

Acceleration = Change in velocity/time interval

Acceleration is generally represented by '$\vec{a}$'. Since velocity is a vector quantity, the acceleration is also a vector quantity.

Suppose, the velocity of a moving object is v_1 at time t_1 and becomes v_2 at time t_2. It means that in the time-interval $(t_2 - t_1)$, the change in the velocity of the object is $(v_2 - v_1)$. Hence, the average acceleration of the object in time-interval $t_2 - t_1$ is

$$\vec{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

If the time interval Δt is infinitesimally small ($\Delta t \to 0$) then the above formula gives acceleration 'at a particular time'. This is called 'instantaneous acceleration' and is given by

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

The unit of acceleration is meter /second^2 and its dimensional formula is (LT^{-2}).

If the velocity of an object undergoes equal changes in equal interval of time, then its acceleration is said to be 'uniform'. If the magnitude of the velocity (speed) of an object is increasing with time, then the acceleration of the object is positive. If the magnitude of the velocity (speed) is decreasing, then the acceleration is negative and it is called 'retardation'.

1.15 RECTILINEAR MOTION

When a particle moves along a straight line, its motion is said to be 'rectilinear'. In this type of motion the acceleration of the particle is either zero (velocity constant) or arises from a change in the magnitude of the velocity.

Let us consider a particle in rectilinear motion, say along the x-axis, under a constant acceleration a. We can derive relations between the kinematical variable x, v, a and t.

Suppose the particle has velocity v_0 at the origin ($x = 0$) at time $t = 0$, and moving with constant acceleration a, acquires a velocity v at time t. From the definition of acceleration, we have

$$a = \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t - 0}$$

or $$v = v_0 + at \qquad ...(1)$$

Since the acceleration is constant, the average velocity v_{av} in time-interval t equal to one half the sum of the velocities at the beginning and the end of the interval.

Thus $$v_{av} = \frac{v_0 + v}{2}$$

Hence, the displacement at time t is

$$x = v_{av} \times t = \frac{v_0 + v}{2} t$$

Substituting for v from Eq.(1), we get

$$x = \frac{v_0 + (v_0 + at)}{2} t$$

$$x = v_o t + ½ at^2 \quad ...(2)$$

Eliminating t between (1) and (2), we can get

$$v^2 = v^2_0 + 2ax \quad ... (3)$$

Eqs. (1), (2) and (3) are equations of rectilinear motion under constant acceleration.

1.16 ACCELERATION OF GRAVITY

When a body is allowed to fall over earth's surface from a certain point above it, it is seen that

(*a*) its velocity increases continuously and the acceleration is found to be uniform and

(*b*) that the acceleration is the same for all bodies.

This uniform acceleration of all the bodies when falling down under the earths action of gravity is called acceleration due to gravity. It is represented by '*g*'.

If the resistance offered by air to the motion of body is assumed negligible, then all the bodies falling freely under gravity have the same acceleration '*g*' acting vertically downwards.

The value of g varies from a value 9.781 ms^{-2} at the equator to the value 9.831 ms^{-2} at the poles.

The equation derived for rectilinear motion along the x-axis can be used for freely falling bodies as well. Let us take y-axis as vertical, here the constant acceleration is g directed vertically downwards. Thus after replacing x by y and a by g in the equation of motion, we obtain equations for vertically downward motion:

$$v = v_o + gt$$

$$y = v_o t + ½ gt^2$$

$$v^2 = v_o^2 + 2gy$$

In case of bodies moving vertically upward, the acceleration due to gravity $\vec{g}$ acts in the opposite direction *i.e.*, as retardation. Hence the equations of motion would be

$$v = v_o - gt$$

$$y = v_o t - ½ gt^2$$

$$v^2 = v_o^2 - 2gy$$

1.17 ACCURACY AND ERRORS IN MEASUREMENT

In many experiments in the physics laboratory, the aim is to determine the value of physical constant. To determine a physical constant, we have to measure the various quantities which are connected with that physical constant by a formula. For example, to determine the density (ρ) of a metal block, we have to measure its mass (*m*) and its volume (V) which are related to ρ by the formula:

$$\rho = m/V$$

The accuracy in the value of ρ obviously depends upon the accuracy in the measurements of *m* and V. Measurements of the quantities in the formula involve errors which are of two types: (*a*) Random errors and (*b*) Systematic errors

Random Errors

Random errors may be due to (*i*) small changes in the conditions of a measurements and (*ii*) the incorrect judgment of the observer in making a measurement. For example, suppose you are determining the weight of a body with the help of a spring balance. You will usually make an error in estimating the coincidence of the pointer with the scale reading or in assessing the correct position of the pointer when it lies between two consecutive graduations of the scale. This error, which is due to incorrect judgment of the observer, is also an example of random errors.

Random errors cannot be traced to any systematic or constant cause of error. They do not obey any well-defined law of action. Their character can be understood or appreciated from the illustration of firing shots at a target using a rifle. The target is usually a bull's eye with concentric rings round it. The result of firing a large number of shots at the target is well known. The target will be marked by a well-grouped arrangement of shots. A large number of shots will be nearer a certain point and other shots will be grouped around it on all sides. These shots which are grouped on both sides of the correct point obey the law of probability which means that large random errors are less probable to occur than small ones. A study of the target will show that the random shots lie with as many to one side of the centre as to the other. They will also show that small derivations from the center are more numerous than large deviation and that a large deviation is very rare.

Method of Minimizing Random Errors

If we make a large number of measurements of the same quantity then it is very likely that the majority of these measurements will have small errors which might be positive or negative. The error will be positive or negative depending on whether the observed measurement is above or below the correct value. Thus random errors can be minimized by taking the arithmetic mean of a large number of measurements of the same quantity. This arithmetic mean will be very close to the correct result. If one or two measurements differ widely from the rest, they should be rejected while finding the mean.

Systematic Errors

During the course of some measurements, certain sources of error operate constantly or systematically making the measurement. Systematically greater or smaller than the correct reading. These errors, whose cause can be traced, are called systematic errors. All instrumental errors belong to this category, such as the zero error in vernier callipers and micrometer screw, the index error in an optical bench, the end error in a meter bridge, faulty graduations of a measuring scale, etc.

Elimination of Systematic Errors

To eliminate systematic errors, different methods are used in different cases.

1. In some cases, the errors are determined previously and the measurements corrected accordingly. For example, the zero error in an instrument is determined before a measurement is made and each measurement is corrected accordingly.
2. In some cases the error is allowed to occur and finally eliminated with the data obtained from the measurements. The heat loss due to radiation is taken into account and corrected for from the record of the temperature at different times.

Order of Accuracy

Even after random and systematic errors are minimized, the measurement has a certain order of accuracy which is determined by the least count of the measuring instruments used in that measurement.

Suppose we determine the value of a physical quantity u by measuring three quantities x, y and z whose true values are related to u by the equation.

$$u = x^{\alpha} y^{\beta} z^{-\gamma} \qquad ...(a)$$

Let the expected small errors in the measurement of quantities x, y and z be respectively $\pm\delta x, \pm\delta y$ and $\pm\delta z$ so that the error in u by using these observed quantities is $\pm\delta u$. The numerical values of δx, δy and δz are given by the least count of the instruments used to measure them.

Taking logarithm of both sides of Eq. (a), we have

$$\log u = \alpha \log x + \beta \log y - \gamma \log z$$

Partial differentiation of the above equation gives

$$\frac{\delta u}{u} = \alpha\frac{\delta x}{x} + \beta\frac{\delta y}{y} - \gamma\frac{\delta z}{z} \qquad ...(b)$$

The proportional or relative error in u is $\frac{\delta u}{u}$. The values of δx, δy and δz may be positive or negative and in some case the terms on the right hand side of Eq. (*b*) may counteract each other. This effect cannot be relied upon and it is necessary to consider the worst case which is the case when all errors add up giving an error δu given by the equation:

$$\left(\frac{\delta u}{u}\right)_{\text{maximum}} = \alpha\frac{\delta x}{x} + \beta\frac{\delta y}{y} + \gamma\frac{\delta z}{z} \qquad ...(c)$$

Thus to find the maximum proportional error in u, multiply the proportional errors in each factor (x, y and z) by the numerical value of the power to which each factor is raised and then add all the terms so obtained.

The sum thus obtained will give the maximum proportional error in the result of u. When the proportional error of a quantity is multiplied by 100, we get the percentage error of that quantity. It is evident from Eq. (c) that a small error in the measurement of the quantity having the highest power will contribute maximum percentage error in the value of u. Hence, the quantity having the highest power should be measured with a great precision as possible. This is illustrated in the following example.

Example: *In an experiment for determining the density* (ρ) *of a rectangular block of a metal, the dimension of the block are measured with callipers having a least count of 0.01 cm and its mass is measured with beam balance of least count 0.1 g. The measured values are: mass of the block (m) = 39.3 gm, length of block (x) = 5.12 cm, Breadth of block (y) = 2.56 cm, Thickness of block (z) = 0.37 cm.*

$$\text{Error in } m = \delta m = \pm 0.1 \text{ g}$$

$$\text{Error in } x = \delta x = \pm 0.01 \text{ cm}$$

$$\text{Error in } y = \delta y = \pm 0.01 \text{ cm}$$

$$\text{Error in } z = \delta z = \pm 0.01 \text{ cm}$$

Find the maximum proportional error in the determination of ρ.

Ans: The density of the block is given by

$$\rho = m/xyz$$

Calculating ρ omitting errors, $\rho = \dfrac{39.3}{5.12 \times 2.56 \times 0.37} = 8.1037 \text{ g cm}^{-3}$

Proportional error in ρ is given by the equation

$$\frac{\delta\rho}{\rho} = \frac{\delta m}{m} - \frac{\delta x}{x} - \frac{\delta y}{y} - \frac{\delta z}{z}$$

The maximum proportional error in ρ is given by the equation

$$\left(\frac{\delta\rho}{\rho}\right)_{\max} = \frac{\delta m}{m} + \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

$$= \frac{0.1}{39.3} + \frac{0.01}{5.12} + \frac{0.01}{2.56} + \frac{0.01}{0.37}$$

$$= 0.0025 + 0.0019 + 0.0039 + 0.0270$$

$$= 0.0353$$

$\therefore$ Maximum percentage error $= 0.0353 \times 100 = 3.53\%$

Hence the error is 3.53% of 8.1037. Clearly it is absurd to give the result for ρ to five significant figures. The error in ρ is given by

$$\delta\rho = 0.0353 \times \rho = 0.0353 \times 8.1037 = 0.286 \cong 0.3$$

Hence the value of $\rho = 8.1037$ is not accurate up to the fourth decimal place. In fact, it is accurate only up to the first decimal place. Hence the value of ρ must be rounded off as 8.1 and the result of measurements is written as

$$\rho = (8.1 \pm 0.3) \text{ g cm}^{-3}$$

It is clear that such a large error in the measurement of ρ is due to a large error (=0.027) in the measurement of z, the smallest of the quantities measured. Hence the order of accuracy of ρ should be increased by measuring z with an instrument having a least count which is smaller than 0.01 cm. Thus a micrometer screw (least count = 0.001 cm), rather than a vernier caliper should be used to measure z.

SUMMARY

Physics is a branch of science that deals with the study of the phenomena of nature. The scientific method used in the study of science involves observation, proposal of a theory, testing the consequences of the proposed theory and modification or refinement of the theory in the light of new facts. The applications of physics have played a very great role in technology and in our daily lives.

Physics is a science of measurement. All quantities which can be measured either indirectly or directly such as length, mass, time, force, temperature, light intensity, electric current etc. are called physical quantities of numerous such quantities, length, mass and time are regarded as fundamental quantities. The measurement of these quantities involves the choice of a unit. The internationally accepted units of length, mass and time respectively are metre, kilogram and second. The units of all other mechanical quantities are derived from these three basic units. These quantities are measured by direct and indirect methods. The measured values of these quantities, show a very wide range variation.

Different physical quantities have different dimensions. The dimensions of a physical quantity are the number of times the fundamental units of mass, length and time appear in that quantity.

2

FORCE AND MOTION

2.1 MECHANICS

Mechanics, the oldest of the physical sciences, is the study of the motion of objects. The calculation of the path of an artillery shell or of a space probe sent from Earth to mars are among its problems.

When we describe motion (or trajectories) we are dealing with that part of mechanics called kinematics, ignoring the forces producing the motion. When we relate motion to the forces associated with it and to the properties of the moving objects, we are dealing with dynamics.

2.2 CAUSE OF MOTION: FORCE

From daily experience we know that the motion of a body is a direct result of its interactions with the other bodies around it (which form its environment). When a batsman hits a ball, he interacts with the ball and modifies its motion. The motion of a freely falling body or a projectile is the result of its interaction with the earth. The motion of an electron around a nucleus is the result of its interaction with the nucleus.

An interaction is quantitatively expressed in terms of a concept called 'force'. An intuitive motion of force is derived in terms of a push or a pull. When we push or pull on a body, we are said to exert a (muscle) force on it. Earth which pulls all bodies towards its centre is said to exert a (gravitational) force on them. A stretched spring pulling a body attached to its end is said to exert a (elastic) force on the body. A locomotive exerts a force on the train it is pulling or pushing. Thus every force exerted on a body is associated with some other body in the environment.

It is not always that an application of force will result in motion or change in motion. For example, we may push a wall, i.e., there is an interaction between us and the wall, and hence there is a force, but the wall may not move at all.

Thus, force may be described as a push or pull, resulting from the interaction between bodies, which produces or tends to produce motion or change in motion.

The analysis of the relation between force and the motion of a body is based on three laws of motion which were first stated by Sir Issac Newton.

2.3 NEWTON'S LAWS OF MOTION

Newton's laws of motion are the basis of mechanics. Galileo's version of inertia was formalized by Newton in a form that has come to be known as Newton's first law of motion. Newton's first law of motion: stated in Newton's words, the first law of motion is:

"Every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it".

Newton's first law is also known as law of inertia and the motion of a body not subject to the action of other forces is said to be inertial motion. With the help of this law we can define force as an external cause which changes or tends to change the state of rest or of uniform motion of a body.

Newton's first law is really a statement about reference frames. We know that the motion of a body can be described only relative to some other body. Its motion relative to one body may be very different from that relative to another. A passenger in an aircraft which is on its take-off run is at rest relative to the aircraft but is in accelerated motion relative to the earth. Therefore we always choose a set of coordinate axes attached to a specified body, relative to which the motion of a given body is described. Such a set of coordinate axes is known as a "reference frame".

The first law tells us that we can find a reference frame relative to which a body remains at rest or in uniform motion along a straight line (*i.e.*, it has no acceleration) when no net external force acts upon it. Such a reference frame is called an 'inertial frame'. Thus an inertial frame is one in which Newton's first law correctly describes the motion of a body not acted on by a net force. Such frames are either fixed with respect to the distant stars or moving at uniform velocity with respect to them.

A reference frame attached to the earth can be considered to be an inertial frame for most practical purposes, although it is not precisely so, due to the axial and orbital motions of the earth. But if a frame of reference is inertial, then every other frame which is in uniform motion relative to it is also inertial.

Newton's first law makes no distinction between a body at rest and one moving with a constant velocity. Both states are "natural" when no net external force or interaction acts on the body. That this is so becomes clear when a body at rest in one inertial frame is observed from a second (inertial) frame moving with constant velocity relative to the first. An observer in the first frame finds the body to be at rest, an observer in the second frame finds the same body to be moving with uniform velocity. Both observers find the body with no acceleration. Thus both observe that Newton's first law is being obeyed.

Newton's Second Law of Motion

Newton's second law tells us what happens to the state of rest or of uniform motion of a body when a net external force acts on the body *i.e.*, when the body interacts with other surrounding bodies.

This law states: **"The change of motion of an object is proportional to the force impressed, and is made in the direction of the straight line in which the force is impressed".**

By "change of motion" Newton meant the rate of change of momentum (p) with time. So mathematically we have

$$\vec{F} \propto \frac{d}{dt}(\vec{p})$$

or $$\vec{F} = k\frac{d}{dt}(\vec{p}) \qquad ...(1)$$

Where $\vec{F}$ is the impressed force and k is a constant of proportionality. The differential operator d/dt indicates the rate of change with time. Now, if the mass of the body remains constant (*i.e.*, neither the body is gaining in mass like a conveyer belt nor it is disintegrating like a rocket), then

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$$

Where $\vec{a} = \frac{d\vec{v}}{dt} =$ the acceleration of the body. Thus Eq. (1) becomes

$$\vec{F} = km\vec{a}$$
$$F = km\,a$$

This law provide a quantitative definition of force.

Thus force is equal to mass times acceleration, if the mass is constant. The force has the same direction as the acceleration. This is an alternative statement of the second law.

We note that the Newton's first law is contained in the second law as a special case, if $\vec{F} = 0$, then $\vec{a} = 0$. In other words, if the net force on a body is zero, the acceleration of the body is zero. Therefore, in the absence of a net force a body will move with constant velocity or be at rest, which is the first law.

Newton's Third Law of Motion

A force acting on a body arises as a result of its ineraction with another body surrounding it. Thus any single force is only one aspect of a mutual interaction between two bodies. We find that whenever one body exerts a force on a second body, the second body always exerts on the first a force which is equal in magnitude but opposite in direction and has the same line of action. A single isolated force is therefore an impossibility.

The two forces involved in every interaction between the bodies are called an 'action' and a 'reaction'. Either force may be considered the 'action' and the other the 'reaction'.

This property of forces was stated by Newton in his third law of motion: **"To every action there is always an equal and opposite reaction".**

If a body A exerts a force F_{AB} on a body B, then the body B in turn exerts a force F_{BA} on A, such that

$$F_{AB} = -F_{BA}$$

So, we have $$F_{AB} + F_{BA} = 0$$

2.4 FREELY FALLING BODIES

It is a matter of common observation that bodies fall towards the ground. Leaves of the tree, fruits from the tree, a body just dropped from the top of a tower, a stone thrown upward etc. all reach the ground. According to Aristotle it was believed that the time taken by a heavier body to reach the ground as less as compared to a lighter body dropped from the same height. This belief continued until the sixteenth century and nobody either challenged it or tried to prove its correctness.

In 1590, Galileo disapproved this idea. He showed from his famous experiments at the leaning tower of Pisa that all bodies dropped from the same height reach the ground at the same instant irrespective of their masses.

Galileo showed that bodies of different masses when dropped from the top of the leaning tower of Pisa, reached the ground at the same time. A large crowd was present to see the correctness of Galileo's idea. When a stone and a piece of paper were dropped from the tower at the same time, it was found that the stone reached the ground earlier. This difference in time was explained by Galileo to be due to the resistance offered by air on the paper. He showed that if a paper and a stone are dropped from the same height in vacuum, both take the same time to fall through the same height.

Laws of Falling Bodies

1. All bodies fall with equal rapidity in vacuum irrespective (Equations of Motion in Free Fall) of their masses
2. The velocity acquired by a body falling freely from rest is directly proportional to the time of its fall

$$v \propto t$$

or

$$v = gt$$

3. The distance moved by a body falling freely from rest is directly proportional to the square of the time of fall.

$$S \propto t^2$$

$$S = \tfrac{1}{2}\, gt^2$$

2.5 MOTION IN A VERTICAL PLANE

Equations of motion in a vertical plane is given by

$$v = u - gt$$

$$S = ut - (gt^2)/2$$

$$v^2 = u^2 - 2\, gs$$

Components of Vectors

In mechanics and other branches of physics, we often need to find the component of a vector in a certain direction. The component is the 'effective part' of the vector in that direction. We can illustrate it by considering a picture held up by two strings OP and OQ each at an angle of 60° to the vertical. Figure 1(*a*). If the force or tension in OP is 6 N, its vertical component S acting upwards at P helps to support the weight W of the picture, which acts vertically downwards. The upward component T of the 6 N force in OQ acting in the direction QT, also helps to support W. Figure 1(*b*) shows how the component value of a vector F can be found in a direction OX. If OR represents F to scale, we draw a rectangle OPRQ which has OR as its diagonal. Now F is the sum of the vectors OP and OQ. The vector OQ has no effect in a direction OX at 90° to itself. So the effective part or component of F in the direction OX is the vector OP.

If θ is the angle between F and the direction OX, then

$$OP/OR = \cos\theta \text{ or } OP = OR\cos\theta = F\cos\theta$$

So the component of any vector F in a direction making an angle θ to F is always given by

$$= F\cos\theta$$

In a direction OY perpendicular to OX, F has a component F cos (90° – θ) which is F sin θ. This component is represented by OQ in the figure (*b*).

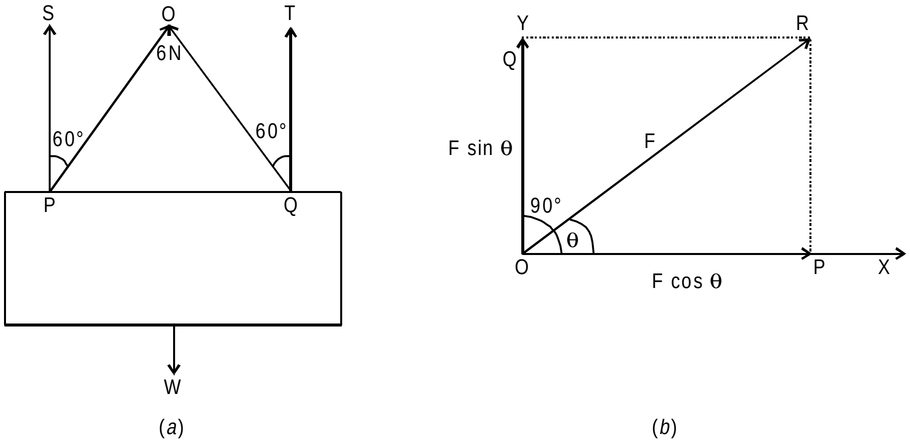


Fig. 1 Components of vectors.

2.6 PROJECTILE MOTION

A body projected in the horizontal direction with a uniform velocity is under the action of (*i*) uniform velocity in the horizontal direction and (*ii*) uniform acceleration due to gravity in the vertically downward direction. Such a body is known as projectile *e.g.*, a bomb released from an aeroplane moving with a uniform speed and a bullet fired from a rifle. The path traversed by the projectile is called its trajectory.

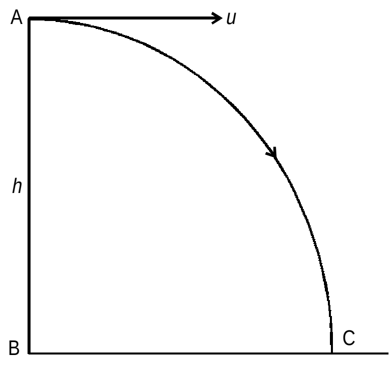


Fig. 2

Consider a tower AB of height *h*. A stone is thrown in the horizontal direction from the top of tower. As soon as the stone is released, it is under the action of (*i*) uniform velocity '*u*' in the horizontal direction and (*ii*) acceleration due to gravity '*g*' in the vertically downward direction. Under the combined effect of the two, the stone does not reach the foot of the tower but it falls at some distance away from the foot of the tower. Suppose the stone reaches the point C. The distance BC is the horizontal range.

Suppose the time taken by stone to reach the ground = *t*

$\therefore$ Vertical distance $h = \frac{1}{2} gt^2$ ...(1)

Since the initial velocity in the vertical direction is zero, the horizontal distance

$$BC = x = u.t$$

$$\therefore \quad t = x/u \quad ...(2)$$

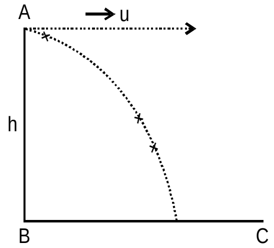


Fig. 3

From Eqs. (1) and (2)

$$h = \frac{1}{2} g (x/u)^2$$

or

$$x^2 = \frac{2u^2}{g} h$$

This is the equation of a parabola. Therefore the path traversed by the stone is a parabola.

Take the case of an aeroplane flying at a height h from the ground. It is to drop a bomb at C. If the bomb is released just at the moment when the aeroplane is vertically above c, the bomb will not fall at C but it will fall at a point some distance away from C. From the knowledge of the height h, time t can be calculated from the relation:

$$h = \frac{1}{2} gt^2, \text{ or } t = \sqrt{\frac{2h}{g}}$$

and

$$BC = ut$$

Therefore the bomb should be released at a point A which is vertically above B and t seconds earlier. Then after traversing a parabolic path the bomb will reach the point C. It is to be remembered that the aeroplane will also reach a point vertically above C just at the moment when the bomb reaches the point C. In calculating the distance BC air resistance is neglected.

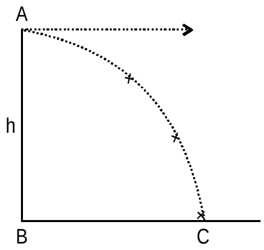


Fig. 4

Show that the horizontal range is maximum when the angle of projection is 45° with the horizontal.

Suppose a body is projected with a velocity u making an angle θ with the horizontal. Resolve this velocity u into two components.

(*i*) $u \cos \theta$ in the horizontal direction,

(*ii*) $u \sin \theta$ in the vertically upward direction.

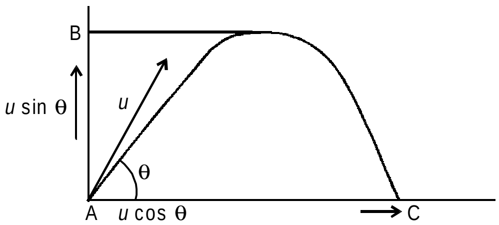


Fig. 5

The horizontal component remains constant throughout the flight of the body in air, whereas the vertical component of the velocity changes continuously. The acceleration due to gravity is negative.

Maximum height: Suppose the maximum vertical height reached is h.

From the equation, $v^2 - u^2 = 2gh$

$$v = 0,\ u = u \sin\theta \text{ and } g \text{ is negative}$$

$$\therefore \quad 0 - u^2 \sin^2\theta = -2gh$$

or
$$h = \frac{u^2 \sin^2\theta}{2g} \qquad ...(i)$$

After time t, the distance travelled in the horizontal direction,

$$x = (u \cos\theta)t \qquad ...(ii)$$

Distance travelled in the vertical direction

$$y = (u \sin\theta)\ t - \frac{1}{2}gt^2 \qquad ...(iii)$$

From equation (ii), $t = x/(u \cos\theta)$

$$y = \frac{u \sin\theta\ x}{u \cos\theta} - \frac{1}{2}g\left(\frac{x}{u \cos\theta}\right)^2$$

$$= x \tan\theta - \frac{g}{2}\left(\frac{x}{u\cos\theta}\right)^2 \qquad ...(iv)$$

This is the equation of a parabola

Time in air: Suppose the projectile remains for time t in air: It means $y = 0$

From equation (iii)

$$0 = (u \sin\theta)\ t - \frac{1}{2}gt^2$$

or
$$t = \frac{2u \sin\theta}{g} \qquad ...(v)$$

Maximum range: The horizontal range

$$x = (u \cos\theta)\ t$$

Putting value of t from Eq. (v), we get

$$x = (u \cos\theta).\frac{2u \sin\theta}{g}$$

$$= \frac{u^2 \sin 2\theta}{g} \qquad ...(vi)$$

The range x will be maximum

When $\sin 2\theta = 1$ or $2\theta = 90°$ *i.e.*, $\theta = 45°$

Hence the horizontal range is maximum when the projectile make an angle of 45° with the horizontal.

2.7 EQUILIBRIUM OF FORCES

When a particle is in equilibrium, then the resultant of all the forces acting on it is zero. It then follows from Newton's first law of motion that a particle in equilibrium is either at rest or is moving in a straight line with constant speed. It is found that for a large number of problems, we have to deal with equilibrium of forces lying in a plane. Therefore, we shall restrict our discussion to the case when a particle is in equilibrium under the influence of a number of coplanar forces $\vec{F}_1, \vec{F}_2, \vec{F}_3$ ... . The required condition is given by

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 .. = 0 \qquad ...(i)$$

Since the forces are coplanar, we can resolve them along two mutually perpendicular directions of x and y-axes, O being the particle. So above equation (i) can be rewritten as

$$\left(F_{1x}\hat{i} + F_{1y}\hat{j}\right) + \left(F_{2x}\hat{i} + F_{2y}\hat{j}\right) + \ldots = 0$$

or
$$\left(F_{1x} + F_{2x}\right)\hat{i} + \left(F_{1y} + F_{2y}\right)\hat{j} = 0$$

$$\left.\begin{aligned} \Sigma(F_{1x} + F_{2x} + \ldots) &= 0 \\ \Sigma(F_{1y} + F_{2y} + \ldots) &= 0 \end{aligned}\right\} \quad (ii)$$

and

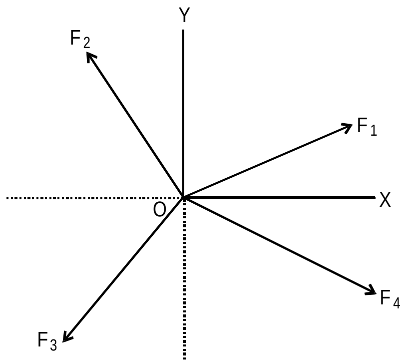


Fig. 6

Equation (ii) can be expressed in a concise form as

$$\Sigma F_x = 0,$$

and
$$\Sigma F_y = 0$$

Where Σ denotes summation of the x- or y-components of the forces.

2.8 FRICTIONAL FORCES

When one body moves in contact with another, its motion is opposed by a force which comes into play at the plane of contact of the two bodies. This resistive force which tends to destroy the relative motion between the two is called the "force of friction". Frictional forces may occur between the surfaces of contact even when there is no relative motion between the bodies. Work is done in overcoming the friction and consequently friction produces power losses in moving machinery whose efficiency falls below hundred percent.

Friction has many useful aspects also. We could not walk without friction and if somehow we start moving we could not stop. It would have been difficult even to fix nail in the wall. Though annoying, friction is a necessity for us as our everyday activities depend upon friction.

Origin of friction

The phenomenon of friction occurring between dry surfaces of two sliding bodies is complicated when it is observed at microscopic level. The force laws for friction do not have simplicity and accuracy of very high order.

Large number of surface irregularities is the key reason of friction. When a pull is exerted on a body so as to let it slide over the other surface; the relative motion is resisted on account of large number of surface irregularities and an opposing force is developed which is the force of friction. As the pull is increased, the opposing force also increases and at a certain pull, the body begins sliding.

2.9 STATIC FRICTION AND COEFFICIENT OF STATIC FRICTION

Suppose a block rests over a horizontal table Fig. 7(*a*). weight of block is balanced by the normal reaction. Next a very small horizontal force F_a is applied to the block Fig. 7(*b*). The block still remains stationary. One can say that "frictional force F_s" has come into existence (F_s – F_a). The frictional force preventing a body from sliding over the surface of other body is termed static friction. Again if applied force F_a is increased a little, the block still remains stationary which suggests that frictional force F_s has also increased with F_a. Increasing F_a, a stage is reached when the block begins to move. Till the motion does not start, the F_s is equal and opposite to applied force in all stages. The greatest value of F_s at the plane of contact of surfaces of two bodies in the stage when one body is just to slide over the other is termed "limiting friction", and is denoted by F_s. Which acts between surfaces at rest with respect to each other is called 'force of static friction'. Again

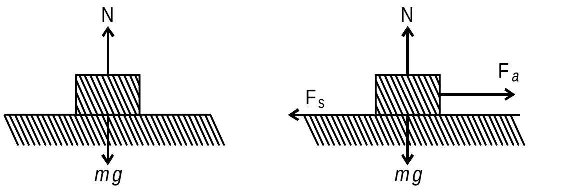
Maximum force of static friction, F_s = Smallest force needed to start motion F_L

The coefficient of static friction (μ_s) is defined as

$$\mu_s = \frac{\text{Magnitude of maximum forces of static friction}}{\text{Magnitude of normal force}}$$

i.e., $$\mu_s = F_s/N$$

or $$F_s = \mu_s N$$



(*a*) No force applied (*b*) Applied force exists, but no motion

Fig. 7

2.10 DYNAMIC FRICTION AND COEFFICIENT OF DYNAMIC FRICTION

In general a smaller force is needed to maintain uniform motion as compared to start the motion of a body over the surface. The force acting between surfaces in relative motion is called forces of sliding friction (or kinetic friction or dynamic friction) and is denoted by F_d.

The kinetic friction is measured by the force necessary to keep the two surfaces in uniform relative motion in contact with one another, coefficient of dynamic friction, (μ_d) is defined as μ_d = Magnitude of force of dynamic friction/magnitude of normal force

i.e., $$\mu_d = F_d/N$$

or $$F_d = \mu_d\ N$$

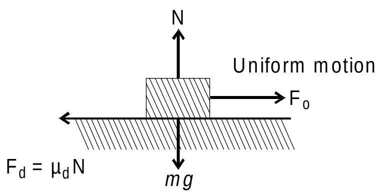


Fig. 8

Dependence of Coefficient of Friction

Coefficient of friction (μ) , depends upon (*i*) finishing of the surface, (*ii*) nature of material, (*iii*) surface films, (*iv*) contamination, (*v*) temperature.

The explanation of the two laws of friction lies in surface adhesion theory. In "rolling"' the frictional force is proportional to the velocity of the body; *i.e.*,

$$F = -kv$$

Where k = constant of proportionality

If, motion of the body is so fast that fluid swirls around (aeroplane moving in air) than frictional drag is given by

$$F = -kv^2$$

For still higher velocities above equation does not hold good.

Angle of Friction and Cone of Friction

Let us consider a block A of mass m. HL is a horizontal rough surface. A force P is applied parallel to surface on the block. F_s is frictional force, N normal reaction force.

If body be at rest, the resultant of F_s and N is a single force R that makes angle θ with normal to surface, we have

$$N = R \cos \theta$$

$$F_s = R \sin \theta$$

and so $$\tan \theta = F_s/N$$

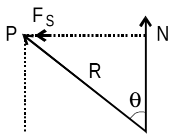


Fig. 9

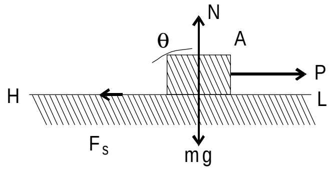


Fig. 10

As P is slowly increased, F_s also increases and thus θ also increases and θ is maximum when P is such that block is just to move. The friction in this case is termed "limiting friction".

The angle of friction (λ) is defined as the angle which the resultant reaction (R) of the surface makes with the normal when the friction is limiting. Thus,

$$(\tan \theta)_{\text{limiting}} = \tan \lambda = \frac{(F_s)_{\text{limiting}}}{N}$$

or $$\tan \lambda = \frac{F_L}{N} = \mu_s$$

As $$F_L = (F_s)_{\text{limiting}}$$

Resultant of F_L and N lies on the surface of cone having λ as the semi-vertical angle and direction of N as its axis. This cone is termed "cone of friction" as shown in figure 11.

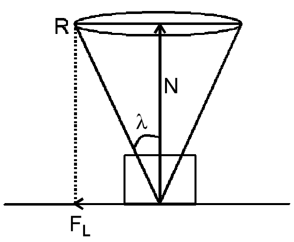


Fig. 11

PROBLEM

Q. 1. *Two blocks of masses* m_1 *and* m_2 *are connected by a massless spring on a horizontal frictionless table. Find the ratio of their accelerations* a_1 *and* a_2 *after they are pulled apart and then released.*

Solution. The blocks are pulled apart by equal and opposite forces. On being released they start moving toward each other under equal and opposite (elastic) force exerted by the *massless* spring. Since force is equals to mass multiplied by acceleration, we have

$$F = m_1 a_1 = m_2 a_2$$

$$\therefore \qquad \frac{a_1}{a_2} = \frac{m_2}{m_1}$$

Q. 2. *A car of mass 1200 kg moving at 22 m/s is brought to rest over a distance of 50m. Find the breaking force and the time required to stop.*

Solution. As soon as the brakes are applied, the car is decelerated. If a be the (negative) acceleration, then from the relation $v^2 = v_o^2 + 2ax$, we have

$$0 = (22)^2 + 2a\,(50)$$

so that $$a = -\frac{22 \times 22}{2 \times 50} = -4.84 \text{ m/s}^2$$

The braking (retarding) force on the car is, therefore,

$$F = \text{mass} \times \text{acceleration}$$
$$= 1200 \times (-4.84)$$
$$= -5808 \text{ nt.}$$

The negative sign signifies retardation.

Let t be the time the car takes to stop. Then, from the relation $v = v_0 + at$, we have

$$0 = 22 + (-4.84)t$$

$$\therefore \qquad t = 22/4.84 = 4.55 \text{ sec.}$$

Q. 3. *A car having a mass of 150 kg is moving at 60 km/hour. When the brakes are applied to produce a constant deceleration, the car stops in 1.2 min. Determine the force applied to the car.* [**Ans.** –347 nt.]

Q. 4. *A gun fires ten 2 gm bullets per second with a speed of 500 m/sec. The bullets are stopped by a rigid wall. (a) What is the momentum of each bullet? (b) What is the kinetic energy of each bullet? (c) What is the average force exerted by the bullets on the wall?*

Solution: (*a*) The momentum of each bullet is

$$P = mv$$
$$= (2 \times 10^{-3} \text{ kg})\,(500 \text{ m/s}) = 1 \text{ Kg-m/s.}$$

(*b*) The kinetic energy of each bullet is

$$K = \tfrac{1}{2}\, mv^2$$
$$= \tfrac{1}{2}\,(2 \times 10^{-3} \text{ kg})\,(500 \text{ m/s})^2 = 250 \text{ joule.}$$

(*c*) The bullets are stopped by a rigid wall. The magnitude of the change of momentum for each bullet is therefore 1 kg-m/s. Since 10 bullets are fired per second, the rate of change of momentum is

$$dP/dt = 10 \text{ kg-m/s}^2$$

But this must be equal to the average force exerted on the wall. Thus

$$F = dP/dt = 10 \text{ kg-m/s}^2 = 10 \text{ nt.}$$

Q. 5. *A body of mass 10 gm falls from a height of 3 meter into a pile of sand. The body penetrates the sand a distance of 3 cm before stopping. What force has the sand exerted on the body?*

Solution. Let v be the velocity of the body at the instant it reaches the pile of sand. Then from the relation $v^2 = v_o^2 + 2gy$, we have

$$v^2 = 0 + 2 \times (9.8 \text{ meter/sec}^2) \times 3 \text{ meter.}$$
$$= 58.8 \text{ (meter/sec)}^2$$

This velocity is reduced to zero due to the deceleration 'a' produced by the sand.

Thus, from the relation $v^2 = v_o^2 + 2ay$, we have

$$0 = 58.8 + 2a \text{ (0.03 meter)}$$

$$a = -\frac{58.8}{2 \times 0.03} = -980 \text{ meter/sec}^2$$

The mass of the body is 10 gm = 0.01 kg. Hence the (retarding) force exerted by the sand on it is

$$F = ma$$
$$= 0.01 \text{ kg} \times (-980 \text{ m/s}^2)$$
$$= -9.8 \text{ nt}$$

Q.6. *An electron (mass 9.0×10^{-31} kg) is projected horizontally at a speed of 1.2×10^7 m/sec into an electric field which exerts a constant vertical force of 4.5×10^{-15} nt on it. Determine the vertical distance the electron is deflected during the time it has moved forward 3.0 cm horizontally.*

Solution. The time taken by the electron to move 3.0 cm (= 0.03 meter) horizontally is given by

$$t = \frac{0.03 \text{ meter}}{1.2 \times 10^7 \text{ meter/sec}} = 2.5 \times 10^{-9} \text{ sec.}$$

The vertical acceleration of the electron is

$$a = \frac{\text{Force}}{\text{Mass}} = \frac{4.5 \times 10^{-15}}{9.0 \times 10^{-31}} = 5.0 \times 10^{15} \text{ nt/kg}$$

The vertical distance moved under this acceleration in time t is

$$y = \frac{1}{2}at^2$$
$$= \frac{1}{2} \times (5.0 \times 10^{15}) \times (2.5 \times 10^{-9})^2$$
$$= 1.56 \times 10^{-2} \text{ meter} = 1.56 \text{ cm.}$$

Q. 7. *A block of mass m = 2.0 kg is pulled along a smooth horizontal surface by a horizontal force F. Find the normal force exerted on the block by the surface. What must be the force F if the block is to gain a horizontal velocity of 4.0 meter/sec in 2.0 sec starting from rest?*

Solution. Let N be the normal force exerted by the smooth surface on the block. The net vertical force on the block is N – mg. Since there is no vertical acceleration in the block, we have by Newton second law (net force = mass × acceleration)

$$N - mg = 0$$

or $$N = mg = 2.0 \times 9.8 = 19.6 \text{ nt}$$

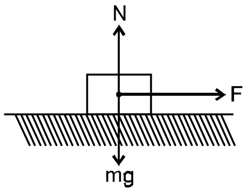


Fig. 12

The horizontal acceleration 'a' may be obtained from the relation $v = v_0 + at$:

$$4.0 = 0 + a\,(2.0)$$

$\therefore$ $$a = 2.0 \text{ m/sec}^2$$

The required horizontal force is therefore, by Newton's Law, given by

$$F = ma$$
$$= 2.0 \times 2.0 = 4.0 \text{ nt.}$$

Q. 8. *A block of mass M is pulled along a smooth horizontal surface by a rope of mass m. A force F is applied to one end of the rope. Find the acceleration of the block and the rope. Also deduce the force exerted by the rope on the block.*

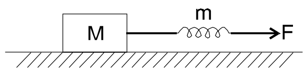


Fig. 13

[**Ans.** F/(M + m), MF/M + m)]

Q. 9. *Two blocks of masses m_1 = 2.0 kg and m_2 = 1.0 kg are in contact on a frictionless table. A horizontal force F = 3.0 nt is applied to m_1. Find the force of contact between m_1 and m_2. Again, find the force of contact if F is applied to m_2.*

Solution. Let F′ be the force of contact between m_1 and m_2. The block m_1 exerts a force F on m_2. and by the law of action and reaction, m_2 also exerts force F′ on m_1, as shown in the figure.

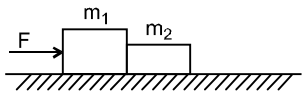


Fig. 14

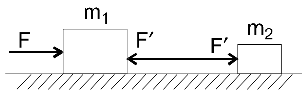


Fig. 15

The new force on m_1 is F – F′, and that on m_2 is F′. Let a be the (common) acceleration produced. Then, by Newton's second law, we have

$$F - F' = m_1 a$$

and $$F' = m_2 a$$

Solving: $$F' = \frac{m_2}{m_1 + m_2} F$$

Putting the given values:

$$F' = \frac{1.0 \text{ kg}}{2.0 \text{ kg} + 1.0 \text{ kg}} (3.0 \text{ nt}) = 1.0 \text{ nt.}$$

If F is applied to m_2., then we would have $F' = 2.0$ nt.

Q. 10. *Three blocks of masses m_1 = 10 kg, m_2 = 20 kg and m_3 = 30 kg are connected as shown in the Fig. 16 on a smooth horizontal table and pulled to the right with a force T_3 = 60 nt. Find the tension T_1 and T_2.* **(Lucknow, 1985)**

Solution. The force T_3 is exerted on the entire system. The block m_3 exerts a forward force T_2 on m_3; while by Newton's third law the block m_2 exerts a backward force T_2 on m_3. Similarly, m_3 exerts a forward force T_1 on m_1 and m_1 exerts a backward force T_1 on m_1. This is shown below.

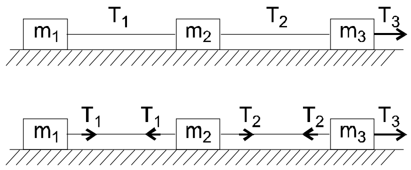


Fig. 16

Now, suppose all the blocks acquire an acceleration a. First consider the system as a whole.

Net force on the system = mass × acceleration

$$T_3 = (m_3 + m_2 + m_1)a \qquad ...(i)$$

Again, we consider m_2 and m_1 together.

Net force on m_2 and m_1 = mass × acceleration.

$$T_2 = (m_2 + m_1)a \qquad ...(ii)$$

Finally, we consider m_1 only.

$$T_1 = m_1 a \qquad ...(iii)$$

Here, T_3 = 60 nt; m_1 = 10 kg, m_2 = 20 kg and m_3 = 30 kg. Substitution in eq. (*i*) gives

$$60 = (30 + 20 + 10)\ a$$

$$a = 60/60 = 1 \text{ m/sec}^2$$

Substituting the value of a in (*ii*) and (*iii*), we get

$$T_2 = 30 \text{ nt.}$$

$$T_1 = 10 \text{ nt.}$$

Q. 11. *An inextensible string connecting blocks m_1 = 10 kg and m_2 = 5 kg passes over a light frictionless pulley as shown. m_1 moves on a frictionless surface while m_2 moves vertically. Calculate the acceleration a of the system and the tension T in the string.*

Solution: The diagram shows the forces on each block. The hanging block m_2 (due to gravity) exerts through the tension force T on m_2.

Now, the forces on m_1 are: its weight mg vertically downward, normal force N exerted by the smooth surface and the tension T. If a be the acceleration of the block in the horizontal direction then by Newton's law, we have

$$T = m_1 a \qquad ...(i)$$

And $$N - m_1 g = 0 \qquad ...(ii)$$

Since the string is inextensible, the acceleration of the block m_2 is also a. The new force on m_2 is $m_2 g$ – T. Hence by Newton' second law, we have

$$m_2 g - T = m_2 a \qquad ...(iii)$$

Adding (*i*) and (*iii*), we get

$$m_2 g = (m_1 + m_2)a$$

or $$a = \frac{m_2}{m_1 + m_2} g$$

Substituting the given value of a in (i) we get

$$T = \frac{m_1 m_2}{m_1 + m_2} g$$

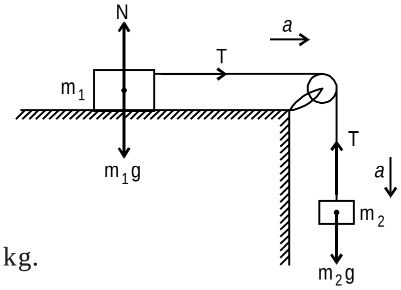


Fig. 17

Substituting the given values: $m_1 = 10$ kg and $m_2 = 5$ kg.

$a = (1/3)\ g = (1/3) \times 9.8 = 3.27$ nt/kg

and $T = (10/3)\ g = (10/3) \times 9.8 = 32.7$ nt.

Q. 12. *An inextensible string connecting unequal masses $m_1 = 1.0$ kg and $m_2 = 2.0$ kg passes over a massless and frictionless pulley as shown in figure 18. Find the tension in the string and the acceleration of the masses.*

Solution. As the string is inextensible, both masses have the same acceleration a. Also, the pulley is massless and frictionless, hence the tension T at both ends of the string is the same. The heavier mass m_2 is accelerated downward and the lighter mass m_1 is accelerated upward.

The new (upward) force on m_1 is $T - m_1 g$, and the net (downward) force on m_2 is $m_2 g - T$. Therefore, by Newton's second law, we have

$$T - m_1 g = m_1 a \quad ...(i)$$

and $$m_2 g - T = m_2 a \quad ...(ii)$$

Adding (*i*) and (*ii*), we get

$$(m_2 - m_1)\ g = (m_1 + m_2) a$$

or $$a = \frac{m_2 - m_1}{m_1 + m_2} g \quad ...(iii)$$

Substituting for a in eq. (*i*) and solving , we get

$$T = \frac{2 m_1 m_2}{m_1 + m_2} g$$

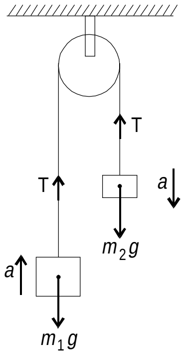


Fig. 18

Thus, for $m_1 = 1.0$ kg and $m_2 = 2.0$ kg, we have

$a = (1/3) \times 9.8 = 3.27$ m/sec^2

and $T = (4/3)\ g = (4/3) \times 9.8 = 13.1$ nt.

Q. 13. *A uniform flexible chain of length l with mass per unit length λ, passes over a small frictionless, massless pulley. It is released from a rest position with a length of chain x hanging from one side and a length $l - x$ from the other side. (a) Under what condition will it accelerate? (b) If this condition is met, find the acceleration a as a function of x.* **(Lucknow 1997)**

[**Ans.** (*a*) $l \neq 2x$, (*b*) $a = g\ (1 - 2x/l)$]

Q. 14. *A block of mass $m_1 = 3.0$ kg on a smooth inclined plane of 30° is connected by a cord over a small, frictionless pulley to a second block of mass $m_2 = 2.0$ kg hanging vertically* *as shown in figure 19. Calculate the acceleration with which the blocks move, and also tension in the cord.* (*take g = 10 m/s²*).

Solution. Let a be the acceleration of the blocks m_1 and m_2, and T the tension in the cord, which is uniform throughout as the cord is massless and the pulley is frictionless.

The force acting on the block m_1 are: (*i*) its weight $m_1 g$ which is the force exerted by the earth, (*ii*) the tension T in the string, and (*iii*) the normal force N exerted by the inclined surface (the force is normal because there is no frictional force between the surfaces).

The net force on m_1 along the plane is $T - m_1 g \sin\theta$, and that perpendicular to the plane is $N - m_1 g \cos\theta$. The block m_1 is accelerated along the incline while the acceleration perpendicular to the plane is zero. Therefore, by Newton's second law, we have

$$T - m_1 g \sin\theta = m_1 a \qquad ...(i)$$

and

$$N - m_1 g \cos\theta = 0 \qquad ...(ii)$$

Again, the forces on m_2 are : (*i*) its weight $m_2 g$ and (*ii*) the tension T in the string. The net vertical (downward) force is $m_2 g - T$. As the block is accelerated downward, Newton's second law gives

$$m_2 g - T = m_2 a \qquad ...(iii)$$

Solving (*i*) and (*iii*), we get

$$a = \frac{m_2 - m_1 \sin\theta}{m_1 + m_2} g$$

and

$$T = \frac{m_1 m_2 (1 + \sin\theta)}{m_1 + m_2} g$$

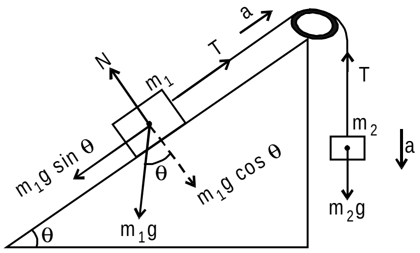


Fig. 19

Here $m_1 = 3.0$ kg, $m_2 = 2.0$ kg, $\sin\theta = \sin 30° = ½$ and $g = 10$ m/s²

$$a = \frac{2.0 - \left(3.0 \times \frac{1}{2}\right)}{3.0 + 2.0} 10 = 1.0 \text{ m/s}^2$$

and

$$T = \frac{(3.0)\,(2.0)\left(1 + \frac{1}{2}\right)}{3.0 + 2.0} 10 = 18 \text{ nt.}$$

Q. 15. *A car, mass 1000 kg moves uphill along a smooth road inclined 30°. Determine the force which the engine of the car must produce if the car is to move (a) with uniform motion* *(b) with an acceleration of 0.2 m/s*^{2}*. (c) Find also in each case the force exerted on the car by the road.* [**Ans.** (*a*) 4900 nt, (*b*) 5100 nt, (*c*) 4900 √3 in each case]

Q. 16. *A block of mass m = 2.0 kg is kept at rest on a smooth plan, inclined at an angle* $\theta = 30°$ *with the horizontal, by means of a string attached to the vertical wall. Find the tension in the string and the normal force acting on the block. If the string is cut, find the acceleration of the block. Neglect friction* [**Ans.** 9.8 nt, 17 nt, 4.0 m/s^2]

Q. 17. *A space traveller, whose mass is 100 kg leaves the earth. Compute his weight on earth (g = 9.8 m/s*^{2}*) and on Mars (g = 3.8 m/s*^{2}*).*

Solution: Weight = mass × acceleration due to gravity. Therefore, weight on earth

$$W_E = 100 \text{ kg} \times 9.8 \text{ m/sec}^2$$
$$= 980 \text{ nt},$$

and weight on Mars

$$W_M = 100 \text{ kg} \times 3.8 \text{ m/sec}^2$$
$$= 380 \text{ nt}.$$

The weight in interplanetary space would be zero. Of course, the mass will remain 100 kg at all locations.

Q. 18. *A body has a mass of 100 kg on the Earth. What would be it (i) mass and (ii) weight on the Moon where the acceleration due to gravity is 1.6 m/sec*^{2}*.*

[**Ans.** (*i*) 100 kg, (*ii*) 160 nt.]

Q. 19. *A massless string pulls a mass of 50 kg upward against gravity. The string would break if subjected to a tension greater than 600 Newtons. What is the maximum acceleration with which the mass can be moved upward?* **(Lucknow 1990, 1996)**

Solution. The force acting upon the mass are (*i*) its weight mg and (*ii*) tension T in the string. The net (upward) force, T – *mg*, is responsible for the upward acceleration *a* of the mass. By Newton's second law (net force = mass × acceleration), we have

$$T - mg = ma$$

Here (maximum) tension T = 600 nt and m = 50 kg. Therefore; the (maximum) acceleration a is given by

$$a = \frac{T - mg}{m}$$

$$a = \frac{600 - (50 \times 9.8)}{50} = 2.2 \text{ m/sec}^2$$

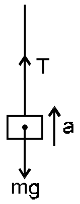


Fig. 20

Q. 20. *A body of mass 50 kg is hanging by a rope attached to the ceiling of an elevator. Calculate the tension in the rope if the elevator is (i) accelerating upward at 4 m/s*^{2}*,(ii) downward at 4 m/s*^{2}*, (iii) falling freely, (iv) moving with uniform velocity of 5 m/s upward or downward.*

Solution. The forces of the body are its weight *mg* and the tension T in the rope which is the upward force exerted on the body by the rope. (If the body were suspended by a spring balance, then T would be the reading of the balance. T is also known as the 'apparent weight' of the body).

(*i*) The body is at rest relative, to the elevator, and hence has an upward acceleration '*a*' relative to the earth. Taking the upward direction as positive, the resultant force on the body is T – *mg*. Hence from Newton's law (force = mass × acceleration), we have

$$T - mg = ma$$

$$\therefore \quad T = mg + ma$$

Thus the apparent weight T is greater than the true weight *mg*, and the body appears "heavier"

Substituting the given values: $m = 50$ kg, $a = 4\ m/s^2$

$$T = m\,(g + a) = 50\,(9.8 + 4) = 690 \text{ nt.}$$

(*ii*) When the elevator is accelerating downward ('*a*' negative), we may write

$$T - mg = -\,ma$$

$$\therefore \quad T = mg - ma$$

In this case the apparent weight T is less than true weight *mg*, and the body appears "lighter".

Substituting the given values:

$$T = m(g - a) = 50\,(9.8 - 4) = 290 \text{ nt}$$

(*iii*) If the elevator falls freely , $a = g$, then we have

$$T = m(g - a) = m\,(g - g) = 0$$

In this case the tension (apparent weight) is zero and the body appears "weightless". If the body were suspended by a spring balance in a freely falling elevator, the balance would read zero.

(*iv*) If the elevator is at rest, or moving vertically (either up or down) with constant velocity, $a = 0$. In this case

$$T - mg = 0$$

$$T = mg$$

The apparent weight equals the true weight.

$$T = mg = 50 \times 9.8 = 490 \text{ nt.}$$

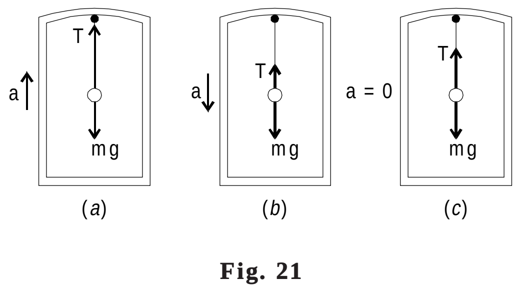


Fig. 21

Q. 21. *A body hangs from a spring balance supported from the roof of an elevator. (a) If the elevator has an upward acceleration of 2.45* m/s^2 *and the balance reads 50 nt, what is the true weight of the body? (c) What will the balance read if the elevator cable breaks?*

[**Ans.** (*a*) 40 nt (*b*) 2.45 m/s^2 downward, (*c*) zero]

Q. 22. *An elevator weighing 500 kg is pulled upward by a cable with an acceleration of 2.0 m/s*^{2} *(a) What is the tension in the cable? (b) What is the tension when the elevator is accelerating downward at 2.0 m/s*^{2}. [**Ans.** (*a*) 5900 nt, (*b*) 3900 nt]

Q. 23. *An elevator and its load have a total mass of 800 kg. Find the tension T in the supporting cable when the elevator, moving downward at 10 m/s, is brought to rest with constant acceleration in a distance of 25 m.*

Solution. Let us take the upward direction as positive. The elevator having an initial downward velocity of 10 m/s is brought to rest within a distance of 25 *m*. Let us use the relation

$$v^2 = v_0^2 + 2ax$$

Here $v = 0$, $v_0 = -10$ m/s and $x = -25$ m. Thus

$$0 = (-10)^2 + 2a\,(-25)$$

or $$a = \frac{-(-10)^2}{2\,(-25)} = 2 \text{ m/s}^2$$

The acceleration is therefore positive (upward). The resultant upward force on the elevator is T – mg, so that by Newton's law

$$T - mg = ma$$
$$T = mg + ma$$
$$= m\,(g + a)$$
$$= 800\,(9.8 + 2) = 9440 \text{ nt.}$$

Q. 24. *An elevator is moving vertically upward with an acceleration of 2 m/s*^{2}. *Find the force exerted by the feet of a passenger of mass 80 kg on the floor of the elevator. What would be the force if the elevator were accelerating downward?*

Solution. The force exerted by the passenger on the floor will always be equal in magnitude but opposite in direction to the force exerted by the floor on the passenger. We can therefore calculate either the force exerted on the floor (action force) or the force exerted on the passenger (reaction force).

Let us take the upward direction as positive, and consider the forces acting on the passenger. These are the passenger's true weight *mg* acting downward, and the force P exerted on him by the floor which is acting upward. P is the apparent weight of the passenger. The resultant upward force on the passenger is thus P – *mg*. If a be the upward acceleration of the passenger (and of the elevator), we have by Newton's second law

$$P - mg = ma$$

or $$P = mg + ma$$

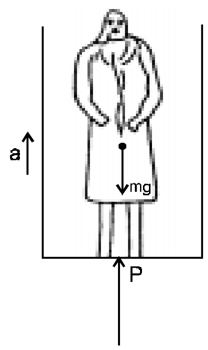


Fig. 22

Thus the apparent weight P is greater than the true weight *mg*. The passenger feels himself pressing down on the floor with greater force (the floor is pressing upward on him with greater force) than when he and the elevator are at rest (or moving with uniform velocity).

Substituting the given values: m = 80 kg and a = 2 m/s^2

$$P = m\,(g + a) = 80\,(9.8 + 2) = 944 \text{ nt.}$$

If the elevator is acelerating downward (a negative),we may write

$$P = mg - ma.$$

In this case the apparent weight P is less than the true weight mg. The passenger feels himself pressing down on the floor with less force than when he and the elevator are at rest (or moving with uniform velocity).

Substituting the given values:

$$P = m\,(g - a) = 80\,(9.8 - 2) = 624 \text{ nt}$$

If the elevator cable breaks, the elevator falls freely ($a = g$). In the case

$$P = m(g - a) = m(g - g) = 0$$

Then the passenger and floor would exert no forces on each other. Then passenger's apparent weight would be zero.

Q. 25. *A 90-kg man is in an elevator. Determine the force exerted on him by the floor when: (a) the elevator goes up/down with uniform speed, (b) the elevator accelerates upward at 3 m/s^2, (c) the elevator accelerates downward at 3 m/s^2, and (d) the elevator falls freely.*

[**Ans.** (*a*) 882 nt, (*b*) 1152 nt, (*c*) 612 nt, (*d*) zero]

Q. 26. *The total mass of an elevator with a 80-kg man in it is 1000 kg. This elevator moving upward with a speed of 8 m/s, is brought to rest over a distance of 20 m. Calculate (a) the tension T in the cables supporting the elevator, and (b) the force exerted on the man by the elevator floor.*

Solution. (*a*) Let us take the upward direction as positive. The elevator having an initial upward speed of 8 m/s is brought to rest within a distance of 20 m. Let use the relation

$$v^2 = v_0^2 + 2ax$$

Here $v = 0$, $v_0 = 8$ m/s, and $x = 20$ m. Thus

$$0 = (8)^2 + 2a\,(20)$$

$$a = -\frac{(8)^2}{2 \times 20} = -1.6 \text{ m/s}^2$$

The acceleration is therefore negative (downward). The resultant upward force on the elevator is $T - mg$, where mg is the total weight of the elevator. By Newton's law

$$T - mg = ma$$

or

$$\begin{aligned} T &= mg + ma \\ &= m\,(g + a) \\ &= 1000\,(9.8 - 1.6) = 8200 \text{ nt.} \end{aligned}$$

(*b*) Let P be the (upward) force exerted on the man by the elevator floor. If m' be the mass of a man, its weight acting downward is $m'g$. Thus the net upward force on the man is $P - m'g$. Since a is the acceleration of the man (and of the elevator), we have by Newton's law

$$P - m'g = m'a$$

or

$$\begin{aligned} P &= m'g + m'a = m'\,(g + a) \\ &= 80\,(9.8 - 1.6) = 656 \text{ nt.} \end{aligned}$$

Q. 27. *A bob hanging from the ceiling of a car acts as an accelerometer. Derive the general expression relating the horizontal acceleration a of the car to the angle θ made by the bob with the vertical.*

Solution. When the car has an acceleration 'a' toward the right, the rope carrying the ball makes an angle θ with the vertical.

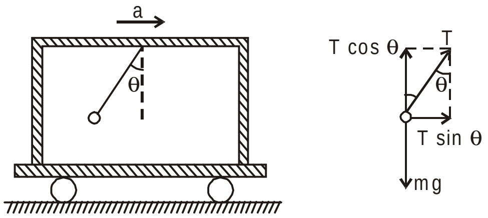


Fig. 23

Two forces are exerted on the ball: its weight mg and the tension T in the rope (shown separately). The resultant vertical force is T cos θ – mg, and the resultant horizontal force is T sin θ. The vertical acceleration of the system is zero, while the horizontal acceleration is a. Hence by Newton's second law, we have

$$T \cos \theta - mg = 0$$

and

$$T \sin \theta = ma$$

Eliminating T, we get

$$\tan \theta = a/g$$

or

$$a = g \tan \theta$$

This is the required result

Q. 28. *Consider free fall of an object of mass m from rest through the air which exerts a frictional force on it, proportional to its velocity. If the proportionality constant is k : (a) Write down its equation of motion. (b) Show that it ceases to accelerate when it reaches a terminal velocity* $v_T = mg/k$. *(c) Verify that the velocity of the object varies with time as* $v = v_T(1 - e^{-kt/m})$.

Solution. (*a*) The forces acting on the object are (*i*) the force of gravity mg in the direction of motion and (*ii*) the frictional force kv against the motion. The net force in the direction of motion is therefore $mg - kv$. This, by Newton's second law, is equal to mass m multiplied by the acceleration a of the body. Thus

$$mg - kv = ma \qquad ...(i)$$

If y represents the (vertical) displacement, then $v = \frac{dy}{dt}$ and $a = \frac{dv}{dt} = \frac{d^2y}{dt^2}$. The above equation may therefore be written as

$$mg - k\frac{dy}{dt} = \frac{md^2y}{dt^2}$$

$$\frac{d^2y}{dt^2} = -\frac{k}{m}\frac{dy}{dt} + g \qquad ...(ii)$$

This is the equation of motion of the object

(*b*) Let v_T be the 'terminal' velocity attained by the object when the acceleration a becomes zero. Making this substitution in eq. (*i*), we have

$$mg - kv_T = 0$$

$$v_T = \frac{mg}{k} \qquad \text{...}(iii)$$

(*c*) The equation of motion (*ii*) can be written as

$$\frac{dv}{dt} = -\frac{k}{m}v + g$$

But $g = (k/m)\, v_T$ from equation (*iii*). Therefore

$$\frac{dv}{dt} = -\frac{k}{m}v + \frac{k}{m}v_T = \frac{k}{m}(v_T - v)$$

$$\frac{dv}{v_T - v} = \frac{k}{m}dt$$

Integrating, we get

$$-\log_e (v_T - v) = \frac{k}{m}t + C \text{ (constant)}$$

When $t = 0$, $v = 0$. Thus

$$-\log_e v_T = C$$

Therefore, the above equation becomes

$$-\log_e (v_T - v) = \frac{k}{m}t - \log_e v_T$$

or $$\log_e \frac{(v_T - v)}{v_T} = \frac{-k}{m}t$$

or $$\frac{(v_T - v)}{v_T} = e^{-\frac{kt}{m}}$$

or $$1 - \frac{v}{v_T} = e^{-\frac{kt}{m}}$$

or $$\frac{v}{v_T} = 1 - e^{-\frac{kt}{m}}$$

or $$v = v_T\left(1 - e^{-\frac{kt}{m}}\right)$$

This is the required expression.

Q. 29. *A block weighing 20 nt rests on a rough surface. A horizontal force of 8 nt is required before the block starts to slide, while a force of 4 nt keeps the block moving at constant speed once it has started sliding. Find the coefficients of static and sliding (kinetic) friction.*

Solution. The smallest (horizontal) force F required to start the motion is equal to the maximum force of static friction f_s , *i.e.*,

$$F = f_s$$

But we know that (maximum) $f_s = \mu_s N = \mu_s W$,

Where μ_s is the coefficient of static friction and N is the normal reaction which is equal to the weight W of the block. Thus

$$F = \mu_s W,$$

or

$$\mu_s = \frac{F}{W} = \frac{8 \text{ nt}}{20 \text{ nt}} = 0.4$$

Again; the force F′ required to maintain a uniform motion is equal to the force of kinetic friction, f_k . That is

$$F' = f_k = \mu_k N = \mu_k W$$

or

$$\mu_k = \frac{F'}{W} = \frac{4 \text{ nt}}{20 \text{ nt}} = 0.2$$

Q. 30. *A block weighing 10 nt is rest on a horizontal table. The coefficient of static friction between block and table is 0.50. (a) What is the magnitude of the horizontal force that will just start the block moving? (b) What is the magnitude of a force acting upward 60° from the horizontal that will just start the block moving? (c) If the force acts down at 60° from the horizontal, how large can it be without causing the block to move?*

Solution. (*a*) As shown in Fig. 24, the horizontal force F that will just start the block moving is equal to the maximum force of static friction. Thus

$$F = f_s = \mu_s N$$

Where μ_s is the coefficient of static friction and N is the normal reaction. Also

$$N = W,$$

So that

$$F = \mu_s W = 0.50 \times 10 \text{ nt} = 5.0 \text{ nt}.$$

(*b*) In this case the forces on the block are shown in Fig. The applied force is inclined at ϕ upward from the horizontal. Its horizontal and vertical components are $F_x = F \cos \phi$ and $F_y = F \sin \phi$. In this case, we have, for equilibrium

$$F_x = f_s = \mu_s N$$

and

$$F_y = W - N$$

Thus

$$F_x = \mu_s (W - F_y)$$

or

$$F \cos \phi = \mu_s (W - F \sin \phi)$$

or

$$F = \frac{\mu_s W}{\cos \phi + \mu_s \sin \phi}$$

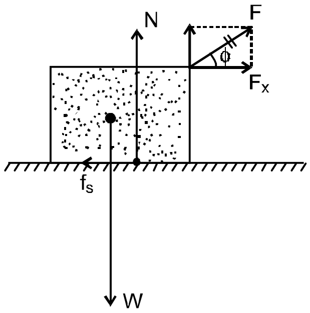


Fig. 24

Here

$$\mu_s = 0.50, \; W = 10 \text{ nt},$$

$$\cos \phi = \cos 60° = 0.50 \text{ and } \sin 60° = 0.866$$

$$F = \frac{0.50 \times 10}{0.50 + (0.50 \times 0.866)} = 5.36 \text{ nt}.$$

Note that in this example the normal reaction N is not equal to the weight of the block, but is less than the weight by the vertical component of the force F.

(*c*) In this case (Fig. 25), the equilibrium equations will be:

$$F_x = f_s = \mu_s N$$

and $$F_y = N - W$$

From these we can see that

$$F = \frac{\mu_s W}{\cos\phi - \mu_s \sin\phi}$$

$$F = \frac{0.50 \times 10}{0.50 - (0.5 \times 0.866)} = 74.6 \text{ nt}$$

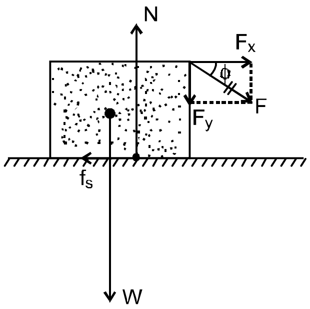


Fig. 25

In this case the normal reaction is greater than the weight of the block.

Q. 31. *A block is pulled to the right at constant velocity by a 10-nt force acting 30° above the horizontal. The coefficient of sliding friction between the block and the surface is 0.5. What is the weight of the block?* [**Ans.** 22.3 nt]

Q. 32. *A block of mass 10 kg rests on a horizontal surface. What constant horizontal force F is needed to give it a velocity of 4 m/s in 2 sec, starting from rest, if the friction force between the block and the surface is constant and is equal to 5 Newton?*

Solution. Let a be the acceleration in the block when in motion. From the relation $v = v_0 + at$, we have

$$a = \frac{v - v_0}{t}$$

Here $v = 4$ m/s, $v_0 = 0$ and $t = 2s$. Thus

$$a = \frac{4 - 0}{2} = 2 \text{ m/s}^2$$

If F be the force applied on the block, then the net force producing the above acceleration would be $F - f_s$, where f_s is the static force of friction between the block and the surface. Thus, by Newton's second law,

$$F - f_s = ma$$

or $$F = ma + f_s$$

Here $m = 10$ kg, $a = 2$ m/s^2 and $f_s = 5$ nt.

∴ $F = (10 \times 2) + 5 = 25$ nt.

Q. 33. *A body of mass 2000 kg is being pulled by a tractor with a constant velocity of 5 m/s on a rough surface. If the coefficient of kinetic friction is 0.8, what are the magnitudes of the frictional force and of the applied force?*

Solution. The force of kinetic friction is given by

$$f_k = \mu_k N,$$

where μ_k is the coefficient of kinetic friction and N is the normal reaction which is equal to the weight W (= mg) of the body. Then

$$f_k = \mu_k mg$$

$$= 0.8 \times 2000 \text{ kg} \times 9.8 \text{ nt/kg}$$

$$= 15680 \text{ nt, in the direction opposite to motion.}$$

In order to pull the body with a constant velocity, the applied force must be just equal to the frictional force, and applied in the direction of motion.

Q. 34. *The coefficient of friction between a block of wood of mass 20 kg and a table surface is 0.25. (a) What force is needed to give it an acceleration of 0.2 m/s^2, (b) What forces are acting when it is moving with a constant velocity of 2 m/s, (c) If no external force is applied, how far will block travel before coming to rest.*

[**Ans.** (*a*) 53 nt, (*b*) applied and frictional force each 49 nt. (*c*) 0.816 m]

Q. 35. *A block sliding initially with a speed of 10 m/s on a rough horizontal surface, comes to rest in a distance of 70 meters. What is the coefficient of kinetic friction between the block and the surface?*

Solution. Let us assume that the force of kinetic friction f_k between the block and the surface is constant. Then we have a uniformly decelerated motion. From the relation

$$v^2 = v_0^2 + 2ax$$

with the final velocity $v = 0$, we obtain

$$0 = (10 \text{ m/s})^2 + 2a\ (70 \text{ m})$$

so that

$$a = -\frac{10 \times 10}{2 \times 70} = -0.71 \text{ m/s}^2$$

The negative sign indicates deceleration. The frictional force on the block is, by the second law of motion, given by

$$f_k = -ma$$

where m is the mass of the block. But we know that

$$f_k = \mu_k N$$

where μ_k is the coefficient of kinetic friction and N is the normal reaction which is equal to the weight W (= mg) of the block. Thus

$$f_k = \mu_k\ mg$$

$$\mu_k = \frac{f_k}{mg} = \frac{-ma}{mg} = \frac{-a}{g} \qquad [f_k = -ma]$$

$$= \frac{0.71 \text{ m/s}^2}{9.8 \text{ m/s}^2} = 0.072.$$

Q. 36. *If the coefficient of static friction between the tyres and the road is 0.5, what is the shortest distance in which an automobile can be stopped when traveling at 72 km/hour?*

Solution. Let f_s be the constant force of static friction between the tyres and the road. The automobile is under uniformly decelerated motion. From the relation

$$v^2 = v_0{}^2 + 2ax$$

with the final velocity $v = 0$, we obtain $x = -\dfrac{v_0^2}{2a}$, ...(*i*) where the minus sign means that the acceleration a caused by f_s is opposite to the direction of motion.

Now, by Newton's second law, we have

$$f_s = -ma,$$

Where m is the mass of the automobile. But we know that

$$f_s = \mu_s N = \mu_s mg$$

where μ_s is the coefficient of static friction and N (= mg) is the normal reaction.

From the last two expression, we have

$$-ma = \mu_s mg$$

or $$a = -\mu_s g.$$

Then from eq. (i) the distance of stopping is

$$x = \frac{v_0^2}{2\mu_s g}$$

Here v_0 = 72 km/hour = 20 m/sec and $\mu_s = 0.5$

$$\therefore \quad x = \frac{(20 \text{ m/s})^2}{2 \times 0.5 \times (9.8 \text{ m/s}^2)} = 40.8 \text{ meter}.$$

Q. 37. Block A weight 100 nt. The coefficient of static friction between the block and table is 0.30. The block B weights 20 nt and the system is in the equilibrium. Find the friction force exerted on block A. Also find the maximum weight of block for which the system will be in equilibrium. **(Lucknow 1991, 1997)**

Solution. The forces acting on the knot O are the tension T, T′ and W (weight of B) in the cords. (The reactions to T and T′ which act on A and on the wall are also shown). Since the knot is in equilibrium, we have

$$T = T' \cos 45°$$

and $$T' \sin 45° = W$$

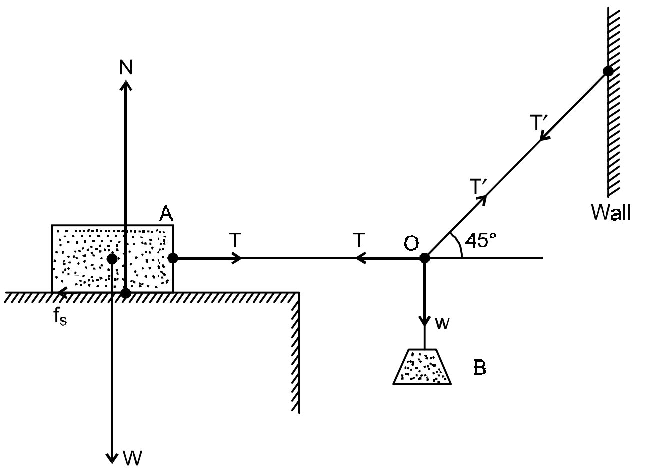


Fig. 26

These equations give

$$T = W = 20 \text{ nt.}$$

Now, the forces on the block A are its weight W, normal reaction N, tangential force of static friction f_s and the tension T. Again, since the block is in equilibrium, we have

$$N = W$$

and $$f_s = T = 20 \text{ nt.}$$

As W (and hence T) increases, the friction force f_s also increases until it acquires its maximum value, at which

$$f_s = \mu_s N \qquad \text{(where } \mu_s \text{ is coeff. of static friction)}$$

$$= \mu_s W \qquad (\because N = W)$$

$$= 0.30 \times 100 = 30 \text{ nt.}$$

At this limiting state of equilibrium, W, T and f_s all are equal and maximum. Thus

$$W_{max} = 30 \text{ nt.}$$

Q. 38. *A 4.0 kg block A is put on top of a 5.0 kg block B; which is placed on a smooth table. It is found that a horizontal force of 12 nt is to be applied on A in order to slip it on B. Find the maximum horizontal force F which can be applied to B so that both A and B move together, and the resulting acceleration of the blocks.*

Solution. Let m_1 and m_2 be the masses of the blocks A and B. If μ_k be the coefficient of sliding friction between A and B, then the frictional force on A exerted by B is

$$f_k = \mu_k N = \mu_k m_1 g.$$

A force of 12 nt maintains a slip of A on B against the force of friction f_k. Thus

$$12 = f_k = \mu_k m_1 g$$

Here $$m_1 = 4.0 \text{ kg.}$$

$$\mu_k = \frac{12}{m_1 g} = \frac{12}{4.0 \times g} = \frac{3}{g}$$

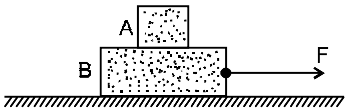


Fig. 27

B is placed on a smooth table. Hence in order to move A and B together we require a force F on B which just overcomes the frictional force on B exerted by A and that on A exerted by B. Thus

$$F = \mu_k m_1 g + \mu_k m_2 g$$

$$= \mu_k (m_1 + m_2) g$$

$$= 3/g\ (4.0 + 5.0)\ g = 27 \text{ nt}$$

This force is causing both A and B to move on the smooth table. Hence the acceleration produced in them is

$$a = \frac{F}{m_1 + m_2} = \frac{27}{4.0 + 5.0} = 3 \text{ m/s}^2$$

Q. 39. *Block A weight 4 nt and block B weight 8 nt. The coefficient of sliding friction between all surfaces is 0.25. Find the force F to slide B at a constant speed when (a) A rests on B and moves with it, (b) A is held at rest, (c) A and B are connected by a light cord passing over a smooth pulley.* **(Lucknow, 1994)**

[**Ans.** (*a*) 3 nt, (*b*) 4 nt, (*c*) 5 nt]

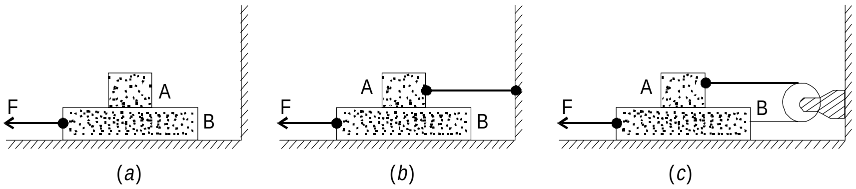


Fig. 28

Q. 40. *A block of mass 5 kg resting on a horizontal surface is connected by a cord passing over a light frictionless pulley to a hanging block of mass 5 kg. The coefficient of kinetic friction between the block and the surface is 0.5. Find the tension in the cord and the acceleration of each block.*

Solution. Let m_1 and m_2 be the masses of the blocks A and B. The diagram shows the forces on each block. The block B exerts through the tension in the cord of force T on block. A, while A exerts an equal reaction force T on B.

Now, the forces on A are: its weight W_1 (= $m_1 g$) vertically downward, normal reaction N exerted by surface, tangential frictional force f_k exerted by the surface, and the tension T. If a be the acceleration of the block A in the horizontal direction, then by second law of motion, we have

$$T - f_k = m_1 a$$

and
$$N - W_1 = 0$$

or
$$N = W_1 = m_1 g.$$

But
$$f_k = \mu_k N = \mu_k m_1 g$$

∴
$$T - \mu_k m_1 g = m_1 a \quad ...(i)$$

Since the cord is inextensible, the acceleration of the block B is also a. The net force on B is $W_2 -$ T. Hence by law of motion, we have

$$W_2 - T = m_2 a \quad \text{or} \quad m_2 g - T = m_2 a \quad ...(ii)$$

Substituting the value of T from (*i*) into (*ii*), we get

$$m_2 g - (m_1 a + \mu_k m_1 g) = m_2 a$$

or
$$m_2 g - \mu_k m_1 g = (m_1 + m_2) a$$

or
$$a = \frac{(m_2 - \mu_k m_1)}{m_1 + m_2} g \quad ...(iii)$$

Substituting this value of a in Eq. (*ii*), we get

$$m_2 g - T = m_2 \frac{(m_2 - \mu_k m_1)}{m_1 + m_2} g$$

$$T = m_2 g \left[1 - \frac{m_2 - \mu_k m}{m_1 + m_2}\right]$$

or
$$T = \frac{m_1 m_2 (1 + \mu_k)}{m_1 + m_2} g \quad ...(iv)$$

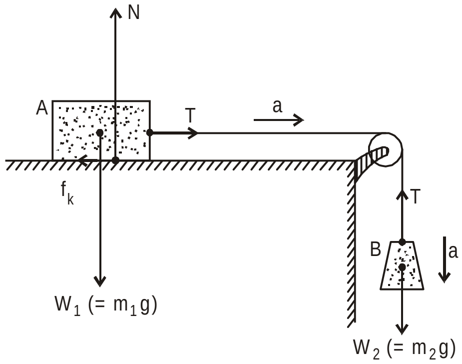


Fig. 29

Substituting the given values in (*iii*) and (*iv*); m_1 = 5 kg, m_2 = 5 kg, μ_k = 0.5

$$a = \frac{(5 - 0.5 \times 5)}{5 + 5}(9.8) = 2.45 \text{ m/s}^2$$

and
$$T = \frac{5 \times 5\,(1 + 0.5)}{5 + 5}(9.8) = 36.75 \text{ nt.}$$

Q. 41. *The masses of A and B in Figure 30 are 10 kg and 5 kg. Find the minimum mass of C that will prevent A from sliding, if μ_s between A and the table is 0.20. Compute the acceleration of the system if C is removed. Take μ_k between table and A also to be 0.20.*

[**Ans.** 15 kg, g/5]

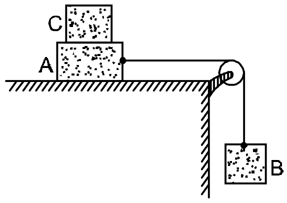


Fig. 30

Q. 42. *A 10 kg block is sliding down a plane inclined at an angle of 30° with the horizontal. Find the normal reaction and the acceleration of the block. The coefficient of kinetic friction is 0.5.*

Solution. The forces on the block (Fig. 31) are its weight W and the normal and the (kinetic) frictional components of the force exerted by the plane, namely, N and f_k respectively.

The weight W may be resolved into two components, W sin θ parallel to the plane and W cos θ perpendicular to the plane. Thus the net force along the plane is W sin θ – f_k and that perpendicular to the plane is N – W cos θ. If a be the acceleration down the plane, we have by second law of motion

$$W \sin\theta - f_k = ma \qquad ...(i)$$

Since there is no motion perpendicular to the plane, we have

$$N - W \cos\theta = 0 \qquad ...(ii)$$

Also
$$f_k = \mu_k N \qquad ...(iii)$$

From eq. (*ii*), we get

$$N = W \cos\theta$$

Here
$$W = mg = 10 \times 9.8 = 98 \text{ nt and } \cos\theta = \cos 30° = 0.866$$

Substituting this value of N and μ_k = 0.5 in (*iii*), we get

$$f_k = 0.5 \times 84.87 = 42.43 \text{ nt}$$

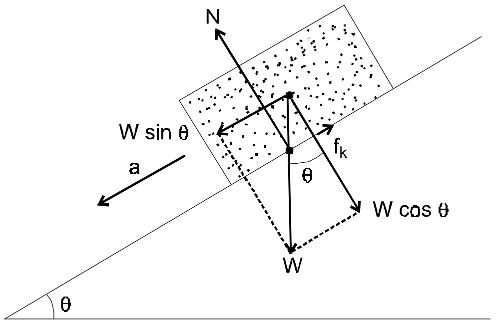


Fig. 31

Now, Eq. (*i*)

$$a = \frac{W \sin\theta - f_k}{m}$$

$$= g \sin \theta - (f_k/m) \qquad [\therefore W = mg]$$

$$= (9.8 \times 0.5) - (42.43/10) \qquad [\sin \theta = \sin 30° = 0.5]$$

$$= 4.9 - 4.243 = 0.657 \text{ m/s}^2$$

Q. 43. *A piece of ice slides down a 45° inclined plane in twice the time it takes to slide down a frictionless 45° inclined plane. What is the coefficient of kinetic friction between ice and inclined plane?* **(Lucknow, 1983)**

Solution. Let a be the acceleration of the ice piece down the first (rough) plane. The forces on the piece are shown in Fig. 31. The net force down the plane is W sin θ – f_k and that perpendicular to the plane is N – W cos θ. By second law of motion, we have

$$W \sin\theta - f_k = ma \qquad ...(i)$$

and $$N - W \cos \theta = 0$$

so that $$N = W \cos \theta$$

Also, $$f_k = \mu_k N = \mu_k W \cos \theta$$

Putting this value of f_k in (*i*), we get

$$W \sin \theta - \mu_k W \cos \theta = ma$$

or $$a = g \sin \theta - \mu_k g \cos \theta, \qquad ...(ii)$$

because W = *mg*, here θ = 45°, so that sin θ = cos θ = $1/\sqrt{2}$. Thus

$$a = \frac{g}{\sqrt{2}}(1 - \mu_k)$$

Let *t* be the time taken by the piece to slide down the plane. Then from $x = v_0 t + \frac{1}{2} a t^2$, we have (here v_0 = 0)

$$x = \frac{1}{2} \frac{g}{\sqrt{2}}(1 - \mu_k) t^2 \qquad ...(iii)$$

For the frictionless plane ($\mu_k = 0$), we have

$$a = g \sin\theta = g/\sqrt{2}$$

The time is now $t/2$. Thus

$$x = \frac{1}{2}\frac{g}{\sqrt{2}}\left(\frac{t}{2}\right)^2 \qquad ...(iv)$$

Comparing (*iii*) and (*iv*), we get

$$1 - \mu_k = ¼$$

$$´\mu_k = 1 - ¼ = 3/4 = 0.75.$$

Q. 44. *A block slides down an inclined plane of slope angle θ with constant velocity. It is then projected up the same plane with an initial speed v_0. How far up the incline will it move before coming to rest? Will it slide down again?* **(Lucknow 1986, 1990, 1993)**

Solution. Refer to Fig. 31. The net force on the block down the plane is W sin θ – f_k which is zero because the block slides down with constant velocity (zero acceleration). Thus

$$W \sin\theta - f_k = 0$$

or

$$f_k = W \sin\theta$$

When the block is projected up (now f_k will be down the plane), the net force down the plane becomes W sinθ + f_k = W sin θ + W sin θ = 2 W sin θ. Hence the acceleration down the plane would be

$$a = \frac{\text{force}}{\text{mass}} = \frac{2W \sin\theta}{m} = 2g \sin\theta$$

Now, using the relation

$$v^2 = v_0^2 + 2ax$$

with the final velocity $v = 0$, we have

$$0 = v_0^2 + 2(2g \sin\theta)x$$

$$x = -\frac{v_0^2}{4g \sin\theta}$$

Hence the distance covered up the plane

$$-x = \frac{v_0^2}{4g \sin\theta}$$

As soon as the block stops, the frictional force becomes static, which is larger than the kinetic. Hence the block will not slide down again.

Q. 45. *A block of mass 0.2 kg starts up a plane inclined 30° with the horizontal with a velocity of 12 m/s. If the coefficient of sliding friction is 0.16, how far up the plane the block travels before stopping? What would be the block's speed when it is made to return to the bottom of the plane?* [**Ans.** 11.5 m, 9.01 m/s]

Q. 46. *A body with a mass of 0.80 kg is on a plane inclined at 30° to the horizontal. What force must be applied on the body so that it slides with an acceleration of 0.10 m/s^2 (a) uphill, (b) downhill? The coefficient of sliding friction with the plane is 0.30.*

Solution. Let us first consider the body moving uphill (Fig. 32*a*). The forces acting on the body are its weight W (= mg) acting vertically downward, the normal force N and the kinetic frictional force f_k exerted by the plane, and the force F applied uphill. The frictional force f_k is always against the motion and hence in this case downhill. The weight W may be resolved into two components, W sin θ parallel to the plane and W cos θ perpendicular to the plane. Thus the net force on the body along the plane directed uphill is F – W sin θ – f_k, and that perpendicular to the plane is N – W cos θ. If a be the acceleration up the plane, the second law of motion gives

$$F - W \sin\theta - f_k = ma \quad ...(i)$$

Since there is no motion perpendicular to the plane, we have

$$N - W\cos\theta = 0$$

or

$$N = W\cos\theta$$

Also, we know that

$$f_k = \mu_k N = \mu_k W\cos\theta.$$

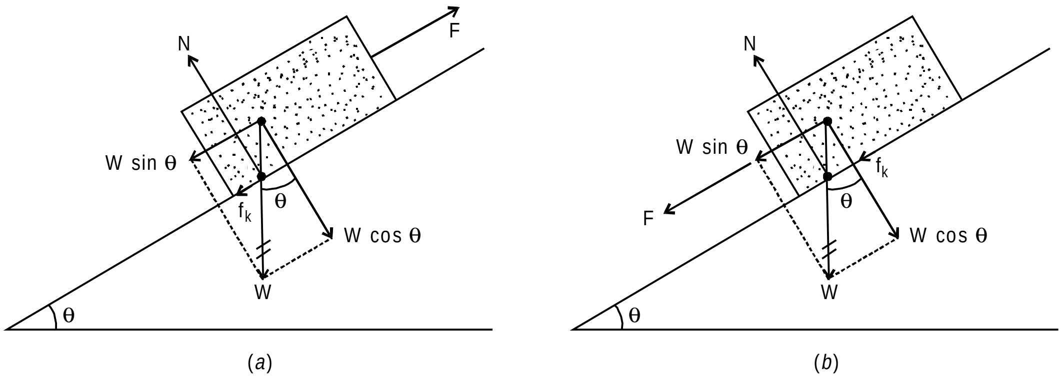


Fig. 32

Where μ_k is the coefficient of kinetic friction. Substituting this value of f_k in eq. (*i*), we get

$$F - W\sin\theta - \mu_k W\cos\theta = ma$$

Replacing W by mg, we get

$$F - mg(\sin\theta + \mu_k\cos\theta) = ma$$

or

$$F = m[a + g(\sin\theta + \mu_k\cos\theta)] \quad ...(ii)$$

Here m = 0.80 kg, a = 0.10 m/s^2, μ_k = 0.30 and θ = 30°, thus

sin θ = sin 30° = 0.5 and cos θ = cos 30° = 0.866. Whence

$$F = 0.80\,[0.10 + 9.8\,(0.50 + 0.30 \times 0.866)]$$
$$= 6.04 \text{ nt.}$$

(*b*) When the body is moving down hill, the forces are as shown in Fig. (*b*). We now assume that the applied force F is downhill. f_k is always against the motion and hence now it is uphill. If a be the acceleration down the plane, then the equation of motion is

But $f_k = \mu_k \text{ W} \cos\theta$ (as before), therefore

$$F + W \sin\theta - \mu_k W \cos\theta = ma$$

or $$F = m[a - g(\sin\theta - \mu_k \cos\theta)] \quad ...(iv)$$

Putting the values:

$$F = 0.80\,[0.10 - 9.8\,(0.5 - 0.30 \times 0.866)] = -1.80 \text{ nt.}$$

The negative sign means the force F is uphill instead of downhill as we had assumed. Still the motion is downhill.

Q. 47. *Forces required to move a body on a rough inclined plane with a uniform velocity in the upward and downward directions are in the ratio 2:1. Find the inclination of the plane if the coefficient of friction is 3.* **(Lucknow, 1987)**

Solution: Let F_1 and F_2 be the required forces. Putting $a = 0$ (because velocity is uniform) in eqs. (*ii*) and (*iv*) in the last problem, we get

$$F_1 = mg\,(\sin\theta + \mu_k \cos\theta)$$

and $$F_2 = -mg\,(\sin\theta - \mu_k \cos\theta)$$

But $F_1 = 2\,F_2$. This gives

$$\sin\theta + \mu_k \cos\theta = -2\,(\sin\theta - \mu_k \cos\theta)$$

or $$\tan\theta = \frac{\mu_k}{3}$$

Now $$\mu_k = 3$$

$$\tan\theta = 1$$

or $$\theta = 45°$$

Q. 48. *A block A weighing 100 nt is placed on an inclined plane of slope angle 30° and is connected to a second hanging block B by a cord passing over a small smooth pulley. The coefficient of sliding friction is 0.30. Find the weight of the block B for which the block A moves at constant speed (a) up the plane, (b) down the plane.*

[**Ans.** (*a*) 76 nt, (*b*) 24 nt]

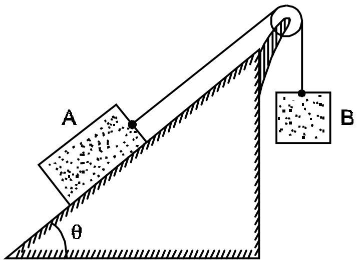


Fig. 33

Q. 49. *Two blocks with masses m_1 = 1.65 kg and m_2 = 3.30 kg are connected by a string and slide down a plane inclined at an angle θ = 30° with the horizontal. The coefficient of sliding friction between m_1 and the plane is 0.226 and that between m_2 and the plane is 0.113. Calculate the common acceleration of the two blocks and the tension in the string.*

[**Ans.** 3.62 meter/sec^2, 1.06 nt]

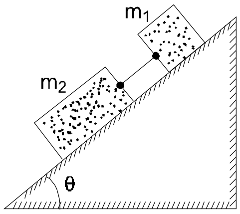


Fig. 34

Q. 50. A *block is at rest on an inclined plane making an angle θ with the horizontal. As the angle of incline is increased, slipping just starts at an angle of inclination θ_s. Find the coefficient of static friction between block and incline in terms of θ_s.*

Solution. The forces on the block are shown in Fig. 35. W is the weight of the block, N is the normal reaction and f_s the tangential force of static friction exerted on the block by the inclined surface. The weight W may be resolved into two components, W sin θ parallel to the inclined plane and W cos θ perpendicular to it. Since the block is in equilibrium, the net force along the plane as well as perpendicular to the plane is zero. That is

$$f_s - W \sin\theta = 0$$

and $$N - W \cos\theta = 0$$

Also $$f_s \leq \mu_s N.$$

When we increase θ to θ_s, slipping just begins. Thus for $\theta = \theta_s$, we can use $f_s = \mu_s$ N. Substituting this above, we get

$$\mu_s N = W \sin\ \theta_s$$

and $$N = W \cos\ \theta_s$$

Dividing, we get

$$\boxed{\mu_s = \tan\theta_s}$$

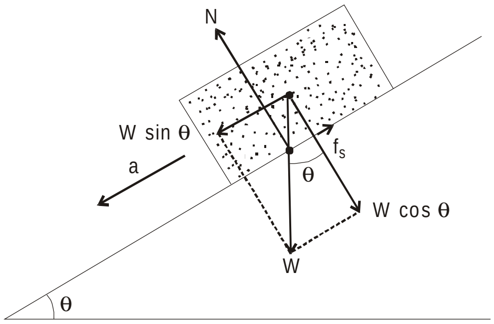


Fig. 35

Hence measurement of the angle of inclination at which slipping just starts provides us a method for determining the coefficient of static friction between two surfaces.

In a similar way, we can show that the angle of inclination θ_k required to maintain a constant speed of the block sliding down the plane is given by

$$\mu_k = \tan\theta_k$$

where $\theta_k < \theta_s$ and μ_k is the coefficient of sliding (kinetic) friction.

Q. 51. *A block rests on a plane inclined at θ with the horizontal. The coefficient of sliding friction is 0.50 and that of static friction is 0.75. (a) As the angle θ is increased, find the minimum angle at which the block starts to slip. (b) At this angle find the acceleration once the block has begun to move. (c) How long is required for the block to slip 8 meter along the inclined plane.* [**Ans.** (*a*) 37°, (*b*) 1.96 m/s^2, (*c*) 2.86 s]

Q. 52. *A particle starting from rest moves in a straight line with acceleration a = (0.5 + 0.1 t) m/s^2, where t is the time. Calculate its velocity and the distance travelled after 2 seconds.*

(Lucknow, 1984, 1989)

Solution. Suppose that initial position of particle is at origin *i.e.*, at $t = 0$, $x = 0$, $v = 0$

Given $$a = 0.5 + 0.1t \text{ m/s}^2$$

or $$dv/dt = 0.5 + 0.1t \text{ or; } dv = (0.5 + 0.1\ t)\ dt$$

Integrating we get,

$$v = 0.5\,t + \frac{0.1\,t^2}{2} + k;$$

where k is constant of integration, determined by boundary condition.

At start; $v = 0$ and $t = 0$; so above equation yields; $k = 0$

So, $$v = 0.5t + \frac{0.1\,t^2}{2}$$

At $t = 2$ sec; $v = 0.5 \times 2 + ½ \times 0.1 \times 4$

or $v = 1 + 0.2 = 1.2$ m/sec

Again rewriting v,

$$\frac{dx}{dt} = 0.5t + \frac{0.1\,t^2}{2};\ i.e.,\ dx = \left(0.5t + \frac{0.1\,t^2}{2}\right)dt$$

integrating, one gets

$$x = \frac{0.5\,t^2}{2} + \frac{0.1}{2}\,\frac{t^3}{3} + k$$

at start, $x = 0$, $t = 0$; so above equation gives; $k = 0$

so $$x = \frac{0.5\,t^2}{2} + \frac{0.1}{2}\,\frac{t^3}{3}$$

and at $t = 2$;

$$x = \frac{0.5}{2} \times 4 + \frac{0.1}{2} \times \frac{8}{3}$$

$$= 1 + .133$$

or $x = 1.133$ m

Q. 53. *Two blocks of masses m_1 and m_2 are connected by a massless spring on a horizontal frictionless table. Find the ratio of their acceleration a_1 and a_2 after they are pulled apart and released.* **(Lucknow, 1988)**

Solution. Situation is shown in the Fig. 36 below.

Here F_1 and F_2 are forces in opposite directions on m_1 and m_2.

T_1 and T_2 are resulting tension in the spring due to F_1 and F_2.

T_1, F_1 are one action-reaction pair

T_2, F_2 are other action-reaction pair

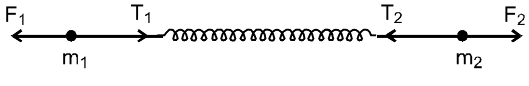


Fig. 36

As the system is in equilibrium

$$F_1 + F_2 = 0 \text{ and } T_1 + T_2 = 0$$

i.e., $$m_1 a_1 + m_2 a_2 = 0$$

i.e., $$\frac{a_1}{a_2} = \frac{-m_2}{m_1}$$

Thus, the ratio of the acceleration is inversely proportional to their masses. The negative sign suggests that acceleration of the block m_1 and m_2 are oppositely directed.

Q. 54. *The displacement (x) of a particle moving in one dimension, under the action of a constant force is related to time (t) by the equation* $t = \sqrt{x} + 3$, *where x is in meter and t in sec. Find displacement of particle when its velocity is zero.*

Solution. Since

$$t = \sqrt{x} + 3$$

$$x = (t - 3)^2 = t^2 - 6t + 9$$

Differentiating

$$v = dx/dt = 2t - 6$$

When $v = 0$, $t = 3$ sec.

∴ Displacement at time $t = 3$ sec. Will be

$$x = (3)^2 - 6(3) + 9$$

$$= 0$$

Q. 55. *A bullet of mass 20 gm, moving with velocity 16 m/s penetrates a sand bag and comes to rest in 0.05 sec. Find (i) depth of penetration, (ii) average retarding force of sand.*

(Lucknow, 1989)

Solution. Given, $u = 16$ m/s, $v = 0$ and $t = 0.05$ sec.

The retardation in sand is given by

$$0 = u - at$$

i.e.,

$$a = 16/.05 = 320 \text{ m/s}^2$$

(*i*) The depth of penetration is obtained by using equation

$$v^2 = u^2 - 2as$$

or

$$0 = (16)^2 - 2(320)s$$

or

$$s = \frac{16 \times 16}{2 \times 320} = 0.4 \text{ m}$$

(*ii*) The average retarding force of sand is

$$F = \frac{mv - mu}{t}$$

$$= m\left(\frac{v - u}{t}\right) = ma$$

$$= (20 \times 10^{-3}) \times 320$$

$$= 6.4 \text{ Newton}$$

Q. 56. *A light string which passes over a frictionless pulley, and hangs vertically on each side of it, carries at each end a mass of 240 gm. When a rider of 10 gm is placed over one of the masses, the system moves from rest a distance of 40 cm in 2 seconds. Calculate g.*

Solution. Suppose P and Q are two masses each, equal to 0.240 kg. They are hanging from the pulley as shown. when rider R of mass 0.010 kg. is placed on mass Q, the system moves through 0.40 m in 2 sec. The final position is shown by dotted lines.

Let T be the tension in string and a be the acceleration either P or Q. Since P moves upward we have for its motion;

$$T - 0.240\ g = 0.240\ a \qquad ...(i)$$

and as Q moves downwards, therefore

$$(0.250\ g - T) = 0.250\ a \qquad ...(ii)$$

from (i) and (ii); eliminating T;

$$0.250\ g - 0.240\ g = 0.240\ a + 0.250\ a$$

or $$0.010\ g = 0.490\ a$$

$$g = 49\ a$$

To find a, we have

$$s = 0.40 \text{ m},\ t = 2,\ u = 0,\ a = ?$$

Using the formula $$s = ut + \tfrac{1}{2}\ at^2$$

$$0.40 = 0 + \frac{1}{2} a \times 4 = 2a$$

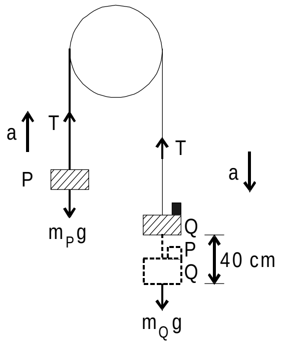


Fig. 36(a)

or $$a = 0.20 \text{ m/sec}^2$$

So, $$g = 49\ a = 49 \times 0.20 \text{ m/sec}^2 = 9.80 \text{ m/sec}^2$$

Q. 57. *A uniform rope of length D resting on a frictionless horizontal surface is pulled at one end by a force P. What is the tension in the rope at a distance 'd' from the end where the force is applied?*

Solution. Suppose the rope to be divided into two segments A and B of length 'd' and (D – d). Let T be the tension at the boundary of A and B. If M is total mass of rope and 'a', the acceleration produced a = P/M in system,

Now the equation of motion of segment B is

$$T = \frac{M}{D}(D - d)\left(\frac{P}{M}\right)\ ;\ \left(\frac{M}{D} = \text{mass/length}\right)$$

$$= P\left[\frac{D - d}{D}\right]$$

$$T = P\left[1 - \frac{d}{D}\right]$$

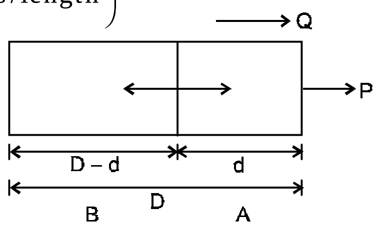


Fig. 37

Q. 58. *A uniform rod of length L and density* ρ *is being pulled along a smooth floor with a horizontal acceleration a. Find the magnitude of stress at the transverse cross-section through the mid-point of rod?* **(I.I.T. 1993)**

Solution. If A is cross sectional area of rod, its mass M = AL.ρ

∴ Force acting forward on rod, F = A Lρα

If p is stress at the mid-cross section of rod, the resultant force on right half length 'b' of rod will be

$$= F - P.A$$

∴ From Newton's second Law,

$$F - PA = \text{mass of length } b \times acc^n$$

$$A\,L\,\rho\alpha - PA = \left(L\frac{A}{2}\rho\right)\alpha$$

i.e., $$P = \frac{1}{2}L\,\rho\alpha$$

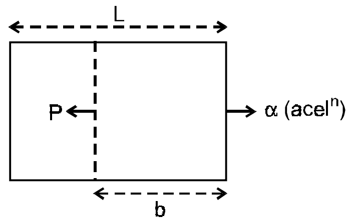


Fig. 38

Q. 59. *A mass of 500 gm is placed on a smooth table with a string attached to it. The string goes over a frictionless pulley and is connected to another mass of 200 gm. At t = 0, the mass of 500 gm is at a distance of 200 cm from the end and moving with a speed of 50 cm/sec towards the left (see Fig. 39). what will be its position and speed at t = 1 sec.*

Solution. Suppose 'a' represent the acceleration of both the block at $t = 0$

The equation of motion of 200 gm is

$$200\,g - T = 200\,a \quad ...(i)$$

The equation of motion of 500 gm is

$$T = 500\,a \quad ...(ii)$$

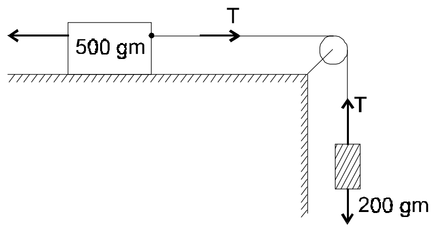


Fig. 39

Adding equation (*i*) and (*ii*) we get

$$200\,g - 500\,a = 200\,a \quad ...(iii)$$

i.e., $$a = 2/7\,g$$

so $$a = (2/7) \times 980 = 280 \text{ cm/sec}^2$$

for displacement of block of 500 gm

Using, $$s = ut + \frac{1}{2}at^2$$

$$u = -50 \text{ cm/sec}, \; a = 280 \text{ cm/sec}^2, \; t = 1 \text{ sec}$$

∴ $$s = -50 \times 1 + \frac{1}{2} \times 280 \times t^2$$

$$= -50 + 140$$

$$s = +90 \text{ cms}$$

Thus, the block of 500 gm has moved 90 cms to the right of its initial position. The velocity v of block of 500 gm is then

$$v = u + at; \quad u = -50 \text{ cm/sec}, \; a = 280 \text{ cm/sec}^2, \; t = 1 \text{ sec}$$

$$= -50 + 280 \times 1$$

$$= 230 \text{ cm/sec (to the right)}$$

Q. 60. *A uniform flexible chain of length L and mass per unit length λ passes over a smooth massless small pulley. At an instant a length x of the chain hangs on one side of the pulley and length (L – x) on the other side. Find the instantaneous acceleration. Also find the time interval in which x changes from L/4 to 0, if the chain moves with a negligible velocity at x = L/2.* **(Lucknow, 1987)**

Solution. Suppose instantaneous common acceleration is 'a'. Weight of chain of length $x = x\,\lambda\,g$. Net upward force on chain of length $x = T - x\lambda g = x\lambda a$

or, $$T = x\lambda a + x\lambda g = x\lambda(a + g) \quad ...(i)$$

weight of chain of length $(L - x) = (L - x)\,\lambda g$

Net force acting downward on chain of length (L – x) is

$$(L - x)\,\lambda g - T = (L - x)\,\lambda a \quad ...(ii)$$

Putting T from Eq. (i) in Eq. (i), we get

$$(L - x)\,\lambda g - x\lambda\,(a + g) = (L - x)\,\lambda a$$

or $$L\lambda g - x\,\lambda g - a\,x\lambda - g\,x\lambda = (L - x)\,\lambda a$$

or $$L\lambda g - 2x\lambda g = (L - x)\,\lambda . a + a . x\lambda$$

or $$\lambda g\,(L - 2x) = a\lambda L$$

or $$a = \frac{\lambda g\,(L - 2x)}{\lambda L} = \frac{g}{L}\,(L - 2x)$$

or $$a = \frac{g}{L}\,(L - 2x)$$

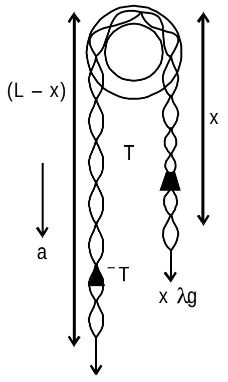


Fig. 40

Q. 61. *Two rough planes, inclined at 30° and 60° to horizontal and of same height are placed back to back. Two masses of 5 kg and 10 kg are placed on the faces and connected by a string passing over the top of the planes. If coefficient of friction is* $1/\sqrt{3}$, *find the resultant acceleration.*

Solution. The 10 kg mass moves down and 5 kg mass moves up the plane.

Let; a = acceleration of system, T = the tension of the string and N_1 and N_2 be the normal reaction of the plane.

For 5 kg mass, ($\Sigma F = ma$)

$$T - (\mu N_1 + 5\,g \sin 30°) = 5a \quad ...(i)$$

and $$5\,g \cos 30° = N_1 \quad ...(ii)$$

For 10 kg mass

$$10\,g \sin 60° - T - \mu\,N_2 = 10\,a \quad ...(iii)$$

$$10\,g \cos 60° = N_2 \quad ...(iv)$$

(ii) and (i) gives

$$T - 5\,g\,(\sin 30° + \mu \cos 30°) = 5\,a \quad ...(v)$$

and (iv) in (iii) yields

$$10g\,(\sin 60° - \mu \cos 60°) - T = 10\,a \quad ...(vi)$$

So, (v) and (vi) give

$$10g\left(\frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{3}}\right) - 5g\left(\frac{1}{2} + \frac{1}{2}\right) = 15\,a$$

or $$a = \frac{2\sqrt{3} - 3}{9}\,g$$

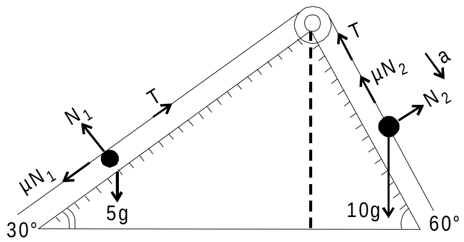


Fig. 41

Q. 62. *A block of mass 2 kg slides on an inclined plane which makes an angle of 30° with the horizontal. The coefficient of friction between the block and surface is* $\sqrt{3/2}$;

(i) *What force should be applied to the block so that block moves down without any acceleration?*

(ii) *What force should be applied to the block so that block moves up without any acceleration.*

(iii) *Calculate the ratio of power, needed in the above two cases: if the block moves with the same speed in both the cases.*

Solution. (*i*) The figure 42 shows the situation, in which various forces are depicted, Resolving the forces along and perpendicular to the plane; we get

$$\mu N = P_1 + mg \sin 30° \qquad ...(i)$$

$$N = mg \cos 30° \qquad ...(ii)$$

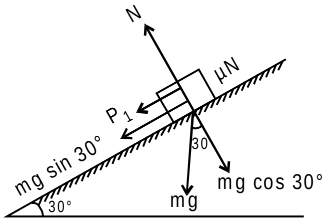


Fig. 42

(*ii*) and (*i*) yields

$$\mu\, mg \cos 30° = P_1 + mg \sin 30°$$

or
$$P_1 = mg\,(\mu \cos 30° - \sin 30°)$$

$$= 2 \times 9.8 \left(\frac{\sqrt{3}}{2} . \sqrt{\frac{3}{2}} - \frac{1}{2}\right)$$

$$= \frac{19.6\left(3 - \sqrt{2}\right)}{2\sqrt{2}} = \frac{19.6 \times 1.586}{2 \times 1.414}; \qquad P_1 = 11 \text{ Newton}$$

(*ii*) The forces acting in this case are shown in figure 43. Resolving along and perpendicular to the rough inclined plane; one gets;

$$P_2 = mg \sin 30° + \mu N \qquad ...(iii)$$

$$N = mg \cos 30° \qquad ...(iv)$$

(*iv*) and (*iii*) gives

$$P_2 = mg\,(\sin 30° + \mu \cos 30°)$$

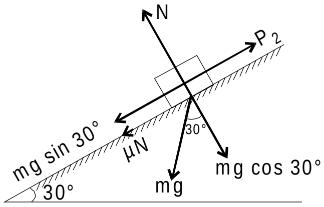


Fig. 43

or
$$= 2 \times 9.8 \left[\frac{1}{2} + \sqrt{\frac{3}{2}} . \frac{\sqrt{3}}{2}\right]$$

$$= 2 \times 9.8 \left[\frac{\sqrt{2} + 3}{2\sqrt{2}}\right] = \frac{9.8}{1.414}\, 4.414$$

$$P_2 = 30.6 \text{ Newtons}$$

(*iii*) As Power = (force × displacement)/time = force × velocity

As the velocity is same in both the cases, the ratio of powers shall be equal to the ratio of forces applies *i.e.*,

$$\frac{P_1}{P_2} = \frac{F_1}{F_2} = \frac{11}{30.6} = 0.36$$

63. *A block weighing 400 kg rests on a horizontal surface and supports on top of it, another block of weight 100 kg as shown. The block W_2 is attached to a vertical wall by a string 6 m long. Find the magnitude of the horizontal force P applied to the lower block as shown in figure which shall be necessary so that slipping of W_2 occurs.*

(μ for each surfaces in contact is ¼)

Solution. Forces on block W_2 are shown below:

It may be seen, that

$$T \cos\theta = \mu N_2 \qquad ...(i)$$

$$N_2 + T \sin\theta = 100g \qquad ...(ii)$$

From figure 44(*a*) :

$$\sin\theta = \sqrt{11}/6 \ : \cos\theta = \frac{5}{6}; \ \tan\theta = \sqrt{11}/5$$

From (*i*) $\quad T = \dfrac{\mu N_2}{\cos\theta}$; then (*ii*) is,

$$N_2 + \mu N_2 \tan\theta = 100\ g$$

or $$N_2 (1 + \mu \tan\theta) = 100\ g$$

or $$N_2 = \frac{100\ g}{1 + \mu \tan\theta} \qquad ...(iii)$$

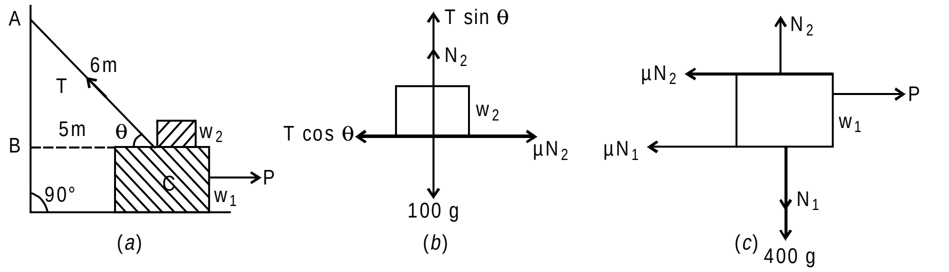


Fig. 44

Forces on block W_1 are as shown in Fig. 44(*b*)

As observed for block W_1:

$$P = \mu N_2 + \mu N_1$$

$$= \left[\frac{\mu \times 100g}{1 + \mu\tan\theta} + \mu\left(\frac{100\ g}{1 + \mu\tan\theta} + 400\ g\right)\right]$$

(as $N_2 = N_1 + 400$ g)

$$P = \mu g \left[\frac{\mu \times 100\ g}{1 + \mu\ \tan\theta} + \left(\frac{100}{1 + \mu\ \tan\theta} + 400\right)\right]$$

$$= \frac{9.8}{4}\left[\frac{200 \times 20}{20 + \sqrt{11}} + 400\right] = 9.8\left[\frac{1000}{20 + 3.316} + 100\right]$$

$$= 142.8 \text{ kg wt}$$

Q. 64. *A block of mass 3 kg rests on a rough horizontal table, the coefficient of friction between the surface is 0.5. A massless string is tied to the block which passes over a smooth light pulley at the edge of the table and supports a block of 5 kg. Find the acceleration of the system, if any.*

Solution. For 3 kg mass, eqn. of motion is,

$$T - \mu N = 3a \qquad ...(i)$$

For 5 kg mass,

$$5g - T = 5a \qquad ...(ii)$$

and

$$\mu N = 0.5 \times 3 \times 9.8$$

$$= 14.7 \text{ Newton} \qquad ...(iii)$$

adding (*i*) and (*ii*) and using (*iii*)

$$5g - \mu N = 8a$$

or

$$5 \times 9.8 - 14.7 = 8a$$

or

$$a = \frac{49 - 14.7}{8} = \frac{34.3}{8}$$

$$a = 4.29 \text{ m/s}^2$$

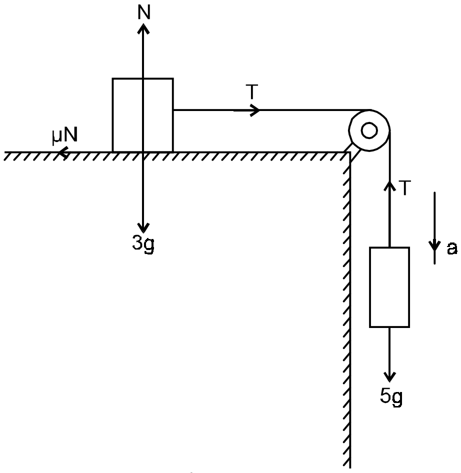


Fig. 45

Q. 65. *Find the least force required to drag a particle of mass m along a horizontal surface.*

Solution. Refer Fig. 46

P = force acting at $\angle\theta$, R = resultant of N and F_s, λ = angle of friction

Resolving the forces horizontally and vertically

$$P \cos\theta - R \sin\lambda = 0 \qquad ...(i)$$

$$P \sin\theta + R \cos\lambda = mg \qquad ...(ii)$$

from (*i*)

$$R = \frac{P \cos\theta}{\sin\lambda} \qquad ...(iii)$$

then (*ii*) is;

$$P \sin\theta + \frac{P\cos\theta}{\sin\lambda} . \cos\lambda = mg$$

or,

$$P \sin\theta + P \cos\theta \cot\lambda = mg \qquad ...(iv)$$

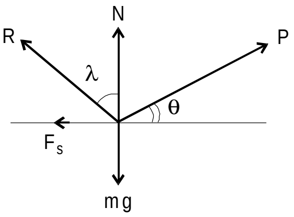


Fig. 46

Differentiating (*iv*) w.r.t. to θ

$$\sin\theta \frac{dP}{d\theta} + P\cos\theta + \cot\lambda \left(\cos\theta \frac{dP}{d\theta} - P \sin\theta\right) = 0$$

$$\frac{dP}{d\theta} (\sin\theta + \cos\theta \cot\lambda) + P(\cos\theta - \cot\lambda \sin\theta) = 0$$

$$\frac{dP}{d\theta} = \frac{P\,(\sin\theta - \cot\lambda - \cos\theta)}{\sin\theta + \cos\theta \cot\lambda} \qquad ...(v)$$

for P to be minimum; $dP/d\theta = 0$ *i.e.*,

$$\sin\theta \cot\lambda - \cos\theta = 0; \text{ or; } \tan\theta = \tan\lambda,$$

i.e.,

$$\theta = \lambda \qquad ...(vi)$$

Putting (*vi*) and (*iv*)

$$P \sin \lambda + P \cos \lambda \cot \lambda = mg$$

Multiplying both sides by sin λ,

$$P \sin^2 \lambda + P \cos^2 \lambda = mg \sin \lambda$$

or $$P = mg \sin \lambda$$

Q. 66. *A block slides down an inclined plane of slope θ, with a constant velocity. It is then projected up the same plane with an initial velocity v_0. How far will it move before coming to rest. Will it slide down again ?* **(Lucknow, 1982, 1986)**

Solution. We have, while sliding as shown in Fig. 47(*a*) $mg \sin \theta = \mu N$, $mg \cos \theta = N$

i.e., $$\mu = \tan \theta \qquad ...(i)$$

Now, in upward motion on same plane shown in Fig. 47(*b*)

$$\text{Retarding Force} = mg \sin \theta + \mu N; \ (N = mg \cos \theta)$$
$$= mg \sin \theta + \mu \, mg \cos \theta$$
$$= mg (\sin \theta + \mu \cos \theta)$$

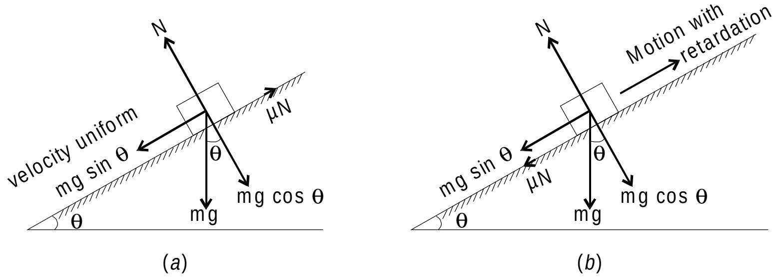


Fig. 47

Hence, retardation $= g (\sin \theta + \mu \cos \theta)$

$$= g (\sin \theta + \tan \theta \,.\, \cos \theta) = 2 g \sin \theta$$

using $$v^2 = u^2 + 2as;$$

with $$u = v_0; \ a = -2g \sin \theta, \ v = 0$$

$$0 = v_0^2 - 4 g \sin \theta \, x \, s$$

or $$s = \frac{v_0^2}{4 g \sin \theta}$$

The block will not slide down again since the component of force along the plane downwards will be just balanced by the force of the motion.

Q. 67. *A uniform rod rests in limiting equilibirum in contact with a floor and a vertical wall (rod in a vertical plane). Supposing the wall and floor to be unequally rough, compute the angle between rod and wall.*

Solution. Situation is depicted in the figure 48.

LK is rod; MS is wall; θ is angle between rod and the wall. Let, $2l$ = length of the rod; μ = coeff. of static friction between the rod and the wall; μ′ = coeff. of static friction between rod and floor.

For equilibrium in horizontal direction,

$$\mu' N' = N \quad \text{or} \quad N' = \frac{N}{\mu'} \qquad ...(i)$$

For equilibrium in vertical direction

$$N' + \mu N = W$$

or $$\frac{N}{\mu'} + \mu N = W \quad \text{or,} \quad N\left(\frac{1}{\mu'} + \mu\right) = W$$

or $$N = \left(\frac{1 + \mu\mu'}{\mu'}\right) = W \qquad ...(ii)$$

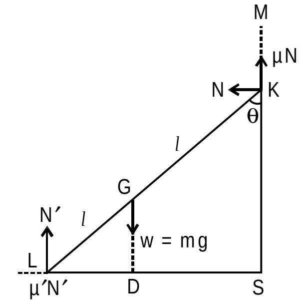


Fig. 48

Now taking moment about point L, we have

$$W.LD = N.KS + \mu\ N.LS$$

but as LD = $l \sin\theta$; KS = $2l\cos\theta$; and LS = $2l\sin\theta$

above equation becomes

$$W.l \sin\theta = N.2l\cos\theta + \mu N\ 2l \sin\theta$$

and putting the value of N from (*ii*)

$$W.l\sin\theta = \frac{\mu' W}{1 + \mu\mu'}.2l\cos\theta + \mu.\frac{\mu' W}{1 + \mu\mu'}.2l\sin\theta$$

or $$W.l\sin\theta - \frac{2\mu\mu' W \sin\theta\, l}{1 + \mu\mu'} = \frac{2\mu' Wl\cos\theta}{1 + \mu\mu'}$$

$$Wl\left(1 - \frac{2\mu\mu'}{1 + \mu\mu'}\right)\sin\theta = \frac{2\mu' Wl\cos\theta}{1 + \mu\mu'} \quad \text{or} \quad \tan\theta = \frac{2\mu'}{1 - \mu\mu'} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{2\mu'}{1 - \mu\mu'}\right)$$

Q. 68. *Forces required to move a body on a rough inclined plane with a uniform velocity in the upward and downward direction are in the ratio 2:1. Find the inclination of the plane if the coefficient of friction is 0.3.* **(Lucknow, 1987)**

Solution. (*i*) Uphill uniform motion: Net force in uphill direction is zero because body moves with constant velocity; *i.e.*,

$$F_1 - (mg \sin\theta + \mu N) = 0$$

or $$F_1 = mg\sin\theta + \mu\ mg\cos\theta \quad (\text{or } N = mg\cos\theta)$$

or $$F_1 = mg(\sin\theta + \mu\cos\theta)$$

(*ii*) Downhill uniform motion: Again, resultant force in downhill direction is zero as the body moves with constant velocity. If F_2 is applied downward force, then

$$(F_2 + mg\sin\theta) - \mu N = 0$$

or $$F_2 = \mu\, mg\cos\theta - mg\sin\theta = 0$$

or $$F_2 = mg(\mu\cos\theta - \sin\theta)$$

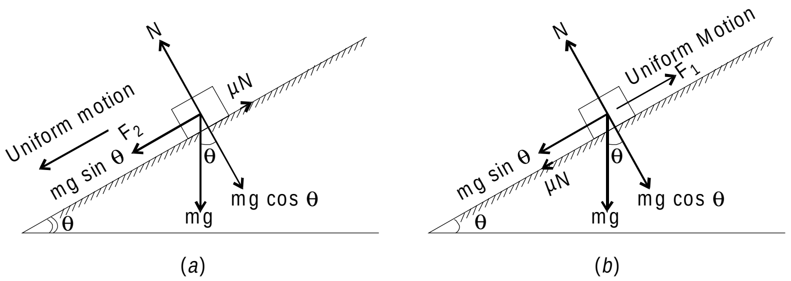


Fig. 49

According to question

$$\frac{F_1}{F_2} = \frac{2}{1}, \text{ i.e., } F_1 = 2F_2$$

$$mg\,(\sin\theta + \mu\cos\theta) = 2\,mg\,(\mu\cos\theta - \sin\theta)$$

or $$\sin\theta + \mu\cos\theta = 2\,\mu\cos\theta - 2\sin\theta$$

or $$3\sin\theta = \mu\cos\theta$$

or $$\tan\theta = \mu/3 = 0.3/3 = 0.1;\ \theta = \tan^{-1}(0.1)$$

Q. 69. *A piece of ice slides down a 45° incline in thrice the time it takes to slide down on a frictionless 45° incline. Find the coefficient of kinetic friction between ice and the incline.*

(Lucknow, 1983, 1988)

Solution. Frictionless incline:

Let length of incline be s, initial velocity = 0

$$\text{downward force} = mg\sin 45° = ma$$

$$a = \frac{mg\sin 45°}{m} = \frac{g}{\sqrt{2}}$$

Now, if t_1 is the time taken to slide down the incline, using

$$S = ut + \frac{1}{2}at^2; \text{ with } u = 0,\ t = t_1,\ a = g/\sqrt{2}$$

We have,

$$s = \frac{1}{2}\left(\frac{g}{\sqrt{2}}\right)t_1^2 \quad \ldots(i)$$

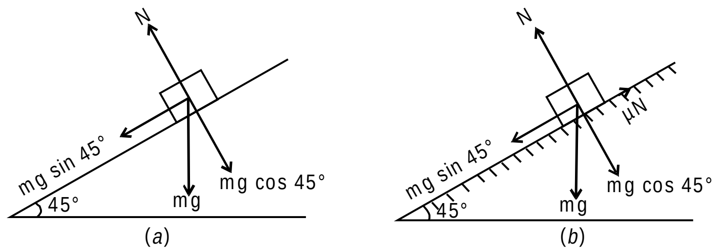


Fig. 50

For incline having friction:

$$\text{Downward force} = mg \sin 45° - \mu N$$
$$= mg \sin 45° - \mu\ mg \cos 45°$$
$$ma = mg (\sin 45° - \mu \cos 45°)$$
$$= mg\left(\frac{1}{\sqrt{2}} - \mu \frac{1}{\sqrt{2}}\right)$$

so, downward acceleration; $a = \frac{g}{\sqrt{2}}(1 - \mu)$

if t_2 is the time the body takes to slide down the incline using, $s = ut + \frac{1}{2}at^2$; with $u = 0$, $t = t_2$

and $a = \frac{g}{\sqrt{2}}(1 - \mu)$

$$s = 0 + \frac{1}{2} \times \frac{g}{\sqrt{2}}(1 - \mu)t_2^2 \quad ...(ii)$$

from (*i*) and (*ii*)

$$\frac{1}{2}\left(\frac{g}{\sqrt{2}}\right)t_1^2 = \frac{g}{2\sqrt{2}}(1 - \mu)t_2^2$$

or $t_1^2 = (1 - \mu)t_2^2$ ...(*iii*)

But according to question, $t_2 = 3\ t_1$ then (*iii*) is

$$t_1^2 = (1 - \mu)(3t_1)^2 \text{ or } 1 - \mu = 1/9$$

or $\mu = 8/9 = 0.889$

Q. 70. *Two blocks are kept on a horizontal smooth table as shown. Find out the minimum force that must be applied on 10 kg block to cause 6 kg block just to slide. Coeff. of static friction between two blocks is 0.3.* **(Lucknow, 1994)**

Solution. Let F be the minimum force applied on 10 kg block, the acceleration produced in system is, $\alpha = F/(10 + 6) = F/16$. This gives a reactionary force 6 α which will tend to slide the 6 kg block where as frictional force will tend to stop it. Hence for 6 kg not to move relative to 10 kg.

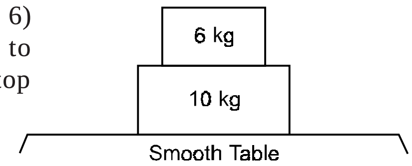


Fig. 51

$$6\alpha = \mu_s \times 6 \times g$$
$$(6F/16) = 0.3 \times 6 \times 9.8 \text{ or}$$
$$F = 0.3 \times 16 \times 9.8 = 47.04 \text{ Newton}$$

Q. 71. *Determine the frictional force of air on a body of mass 1 kg falling with an acceleration of 8 m/sec*^{2} *(g = 10 m/sec*^{2}*).*

$$W' = mg' = mg - F_r$$

so $$F_r = mg - mg' = 10 - 8 = 2 \text{ Newton}$$

Q. 72. *Block B in the figure 52 has mass 160 kg. The coefficient of static friction between block and table is 0.25, what is the maximum mass of block A for which the system will be in equilibrium ?* **(Lucknow, 1988)**

Solution. Let m be the maximum mass of block A for which system is in equilibrium. Various forces are shown in the figure 52. For equilibrium;

$$T \cos 45° = \mu N \quad ...(i)$$

$$N = Mg \quad ...(ii)$$

and $$mg = T \sin 45° \quad ...(iii)$$

(i) and (ii) give, $T \cos 45° = \mu M g$

$$T/\sqrt{2} = 0.25 \times 160 \times 9.8$$

or $$T = 0.25 \times 160 \times 9.8 \times \sqrt{2} \quad ...(iv)$$

Given $M = 160$ kg; $g = 9.8$ m/s^2; $\mu = 0.25$

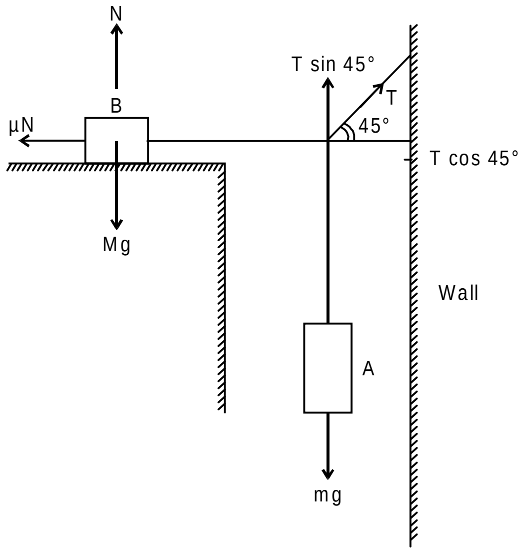


Fig. 52

Putting (iv) in (iii)

$$mg = 0.25 \times 160 \times 9.8 \times \sqrt{2} \times 1/\sqrt{2}$$

or $$m = 0.25 \times 160 \text{ kg or } m = 40 \text{ kg.}$$

Q. 73. *A body rests upon inclined plane and will just slide down the plane when the slope of the plane is 30°. Find the acceleration of the body down the plane when the slope is increased to 60°.*

Solution. When body just slides down the inclined plane, the inclination of the plane is 30°;

so $$mg \sin 30° = \mu N$$

or $\mu = \tan 30°$ (as N = $mg \cos 30°$)

When slope is 60°; the body moves down the plane.
The force acting down is:

$$mg \sin 60° - \mu N = ma$$

(with N = $mg \cos 60°$)

so, $m(g \sin 60° - \mu\, g \cos 60°) = ma$

where $a = acc^n$

or $a = g(\sin 60° - \tan 30° \cos 60°)$

$$= 9.8\left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}} \times \frac{1}{2}\right) = 5.66 \text{ m/s}^2$$

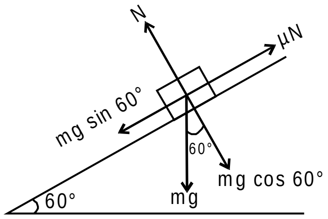


Fig. 53

Q. 74. *A body with a mass of 0.84 kg is on a plane inclined at 30° to the horizontal. What force must be applied on the body so that it moves with uniform acceleration of 0.12 m/sec². (a) uphill (b) downhill; (The coefficient of sliding friction with the plane is 0.32).*

(Lucknow, 1978, 1982)

Solution. (*a*) Uphill motion: Let F be the applied force so that body moves uphill with an acceleration = 0.12 m/sec².

$$F - (\mu N + mg \sin 30°) = ma$$

or $F = ma + (\mu\, mg \cos 30° + mg \sin 30°)$

$= ma + mg(\mu \cos 30° + \sin 30°)$

$= 0.84 \times 0.12 + 0.84 \times 9.8\,[0.32 \times (\sqrt{3}/2) + (1/2)]$

or $F = 0.1008 + 6.397 = 6.498$ Newton

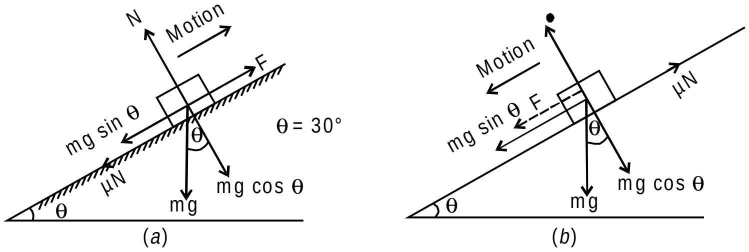


Fig. 54

(*b*) Downhill motion: Suppose F is the applied force down the hill so that body moves with an acceleration = 0.12 m/s² then,

$$F + mg \sin\theta - \mu N = ma$$

or, $F = ma - mg \sin 30° + \mu\, mg \cos 30°$

(as N = $mg \cos 30°$)

$= ma - mg(\sin 30° - 0.32 \cos 30°)$

$= 0.84 \times 0.12 - 0.84 \times 9.8\,[(1/2) - 0.32\,(\sqrt{3}/2)]$

$= 1.7330$ Newtons; *i.e.*, in upward direction.

Q. 75. *A body of 5 kg weight is just prevented from sliding down a rough inclined plane by a force of 2 kg weight acting up the line of greatest slope. When this force is increased to 3 kg weight; the body just begins to slide up the plane. Prove that coefficient of friction between the body and the plane is* $\sqrt{3}/15$.

Solution. Body is just prevented from sliding down the plane when 2 kg force acts up the inclined plane; it suggests that friction is limiting and acts up the plane. Resolving the forces on the body, along and perpendicular to the plane;

$$N = 5\,g\cos\theta;\ \text{and}\ \mu N + 2g = 5\,g\sin\theta \qquad ...(i)$$

i.e., $$\mu\, 5g\cos\theta + 2g = 5\,g\sin\theta$$

or $$5\sin\theta - 5\,\mu\cos\theta = 2 \qquad ...(ii)$$

When 3 kg weight is applied; the body just slides up the plane. The friction is again limiting but now will act down the inclined plane.

Resolving the forces along and perpendicular to the inclined plane;

$$3g = 5g\sin\theta + \mu N \ \text{and}\ N = 5\,g\cos\theta \qquad ...(iii)$$

i.e., $$3g = 5g\sin\theta + \mu\, 5g\cos\theta$$

or $$5\sin\theta + \mu\, 5\cos\theta = 3 \qquad ...(iv)$$

adding (*ii*) and (*iv*)

$$10\sin\theta = 5 \ \text{or}\ \sin\theta = \frac{1}{2} \ \text{or}\ \theta = 30°$$

then from (*iv*)

$$5 \times \frac{1}{2} + 5\mu \times (\sqrt{3}/2) = 3$$

or $$\mu = \frac{1}{5\sqrt{3}} = 0.115$$

Q. 76. *A block is released from the rest at the top of a frictionless inclined plane 16 m long. It reaches the bottom 4.0 sec later. A second block is projected up the plane from the bottom at the instant the first block is released in such a way that is returns to the bottom simultaneously with the first block.*

(*i*) *Find the acceleration of each block on the incline.*

(*ii*) *What is the initial velocity of the second block.*

(*iii*) *How far up the inclined plane does it travel?*

Solution. Taking the displacement down the incline to be positive the acceleration of both the blocks when they return simultaneously is, $a = g\sin\theta$; from $s = ut + \frac{1}{2}at^2$ with $u = 0$, $t = 4$ sec., $s = 16$ m.

$$16 = 0 + \frac{1}{2}a4^2$$

$$a = (2 \times 16)/16 = 2\ \text{m/s}^2$$

direction of v_0 (up the plane) is taken as positive, then acceleration is negative *i.e.*, $a = -2$ m/s^2 using $t = 4$, $u = v_0$

then $$s = ut + \frac{1}{2}at^2; \text{ with } a = -2 \text{ m/s}^2,\ t = 4,\ u = v_0$$

i.e., $$0 = v_0 \times 4 - \frac{1}{2} \times (-2) \times 4^2$$

or $$v_0 = 4 \text{ m/s}$$

(*iii*) Supposing the second block to travel a distance *s*, up the incline where it comes to rest, gives; $$v^2 = u^2 + 2as$$

with $$v = 0,\ u = v_0 = 4 \text{ m/s},\ a = -2 \text{ m/s}^2$$

then $$0 = 4^2 + 2 \times (-2)s$$

or $$s = 4 \text{ m}$$

Q. 77. *Two blocks of mass 2.9 kg and 1.9 kg are suspended from rigid support by two extensible wires each 1 meter long. The upper wire has negligible mass and lower wire has uniform mass of 0.2 kg/m. The whole system has upward acceleration 0.2 m/s^2 as shown.*

Find (i) Tension at mid point of lower wire

(ii) Tension at mid point of upper wire **(I.I.T., 1989)**

Solution. Since upper wire has negligible mass, the tension T_1 in upper wire will be same all along its length.

If T_2 is tension at the upper point of lower wire, for equilibrium of m_1.

$$T_1 - T_2 - m_1 g = m_1 a \quad ...(i)$$

and taking m_3 as mass of lower wire, for equilibrium of m_2,

$$T_2 - m_2 g - m_3 g = (m_2 + m_3)\, a \quad ...(ii)$$

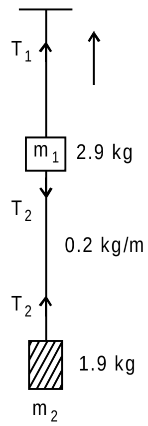


Fig. 55

Adding the two equations

$$T_1 - (m_1 + m_2 + m_3)\, g = (m_1 + m_2 + m_3)\, a$$

i.e., $$T_1 = (m_1 + m_2 + m_3)\,(g + a)$$
$$= (2.9 + 1.9 + 0.2)\,(9.8 + 0.2)$$
$$= 5 \times 10 = 50 \text{ Newton}$$

This will be the tension at mid point of upper wire.

Putting value of T_1 in equation (*i*)

$$50 - T_2 = 2.9\,(9.8 + 0.2) = 29$$

i.e., $$T_2 = 21 \text{ Newton}$$

At mid point tension will be less than at upper point since only half length of lower wire will contribute to tension due to its own weight.

i.e., $$T_m + \left(0.2 \times \frac{1}{2}\right) \times (9.8 + 0.2) = 21$$

or $$T_m + 0.1 \times 10 = 21$$

i.e., $$T_m = 20 \text{ Newton}$$

Q. 78. *How can a mass of 100 kg be lowered with a massless rope which can support, without breaking, a mass of 80 kg only.* **(Lucknow, 1994)**

Solution. Let, the 100 kg mass be lowered with aceleration 'a' then taking T $\leq$ 80 × 9.8 Newton,

the equilibrium of motion for the system will be $100g - T = 100\ a$

taking maximum possible of T, we get

$$100\ g - 80\ g = 100\, a$$

or
$$a = (1/5)\ g$$

Hence, the given mass can be lowered with acceleration of $g/5$ or more than this.

Q. 79. *A stone weighing 2 kg and tied at the end of string of length 2.5 meters, revolves with frequency of 10 revolutions/sec. Find out the force on the stone as measured (i) in inertial frame (ii) in frame rotating with the string.*

Solution. (*i*) The actual force which makes body to rotate, in inertial frame, is the centripetal force $= -\dfrac{mv^2}{r} = -\,mr\omega^2$

(–*ve* sign shows that force is towards center).

So, force on stone, in inertial frame,

$$= -m\ \omega^2\ r = -2\ (2\pi \times 10)^2 \times 2.5$$

$$= -19719.2 \text{ Newton}$$

force being supplied by tension of string

(*ii*) The frame rotating with the string is non-inertial and in this frame the acceleration of the stone is 0. So total force in this frame = 0.

The equilibrium of the stone in rotating frame is obtained with the help of centrifugal force (Fictious force).

Q. 80. *Two blocks A and B are joined to each other by a string and a spring of force constant 1960 N/m which passes over a pulley, attached to another block C as shown. Block B slides over block C and A moves down along C with same speed. Coefficient of friction between blocks and surface of C is 0.2. Taking mass of block A as 2 kg, calculate.*

(*a*) *mass of block B*

(*b*) *energy stored in spring.*

Solution. When both block B and A at same speed the spring is in condition of maximum extension. Taking T as tension of spring and the string, and 'm' as the mass of block B, we have, from Fig. 56

For motion of B, $\quad T - \mu\ mg = 0$ ...(*i*)

For motion of A, $\quad T = 2g$ ...(*ii*)

combining (*i*) and (*ii*) we get

$$\mu\ mg = 2g$$

or, $\quad 0.2\ m = 2$ so, $m = 10$ kg

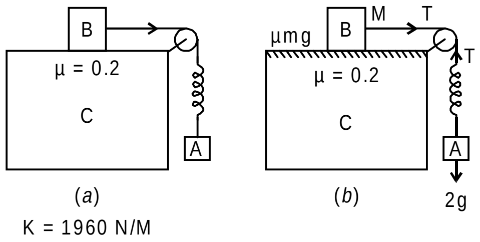


Fig. 56

Putting value of '*m*' in equation (*i*) or using equation (*ii*) we get

$$T = 2 \times 9.8 = 19.6 \text{ N}$$

The elongation of a string is therefore

$$x = \frac{\text{Tension}}{\text{Force} - \text{constant}} = \frac{19.6}{1960} = 0.01 \text{ m}$$

Hence, energy stored in spring

$$U = \frac{1}{2}kx^2 = \frac{1}{2} \times 1960 \times (0.01)^2$$

$$= 98 \times 10^{-3} = 0.098 \text{ joule}$$

Q. 81. *A body falling freely from a given height H hits an inclined plane in its path at a height 'h'. As a resutl of impact the direction of velocity of body becomes horizontal. For what value of (h/H), the body will take maximum time to reach the ground.*

Solution. Let t_1 be the time of fall through height (H – h) and t_2 the time of free fall through height 'h' then

$$H - h = \frac{1}{2}gt_1^2$$

and

$$h = \frac{1}{2}gt_2^2 \text{ (since after impact there is no initial velocity along vertical)}$$

$$t = t_1 + t_2 = \sqrt{2(H - h)/g} + \sqrt{2h/g}$$

or

$$t = \sqrt{\frac{2}{g}}\left[(H - h)^{1/2} + h^{1/2}\right]$$

for time to be maximum the condition is $dt/dh = 0$

$$-\frac{1}{2}(H - h)^{-\frac{1}{2}} + \frac{1}{2}h^{-\frac{1}{2}} = 0$$

or

$$\sqrt{h} = \sqrt{(H - h)}$$

which yields,

$$\frac{h}{H} = \frac{1}{2}$$

Q. 82. *A projectile is launched with an initial speed v in direction θ to the horizontal which hits an inclined plane of inclination β with horizontal (β < θ) and passing through the launching point. Derive an expression for the range R on this inclined plane. When will the projectile hit this plane?*

Solution. Let the projectile hit the inclined plane at M and OM = R (range). At the instant of hitting the plane, the x and y displacements are ON and NM respectively as shown in Fig. 57.

Now $$ON = R\cos\beta;\ NM = R\sin\beta$$

Clearly time taken to cover horizontal distance ON, $t = \dfrac{ON}{v\cos\theta} = \dfrac{R\cos\beta}{v\cos\theta}$ ...(*i*)

For vertical distance NM; we have s = NM; $u = v\sin\theta$, $a = -g$, $t = R\cos\beta / v\cos\theta$

Then $$s = ut + \frac{1}{2}at^2 \text{ gives;}$$

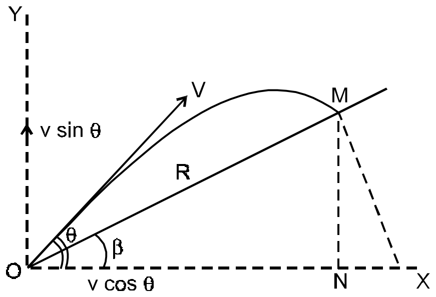


Fig. 57

$$R\sin\beta = v\sin\theta\,\frac{R\cos\beta}{v\cos\theta} - \frac{1}{2}g\frac{(R\cos\beta)^2}{(v\cos\theta)^2}$$

$$\frac{1}{2}g\,\frac{R^2\cos^2\beta}{v^2\cos^2\theta} = R(\tan\theta\cos\beta - \sin\beta)$$

or $$R = \frac{2v^2\cos^2\theta}{g\cos^2\beta}\,\frac{[\sin\theta\cos\beta - \cos\theta\sin\beta]}{\cos\theta}$$

or $$R = \frac{2v^2\cos\theta\sin(\theta-\beta)}{g\cos^2\beta}$$

Putting above in (*i*)

$$t = \frac{2v^2\cos\theta\sin(\theta-\beta)}{g\cos^2\beta}\,\frac{\cos\beta}{v\cos\theta} = \frac{2v\sin(\theta-\beta)}{g\cos\beta}$$

Q. 83. *A block Q (refer figure 58) having mass 0.2 kg. is placed on the top of block P of mass 0.8 kg. Coefficient of sliding friction between P and the table is 0.2 and that between P and Q is 0.5. The pulley is light and smooth. What horizontal force A will maintain the motion of P with uniform speed?*

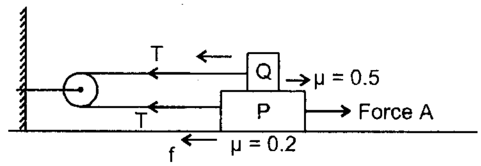


Fig. 58

Solution. Frictional force between P and Q = μ mg = 0.5 × 0.2 × 9.8 N which is equal to tension T. Its direction is towards left. Frictional force due to table on P; directed towards left is

$$f = (0.2 + 0.8) \times g \times 0.2$$
$$= 1.96 \text{ N}$$

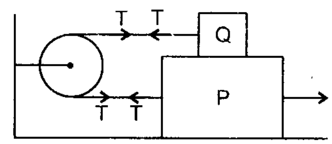


Fig. 59

so, far uniform motion of P; total left directed force on P should equal, right directed force *i.e.*, A = T + f

so;
$$A = (0.5 \times 0.2 \times 9.8) + 1.96 \text{ N}$$
$$= 2.94 \text{ N}$$

Q. 84. *A man of mass m climbs a rope which passes over a fixed pulley. The other end of the rope is attached to a block of mass M lying on a horizontal basement. The coefficient of friction between M and basement is μ. Find the acceleration of M, the acceleration of the man and the tension in the rope when the man moves (i) upward with an acceleration 'b' relative to the rope (ii) downward with same acceleration.*

Solution. The rope and the mass M have a common acceleration a_2. The acceleration of man relative to fix pulley in this case will be $(a_2 - b)$ downwards;

so;
$$T_2 - \mu Mg = Ma_2 \quad ...(i)$$
$$mg - T_2 = m(a_2 - b) \quad ...(ii)$$

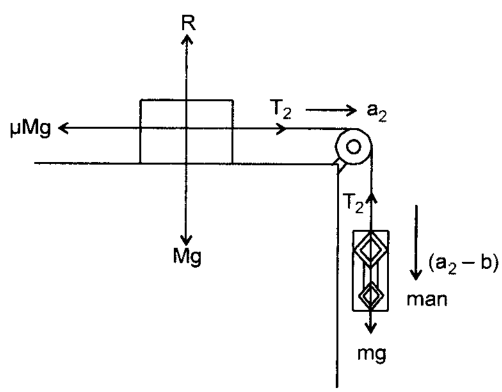


Fig. 60

equation $(i) + (ii)$ gives; $g(m - \mu M) = a_2(m + M) - mb$

or
$$a_2 = \frac{m(b+g) - \mu gM}{m+M} \quad ...(iii)$$

Putting (iii) in (ii)
$$T_2 = \frac{mMg\left[g(1+\mu) + 11b\right]}{m+M}$$

(iii) In this case b is replaced by $-b$ in above treatment.

Q. 85. *The water of 0.5 km wide river is flowing with a velocity of 4 km/hr. A boat man standing on one of the bank of the river wishes to take his boat to a point on the opposite bank exactly infront of his present position. He can row his boat with a velocity of 8 km/hr relative to water. In which direction should he row his boat. Obtain the time to cross the river.*

Solution. Let the boatman start from P and Q be his destination. Then resultant velocity (relative to earth *i.e.*, bank) should be in direction PQ. So boat should be rowed along PR. Let resultant velocity be v.

$$\text{Sin } \alpha = RQ/PR = 4/8 \quad i.e., \ \alpha = 30°$$

and
$$PQ = \sqrt{8^2 - 4^2} = \sqrt{48} = 6.9$$

i.e.,
$$v = 6.9 \text{ km/hr}$$

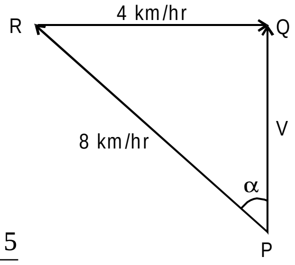


Fig. 61

So boat man should row in direction 30° with PQ, upstream,
Time to cross the river;

$$t = \frac{PQ}{\text{Speed-in-direction PQ}} = \frac{0.5}{6.9}$$

$$= \frac{1}{13.8} \text{ hr} = \frac{60}{13.8} \text{ min} = 4.35 \text{ min}$$

Q. 86. *An aeroplane is flying with velocity 70 km/hr in northeast direction. Wind is blowing at 30 km/hr from north to south. What is the resultant displacement of aeroplane in 4 hours?*

Solution. Let v_1 = velocity of aeroplane; v_2 = velocity of wind then resultant velocity of aeroplane, v is, $v = v_1 + v_2$

Resolving v_1 and v_2 in x and y direction;

$$v_1 = v_0 \cos 45° \ i + v_0 \sin 45° \ j$$

$$v_1 = \frac{70}{\sqrt{2}} (\hat{i} + \hat{j})$$

$$v_2 = -30 \ j$$

$$= 35\sqrt{2}\,\hat{i} + (35\sqrt{2} - 30)\,\hat{j}$$

$$v = (49.5 \ i + 19.5 \ j) \text{ km/hr.}$$

The result and displacement $r = (49.5 \ i + 19.5j) \times 4 = (198 \ i + 78j)$

and
$$\left[\vec{r}\right] = \sqrt{189^2 + 78^2} = 204.46 \text{ km.}$$

if ϕ = angle made by r with x-axis (*i.e.*, east)

$$\tan \phi = y/x = 19.5/49.5 \text{ or } \phi = 23° \text{ nearly.}$$

Q. 83 *An aeroplane is flying in a horizontal direction with a velocity of 600km per hour and a height of 1960 meters. When it is vertically above the point A on the ground, a body is dropped from it. The body strikes the ground at a point B. Calculate the distance AB.*

(IIT, 1973)

Ans. When a body is dropped, it is acted upon by

(*i*) Uniform velocity = 600 km/hour = $\frac{600000}{3600}$ m/s in the horizontal direction.

(*ii*) Acceleration due to gravity in the vertical direction.

Here $h = 1960$ m

Suppose the time to reach the ground = t second

$$h = ½\ gt^2;\ \text{here } g = 9.80 \text{ m/s}^2$$

$$1960 = \frac{9.80}{2} t^2$$

$$t^2 = \frac{1960}{4.90} = 400 \text{ second}^2$$

$$t = 20 \text{ second}$$

Distance $AB = S = ut$

$$S = \frac{600000}{3600} \times 20 = 3333.33 \text{ m.}$$

SELECTED PROBLEMS

1. What do you understand by a frame of reference. Define inertial frame. Comment on the reference frame attached with earth. (*Lucknow, 1994, Agra 1996*)

2. What is a coriolis force? Find an expression for it. Exaplain its importance. (*Lucknow, 1995*)

3. Explain

(*a*) Fictitious force, (*Agra*)

(*b*) Galilean transformations,

(*c*) Coriolis force. (*IAS*)

4. A reference frame '*a*' rotates with respect to another reference frame '*b*' with uniform angular velocity ω. If the position, velocity and acceleration of a particle in frame '*a*' are represented by 'R', v_a and f_a show that acceleration of that particle in frame '*b*' is given by where:

$$f_B = f_a + 2\omega \times v_a + \omega \times (\omega \times R)$$

Integrate this equation with reference to the motion of bodies on earth surface.

5. Show that the motion of one projectile as seen from another projectile will always be a straight line motion. (*Agra, 1970*)

6. A plane flies across the north pole at 400 km/hr and follow a longitude 450 (rotating with earth) all the times. Compute what angle does a freely suspended plumb line in the plane makes as it passes over the pole, make with a plumb line which is situated on the earth surface at north pole.

7. A thin uniform rod AB of mass m and length 2λ can rotate in a vertical plane about A as pendulum. A particle of mass 2 m is also fixed on the rod at a diatance x from A. Find x so that the periodic time of small swing may be minimum.

8. A person is running towards a bus with a velocity of constant magnitude $\vec{u}$ and always directed towards the bus. The bus is moving along a straight road with a velocity of constant magnitude $\vec{v}$. Initially at $t = 0.\vec{a}$ is perpendicular to $\vec{v}$ and the distance between the person and the bus l. After what time will the person catch the bus?

$$\left[\textbf{Ans. } \frac{ul}{v^2 - u^2}\right]$$

3

DYNAMICS OF CIRCULAR MOTION AND THE GRAVITATIONAL FIELD

3.1 UNIFORM CIRCULAR MOTION

Centripetal Force: suppose a body is moving on a horizontal circle (radius r) with a uniform velocity v. At any point on the circle, the direction of the velocity is directed along the tangent to the circle at that point *i.e.*, at A the direction of the velocity AX, at B along BY, at C along CZ and so on.

If the body is free at any point it will move tangential to the circle at that point. But as the body is moving along the circumference of the circle, there must be a force acting towards the center of the circular path and this force is called the centripetal force.

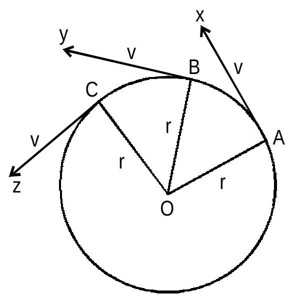


Fig. 1

Centripetal force is defined as that force which acts towards the center along the radius of a circular path on which the body is moving with a uniform velocity.

In figure 1 let A and B be two positions of the body after an interval of time t.

Then AB = velocity. time = vt

Let $o'a$ and $o'b$ be vectors representing the velocities at A and B respectively in figure 2(*ii*). Then $\angle ao'b = \theta$ (the angle between the tangents equal to the angle between the radii). ab is the change in velocity from A to B.

$$\text{Acceleration} = \frac{\text{Change in Velocity}}{\text{Time}} = \frac{ab}{t}$$

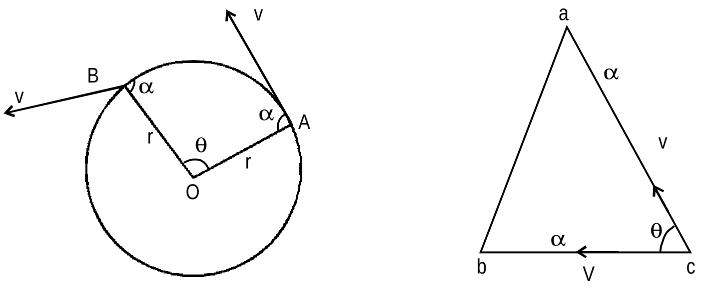
Since A and B are very close to each other, therefore arc AB can be taken as a straight line.

ΔOAB and $\Delta o'ab$ are similar

$$\therefore \qquad \frac{AB}{OA} = \frac{ab}{o'a}$$

$$\therefore \qquad \frac{vt}{r} = \frac{ab}{v} \text{ or } \frac{ab}{t} = \frac{v^2}{r}$$

$$\therefore \quad \text{Acceleration} = \frac{ab}{t} = \frac{v^2}{r}$$



(i) (ii)

Fig. 2

∴ The acceleration on the body acting towards the center of the circular path = $\frac{v^2}{r}$

$$\therefore \quad \text{Centripetal force} = \text{mass} \times \text{acceleration} = m\frac{v^2}{r}$$

$$\boxed{F = \frac{mv^2}{r}}$$

Direction of the Centripetal Force

When a body moves along a circular path with uniform speed, the magnitude of the velocity remains the same but its direction changes at every point. It means there is change in velocity whenever there is a change in direction, there must be some acceleration. As a body has mass also, a force must act upon the body. This force must act along such a direction that the magnitude of the velocity does not change. As a force has no component at right angles to its direction, the force must act at every point in a direction perpendicular to the direction of the velocity. As the velocity is tangential to the circle at every point, the force must be acting along the radius and towards the centre of the circular path. Due to this reason it is called centripetal force.

3.2 CENTRIFUGAL FORCE

When a body is rotating on a circular path, it has a tendency to move along a tangent. If a body A leaves the circular path at any instant, for an observer A who is not sharing the motion along the circular path (*i.e.*, a body B standing outside the reference circle), the body A appears to fly off tangentially at the point of release. For an observer C, who is sharing the same circular motion as that of the body A, the body A appears to be at rest before it is released. According to C, when A is released, it appears to fly off radially away from the center. It appears to the body C as if the body A has been thrown off along the radius away from the center by some force. This inertial force is known as centrifugal force. Its magnitude is mv^2/r. It is not a force of reaction. Centrifugal force is a fictitious force and holds good in a rotating frame of reference.

When a car is on turning round a corner, the persons sitting inside the car experience an outward force. This is due to the fact that no centripetal force is provided by the passengers. Therefore to avoid this outward force, the passengers are to exert an inward force.

3.3 THE CENTRIFUGE

It is an appliance to separate heavier particles from the lighter particles in a liquid. The liquid is rotated in a cylindrical at a high speed with the help of an electric motor. The heavier particles move away from the axis of rotation and the lighter particles moves near to the axis of rotation.

Application

(*a*) Sugar crystals are separated from molasses with the help of a centrifuge.

(*b*) In cream separators, when the vessel containing milk is rotated at high speed, the lighter cream particles collect near the axle while the skimmed milk moves away from the axle.

(*c*) In drying machines, the wet clothes are rotated at high speed. The water particles fly off tangentially through the holes in the wall of the outer vessel.

(*d*) Honey is also separated from bees with the help of a centrifuge.

(*e*) Precipitates, sediments, bacteria etc., are also separated in a similar way.

In an ultra-centrifuge, the speed of rotation is very high and is of the order of 30 to 40 thousand rotations per minute.

3.4 BANKING OF CURVED ROADS AND RAILWAY TRACKS (BANKED TRACK)

A car moving on road, or a train moving on rails, requires a centripetal force while taking a turn. As these vehicles are heavy, the necessary centripetal force may not be provided by friction, and moreover, the wheels are likely to suffer considerable wear and tear. In this case, the centripetal force is produced by slopping down the road inward at the turns. Similarly, while laying the railways tracks, the inner rail is laid slightly lower than the outer rail at turns. By doing so, the car, or the train leans inward while taking turn and the necessary centripetal force is provided. This force is produced by the normal reaction of the earth, or the rail.

In figure 3, is shown a car taking a turn on a road while is given a slope of angle θ, G is the center of gravity of the car at which the weight mg of the car acts. When the car leans the total normal reaction R of the road exerted on the wheels makes an angle θ with the vertical. The vertical component R cos θ balances the weight mg of the car, while the horizontal component R sin θ provides the necessary centripetal force mv^2/r. Thus

$$R \cos \theta = mg \qquad ...(i)$$

and

$$R \sin \theta = \frac{mv^2}{r} \qquad ...(ii)$$

Dividing Eq. (*ii*) by (*i*), we get

$$\boxed{\tan \theta = \frac{v^2}{rg}}$$

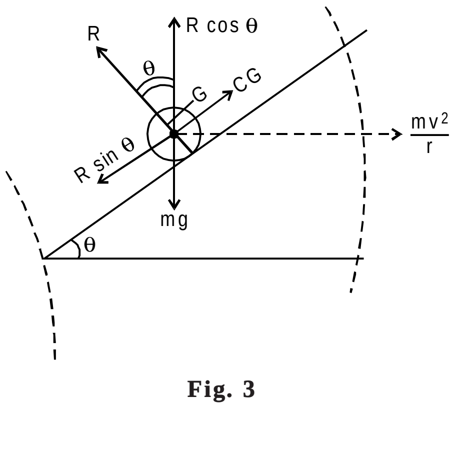


Fig. 3

From this formula the angle θ can be calculated for given values of v and r. Thus the slope θ is proper for a particular speed of the car. Therefore, the driver drives the car with that particular speed at the turn. Since m does not appear in the formula, hence this speed does not depend upon the weight of the car.

If a car is moving at a speed higher than the desired speed, it tends to slip outward at the turn, but then the frictional force acts inward and provides the additional centripetal force. Similarly, if the car is moving at a speed lower than the desired speed it tends to slip inward at the turn, but now the frictional force acts outward and reduces the centripetal force.

The hilly roads are made slopping inward throughout so that the vehicles moving on them lean automatically towards the center of the turn and are acted upon by the centripetal force. If these road were in level then, at the turns, the outer wheels would be raised up thus overturning the vehicle.

The flying aeroplane leans to one side while taking a turn in the horizontal plane. In this situation the vertical component of the force acting on the wings of the aeroplane balances its weight and the horizontal component provides the necessary centripetal force. In the "well of death" the driver while driving the motor-cycle fast on the wall of the well, leans inward. The vertical component of the reaction of the wall balances the weight of the motor-cycle and the driver, while the horizontal component provides the required centripetal force.

3.5 BICYCLE MOTION

As shown in figure 4, a rider taking a turn towards his left hand. Let m be the mass of the rider and the bicycle, v the speed of the bicycle and r the radius of the (circular) turn. The centripetal force mv^2/r necessary to take a turn is provided by the friction between the tyres and the road. Therefore, when the bicycle is turned, a frictional force F (= mv^2/r) towards the center of the turn is exerted at the point A on the road. Let us imagine two equal and opposite forces F_1 and F_2, each equal and parallel to F ($F_1 = F_2 = F$), acting at the center of gravity G of the rider and the bicycle. The force F acting at A can now be replaced by the force F_1 acting at G, and the anticlockwise couple formed by F_1 and F_2 (= F). The force F_1(= F) is the necessary centripetal force. If the rider remains straight while turning, then his weight mg acting vertically downward at G and the earth's normal reaction R (= mg) acting vertically upward at A would cancel each other. In this situation the anticlockwise couple would overturn the bicycle outward. If, however, the rider leans inward towards the center of the turn, then his weight mg and the normal reaction R (= mg) of the earth from a clockwise couple which balances the anticlockwise couple (Fig. 5). Hence the rider moves on the turn without any risk of overturning.

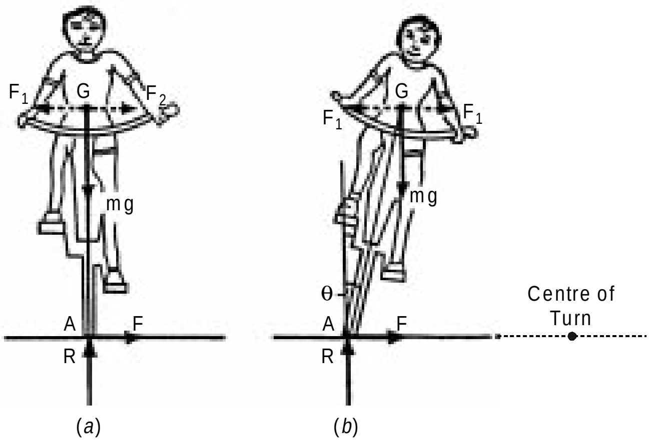


Fig. 4

Suppose the rider leans through an angle q from the vertical for balancing the two couples then:

couple formed by mg and R (= mg) = couple formed by F_1 and F_2 (= F)

$$mg \times \text{GA} \sin\theta = \text{F} \times \text{GA} \cos\theta$$

or $$\tan\theta = \frac{\text{F}}{mg}$$

But $$\text{F} = \frac{mv^2}{r}$$

$\therefore$ $$\tan\theta = \frac{v^2}{rg}$$

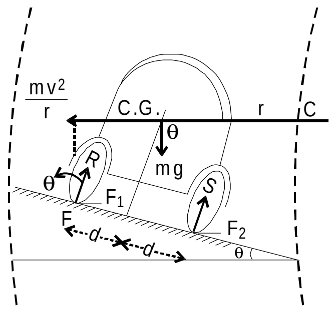


Fig. 5

From this formula, the angle θ can be calculated.

Since the centripetal force mv^2/r is provided by the frictional force, the value of mv^2/r should be less than the limiting frictional force μr otherwise the bicycle would slip. This is why the rider slow down the speed of the bicycle while taking a turn and follows the path of a larger radius. During rainy season the frictional force decrease appreciably and cannot provide the centripetal force. Hence during rains bicycle rider usually slip on the roads while taking turn.

3.6 CONICAL PENDULUM

Suppose a small object A of mass m is tied to a string OA of length l and then whirled round in a horizontal circle of radius r, with O fixed directly above the center B of the circle, If the circular speed of A is constant, the string turns at a constant angle θ to the vertical. This is called a conical pendulum.

Since A moves with a constant speed v in a circle of radius r, there must be a centripetal force $\frac{mv^2}{r}$ acting towards the center B. The horizontal component, T sin θ, of the tension T in the string provides this force along AB. So

$$\text{T} \sin\theta = \frac{mv^2}{r} \qquad ...(i)$$

Also, since the mass does not move vertically, its weight mg must be counter balanced by the vertical component T cos θ of the tension. So

$$\text{T} \cos\theta = mg \qquad ...(ii)$$

Dividing (*i*) by (*ii*), then

$$\tan\theta = \frac{v^2}{rg}$$

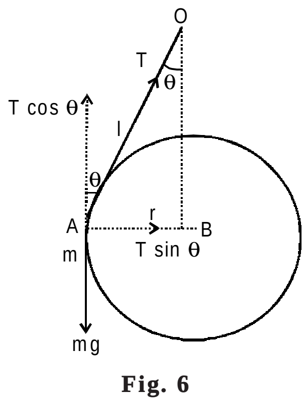


Fig. 6

A similar formula for θ was obtained for the angle of banking of a track, which prevented side-slip.

A pendulum suspended from the ceiling of a train does not remain vertical while the train goes round a circular track. Its bob moves outwards away from the center and the string becomes inclined at an angle θ to the vertical, as shown in figure 6. In this case the centripetal force is provided by the horizontal component of the tension in the string.

3.7 MOTION IN A VERTICAL CIRCLE

When a body tied to the end of a string is rotated in a vertical circle, the speed of the body is different at different points of the circular path. Therefore, the centripetal force on the body and the tension in the string change continuously.

Let us consider a body of mass '*m*' tied to the end of a string of length R and whirled in a vertical circle about a fixed point O to which the other end of the string is attached. The motion is circular but not uniform, since the body speeds up while coming down and slows down while going up.

The force acting on the body at any instant are its weight *mg* directed vertically downward, and the tension $\vec{T}$ in the string directed radially inward. The weight $m\vec{g}$ can be resolved into tangential component $mg \sin\theta$ and a radical (normal) component $mg \cos\theta$. Thus the body has tangential force $mg \sin\theta$ and a resultant radial force $T - mg\cos\theta$ acting on it.

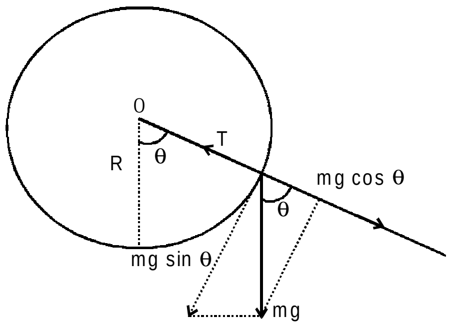


Fig. 7

The tangential force gives to the body a tangential acceleration, which is responsible for the variation in its speed. The radial force provides the necessary centripetal force.

Thus $$T - mg\cos\theta = \frac{mv^2}{R}$$

The tension in the string is therefore

$$\boxed{T = m\left[\frac{v^2}{R} + g\cos\theta\right]} \quad \text{...}(i)$$

Let us consider two special cases of the equation:

(*i*) At the lower point A of the circle, $\theta = 0$ so $\cos\theta = 1$

$$T = T_A \text{ and } v = v_A \text{ (say)}.$$

Equation (*i*) takes the form

$$\boxed{T_A = m\left[\frac{v_A^2}{R} + g\right]}$$

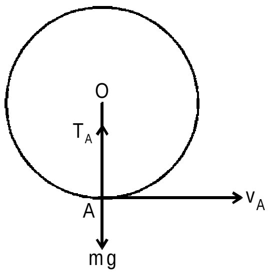


Fig. 8

Which means that the tension must be large enough not only to overcome the weight *mg* but also provide the centripetal force $\frac{mv_A^2}{R}$.

(*ii*) At the highest point B, θ = 180°, so cos θ = –1, T = T_B and $v = v_B$ (say)

Then we have Equation (*i*)

$$T_B = m\left[\frac{v_B^2}{R} - g\right]$$

T_B is less than T_A, because the weight *mg* provides a part of the centripetal force.

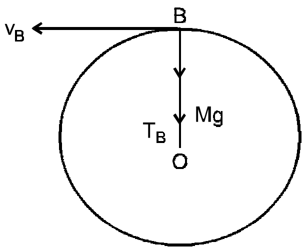


Fig. 9

If v_B decreases, the tension T_B would decrease and vanish at a certain critical speed v_C (say). To determine this critical speed, we put T_B = 0 and $v_B = v_C$ in the last expression. Then

$$0 = m\left[\frac{v_C^2}{R} - g\right]$$

$$\boxed{v_c = \sqrt{Rg}}$$

This speed of the body is called the "critical speed". In this state the centripetal force is provided simply by the weight of the body. If the speed of the body at the highest point B is less than the critical speed $\sqrt{Rg}$, then the required centripetal force would be less than the weight of the body which will therefore fall down (the string would slack).

If a bucket containing water is rotated fast in a vertical plane, the water does not fall even when the bucket is completely inverted. This can be explained in the following manner. The bucket rotates in a vertical plane under a changing centripetal force. The water contained in the rotating bucket experiences a centrifugal force which is always equal and opposite to the centripetal force. When the bucket is at the highest point B of this circular path, then the centrifugal force $\frac{mv_B^2}{R}$ on the water is directed upward. If the speed of rotation of the bucket is quite fast, the centrifugal force is greater than the weight W of the water. The difference $\left(\frac{mv_B^2}{R} - W\right)$ keeps the water pressed to the bottom of the bucket.

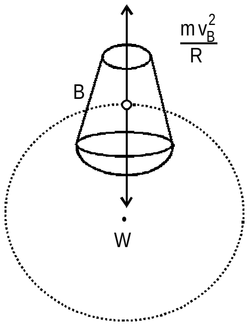


Fig. 10

3.8 MOTION OF PLANET

Our solar system consists of a sun which is stationary at the center of the Universe and nine planets which revolve around the sun in separate orbits. The name of these planets are: Mercury, Venus, Earth, Mass, Jupiter, Saturn, Uranus, Neptune and Pluto. The planet Mercury is closest to the Sun and Pluto is farthest.

These are certain celestial bodies which revolve around the planets. These are called 'satellites'. For example, moon revolves around the earth, hence moon is a satellite of the earth. Similarly, Mars has two satellites, Jupiter has twelve satellites, Saturn has ten satellites, and so on.

3.9 KEPLER'S LAWS OF MOTION

Kepler found important regularities in the motion of the planets. These regularities are known as 'Kepler's three laws of planetary motion'.

(*i*) **Shape of the Orbit:** All planets move around the sun in elliptical orbits having the sun at one focus of the orbit. This is the law of orbits.

(*ii*) **Velocity of the orbit:** A line joining any planet to the sun sweeps out equal areas in equal times, that is, the areal speed of the planet remains constant. This is the law of areas. When the planet is nearest the sun then its speed is maximum and when it is farthest from the sun then its speed is minimum. In Figure 11, if a planet moves from A to B in a given time-interval, and from C to D in the same time-interval, then the areas ASB and CSD will be equal.

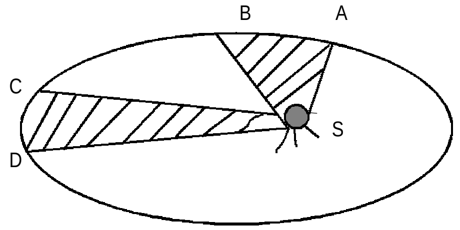


Fig. 11

(*iii*) **Time periods of Planets:** The square of the period of revolution of any planet around the sun is directly proportional to the cube of its mean distance from the sun. This is the law of periods.

If the period of a planet around the sun is T and the mean radius of its orbit is *r*, then

$$T^2 \propto r^3$$

or

$$T^2 = Kr^3$$

where K is a constant. Thus, larger the distance of a planet from the sun, larger will be its period of revolution around the sun.

3.10 DERIVATION OF LAW OF GRAVITATION

Suppose the mass of the planet A is M_1, the radius of its orbit is R_1 and time period of revolution is T_1. It is assumed that the orbit is circular. The force of attraction exerted by the sun on the planet (centripetal force)

$$F_1 = M_1R_1\omega_1^2 = M_1R_1\left[\frac{2\pi}{T_1}\right]^2 \quad ...(i)$$

Similarly for a second planet B of mass M_2, Radius R_2 and period of revolution round the sun T_2

$$F_2 = M_2R_2\omega_2^2 = M_2R_2\left[\frac{2\pi}{T_2}\right]^2 \quad ...(ii)$$

$$\frac{F_1}{F_2} = \left(\frac{M_1}{M_2}\right)\left(\frac{R_1}{R_2}\right)\left(\frac{T_2}{T_1}\right)^2 \qquad ...(iii)$$

But according to Kepler's third law,

$$\left(\frac{T_2}{T_1}\right)^2 = \left(\frac{R_2}{R_1}\right)^3 \qquad ...(iv)$$

Substituting these values in equation (*iii*)

$$\frac{F_1}{F_2} = \left(\frac{M_1}{M_2}\right)\left(\frac{R_2}{R_1}\right)^2$$

$$\frac{F_1 R_1^2}{M_1} = \frac{F_2 R_2^2}{M_2} = \text{constant}$$

$$\therefore \qquad F \propto M/R^2$$

or (*i*) $F \propto M$ and

(*ii*) $F \propto 1/R^2$

Thus, the force of attraction exerted by the sun on a planet is proportional to its mass and inversely proportional to the square of its distance from the sun.

3.11 NEWTON'S CONCLUSIONS FROM KEPLER'S LAWS

Newton found that the orbits of most of the planets (except Mercury and Pluto) are nearly circular. According to Kepler's second law, the areal speed of a planet remains constant. This means that in a circular orbit the linear speed of the planet will be constant. Since the planet is moving on a circular path; it is being acted upon by a centripetal force directed towards the center (Sun). This force is given by

$$F = \frac{mv^2}{r}$$

Where m is the mass of the planet, v is its linear speed and r is the radius of its circular orbit. If T is the period of revolution of the planet, then

$$v = \frac{\text{linear distance travelled in one revolution}}{\text{period of revolution}} = \frac{2\pi r}{T}$$

$$\therefore \qquad F = \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2 = \frac{4\pi^2 mr}{T^2}$$

But according to Kepler's third law, $T^2 = K\,r^3$

$$\therefore \qquad F = \frac{4\pi^2 mr}{K\,r^3} = \frac{4\pi^2}{K}\left(\frac{m}{r^2}\right) \qquad ...(i)$$

or

$$F \propto \frac{m}{r^2}$$

Thus, on the basis of Kepler's laws, Newton drew the following conclusions:

(*i*) A planet is acted upon by a centripetal force which is directed towards the sun.

(*ii*) This force is inversely proportional to the square of the distance between the planet and the sun $\left(F \propto \frac{1}{r^2}\right)$.

(*iii*) This force is directly proportional to the mass of the planet ($F \propto m$). Since the force between the planet and the sun is mutual, the force F is also proportional to the mass M of the sun ($F \propto M$). Now, we can replace the constant $\frac{4\pi^2}{K}$ in Equation (*i*) by GM, where G is another constant. Then, we have

$$F = G\frac{Mm}{r^2}$$

Newton stated that the above formula is not only applied between sun and planets, but also between any two bodies (or particles) of the universe. If m_1 and m_2 be the masses of two particles, then the force of attraction between them is given by

$$F = G\frac{m_1 m_2}{r^2}$$

This is Newton's Law of Gravitation.

3.12 NEWTON'S UNIVERSAL LAW OF GRAVITATIONS

In 1686, Newton stated that in the Universe each particle of matter attracts every other particle. This universal attractive-force is called 'gravitation':

The force of attraction between any two material particles is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them. It acts along the line joining the two particles.

Suppose two particles of masses m_1 and m_2 are situated at a distance r apart. If the force of attraction acting between them is F, then according to Newton's law of gravitation, we have

$$F \propto \frac{m_1 m_2}{r^2}$$

or $$F = G\frac{m_1 m_2}{r^2}$$

where G is the constant of proportionality which is called 'Newton's gravitation constant'. Its value is same for all pairs of particles. So it is a universal constant.

If $m_1 = m_2 = 1$ and $r = 1$, then $F = G$

Hence, gravitation constant is equal in magnitude to that force of attraction which acts between two particles each of unit mass separated by a unit distance apart.

From the above formula $$G = \frac{Fr^2}{m_1 m_2}$$

Thus, if force (F) be in Newton, distance (r) in meter and mass (m_1 & m_2) in kg, then G will be in Newton-meter^2/kg^2. It dimensional formula is [$M^{-1}L^3T^{-2}$].

In rationalized MKS units G = 6.670 × 10^{-11} newton-m^2/kg^2.

3.13 GRAVITY AND THE EARTH

In Newton's law of gravitation, the gravitation is the force of attraction acting between any two bodies. If one of the bodies is earth then the gravitation is called 'gravity'. Hence gravity is the force by which earth attracts a body towards its center. Clearly gravity is a special case of gravitation. It is due to gravity that bodies thrown freely ultimately fall on the surface of the earth.

3.14 ACCELERATION DUE TO GRAVITY

When a body is dropped down freely from a height, it begins to fall towards the earth under gravity and its velocity of fall continuously increases. The acceleration developed in its motion is called 'acceleration due to gravity'. Thus, the acceleration due to gravity is the rate of increase of velocity of a body falling freely towards the earth. It is denoted by 'g'. It does not depend upon the shape, size, mass, etc. of the body. If m be the mass of a body then force of gravity acting on it is mg (weight of the body). Therefore, the acceleration due to gravity is equal in magnitude to the force exerted by the earth on a body of unit mass. The unit of acceleration due to gravity is meter/second^2 or Newton/kg.

The universal constant G is different from $\vec{g}$. The constant G has the dimension $M^{-1}\,L^3\,T^{-3}$ and is a scalar, $\vec{g}$ has the dimension LT^{-2} and is a vector, and is neither universal nor constant.

3.15 EXPRESSION OF ACCELERATION DUE TO GRAVITY g IN TERMS OF GRAVITATIONS CONSTANT G

Suppose that the mass of the earth is M_e, its radius is R_e and the whole mass M_e is concentrated at its center. Let a body of mass m be situated at the surface of the earth or at a small height above the surface, this height being negligible compared to the radius of the earth. Hence the distance of the body from the center of earth may be taken as R_e. According to the law of gravitation, the force of attraction acting on the body due to the earth is given by

$$F = \frac{GM_e m}{R_e^2} \qquad ...(i)$$

The acceleration due to gravity g in the body arises due to the force F. According to Newton's second law of motion, we have

$$F = mg \qquad ...(ii)$$

From Eqs. (i) and (ii), we have

$$mg = G\frac{M_e m}{R_e^2}$$

$$g = \frac{GM_e}{R_e^2}$$

This expression is free from m. This means that the value of g does not depend upon the mass of the body. Hence if two bodies of different masses be allowed to fall freely (in the absence of air), they will have the same acceleration. If they are allowed to fall from the same height, they will reach the earth simultaneously.

In the presence of air, however, the buoyancy effect and the viscous drag will cause different accelerations in the bodies. In this case the heavier body will reach the earth earlier.

3.16 DIFFERENCE BETWEEN MASS AND WEIGHT

Mass is the amount of matter contained in a body. This is fixed for a given body. Its value is the same at all places. Weight is the force with which a body is attracted towards the center of the earth. It is different at different places. It depends on the value of g. A body has the same mass at the poles and at the equator whereas its weight will be more at the poles than at the equator.

With the help of a beam balance, the mass of the body is determined. With the help of a spring balance the weight of the body is determined. The units of mass are gram or Kg. The units of weight are dyne or Newton.

3.17 INERTIAL MASS AND GRAVITATIONAL MASS

Inertial Mass: According to Newton's second law of motion, when a force is applied on a body, the body moves with an acceleration

$$F = ma \qquad \text{or} \qquad m = F/a$$

This mass is called the inertial mass of the body. If the force is increased, the acceleration also increases and F/a = constant for a given body. If the same force is applied on two different bodies and the acceleration produced are equal, then the inertial masses of the two bodies are equal. If the same force is applied on two different bodies, the inertial mass of that body is more in which the acceleration produced is less and vice versa.

Gravitational Mass: According to the law of gravitation, the Gravitational force of attraction of a body towards the center of the earth is equal to the weight of the body.

Let the weight of the body be W

$$W = mg$$

$$m = W/g$$

This mass is called the gravitational mass of the body. This is determined with the help of a beam balance. It will be the same even on the surface of the moon. If the value of g is less at the moon, the weight of the body will also be less but the gravitational mass is the same.

3.18 GRAVITATIONAL FIELD AND POTENTIAL

The intensity of gravitational field at a point due to a mass is defined as the force experienced by unit mass, placed at that point. It is, however, supposed that the introduction of the unit mass at the point does not change the configuration of the field. If m be the mass of a gravitating particle, the intensity of gravitational field at a distance r from it is given by

$$E = -\frac{Gm}{r^2}\hat{r}$$

where $\hat{r} = \frac{\vec{r}}{r}$ is a unit vector along $\vec{r}$.

Here the negative sign indicates that the direction of force is opposite to $\hat{r}$ *i.e.*, towards the mass m. Thus, the force per unit mass is a measure of the field intensity.

The gravitational potential at a point in a gravitational field of a body is the amount of work, required to be done on a unit mass in bringing it from infinity (*i.e.*, the state of zero potential) to that point. In other words, the potential at a point is equal to the potential energy per unit mass. Hence, the potential at a distance $\vec{r}$ from a mass m is given by

$$V = -\int_{\infty}^{r} \vec{E}.d\vec{r} = \int_{\infty}^{r} \frac{Gm}{r^2}\, dr = -\frac{Gm}{r}$$

Therefore, the potential energy of a mass m', placed at that point, is given by

$$U = m'V = -\frac{Gmm'}{r}$$

Similar to the case of an electric field, the expression for the intensity of the gravitational field is

$$E = -\frac{dV}{dr}$$

3.19 EQUIPOTENTIAL SURFACE

An equipotential surface is the surface, at all the points of which the potential is constant. The potential due to a point mass m at a distance r is

$$V = -\frac{Gm}{r}$$

and it is constant on a spherical surface of radius r. This is the equipotenial surface. The intensity of the field E at r is given by

$$E = -\nabla V = -\frac{dV}{dr}\hat{r}$$

where $\hat{r} = \frac{\vec{r}}{r}$ is along OA

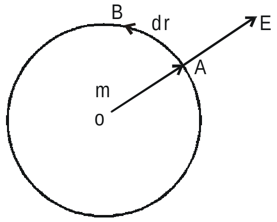


Fig. 12

Now if a unit mass moves a small distance $\vec{AB} = dr$ on the surface,

then $$\vec{E}.d\vec{r} = -\frac{dV}{dr}\hat{r}.dr = 0$$

because vector $\hat{r}$ is perpendicular to the vector dr at the surface. This means that for equipotential surface, the intensity of the field E is perpendicular to the surface at any point A. If a unit mass is moved from one point to the other on an equipotential surface, the amount of work done is equal to zero (because $\int_A^B \vec{E}\left(d\vec{r}\right) = V_B - V_A = 0$)

If we have a mass m' at A, then force on the mass is given by

$$F = -\nabla U = -\nabla(mV) = -m\nabla V = mE$$

Thus the force is also directed perpendicular to the equipotenial surface at A. Hence, if any mass is moved on an equipotential surface, no work will be done.

3.20 WEIGHTLESSNESS IN SATELLITES

The weight of a body is felt due to a reactionary force applied on the body by some other body (which is in contact with the first body). For example, when we stand on a plane we feel our weight due to the reaction of the plane on our feet. If under some special circumstances the reaction of this plane becomes the zero then we shall feel as our weight has also become zero. This is called "State of weightlessness". If the ropes of a descending lift are broken, then persons standing in the lift will feel this state.

Weightlessness is also felt by a space-man inside an artificial satellite. Suppose an artificial satellite of mass m is revolving around the earth (mass M_e) with speed v_0 in an orbit of radius r. The necessary centripetal force is provided by the gravitational force.

$$G\frac{M_e m}{r^2} = \frac{mv_0^2}{r}$$

or

$$\frac{GM_e}{r^2} = \frac{v_0^2}{r} \qquad ...(i)$$

If there is a space-man of mass m' inside the satellite, he is acted upon by two forces: (*i*) gravitational force $\frac{GM_e m'}{r^2}$, (*ii*) reaction R of the base of the satellite, in the opposite direction. Thus there is a net force $\left[\frac{GM_e m'}{r^2} - R\right]$ on the man. It is directed towards the center of the orbit and is the necessary centripetal force on the man. That is,

$$\frac{GM_e m'}{r^2} - R = \frac{m' v_0^2}{r}$$

or

$$\frac{GM}{r^2} - \frac{R}{m'} = \frac{v_0^2}{r}$$

Substituting the value of v_0^2/r from equation (*i*), we get

$$\frac{GM_e}{r^2} - \frac{R}{m'} = \frac{GM_e}{r^2}$$

$$\therefore \qquad R = 0$$

Thus, the reactionary force on the man is zero. Hence he feels his weight zero. If he stands on a spring-balance, the balance will read zero. In fact, every body inside the satellite is in a state of weightlessness. If we suspend a body by a string, no tension will be produced in the string. Space-man cannot take water from a glass because on tilting the glass, water coming out of it will float in the form of drops. Space-man take food by pressing a tube filled with food in the form of paste.

Although moon is also a satellite of the earth, but a person on moon does not feel weightlessness. The reason is that the moon has a large mass and exerts a gravitational force on the person (and this is the weight of person on the moon). On the other hand, the artificial satellite having a smaller mass does not exert gravitational force on the space-man.

3.21 VELOCITY OF ESCAPE

It is a thing of common experience that when a body is projected in the upward direction, it returns back due to the gravitational pull of the earth on it. Now, if the body is projected upwards with such a velocity which will just take the body beyond the gravitational field of the earth, then it will never come back. This velocity of the body is called the velocity of escape.

Let us consider a body of mass m lying at a distance r (>R) from the center of the earth (or planet or sun). The force with which the earth attracts the body is given by

$$F = -\frac{GMm}{r^2}$$

where M is the mass of the earth.

If this mass be moved through a distance dr away from the earth, then the small work done on the body is

$$dW = \frac{GMm}{r^2}\,dr$$

In order that the body may not return on the earth, sufficient work must be done on it to import the body so much kinetic energy that it moves to infinity. Therefore the work done in moving the body from the surface of the earth to infinity is

$$W = \int_R^\infty \frac{GMm}{r^2}\,dr = \left[-\frac{GMm}{r}\right]_R^\infty = \frac{GMm}{R}$$

Evidently, a body of K.E. greater than this GMm/R would escape completely from the earth. Hence the minimum velocity v of projection of the body to escape into space is given by the relation

$$\frac{1}{2}mv^2 = \frac{GMm}{R} \quad \text{or} \quad \boxed{v = \sqrt{\frac{2GM}{R}}} \qquad ...(i)$$

But $$g = GM/R^2$$

$$\therefore \qquad \boxed{v = \sqrt{2gR}} \qquad ...(ii)$$

Expressions (*i*) and (*ii*) are for the escape velocity of a body from earth.

If at the surface of the earth, we take g = 9.8 m/sec^2 and its radius R = 6.4 × 10^6 m, then escape velocity

$$v = \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$$

$$v = 11.2 \times 10^3 \text{ m/sec}$$

$$v = 11.2 \text{ km/sec}$$

Hence, if a body is projected in the upward direction with a velocity 11.2 km/sec or more than this, it will never return to the earth. Escape velocity is the same for all bodies of different masses.

3.22 RELATION BETWEEN ORBITAL VELOCITY AND ESCAPE VELOCITY

The orbital velocity of a satellite close to the earth is $v_0 = \sqrt{gR_e}$, and the escape velocity for a body thrown from the earth's surface is $v_0 = \sqrt{2gR_e}$. Thus

$$\frac{v_0}{v_e} = \frac{\sqrt{gR_e}}{\sqrt{2gR_e}} = \frac{1}{\sqrt{2}}$$

$$\therefore \qquad v_e = \sqrt{2}\, v_0$$

If the orbital velocity of a satellite revolving close to the earth happens to increase to $\sqrt{2}$ times, the satellite would escape.

3.23 SATELLITES

In the solar system, different planets revolve round the sun. The radii of the orbits and their time periods of revolution are different for different planets. In these cases, the force of gravitation between the sun and the planet provides the necessary centripetal force. Similarly the moon revolves around the earth and the force of gravitation between the earth and the moon provides the necessary centripetal force for the moon to be in its orbit. Here moon is the satellite of the earth.

From 1957, many artificial satellites around the earth have been launched. These satellites are put into orbits with the help of multi-stage rockets.

3.24 ORBITAL VELOCITY OF SATELLITE

When a satellite (such as moon) revolves in a circular orbit around the earth, a centripetal force acts upon the satellite. This force is the gravitational force exerted by the earth on the satellite.

In figure, a satellite of mass m is revolving around the earth with a speed v_0 in a circular orbit of radius r. The centripetal force on the satellites is $\frac{mv_0^2}{r}$.

Let M_e be the mass of the earth. The gravitational force exerted by the earth on the satellite will be $\frac{GM_e m}{r^2}$, where G is gravitation constant. As the gravitational force provides the required centripetal force, we have

$$G\frac{M_e m}{r^2} = \frac{mv_0^2}{r}$$

or $$GM_e = v_0^2 r$$

or $$v_0^2 = \frac{GM_e}{r}$$

or $$v_0 = \sqrt{\frac{GM_e}{r}}$$

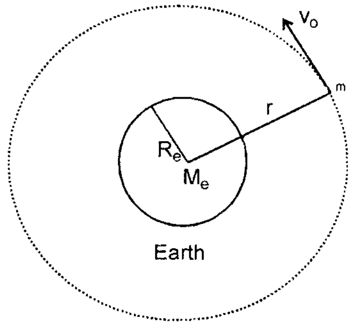


Fig. 13

Let R_e be the radius of the earth and h the height of the satellite above the earth's surface. Then the distance of the satellite from the center of the earth will be

$$r = R_e + h$$

Substituting this value of r in the above equation, we have

$$v_0 = \sqrt{\frac{GM_e}{R_e + h}} \quad ...(i)$$

If the acceleration due to gravity on the earth's surface is g, then

$$g = \frac{GM_e}{R_e^2}$$

or $$GM_e = gR_e^2$$

Substituting this value of GM_e in Eq. (*i*), we get

$$v_0 = R_e\sqrt{\frac{g}{R_e + h}} \quad ...(ii)$$

Eqs. (*i*) and (*ii*) give the speed of revolution of the satellite in its orbit. It is evident from these equations that the speed of a satellite depends only upon its height above the earth's surface. Greater is the height h of the satellite above earth's surface, smaller is the speed of the satellite. Because the speed of a satellite does not depend upon the mass of the satellite, therefore two satellites of different masses revolving in the same orbit around the earth will have the same speed.

Period of Revolution: Let T be the time of one revolution of the satellite. Then

$$T = \frac{2\pi r}{v_0} = \frac{2\pi (R_e + h)}{v_0}$$

Substituting the value of v_0 from Eq. (*i*), we have

$$T = \frac{2\pi (R_e + h)}{\{GM_e/(R_e + h)\}^{\frac{1}{2}}} = 2\pi\sqrt{\frac{(R_e + h)^3}{GM_e}} \quad ...(iii)$$

But $$GM_e = g\, R_e^2$$

$\therefore$ $$\boxed{T = 2\pi\sqrt{\frac{(R_e + h)^3}{gR_e^{\,2}}}}$$

From Eq. (*iii*) or (*iv*) the period of revolution of the satellite can be calculated. It is evident from these equations that the period of revolution of a satellite depends only upon its height above the earth's surface. Greater is the distance of a satellite above the earth's surface, greater is its period of revolution. This is why the moon, which is at a height of 3,80,000 km above earth, completes one revolution of earth in nearly 27 days, while an artificial satellite revolving near the earth's surface completes 10 to 20 revolution in a day. If the height of an artificial satellite above earth's surface be such that its period of revolution is exactly equal to the period of revolution (24 hours) of the axial motion of the earth, then the satellite would appear stationary over a point on earth's equator. It would be synchronous with earth's spin. Such a satellite is known as "geostationary satellite". It is used to reflect T.V. signals and telecast T.V. programs from one part of the world to another.

Now, from Eq. (*iv*), we have

$$(R_e + h)^3 = \frac{T^2}{4\pi^2} \times gR_e^2$$

$$(R_e + h) = \left[\frac{T^2 g R_e^2}{4\pi^2}\right]^{1/3} \qquad \ldots(v)$$

Substituting T = 24 h = 24 × 60 × 60 = 86400 s, $R_e = 6.37 \times 10^6$ m and $g = 9.8$ m/s^2 in Eq. (*v*), we get

$$R_e + h = 42{,}200 \text{ Km}$$

This is the orbital radius of a geostationary satellite.

The height of the satellite above the earth's surface is

$$h = 42{,}200 - 6370 = 35{,}830 \text{ km.}$$

Thus, for a satellite to appear stationary, it must be placed in an orbit around the earth at a height of 35,830 km from the earth's surface. This is often called 'parking orbit' of the satellite. Artificial satellites used for telecasting are placed in parking orbits.

The orbiting speed of the geostationary satellite is given by

$$v = \frac{2\pi\,(R_e + h)}{T}$$

$$v = \frac{2 \times 3.14 \times 42,200 \text{ km}}{24\,h} = 11042 \text{ km/hr}$$

3.25 ORBITAL SPEED AND PERIOD OF REVOLUTION OF A SATELLITE VERY CLOSE TO EARTH

If a satellite is very close to the earth's surface ($h << R_e$), then h will be negligible compared to R_e. In this case putting $h = 0$ in Eq (*ii*), the orbital speed of the satellite is given by

$$v_0 = R_e\sqrt{\frac{g}{R_e}} = \sqrt{gR_e}$$

Putting g = 9.8 meter/sec^2 and R_e = 6.37 × 10^6 meter, we have

$$v_0 = \sqrt{9.8 \times (6.37 \times 10^6)} = 7.9 \times 10^3 \text{ m/s} \cong 8 \text{ km/s}$$

Similarly, putting h = 0 in Eq. (*iv*), the period of revolution of the satellite is given by

$$T = 2\pi\sqrt{\frac{R_e}{g}} = 2 \times 3.14 \times \sqrt{(6.37 \times 10^6)/9.8}$$

$$= 5063 \text{ seconds}$$

$$\approx 84 \text{ minutes.}$$

Thus, the speed of a satellite revolving very close to the earth's surface is nearly 8 km/sec and its period of revolution is nearly 84 minutes.

3.26 ARTIFICIAL SATELLITES

We have seen above that when a satellite revolves around the earth in an orbit near the earth's surface then its orbital velocity is about 8 km/s. Therefore, if we send a body a few hundred kilometers above the earth's surface and give it a horizontal velocity of 8 km/s, then the body is placed in an orbit around the earth. Such a body is called an artificial satellite.

Like moon, an artificial satellite also revolves, around the earth under the gravitational attraction exerted by the earth which acts as the centripetal force. One may ask, how is it that the satellite continues to revolve in an orbit at a definite height above the earth, instead of falling towards the centre of the orbit under the centripetal force. In fact, the satellite continuously falls towards the centre of the earth, but due to the curvature of the earth it is maintained at the same height. This we can see in Fig. 14. If there were no centripetal force on the satellite then it would have moved along a straight-line path PQR————. But due to the presence of centripetal force it moves along a circular path PST————. Thus it is continuously falling towards the center of the earth through the distances QS, RT——.

An artificial satellite is placed in an orbit by means of a multi-stage rocket. The satellite is placed on the rocket. On being fired, the rocket moves vertically upward with an increasing velocity. When the fuel of the first stage of the rocket is exhausted, its casing is detached and the second stage comes in operation. The velocity of the rocket increases further. This process continues. When the rocket, after crossing the dense atmosphere of the earth attains proper height, a special mechanism gives a thrust to the satellite producing a pre-calculated horizontal velocity. A satellite carried to a height (<< earth's radius) and given a horizontal velocity of 8 km/s is placed, by earth's gravity, almost in a circular orbit around the earth, It then continues revolving (without using any fuel). Even due to a slight mistake in calculating the velocity, the orbit of the satellite would change considerably. The satellite is always given such a velocity that its orbit remains outside the earth's atmosphere; otherwise the friction of atmosphere would cause so much heat that it will burn.

The satellite revolves around the earth in an orbit with earth as center, or a focus. If a packet is released from the satellite, it will not fall to the earth but will remain revolving in the same orbit with the same speed as the satellite.

Uses of Artificial Satellite

(1) They are used to study the upper regions of the atmosphere.

(2) Information about the space of the earth can be obtained.

(3) Weather forecast can be communicated to the earth.

(4) Radiation from the sun and outer space can be studied.

(5) Distant telecasting can be achieved.

(6) Meteorites can be studied.

(7) Space flights are possible due to artificial satellites.

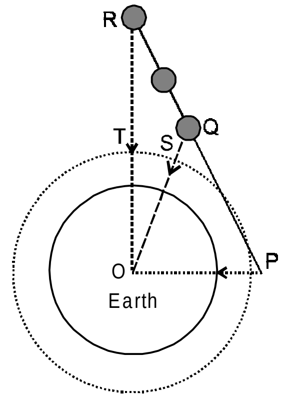


Fig. 14

3.27 BLACK HOLES

Gravitational attraction occurs throughout the universe. It is used in theories of galaxies and stars in addition to explaining the motion of planets and earth satellites as we have done.

Stars appear to have formed after a 'big bang' or explosion of matter when the universe first began. Initially, the hydrogen gas atoms produced moved towards each other under gravitational attraction, the speed and energy of the atoms increased and their temperature rose. Many atoms then had sufficient energy to fuse together and form helium atoms. The fusion of hydrogen atoms to form helium atoms produces nuclear energy and the high temperature rise produces light. A star reaches a stable size or radius when the outward pressure of the gas is balanced by the inward gravitational attraction between the atoms.

When all the hydrogen in the star is used up in the fusion process, which may occur after by millions of years, the unbalanced gravitational attraction between the atoms will result in the star becoming smaller and smaller. Eventually the star will collapses. From the wave-particle duality of light, light can be considered as particles (photons). So as the star collapses in size and becomes denser, the gravitational attraction on light from its surface increases. This make the light bend more and more towards the inside of the star and soon no light escapes from the star.

So a collapsed star has such a strong gravitational field that not even light can escape from it, which is why it appears black. The region round the star from which no light can escape is called a black hole. Fortunately, our star, the sun, is estimated to continue producing nuclear energy for about 5000 million years. In 1994 the Hubble Space Telescope discovered one of the biggest black holes in the universe, about 500 light years across and in a galaxy 52 million light years away from the earth. (1 light year = 9.46×10^{12} km approx.).

PLANETS AND SATELLITES

Q. 1. *The maximum and minimum distances of a comet from the sun are 1.6×10^{12} m and 8.0×10^{10} m respectively. If the speed of the comet at the nearest point is 6.0×10^4 m/sec, calculate the speed at the farthest point.*

Ans. The speed of a satellite round a planet in elliptical orbit varies constantly. In order to conserve angular momentum. $\vec{L}\ (= \vec{r} \times m\vec{v})$ at these points is

$$\vec{L} = mv_1r_1 = mv_2r_2$$

or

$$v_1r_1 = v_2\ r_2$$

Putting the given values, we have

$$v_1 \times (1.6 \times 10^{12}) = (6.0 \times 10^4) \times (8.0 \times 10^{10})$$

$$\therefore \quad v_1 = \frac{(6.0 \times 10^4) \times (8.0 \times 10^{10})}{1.6 \times 10^{12}}$$

$$= 3.0 \times 10^3 \text{ meter/sec} \quad \textbf{Ans.}$$

Q. 2. *Earth's revolution round the sun has radius 1.5×10^{11} meters and period 3.15×10^7 second (one year). If the gravitational constant is 6.67×10^{-11} newton-m^2/kg^2, deduce the mass of the sun. The formula is to be deduced.*

Ans. The period of revolution T of the earth round the sun (mass M_s) in a circular orbit of radius r is given by

$$T^2 = \frac{4\pi^2 r^3}{GM_s}$$

$$\therefore \quad M_s = \frac{4\pi^2 r^3}{GT^2}$$

Substituting the given values:

$$M_s = \frac{4 \times (3.14)^2 \times (1.5 \times 10^{11}\ \text{m})^3}{(6.67 \times 10^{-11}\ \text{newton-m}^2/\text{kg}^2)\,(3.15 \times 10^7\ \text{s})^2}$$

$$= 2.0 \times 10^{30} \text{ kg} \quad \textbf{Ans.}$$

Q. 3. *Determine the mass of the earth from the moon's revolution around the earth in a circular orbit of radius 3.8×10^5 km with period 27.3 days, G = 6.67×10^{-11} newton-m^2/kg^2.*

Ans. The moon revolves round the earth. If r be the radius of its orbit and T the orbiting period, then the mass of the earth is given by

$$M = \frac{4\pi^2 r^3}{GT^2}$$

Here $r = 3.8 \times 10^5$ km $\mp$ 3.8×10^8 meter, T = 27.3 days = 27.3 × 24 × 60 × 60 sec.

$$\therefore \qquad M = \frac{4 \times (3.14)^2 \times (3.8 \times 10^8)^3}{(6.67 \times 10^{-11}) \times (27.3 \times 24 \times 60 \times 60)^2}$$

$$= 5.8 \times 10^{24} \text{ kg.}$$

Q. 4. *With what horizontal velocity must a satellite be projected at 800 km above the surface of the earth so that it will have a circular orbit about the earth. Assume earth's radius 6400 km. What will be the period of rotation? g = 9.8 m/sec²*.

Ans. The orbital velocity of a satellite revolving round the earth at a height h from earth's surface is given by

$$v = R\sqrt{\left(\frac{g}{R+h}\right)}$$

Putting the given values $\quad R = 6400 \text{ km} = 6.4 \times 10^6$ meter

and $\quad R + h = 6400 + 800 = 7200$ km

$$= 7.2 \times 10^6 \text{ meter}$$

we get $\quad v = (6.4 \times 10^6) \times \sqrt{\dfrac{9.8}{7.2 \times 10^6}}$

$$= 7.5 \times 10^3 \text{ m/sec} = 7.5 \text{ km/sec.}$$

The satellite must be given a horizontal velocity of 7.5 km/sec after taking it to a stable orbit round the earth.

The period of revolution is $\quad T = \dfrac{2\pi(R+h)}{v} = \dfrac{2 \times 3.14 \times 7200 \text{ km}}{7.5 \text{ km/sec}}$

$$= 6029 \text{ sec} = 100.5 \text{ min } \textbf{Ans.}$$

Q. 5. *The period of a satellite in circular orbit of radius 12000 km around a planet is 3 hours. Obtain the period of a satellite in circular orbit of radius 48000 km around the same planet.*

Ans. The period of a satellite is proportional to the $\frac{3}{2}$th power of its orbital radius.

i.e., $\quad T \propto (r)^{3/2}$

Thus, if T′ be the period in an orbit of radius r', thus

$$\frac{T}{T'} = \left(\frac{r}{r'}\right)^{3/2}$$

Here T = 3 hours, r = 12000 km and r' = 48000 km. Therefore

$$\frac{3}{T'} = \left(\frac{12000}{48000}\right)^{3/2} = \left(\frac{1}{4}\right)^{3/2} = \frac{1}{8}$$

$$\boxed{T' = 24 \text{ hours}} \textbf{ Ans.}$$

Q. 6. *Deduce an expression for the period T of an earth satellite orbiting in a circular path at height nR from earth's surface, where R is earth's radius. Compute the value of n which makes T equal to 1 day, given that T = 1.41 hours for n = 0.*

Ans. The orbiting period of an earth's satellite at a height h from earth's surface is

$$T = \frac{2\pi(R+h)}{R}\sqrt{\left(\frac{R+h}{g}\right)}$$

where R is earth's radius,

If $h = nR$, we have

$$T = \frac{2\pi(R+nR)}{R}\sqrt{\left(\frac{R+nR}{g}\right)}$$

or
$$T = 2\pi\sqrt{\frac{R}{g}}(1+n)^{3/2}$$

Given that for $n = 0$, T = 1.41 hours. Substituting it in the last expression, we get

$$2\pi\sqrt{\frac{R}{g}} = 1.41 \text{ hours}$$

$$\therefore \quad T = 1.41\,(1+n)^{3/2} \text{ hours}$$

For T = 1 day = 24 hours, we have

$$24 = 1.41\,(1+n)^{3/2}$$

or
$$(1+n)^{3/2} = \frac{24}{1.41} = 17$$

or
$$(1+n) = (17)^{2/3} = 6.61$$

$$\therefore \quad n = 5.61$$

A satellite orbiting the earth at height 5.61 R from earth's surface would appear stationary over a point on the equator.

Q. 7. *The mean distance of Mars from sun is 1.524 times the distance of Earth from sun. Compute the period of revolution of Mars around sun.*

Ans. The period T of a planet around the sun is proportional to the $\left(\frac{3}{2}\right)^{th}$ power of its distance from the sun. Thus

$$T \propto (r)^{3/2}$$

Therefore
$$\frac{T_{mars}}{T_{earth}} = \left(\frac{r_{mars}}{r_{earth}}\right)^{3/2} = (1.524)^{3/2} = 1.88$$

Now
$$T_{earth} = 1 \text{ year (earth revolves round sun once in a year).}$$

$$\therefore \quad T_{mars} = T_{earth} \times 1.88 = 1.88 \text{ earth-years}$$ **Ans.**

Q. 8. *A satellite moves in a circular orbit around the earth at a height $\frac{R}{2}$ from earth's surface, where R is the radius of the earth, calculate its period of revolution ($R = 6.38 \times 10^6$ meter).*

Ans. The period of revolution of a satellite at a height h from earth's surface is given by

$$T = \frac{2\pi (R + h)}{R} \sqrt{\left(\frac{R + h}{g}\right)}$$

For $h = \frac{R}{2}$, we have

$$T = \frac{2\pi \times 1.5\ R}{R} \sqrt{\frac{1.5\ R}{g}}$$

$$= 2\pi \sqrt{\left(\frac{R}{g}\right)}\ (1.5)^{3/2}$$

$$= 2 \times 3.14 \times \sqrt{\frac{6.38 \times 10^6 \text{ meter}}{9.8 \text{ meter/sec}^2}} \times (1.5)^{3/2}$$

$$= 5067 \text{ sec} \times (1.5)^{3/2} = 9310 \text{ sec}$$

$$= 2 \text{ hours } 35 \text{ min} \quad \textbf{Ans.}$$

Q. 9. *Calculate the limiting velocity required by an artificial satellite for orbiting very closely round the earth ($R = 6.4 \times 10^6$ meter, $g = 9.8$ meter/sec^2). What would be the period.*

Ans. The orbital velocity of a satellite revolving round the earth (radius R) at a height h from earth's surface is given by

$$v = R \sqrt{\frac{g}{R + h}}$$

For a satellite orbiting very closely the earth's surface, $h << R$, so that

$$v = R \sqrt{\frac{g}{R}} = \sqrt{gR}$$

$$= \sqrt{9.8 \times 6.4 \times 10^6} = 7.92 \times 10^3 \text{ m/sec}$$

$$= 7.92 \text{ km/sec.}$$

The period is

$$T = \frac{2\pi (R + h)}{R} \sqrt{\frac{R + h}{g}}$$

For $h << R$

$$T = 2\pi \sqrt{\frac{R}{g}} = 2 \times 3.14 \times \sqrt{\frac{6.4 \times 10^6}{9.8}} = 5.075 \times 10^3 \text{ sec}$$

$$= 84.6 \text{ min} = 1.41 \text{ hours} \quad \textbf{Ans.}$$

Q. 10. *A satellite is revolving round near the equator of a planet of mean density ρ. Show that the period T of such an orbit depends only on the density of the planet.*

Ans. Let m be the mass and v the velocity of satellite which orbits round the planet just above its surface. The radius of the orbit is practically equal to the radius R of the planet. Therefore, the centripetal force acting upon the satellite is mv^2/R.

If M be the mass of planet, the gravitational force between the planet and the satellite is GMm/R^2 and this supplies the required centripetal force. Thus

$$\frac{mv^2}{R} = G\frac{Mm}{R^2}$$

or

$$v = \sqrt{\frac{GM}{R}}$$

The period of the satellite is

$$T = \frac{2\pi R}{v} = 2\pi\sqrt{\frac{R^3}{GM}}$$

Now, the mass M of the planet (mean density suppose ρ) is

$$M = \frac{4}{3}\pi r^3 \rho$$

Making the substitution in the last expression, we get

$$T = \sqrt{\frac{3\pi}{G\rho}}$$

Thus the period depends only on the density of the planet.

Q. 11. *A satellite is launched into a circular orbit 1600 km above the surface of the earth. Find the period of revolution if the radius of the earth is R = 6400 km and the acceleration due to gravity is 9.8 m/sec^2. At what height from the ground, should it be launched so that it may appear stationary over a point on the earth's equator?*

Ans. The orbiting period of a satellite at a height h from earth's surface is

$$T = \frac{2\pi(R + h)}{R}\sqrt{\frac{R + h}{g}}$$

Here R = 6400 km, h = 1600 km = $\frac{R}{4}$. Then

$$T = \frac{2\pi\left(R + \frac{R}{4}\right)}{R}\sqrt{\frac{\left(R + \frac{R}{4}\right)}{g}}$$

$$= 2\pi\sqrt{\left(\frac{R}{g}\right)}\left(1 + \frac{1}{4}\right)^{3/2}$$

$$= 2\pi\sqrt{\frac{R}{g}}\,(1.25)^{3/2}$$

Putting the given values:

$$T = 2 \times 3.14 \times \sqrt{\frac{6.4 \times 10^4 \text{ m}}{9.8 \text{ m/sec}^2}}\,(1.25)^{3/2}$$

$$= 7092 \text{ sec} = 1.97 \text{ hours.}$$

Now, a satellite will appear stationary in the sky over a point on earth's equator if its period of revolution round the earth is equal to the period of revolution of the earth round its own axis which is 24 hours. Let us find the height h of such a satellite above earth's surface in terms of earth's radius. Let it be nR. Then

$$T = \frac{2\pi\,(R + nR)}{R}\sqrt{\frac{R + nR}{g}}$$

$$= 2\pi\sqrt{\frac{R}{g}}\,(1 + n)^{3/2}$$

$$= 2 \times 3.14\sqrt{\frac{6.4 \times 10^6 \text{ meter}}{9.8 \text{ m / sec}^2}}\,(1 + n)^{3/2}$$

$$= (5067 \text{ sec})\,(1 + n)^{3/2}$$

$$= (1.41 \text{ hours})\,(1 + n)^{3/2}$$

For T = 24 hours, we have

$$(24 \text{ hours}) = (1.41 \text{ hours})\,(1 + n)^{3/2}$$

or $$(1 + n)^{3/2} = \frac{24}{1.41} = 17$$

or $$1 + n = (17)^{2/3} = 6.61$$

or $$n = 5.61$$

The height of the geo-stationary satellite above earth's surface is

$$nR = 5.61 \times 6400 \text{ km} = 3.59 \times 10^4 \text{ km} \quad \textbf{Ans.}$$

Q. 12. *Two satellites of same mass are launched in the same orbit round the earth so as to rotate opposite to each other. They collide in elastically and stick together as wreckage. Obtain the total energy of the system before and just after the collision. Describe the subsequent motion of the wreckage.*

Ans. The potential energy of a satellite in its orbit is $-\frac{GMm}{r}$, and the kinetic energy is $\frac{GMm}{2r}$. The total energy is

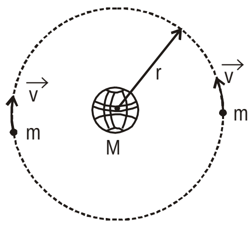


Fig. 15

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$= -\frac{GMm}{2r}$$

where m is the mass of the satellite, M the attracting mass (earth) and r the orbital radius. When there are two satellites, each of mass m, in the same orbit the energy would be

$$-\frac{GMm}{2r} - \frac{GMm}{2r} = -\frac{GMm}{r}$$

Let $\vec{v}$ be the velocity of the wreckage after the collision. Then, by the law of conservation of momentum, we have

$$m\vec{v} + m\vec{v} = (m+m)\vec{v}'$$

or $$mv - mv = (m + m)\, v'$$

$\therefore$ $$v' = 0$$

Therefore, the wreckage of mass $2m$ has no kinetic energy but only potential energy. Hence the total energy just after the collision would be

$$-\frac{GM\,(2m)}{r}$$

As the velocity of the wreckage is zero, the centripetal force disappears and the wreckage falls down under gravity.

Q. 13. *A small satellite revolves round a planet in an orbit just above planet surface. Taking G = 6.66 × 10^{-11} MKS units and mean density of planet as 8.00 × 10^3 MKS units, calculate the time period of the satellite.*

Ans. $$T = \sqrt{\frac{3\pi}{G\rho}}$$

$$= \sqrt{\frac{3 \times 3.14}{6.66 \times 10^{-11} \times 8.00 \times 10^{3}}}$$

$$= 0.420 \times 10^4 \text{ sec} = 70 \text{ min}$$ **Ans.**

Q. 14. *Two earth satellites; A and B, each of mass m are to be launched into circular orbits about earth's centre. Satellite A is to orbit at an altitude of 6400 km and B at 19200 km. The radius of earth is 6400 km.*

(a) What is the ratio of the potential energy of B to that of A, in orbit,

(b) Ratio of kinetic energy,

(c) Which one has the greater total energy.

Ans. (*a*) The potential energy of an earth's satellite in a circular orbit of radius r is

$$U(r) = -\frac{GMm}{r}$$

where M is the mass of the earth and m that of the satellite.

For A: $$r = 6400 + 6400 = 12800 \text{ km}$$

For B: $r = 19200 + 6400 = 25600$ km

$$\therefore \quad \frac{U_B}{U_A} = \frac{r_A}{r_B} = \frac{12800}{25600} = \frac{1}{2}.$$

(*b*) The kinetic energy of an earth's satellite moving with velocity v in a circular orbit of radius r is

$$K = \frac{1}{2} m v^2 = \frac{GMm}{2r}$$

$$\therefore \quad \frac{K_B}{K_A} = \frac{r_A}{r_B} = \frac{1}{2}$$

(*c*) The total energy of the satellite is

$$E = U + K = -\frac{GMm}{r} + \frac{GMm}{2r} = -\frac{GMm}{2r}$$

Which is negative. Clearly the farther the satellite is from the earth, the greater (that is, less negative) is its total energy E. Hence the satellite B has the greater (less negative) total energy.

Q. 15. *A stream of α-particles is bombarded on mercury nucleus (z = 80) with a velocity* 1.0×10^9 *cm/sec. If an α-particle is approaching in head-on direction, calculate the distance of closest approach. The mass of α-particle is* 6.4×10^{-24} *gm and electronic charge is* 4.8×10^{-10} *esu.*

Ans. The distance of closest approach of an α-particle to a nucleus is given by

$$r_o = \frac{2ze^2}{k_i} \text{ cm}$$

where k_i is the initial kinetic energy. Here

$$k_i = \frac{1}{2} m v^2$$

$$= \frac{1}{2} \times (6.4 \times 10^{-24} \text{ gm}) (1.0 \times 10^9 \text{ cm/sec})^2$$

$$= 3.2 \times 10^{-6} \text{ erg.}$$

$$\therefore \quad r_o = \frac{2 \times 80 \times (4.8 \times 10^{-10} \text{ esu})^2}{3.2 \times 10^{-6} \text{ erg}}$$

$$\boxed{r_o = 1.152 \times 10^{-11} \text{ cm}} \quad \textbf{Ans.}$$

Q. 16. *In a double star, two stars (one of mass m and the other of 2m) distant d apart rotate about their common centre of mass. Deduce an expression for the period of revolution. Show the ratio of their angular momenta about the centre of mass is the same as the ratio of their kinetic energies.*

Ans. The centre of mass c will be at distances $\frac{d}{3}$ and $\frac{2d}{3}$ from the masses. $2m$ and m

respectively. Both the stars rotate round c in their respective orbits with the same angular velocity ω. The gravitational force acting on each star due to the other supplies the necessary centripetal force. The gravitational force on either star is $\frac{G(2m)\,m}{d^2}$. If we consider the rotation of the smaller star, the centripetal force ($mr\omega^2$) is $m\left(\frac{2d}{3}\right)\omega^2$.

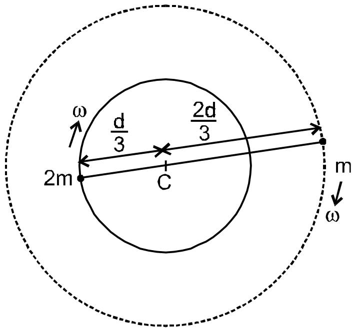


Fig. 16

$$\therefore \qquad \frac{G(2m)\,m}{d^2} = m\left(\frac{2d}{3}\right)\omega^2$$

or

$$\omega = \sqrt{\frac{3Gm}{d^3}}$$

Therefore, the period of revolution is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{d^3}{3\,Gm}}$$

The ratio of the angular momenta is

$$\frac{(I\omega)_{big}}{(I\omega)_{small}} = \frac{I_{big}}{I_{small}} = \frac{(2m)\left(\frac{d}{3}\right)^2}{m\left(\frac{2d}{3}\right)^2} = \frac{1}{2}$$

Since ω is same for both. The ratio of their kinetic energies is

$$\frac{\left(\frac{1}{2}I\omega^2\right)_{big}}{\left(\frac{1}{2}I\omega^2\right)_{small}} = \frac{I_{big}}{I_{small}} = \frac{1}{2}$$

which is the same as the ratio of their angular momenta.

Q. 17. *Estimate the mass of the sun assuming the orbit of the earth around the sun to be circular. The distance between sun and earth is 1.49 × 10^{13} cm, G = 6.66 × 10^{-8} egs units and earth takes 365 days to make one revolution around the sun.*

Q. 18. *An artificial satellite of the earth moves at an altitude of 640 km along a circular orbit. Find the orbital velocity of the satellite. (Radius of earth = 6400 km).*

Q. 19. *Show that the time of revolution of a satellite just above the earths surface is 84.4 min. Density of earth is 5.51 × 10^3 kg/meter^3 and G = 6.67 × 10^{-11} nt-m^2/kg.*

Q. 20. *What is the smallest radius of a circle at which a bicyclist can travel if his speed is 7 m/sec and the coefficient of static friction between the tyres and the road in 0.25. Under these conditions what is the largest angle of inclination to the vertical at which the bicyclist can without falling?*

Ans. The (maximum) force of friction is given by

$$f_s = \mu_s N = \mu_s mg$$

This must supply the required centripetal force. If R be the smallest radius, the maximum required centripetal force would be mv^2/R. Thus

$$\mu_s mg = \frac{mv^2}{R}$$

or

$$R = \frac{v^2}{\mu_s g}$$

here v = 7 m/sec, μ_s = 0.25 and g = 9.8 m/sec^2

$$\therefore \quad R = \frac{(7\text{ m/s})^2}{0.25 \times (9.8\text{ m/sec}^2)} = 200\text{ meter}$$

The (largest) angle of inclination θ is given by

$$\tan\theta = \frac{v^2}{Rg} = \frac{(7\text{ m/sec})^2}{20\text{ m} \times 9.8\text{ m/sec}^2} = 0.25$$

$$\theta = \tan^{-1}(0.25) = 14° \quad \textbf{Ans.}$$

Q. 21. *A cord is tied to a pail of water and the pail is swing in a vertical circle of radius 1 meter. What must be the minimum velocity of the pail at the highest point of the circle if no water is to spill from the pail? (g = 9.8 m/sec^2).*

Ans. At the highest point of the vertical circle, the centripetal force is supplied by the tension in the cord and also by the weight of the pail plus water. The force must be atleast equal to the weight, otherwise water would spill from the pail under gravity. Hence if v_c be the minimum velocity required at the highest point, we have

$$\frac{mv_c^2}{R} = mg$$

or

$$v_c = \sqrt{Rg}$$

Here R = 1 meter

$$v_c = \sqrt{1 \times 9.8} = 3.13\text{ m/sec} \quad \textbf{Ans.}$$

Q. 22. *A circular curve of highway is designed for traffic moving at 15 m/sec. (a) If the radius of the curve is 100 m, what is the correct angle of banking of the road. (b) If the curve is not banked, what is the minimum coefficient of friction between tyres and road that would keep traffic from skidding at this speed?*

Ans. (*a*) Let θ be the correct angle of banking. Then

$$\tan\theta = \frac{v^2}{Rg}$$

Putting the given values, we have

$$\tan\theta = \frac{(15\ \text{m/s})^2}{(100\ \text{m})\,(9.8\ \text{m/s}^2)} = 0.23$$

$$\theta = \tan^{-1}(0.23) = 13°$$

(*b*) If the road is not banked, then the required frictional force *f* (say) must supply the entire centripetal force, $\frac{mv^2}{R}$. Thus

$$f = \frac{mv^2}{R}$$

But $f = \mu N = \mu mg$. Thus

$$\mu mg = \frac{mv^2}{R}$$

or

$$\boxed{\mu = \frac{v^2}{Rg} = 0.23}$$ **Ans.**

Q. 23. *The radius of curvature of a railway line at a place is 800 meter and the distance between the rails is 1.5 meter. What should be the elevation of the outer rail above the inner one for a safe speed of 20 km/hour?*

Ans. Let θ be the correct angle of banking. Then

$$\tan\theta = \frac{v^2}{Rg}$$

Here v = 20 km/hour = 5.56 m/sec and R = 800 m

$$\tan\theta = \frac{5.56 \times 5.56}{800 \times 9.8} = 0.004$$

If x be the elevation of the outer rail and l the distance between the rails, then

$$\tan\theta = \frac{x}{l}$$

or

$$x = l\tan\theta$$

$$= (1.5\ \text{m})\,(0.004) = 0.006\ \text{m} = 0.6\ \text{cm}.$$

Q. 24. *A mass m on a frictionless table is attached to a hanging mass M by a cord through a hole in the table. Find the condition (v and r) with which its must spin for M to stay at rest.*

Ans. The mass M will stay at rest when its weight Mg is used up in supplying the required centripetal force $\frac{mv^2}{r}$ to the mass m, that is when

$$Mg = \frac{mv^2}{r}$$

or
$$\frac{v^2}{r} = \frac{Mg}{m}$$

This is the required condition. The equilibrium will, however, be unstable.

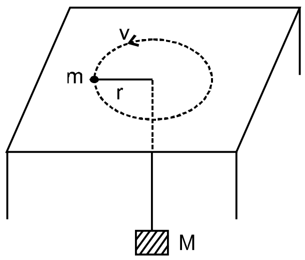


Fig. 17

Q. 25. *A smooth table is placed horizontally and an ideal spring of spring-constant k = 1000 nt/m and unextended length of 0.5 m has one end fixed to its centre the other end is attached to a mass of 5 kg which is moving in a circle with constant speed 10 m/s. Find the tension in the spring and the extension of the spring beyond its normal length.*

Ans. The centripetal force required for the mass m to move in a circle is supplied by the tension T produced in the stretched spring. The streched length of the spring is R, equal to the radius of the circle. If l_o be the unextended length of the spring, then the elongation is $(R - l_o)$ and the tension produced is given by

$$T = k\,(R - l_o), \qquad ...(1)$$

where the spring-constant k is the tension per unit elongation. But this is equal to the centripetal force mv^2/R. Therefore,

$$k(R - l_o) = \frac{mv^2}{R}$$

or
$$kR\,(R - l_o) = mv^2$$

Here $k = 1000$ nt/m, $l_o = 0.5$ m, $m = 5$ kg, $v = 10$ m/sec.

$$\therefore \quad 1000\,R(R - 0.5) = 5 \times 100$$

or
$$R(R - 0.5) = 0.5$$

or
$$2R^2 - R - 1 = 0$$

Solving this quadratic eq. we get R = 1.0 m

The extension of the spring beyond its normal length is

$$R - l_o = 1.0 - 0.5 = 0.5 \text{ m}$$

Substituting this value in (1), the tension is

$$T = 1000 \times 0.5 = 500 \text{ nt.}$$ **Ans.**

Q. 26. *An electron is moving in a circular orbit of radius 5.3×10^{-11} meter around the atomic nucleus at a rate of 6.6×10^{15} rev/sec. Find the acceleration of the electron and centripetal force acting on it. The mass of the electron is 9.1×10^{-31} kg.*

Ans. Let R be the radius of the orbit and f the number of revolutions per second. Then the velocity of the electron is given by

$$v = 2\pi R f$$

Hence its acceleration is

$$a = \frac{v^2}{R} = 4\pi^2 R f^2$$
$$= 4 \times (3.14)^2 \times (5.3 \times 10^{-11}) \times (6.6 \times 10^{15})^2$$
$$= 9.1 \times 10^{22} \text{ m/sec}^2, \text{ towards the nucleus.}$$

The centripetal force is

$$F_c = ma$$
$$= (9.1 \times 10^{-31}) \times (9.1 \times 10^{22})$$
$$= 8.3 \times 10^{-8} \text{ nt towards the nucleus.}$$

Q. 27. *A smooth table is placed horizontal and a spring of unstretched length* l_o *and force-constant k has one end fixed to its centre to the other end of the spring is attached a mass m which is making f revolutions per second around the centre. Show that the radius R of this uniform circular motion is* $kl_o/(k - 4\pi^2 mf^2)$ *and the tension T in the spring in* $4\pi^2 mkl_o f^2/(k - 4\pi^2 mf^2)$.

Ans. The centripetal force required for the mass m to move in a circle is supplied by the tension T produced in the stretched spring. The stretched length of the spring is R, equal to the radius of the circle. Thus the elongation in the spring is $(R - l_o)$ and the tension produced is given by

$$T = k(R - l_o) \qquad \text{...(1)}$$

as k (force-constant) is the tension per unit elongation.

The (linear) velocity of motion is given by

$$v = 2\pi R f$$

Therefore, the required centripetal force is

$$\frac{mv^2}{R} = 4\pi^2 R f^2 m \qquad \text{...(2)}$$

Since $T = \frac{mv^2}{R}$, we have by (1) and (2) we get

$$k(R - l_o) = 4\pi^2 R f^2 m$$

or

$$R = \frac{kl_o}{k - 4\pi^2 f^2 m}$$

Substituting this value of R in Eq. (1) we get

$$T = k\left[\frac{kl_o}{(k - 4\pi^2 f^2 m)} - l_o\right]$$

$$\boxed{T = \frac{4\pi^2 f^2 m l_o k}{k - 4\pi^2 f^2 m}} \quad \textbf{Ans.}$$

Q. 28. *An artificial satellite is revolving round the earth at a distance of 620 km. Calculate the minimum velocity and the period of revolution. Radius of earth is 6380 km and acceleration due to earth's gravity at the surface of the earth is 9.8 metres/sec^2.*

Ans. Radius of earth's satellite orbit r = Radius of earth + Distance of satellite from earth's surface

$$= 6380 + 620 = 7000 \text{ km} = 7 \times 10^6 \text{ m/sec.}$$

Radius of earth $\quad R = 6380 \times 10^3$ m, g = 9.8 m/sec^2

$$\therefore \quad \text{Period of revolution } T = \frac{2\pi r}{R}\sqrt{\frac{r}{g}} = \frac{2\pi \times 7 \times 10^6}{6380 \times 10^3}\sqrt{\frac{7 \times 10^6}{9.8}} = 5775 \text{ sec.}$$

and

$$\text{orbital velocity } v = R\sqrt{\frac{g}{r}} = 6380 \times 10^3 \sqrt{\frac{9.8}{7 \times 10^6}}$$

$$= 7.55 \times 10^3 \text{ m/sec.}$$

Q. 29. *A stone of mass 1 kg is attached to one end of a string 1 m long, of breaking strength 400 nt, and is whirled in a horizontal circle on a frictionless surface the other end of the string is kept fixed. Find the maximum velocity the stone can attain without breaking the string.*

Ans. The required centripetal force $\frac{mv^2}{R}$ is to be supplied by the tension in the string which should not exceed 400 nt. Thus, if v be the maximum velocity the stone can attain, we have

$$\frac{mv^2}{R} = T = 400 \text{ nt}$$

Here m = 1 kg and R = 1 m. Then

$$v^2 = 400$$

$$\therefore \quad \boxed{v = 20 \text{ m/sec}} \quad \textbf{Ans.}$$

Q. 30. *A mass of 1 standard kg is placed at sea level on the earth's equator and is moving with the earth in a circle of radius 6.40 × 10^6 meter (earth's radius) at a constant speed of 465 m/s. Determine the centripetal force needed. Also find the force exerted by the mass on a spring balance from which it is suspended at the equator (its weight). Assume that the mass would weight exactly 9.80 nt the equator of the earth did not rotate about its axis.*

Ans. The centripetal force is given by

$$F_c = \frac{mv^2}{R} = \frac{1 \times 465}{6.40 \times 10^6} = 0.0338 \text{ nt.}$$

The weight

$$w' = w - F_c$$

$$= 9.80 - 0.0338$$

$$= 9.77 \text{ nt} \quad \textbf{Ans.}$$

Q. 31. *Escape velocity from solar system. Show that the escape velocity of a body from solar system, launched from the earth is* $\sqrt{2GM_s/R_{es}}$ *where* M_s *is the mass of the sun and* R_{es} *is the distance of the earth from the sun. Neglect the earth's rotation and influence of earth's gravity. What is the escape velocity, if* $G = 6.67 \times 10^{-11}$ *S.I. unit,* $M_e = 1.33 \times 10^{30}$ *kg and* $R_{es} = 1.49 \times 10^{11}$ *m.*

Ans. In the gravitational field of the sun, the potential energy of a body of mass m situated on the earth is

$$v = -\frac{GM_s m}{R_{es}}$$

If the body is to be escaped from the gravitational influence of the sun, the body should be imparted with kinetic energy $\left(\frac{1}{2}mv^2\right)$ equal in magnitude to $GM_s m/R_{es}$. So that

$$\frac{1}{2}mv^2 = \frac{GM_s m}{R_{es}} \quad \text{or} \quad v = \sqrt{\frac{2GM_s}{R_{es}}}$$

From the given data, the escape velocity (launched from the earth) from the solar system is

$$v = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \times 1.33 \times 10^{30}}{1.49 \times 10^{11}}}$$

$$= 4.2 \times 10^4 \text{ m/sec} = 42 \text{ km/sec.}$$

Q. 32. *Compare the root mean square velocity of oxygen molecules at 27°C with the escape velocity from earth's surface (Boltzmann constant* $k = 1.4 \times 10^{-23}$ *joule/k).*

Ans. If V is the R.M.S. velocity of the oxygen molecular, then the mean kinetic energy per molecule is given by

$$\frac{1}{2}mv^2 = \frac{1}{2}kT$$

where m [= 32 × (mass of a proton) = $32 \times 1.7 \times 10^{-27}$ kg] is the mass of an oxygen molecule and T the absolute temperature.

$$\therefore \quad V = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3 \times 1.4 \times 10^{-23} \times 30°}{32 \times 1.7 \times 10^{-27}}}$$

$$\because \quad T = 273 + 27 = 300 \ k$$

Now, escape velocity from the earth

$$v = \sqrt{2g R_e} = \sqrt{2 \times 9.8 \times 6.4 \times 10^6}$$

$$\therefore \quad \frac{V}{v} = \sqrt{\frac{3 \times 1.4 \times 10^{-23} \times 300}{32 \times 1.7 \times 10^{-27} \times 2 \times 9.8 \times 6.4 \times 10^6}}$$

$$\boxed{\frac{V}{v} \approx 0.04} \textbf{ Ans.}$$

Q. 33. *What is the potential energy of a mass of 1 kg on the surface of the earth, reference to zero potential energy at infinite distance? Calculate also its potential energy at a distance of 10^5 km from the centre of the earth.*

Ans. In case of earth (or solid sphere) the potential energy is given by

$$v(r) = -\frac{GMm}{r}$$

(*i*) For the surface of the earth, $r = R$

$$\therefore \quad v(r) = -\frac{GMm}{R} = \frac{-(6.67 \times 10^{-11}) \times 5.98 \times 10^{24} \times 1}{6.37 \times 10^6}$$

$$= -6.23 \times 10^7 \text{ joules}$$

(*ii*) when $r = 10^5 \text{ km} = 10^8 \text{ m}$

$$v(r) = -\frac{(6.67 \times 10^{-11}) \times 5.98 \times 10^{24} \times 1}{10^8}$$

$$= -3.98 \times 10^6 \text{ joules}$$

Q. 34. *If a body is to be projected vertically upwards from earth's surface to reach a height of 10R, how much velocity should be given?*

Ans. The body should be supplied so much kinetic energy, so as it can reach a height 10R from the surface of the earth or 11R from the centre of earth.

If v is the velocity given to the body initially, then its kinetic energy = $\frac{1}{2}mv^2$

$$\text{Increase in potential energy} = -\frac{gR^2m}{r_2} + \frac{gR^2m}{r_1}$$

$$= -\frac{gR^2m}{11\,R} + \frac{gR^2m}{R} = \frac{10}{11}\,mgR$$

$$\therefore \quad \frac{1}{2}mv^2 = \frac{10}{11}\,mgR$$

or

$$v = \sqrt{\frac{20}{11}gR} = \sqrt{\frac{20}{11} \times 9.8 \times 6.4 \times 10^6}$$

$$= 10.6 \times 10^3 \text{ m/sec.}$$

Q. 35. *For the earth-sun system, calculate (i) the kinetic energy and (ii) the work which will have to be done in doubling the radius of the orbit of the earth.*

Ans. The gravitational potential of the earth due to the sun = $-GM_s/r$

$$\therefore \text{ Potential energy of the earth} = -\frac{GM_sM_e}{r}$$

$$\text{K.E. of the earth} = \frac{1}{2}M_e v^2$$

In case of earth the necessary centripetal force for its rotation is provided by gravitational attraction *i.e.*,

$$\frac{M_e v^2}{r} = \frac{GM_s M_e}{r^2}$$

$$\therefore \quad \frac{1}{2}M_e v^2 = \frac{1}{2}\frac{GM_s M_e}{r}$$

$$\therefore \quad \text{Total energy of the earth} = -\frac{GM_s M_e}{r} + \frac{1}{2}\frac{GM_s M_e}{r} = -\frac{GM_s M_e}{2r}$$

Evidently negative sign indicates that the earth is bound with the earth with this energy *i.e.*,

$$\text{The binding energy} = \frac{GM_s M_e}{2r} = \frac{1}{2}M_e v^2 = \frac{1}{2}M_e r^2\omega^2$$

$$= \frac{1}{2} \times 5.98 \times 10^{24} \times \left(\frac{1.5 \times 10^{11} \times 2 \times 3.14}{365 \times 24 \times 60 \times 60}\right)^2$$

$$= 2.7 \times 10^{33} \text{ joules}$$

$$\left(\because r = 1.5 \times 10^{11} \text{ m and } \omega = \frac{2\pi}{365 \times 24 \times 60 \times 60} \text{ rad/sec}\right)$$

when the radius of the earth's orbit is doubled, then the final binding energy will be

$$= \frac{GM_s M_e}{2\,(2r)} = \frac{1}{2}\frac{GM_s M_e}{2r}$$

$$= \frac{2.7}{2} \times 10^{33} = 1.35 \times 10^{33} \text{ joules.}$$

Therefore the amount of work done in doubling the radius of the orbit

$$= 2.7 \times 10^{33} - 1.35 \times 10^{33} = 1.35 \times 10^{33} \text{ joules} \quad \textbf{Ans.}$$

Q. 36. *Two bodies of masses M_1 and M_2 are placed distant d apart, show that at the position, where the gravitational field due to them is zero, the potential is given by*

$$V = -\frac{G}{d}\left(M_1 + M_2 + 2\sqrt{M_1 M_2}\right)$$

Ans. Let the gravitational field be zero at a point distant r from the mass M_1. Then, the distance of the same point from the mass $M_2 = d - r$. At the point, under consideration, the field due to M_1 and M_2 is zero, *i.e.*,

$$-\frac{GM_1}{r^2} = -\frac{GM_2}{(d-r)^2}$$

$$\text{or} \quad \frac{d-r}{r} = \sqrt{\frac{M_2}{M_1}}$$

Adding 1 on both sides, we have

$$\frac{d}{r} = \frac{\sqrt{M_2} + \sqrt{M_1}}{\sqrt{M_1}} \quad \text{or} \quad r = \frac{d\sqrt{M_1}}{\sqrt{M_1} + \sqrt{M_2}}$$

$$\therefore \qquad d - r = d - \frac{d\sqrt{M_1}}{\sqrt{M_1} + \sqrt{M_2}} = \frac{d\sqrt{M_2}}{\sqrt{M_1} + \sqrt{M_2}}$$

Therefore, potential at the point is

$$V = -\frac{GM_1}{r} - \frac{GM_2}{(d-r)}$$

$$= -\frac{GM_1}{d\sqrt{M_1}}(\sqrt{M_1} + \sqrt{M_2}) - \frac{GM_2}{d\sqrt{M_2}}(\sqrt{M_1} + \sqrt{M_2})$$

$$= -\frac{G}{d}\left(M_1 + \sqrt{M_1 M_2} + M_2 + \sqrt{M_1 M_2}\right)$$

$$\boxed{V = -\frac{G}{d}(M_1 + M_2 + 2\sqrt{M_1 M_2})} \quad \textbf{Ans.}$$

Q. 37. *If a mass 50 kg is raised to a height 2R from the earth's surface, calculate the change in potential energy ($g = 9.8\ m/sec^2$; $R = 6.4 \times 10^6\ m$).*

Ans. The potential energy of a mass m distance r from a solid sphere of a mass M will be given by

$$U(r) = \text{Potential} \times m = -\frac{GMm}{r} \qquad \because V = -\frac{GM}{r}$$

(referring zero potential energy at infinite distance)

But $\qquad g = \frac{GM}{R^2}$, where R is the radius of earth,

$$\therefore \qquad U(r) = -\frac{gR^2 m}{r}$$

Difference of potential energy of mass 'm' between the points at r_1 and r_2 is

$$U(r_2) - U(r_1) = -\frac{gR^2 m}{r_2} + \frac{gR^2 m}{r_1}$$

here $\qquad r_1 = R$ and $r_2 = 2R + R = 3R$

$$\therefore \qquad U(r_2) - U(r_1) = -\frac{gRM}{3} + gRM = \frac{2}{3} gmR$$

$$= \frac{2}{3} \times 9.8 \times 50 \times 6.4 \times 10^6 = 2.1 \times 10^9 \text{ joules} \quad \textbf{Ans.}$$

Q. 38. *(a) A small block of mass 'm' slides along the friction less track. Calculate the height 'h' at which mass must be released on the track to be able to go round the track of radius 'R'.*

(b) If h = 5R, what is the reaction exerted by track on the block when it is at points A, B and C.

Ans. (*a*) Since track is frictionless, no work is done against friction. The mass will go round up to C only when its velocity at point C is such that

$$\frac{mv^2}{R} = mg$$

i.e.,

$$v = \sqrt{Rg}$$

This velocity will be gained due to decrease of potential energy of block in falling through vertical height $(h - 2R)$

$$\therefore \quad \frac{1}{2}mv^2 = mg\,(h - 2R)$$

or

$$\left(\sqrt{Rg}\right)^2 = 2g\,(h - 2R)$$

or

$$\boxed{h = \frac{5R}{2}}$$

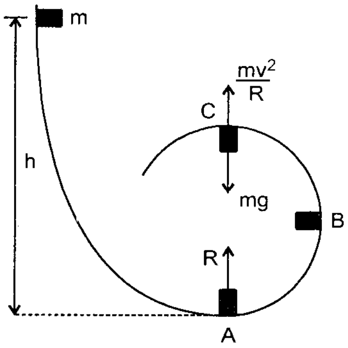


Fig. 17(*a*)

(*b*) At point A block will have speed v_A, given by

$$\frac{1}{2}mv_A^2 = mgh$$

or

$$v_A^2 = 2gh$$

so the reaction of track at the point A will be

$$N_A = mg + \frac{mv_A^2}{R}$$

$$= mg + \frac{m}{R}.2gh$$

But given $h = 5R$, so

$$N_A = mg + \frac{m}{R}.2g.5R = 11\,mg$$

At point B the block exerts on track only force due to reaction of centripetal force

$$N_B = \frac{mv_B^2}{R} = \frac{m}{R}.2g\,(h - R) = \frac{m}{R}\,2g\,(5R - R) = 8\,mg$$

At point C, the reaction of track will be

$$N_C = \frac{mv_c^2}{R} - mg$$

But

$$\frac{1}{2}mv_C^2 = mg\,(h - 2R);\ i.e.,\ v_c^2 = 2g(h - 2R)$$

so
$$N_C = \frac{2mg}{R}(h - 2R) - mg$$
$$= \frac{2mg}{R}(5R - 2R) - mg$$
$$\boxed{N_C = 5\ mg}\ \textbf{Ans.}$$

Q. 39. *A bead can slide without friction on a circular hoop of radius 10 cm in a vertical plane. The hoop rotates at a constant rate of 2 rev/sec, about a vertical diameter.*

(a) Find angle θ at which the bead will be in vertical equilibrium,

(b) Can bead rise up to the height of the centre of hoop,

(c) What will happen if hoop rotates at 1 rev/sec.

Ans. For equilibrium along the hoop

$$mr\omega^2 \cos\theta = mg \sin\theta$$

or
$$(R \sin\theta)\,\omega^2 \cos\theta = g \sin\theta$$

or
$$\cos\theta = \frac{g}{R\omega^2} = \frac{9.8}{0.1\,(2\pi \times 2)^2}$$

or
$$\theta = \cos^{-1}(0.62)$$

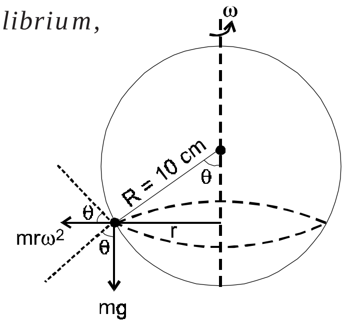


Fig. 18

(*b*) The bead can not rise up to the height of centre of hoop because, then, there will be no vertical component of $mr\omega^2$ to balance the weight mg of bead.

(*c*) If $\omega = 2\pi \times 1$ rev/sec.

$$\cos\theta = \frac{9.8}{0.1\,(2\pi \times 1)^2} = \frac{98}{4\pi^2},\ \text{which is greater than 1.}$$

This means no value of θ is possible. Hence bead will remain stationary at the bottom of hoop.

Q. 40. *A hemispherical bowl of radius R = 0.1 m is rotating about its own axis, which is vertical, with angular velocity ω. A particle of mass m = 10^{-2} kg, on frictionless inner surface of bowl is also rotating with same angular velocity. The particle is at a height 'h' from the bottom of bowl.*

(a) Obtain a relation between h and ω. What is minimum value of ω needed to have non-zero value of h?

(b) It is desired to measure 'g' using this set-up by measuring 'h' accurately. Assuming that R and ω are known and least count in the measurement of 'h' is 10^{-4} m, find minimum possible error in measurement of g.

Ans. (*a*) The forces acting on rotating particle P are

(1) Weight, *mg* acting downwards

(2) Normal reaction, N

(3) Centrifugal force, $mr\omega^2$

For equilibrium of rotating particle

$$N\cos\theta = mg \qquad ...(1)$$

and
$$N\sin\theta - mr\omega^2 = \text{Centripetal force}$$
$$= mr\omega^2$$

i.e.,
$$N\sin\theta = 2\,mr\omega^2 \qquad ...(2)$$

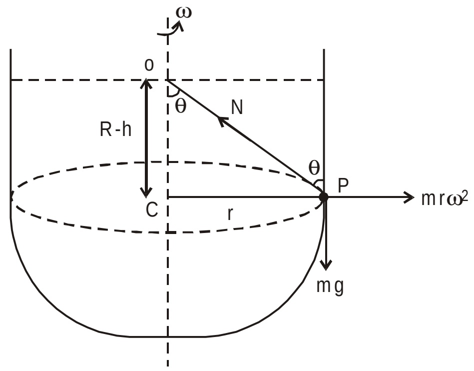


Fig. 19

From (1) and (2);

$$\tan\theta = \frac{2r\omega^2}{g}$$

But $$\tan\theta = \frac{PC}{CO} = \frac{r}{R-h}$$

so $$\frac{r}{R-h} = \frac{2r\omega^2}{g}$$

or $$\omega^2 = \frac{g}{2(R-h)} \quad ...(3)$$

i.e., $$\omega^2 = \frac{9.8}{2\,(0.1-h)} = \frac{4.9}{0.1-h}$$

which is desired relation. For non-zero value of h, $\omega^2 > \frac{4.9}{0.1}$. Hence, $\omega > 7$ radian/sec.

(*b*) Since from Eq. (3), we have

$$g = 2\omega^2\,(R-h)$$

Taking log on both side $\log g = \log 2\omega^2 + \log(R-h)$

differentiating, $$\frac{\Delta g}{g} = 0 - \frac{\Delta h}{R-h}$$

taking $h \to 0$, $$(\Delta g)_{min} = g.\frac{\Delta h}{R} = 9.8 \times \frac{10^{-4}}{0.1} = 9.8 \times 10^{-3}$$

Q 41. *A thread is passing through a hole at the center of a frictionless table. At the upper end, a block of 0.5 kg is tied and another mass 8.0 kg is tied to the lower end which is freely hanging. The smaller mass is rotated about an axis passing through hole so as to balance the heavier mass. If hanging mass is changed to 1.0 kg, what is fractional change in radius and the angular velocity of smaller mass to balance the hanging mass again.*

Ans. Let r = radius of circular path traced by smaller mass and ω = angular velocity of rotation then tension in string

$$T = mr\omega^2$$

Since, the string balances hanging mass M,

Hence $$T = Mg$$

For equilibrium of system, $mr\omega^2 = Mg$

or $$0.5\ r\omega^2 = 8\ g \quad ...(1)$$

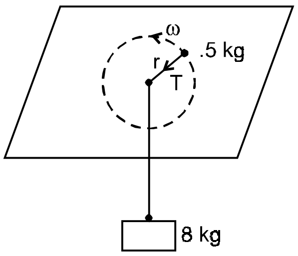


Fig. 20

if r' is the new radius and ω' the angular velocity when 8.0 kg mass is changed to 1.0 kg

$$0.5\ r'\omega'^2 = 1\ g \quad ...(2)$$

Dividing (1) by (2) we get

$$\frac{r'}{r}\left(\frac{\omega'}{\omega}\right)^2 = 8 \qquad ...(3)$$

Again, since angular momentum of system will be conserved

$$mr^2\omega = mr'^2\omega'$$

or

$$\frac{\omega}{\omega'} = \left(\frac{r'}{r}\right)^2$$

putting the value in Eq. (3) we get

$$\left(\frac{r}{r'}\right) = 8 \text{ or } \frac{\delta r}{r} = \frac{r'-r}{r} = \frac{8-1}{1} = 7$$

and

$$\frac{\delta\omega}{\omega} = \frac{\omega' \sim \omega}{\omega} = \frac{\omega'}{\omega} = 1$$

or

$$\frac{\delta\omega}{\omega} = \left(\frac{r}{r'}\right)^2 - 1 = \frac{1}{64} \sim 1 = \frac{63}{64}$$

Q. 42. *A particle describes a horizontal circle on a smooth surface of inverted cone. The height of the plane of circle above the vertex is 9.8 cm. Find speed of the particle.*

Ans. The particle while describing circular path along smooth surface of inverted cone is acted upon by two forces along the inclined surface of cone.

(1) mg cosθ, component of its weight down the inclined surface.

(2) $\frac{mv^2}{r}$ sin θ, the component of reaction of centripetal force up the inclined plane.

Hence for circular motion of particle to be possible, without sliding along surface, condition is,

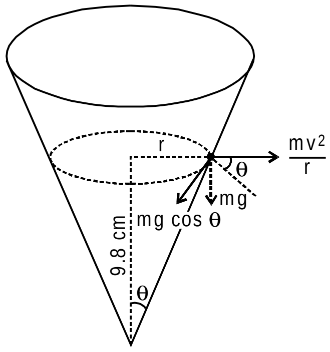


Fig. 21

$$\frac{mv^2}{r}\sin\theta = mg\cos\theta; \text{ where } \theta \text{ is half above the cone angle}$$

$$v = \sqrt{rg\cot\theta}$$

From figure $\quad r = h\tan\theta = 9.8 \times 10^{-2}\tan\theta$

Hence $\quad v = \sqrt{9.8 \times 10^{-2}\tan\theta \times 9.8\cot\theta}$

$$= 0.98\sqrt{\tan\theta\cot\theta}$$

$$\boxed{v = 0.98 \text{ m/sec}}$$ **Ans.**

Q. 43. *A motor car is travelling at 30 m/sec on a circular road of radius 500 meter. It is increasing its speed at the rate of 2 meter/sec*^{2}*. What is acceleration of car?*

Ans. The car possesses two accelerations simultaneously one-tangential (α_T) and two-normal acceleration (α_N).

(1) Tangential acceleration $\alpha_T = 2\text{m/sec}^2$

(2) Normal acceleration $\alpha_N = \frac{v^2}{r} = \frac{(30)^2}{.500} = 1.8 \text{ m/sec}^2$

The first acceleration is responsible for increase of tangential speed while second is causing circular motion. The two accelerations are mutually perpendicular, total acceleration of car will be

$$a = \sqrt{\alpha_T^2 + \alpha_N^2}$$

$$= \sqrt{(2)^2 + (1.8)^2}$$

$$= 2.69 \text{ m/sec}^2 \textbf{ Ans.}$$

Q. 44. *The roadway bridge over a canal is in form of an arc of circle of radius 20 m. What is the minimum speed with which a car can cross the bridge without leaving contact with road at the highest point.*

Ans. At the highest point of bridge, car will be acted upon by centripetal force exerting pressure on the road. This will give rise to equal and opposite reaction $R = \frac{mv^2}{r}$ where v is velocity of car and r is radius of road. For contact not to leave, $\frac{mv^2}{r} \le mg$

∴ Minimum speed which car can have at the highest point would be given by

$$\frac{v_m^2}{r} = g \quad i.e., \ v_m = \sqrt{rg}$$

or

$$v_m = \sqrt{20 \times 9.8} = 14 \text{ m/sec}$$

Q. 45. *Find the maximum speed at which a car can turn round a curve of radius 30 meter on a level road, if coefficient of friction between tyres and road is 0.4.*

Ans. When car passes along the curve it experiences the reaction equal and opposite to centripetal force, horizontally and away from centre of curvature. The frictional force between road and tyres will act horizontally inwards to provide centripetal force and prevents the car from sliding away. Thus, for equilibrium,

$$\frac{mv^2}{r} \le \mu N \quad \text{(where N is normal reaction} = mg)$$

Thus, maximum speed will be given by

$$\frac{mv_{max}^2}{r} \le \mu mg$$

i.e., $$v_{max} = \sqrt{\mu g R} = \sqrt{0.4 \times 9.8 \times 30} = 10.84 \text{ m/sec}$$ **Ans.**

Q. 46. *Calculate the altitude of an artificial satellite if it is always above a certain place on earth's surface, assuming its orbit to be circular. (Mean radius of earth = 6400 km, g = 9.80 m/s^2).*

Ans. When the period of satellite in its orbit is equal to the period of earth rotation about own axis, then the satellite shall always be at a fixed point in reference to earth.

Thus, period of satellite; $T = 24$ hrs

Since $$T = \frac{2\pi}{\omega}$$

or $$\omega = \frac{2\pi}{T} = \frac{2 \times 3.142}{24 \times 60 \times 60} \text{ rad/s}$$

For satellite in circular orbit,

$$\frac{GMm}{r^2} = \frac{mv^2}{r} = mr\omega^2$$

or $$r^3 = \frac{GM}{\omega^2} \quad ...(1)$$

where r = satellite to earth distance,

M = mass of earth,

m = mass of satellite,

ω = angular velocity of satellite.

Again on earth's surface,

$$\frac{GM}{R^2} = g \text{ or } GM = gR^2$$

putting this in eq. (1) we get

$$r^3 = \frac{gR^2}{\omega^2} = \frac{9.8 \times (6400000)^2 \times (24 \times 3600)^2}{(2 \times 3.142)^2}$$

or $$r = 42400 \text{ km}$$

height of satellite from earth surface $h = r - R$

or $$h = 42400 - 6400$$

$$\boxed{h = 36000 \text{ km}}$$ **Ans.**

Q. 47. *A satellite revolves round a planet in an elliptic orbit. Its maximum and minimum distances from the planet are 1.5 × 10^7 meters and 0.5 × 10^7 meters respectively. If the speed of the satellite at the farthest point be 5 × 10^3 m/sec. Calculate the speed at the nearest point.*

Ans. Let, F = Farthest position of satellite from planet situated at focus at the elliptical path

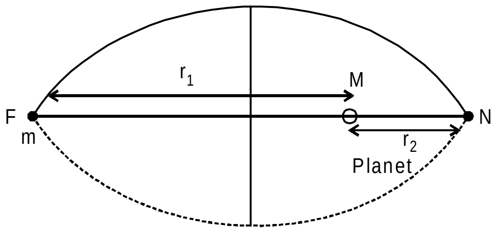


Fig. 22

N = Nearest position of satellite,

m = mass of satellite,

M = mass of planet,

r_1 = farthest distance,

r_2 = nearest distance,

v_1 = velocity at farthest position,

v_2 = velocity at nearest position.

Assuming that path of satellite nearly circular at farthest and nearest position, we have for farthest position,

$$\frac{GMm}{r_1^2} = \frac{mv_1^2}{r_1}$$

or $$\frac{GM}{r_1} = v_1^2 \quad ...(1)$$

at nearest position,

$$\frac{GMm}{r_2^2} = \frac{mv_2^2}{r_2}$$

or $$\frac{GM}{r_2} = v_2^2 \quad ...(2)$$

Dividing (2) by (1), we get

$$\frac{v_2^2}{v_1^2} = \frac{r_1}{r_2}$$

or $$v_2 = v_1 \left(\frac{r_1}{r_2}\right)^{\frac{1}{2}} \quad ...(3)$$

putting values, we get

$$v_2 = 5 \times 10^3 \sqrt{(1.5 \times 10^7)/(0.5 \times 10^7)} = 5 \times 10^3 \sqrt{3}$$

or $$\boxed{v_2 = 8.660 \times 10^3 \text{ m/s}}$$ **Ans.**

Q. 48. *Find the magnitude of centripetal acceleration of a particle on the tip of a fan blade of 0.3 meter diameter, rotating at speed of 1000 rev/min.*

Ans. Radius of circle along which tip of fan blade rotates is $r = 0.15$ meter and angular velocity

$$\omega = 2\pi n = 2\pi \times \frac{1000}{60} = \frac{100\,\pi}{3}$$

$$\therefore \quad \text{Centripetal acceleration} = \frac{v^2}{r} = r\omega^2$$

$$= 0.15 \times \left(\frac{100\pi}{3}\right)^2$$

$$= 1646.26 \text{ m/s}^2$$

Q. 49. *The moon revolves round the earth in a circle of radius 3.8 × 10⁸ m and takes 27.3 days to make a complete revolution. What is the acceleration towards the earth.*

Ans. Given $\quad \text{T} = 27.3 \text{ days} = 27.3 \times 24 \times 60 \times 60 \text{ sec}$

and $\quad r = 3.8 \times 10^8 \text{ m}$

Now, taking the orbit to be approximately circular

$$\omega = \frac{2\pi}{\text{T}} = \frac{2\pi}{27.3 \times 24 \times 60 \times 60}$$

then, centripetal acceleration

$$\alpha = r\omega^2 = 3.8 \times 10^8 \times \left[\frac{2\pi}{27.3 \times 24 \times 60 \times 60}\right]^2$$

or

$$\boxed{\alpha = 15 \times 10^8 \text{ m/sec}}$$

Q. 50. *What would be the period of moon if its mass were doubled but its orbit remained the same. Take g = 9.8 m/sec². (Radius of earth = 6.4 × 10³ m, G = 6.67 × 10⁻¹¹ nt m²/kg²).*

Ans. Suppose, M = mass of earth, m = mass of moon, and r = earth to moon distance, then

$$\frac{\text{GM}m}{r^2} = \frac{mv^2}{r} \qquad ...(1)$$

and

$$\text{T} = \frac{2\pi}{\omega} = \frac{2\pi}{v/r} = \frac{2\pi r}{v} \qquad ...(2)$$

From Eq. (1), $\quad v^2 = \dfrac{\text{GM}}{r} \quad \text{or} \quad v = \sqrt{\dfrac{\text{GM}}{r}}$

Putting v in Eq. (2), $\quad \text{T} = 2\pi r\sqrt{r/\text{GM}}$

The expression for time period of moon is seen to be independent of the mass of moon. Thus, time period of moon does not change when moon's mass is doubled.

Q. 51. *If the earth be at one third its present distance from sun, calculate the number of days in a year.*

Ans. Let R = actual distance of earth from the sun and $R' = \frac{R}{3}$ = supposed distance between sun and earth.

The gravitational force of attraction provides required centripetal force *i.e.*,

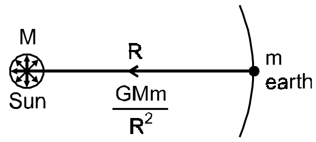


Fig. 23

$$\frac{GMm}{R^2} = \frac{mv^2}{R} \qquad ...(1)$$

But $\qquad v = R\omega = R2\pi h$

where $\qquad M$ = mass of sun,

m = mass of earth,

R = Sun to earth distance,

v = tangential velocity,

T = time taken by earth to complete one revolution.

Since $\qquad v = \frac{2\pi R}{T}$

Putting the value of v in Eq. (1), we get

$$\frac{GMm}{R^2} = \frac{m}{R}\left(\frac{2\pi R}{T}\right)^2$$

or
$$T^2 = \frac{4\pi^2 R^3}{GM} \qquad ...(2)$$

Now, if T′ is time of one revolution of earth when distance is $R' = \frac{R}{3}$, then by analogy, we have,

$$T'^2 = \frac{4\pi^2\left(\frac{R}{3}\right)^3}{GM} \qquad ...(3)$$

Dividing Eq. (3) by Eq. (2), we get

$$\frac{T'^2}{T^2} = \frac{4\pi^2\left(\frac{R}{3}\right)^3}{GM} \times \frac{GM}{4\pi^2 R^3} = \frac{1}{27}$$

or
$$T'^2 = \frac{T^2}{27} \text{ but } T = 1 \text{ year} = 365 \text{ days}$$

So
$$T' = \frac{T}{3\sqrt{3}} = \frac{365}{3 \times 1.732} = 70.3 \text{ days}$$ **Ans.**

Q. 52. *A large mass M and a small mass m hang at the two ends of a string that passes over a smooth tube as shown in the figure 24. The mass m moves around a circular path which* *lies in a horizontal plane. The length of the string from the mass m to the top of the tube is l and θ is the angle which the length makes with the vertical, what should be the frequency of rotation of the mass m so that the mass M remains stationary.*

Ans. Various forces on masses m and M are as shown in figure 24.

For equilibrium of mass m, we have

$$T \cos\theta = mg \qquad ...(1)$$

and

$$T \sin\theta = \frac{mv^2}{r} = m\omega^2 r \qquad ...(2)$$

For equilibrium of mass M,

$$T = Mg$$

From the diagram $r = l \sin\theta$

putting the value of T and r in Eqn. (2)

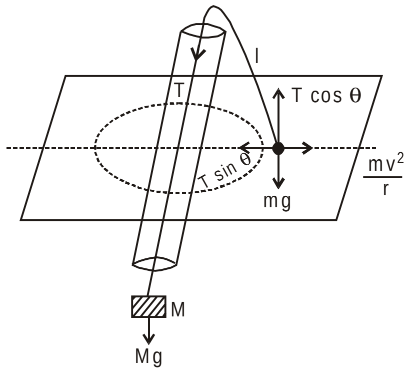


Fig. 24

$$Mg \sin\theta = m\omega^2 \, l \sin\theta$$

or

$$\omega = \sqrt{\frac{Mg}{ml}}$$

and

$$\text{frequency: } n = \frac{\omega}{2\pi}$$

$$n = \frac{1}{2\pi}\sqrt{\frac{Mg}{ml}} \quad \textbf{Ans.}$$

Q. 53. *A boy is sitting on the horizontal platform of a joy wheel at a distance of 7m from the centre. Wheel begins to rotate and when the angular speed exceeds 10 rev/min, the boy just slips. What is the coefficient of friction between boy and platform. (g = 9.8 m/s^2).*

Ans. The boy slips when centrifugal force exceeds the limiting force of friction. If ω is the maximum angular velocity of wheel when boy just slips,

$$\mu mg = mr\omega^2$$

i.e.,

$$\mu = \frac{r\omega^2}{g}$$

given, $\omega = \frac{2\pi \times 10}{60} = \frac{\pi}{3}$ rad/sec

and $r = 7$m

we get $\mu = \left(\frac{\pi}{3}\right)^2 \times \frac{7}{9.8} = 0.78$

$\mu = 0.78$ **Ans.**

Q. 54. *A 40 kg. mass hanging at the end of a rope of length l, oscillates in a vertical plane with an angular amplitude θ_o. What is the tension in the rope when it makes an angle θ with the vertical. If the breaking strength of the rope is 80 kg, what is the maximum amplitude with which the mass can oscillate without breaking the rope?*

Ans. The figure shows the position of the oscillating mass at angular amplitudes θ_0 and θ. The velocity v at θ (after a desent of distance NM from extreme position) is given by

$$v^2 = 2\,g\text{NM}$$

But $\text{NM} = \text{SM} - \text{SN} = l\cos\theta - l\cos\theta_0$

then $v^2 = 2gl\,(\cos\theta - \cos\theta_0)$

If T is tension in the rope at position Q. Then

$$T = mg\cos\theta + \frac{mv^2}{l}$$

Putting the value of v^2

$$T = mg\cos\theta + \frac{m}{l} \times 2gl\,(\cos\theta - \cos\theta_0)$$

or $T = 3mg\cos\theta - 2mg\cos\theta_0$

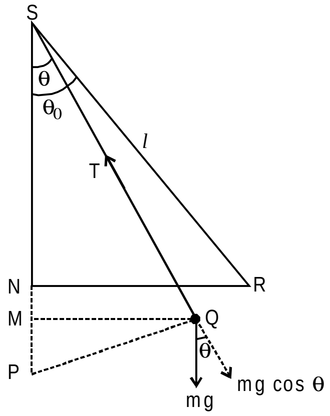


Fig. 25

Putting $m = 40$ kg

$$T = 3 \times 40 \times g\cos\theta - 2 \times 40 \times g \times \cos\theta_0$$

or $T = 40\,g\left[3\cos\theta - 2\cos\theta_0\right]$

The maximum tension in the rope occurs when $\theta = 0$ (mean position), *i.e.*,

$$T_{max} = 40g\,[3 - 2\cos\theta_0]$$

But given $T_{max} = 80$ kg

So, $80\text{ g} = 40g\,[3 - 2\cos\theta_0]$

i.e., $2\cos\theta_0 = 1$

or $\cos\theta_0 = \frac{1}{2}$ *i.e.*, $\boxed{\theta_0 = 60°}$ **Ans.**

Q. 55. *A body slides down from the top of a hemisphere of radius r. The surfaces of block and hemisphere are frictionless. Show that the height at which body loses contact with the surface of the sphere is $\frac{2}{3}r$.*

Ans. Let height at which body loses contact be 'h', then

$$OR = h \text{ and } \angle QOR = \theta$$

If at position Q, velocity is v, then taking limiting case of equilibrium for circular motion

$$mg \cos\theta = \frac{mv^2}{r} \quad ...(1)$$

where $$\cos\theta = \frac{OR}{OQ} = \frac{h}{r} \quad ...(2)$$

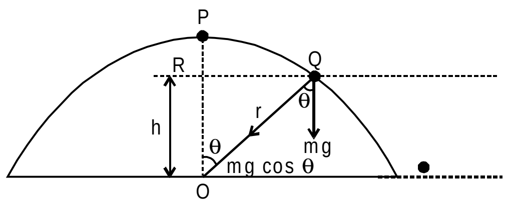


Fig. 26

But the velocity 'v' at Q, in vertical desent PR = ($r - h$), is given by

$$\frac{1}{2}mv^2 = mg\,(r - h)$$

or $$v^2 = 2g\,(r - h) \quad ...(3)$$

putting the values of cos θ from Eq. (2) and value of v^2 from Eq. (3) into Eq. (1), we get for contact upto point Q

$$mg\frac{h}{r} = \frac{m \times 2g\,(r-h)}{r}$$

which gives $$3h = 2r$$

or $$\boxed{h = \frac{2}{3}r} \text{ **Ans.**}$$

Q. 56. *The radius of curvature of a railway track at a place is 800 meter and the distance between the rails is 1.5 meters. What should be the elevation of the outer rail above the inner one for a safe speed of 20 km per hour?*

Ans. Suppose the outer rail is elevated by an angle θ with respect to inner rail *i.e.,* h is height to which outer rail is raised vertically, then

$$\tan\theta = \sin\theta = \frac{h}{d} \text{ or } \tan\theta = \frac{h}{1.5} \text{ (as } \theta \text{ is small)}$$

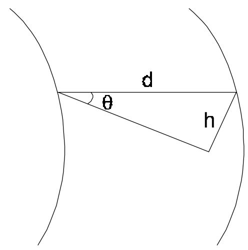


Fig. 27

The banking angle is related to safe speed by

$$\tan\theta = \frac{v^2}{rg} \text{ where } v = 20 \text{ km/hr} = \frac{100}{18} \text{ m/sec}$$

putting values, we get

$$\frac{h}{1.5} = \frac{100 \times 100}{18 \times 18 \times 800 \times 9.8}$$

or $$\boxed{h = 0.59 \text{ cm}} \text{ **Ans.**}$$

Q. 57. *A heavy particle at the end of a tight string (length 20 cm), the other end of which is fixed is allowed to fall from a horizontal position of the string. When the string is vertical, it encounters an obstruction at its middle point and the particle continues its motion in a circle of 10 cm radius. Find the height which the particle will attain before the string slackens.*

Ans. Let OP be the string in horizontal position with end O fixed, and the particle falls from position P. Then the velocity at position Q when string is vertical is given by

$$v^2 = u^2 + 2gh \quad \text{where } u = 0$$

or $$v^2 = 2 \times 980 \times 20 = 39200 \quad ...(1)$$

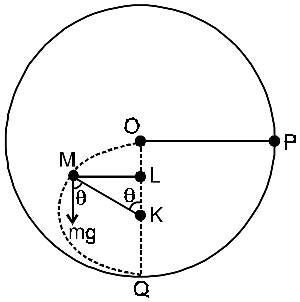


Fig. 28

Let K be mid point of OQ, QM–circular path which particle will describe with K as centre, M–the position where string slackens.

$$r = KQ = KM$$

At M, the component of *mg* along MK provides the centripetal force, *i.e.*,

$$mg \cos\theta = \frac{mv_1^2}{r}; \; v_1 = \text{tangential velocity at M}$$

or $$g\cos\theta = \frac{v_1^2}{r} \; i.e., \; v_1 = \sqrt{rg\cos\theta}$$

again $$\cos\theta = \frac{KL}{KM} = \frac{QL - QK}{QK} = \frac{h-r}{r}$$

we get $$v_1 = \sqrt{rg\frac{(h-r)}{r}}$$

Considering the journey of particle from Q to M, we have

$$v_1^2 = v^2 - 2gh$$

putting the value of v_1 get

$$rg\left(\frac{h-r}{r}\right) = v^2 - 2gh$$

or $$g(h - r) = v^2 - 2gh$$

which gives $$3gh = v^2 + rg$$

or $$h = \frac{v^2 + rg}{3g}$$

or $$h = \frac{39200 + 980 \times 10}{3 \times 980}$$

$$h = 16.6 \text{ cm}$$ **Ans.**

Q. 58. *A sphere of mass 200 gm is attached to an inextensible string of length 130 cm whose upper end is fixed to the ceiling. The sphere is made to describe a horizontal circle of radius 50 cms.*

(1) Calculate the time period of one revolution.

(2) What is the tension in the string.

Ans. Given $$OP = l = 130 \text{ cm}, \; PC = r = 50 \text{ cm}$$

$$r = l\sin\theta$$

The vertical component of tension balances the weight and horizontal component provides the centripetal force; Thus

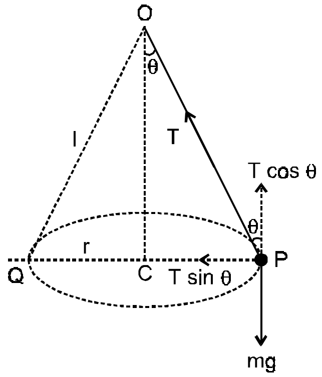


Fig. 29

$$mg = T\cos\theta$$

and
$$\frac{mv^2}{r} = T\sin\theta$$

Dividing we get

$$\tan\theta = \frac{v^2}{rg}$$

or
$$v = \sqrt{rg\tan\theta} \qquad ...(1)$$

again
$$\tan\theta = \frac{CP}{CO} = \frac{r}{\sqrt{OP^2 - CP^2}} = \frac{50}{\sqrt{(130)^2 - (50)^2}} = \frac{5}{12}$$

Putting values of r, g, tan θ in Eq. (1) we get

$$v = \sqrt{50 \times 980 \times \frac{5}{12}} = \frac{350}{\sqrt{6}} \text{ cm/sec.}$$

The time period of one revolution,

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega}\cdot\frac{r}{r} = \frac{2\pi r}{v}$$

$$\boxed{T = \frac{2\pi \times 50}{350/\sqrt{6}} = 2.2 \text{ sec.}} \quad \textbf{Ans.}$$

(2) Using the condition for vertical equilibrium, $T = \frac{mg}{\cos\theta}$

where
$$\cos\theta = \frac{12}{\sqrt{25 + 144}} = \frac{12}{13}$$

then
$$T = \frac{200 \times 980}{12/13} = 2.12 \times 10^5 \text{ dynes} \quad \textbf{Ans.}$$

Q. 59. *A nail is locked at a certain distance vertically below the point of suspension of a simple pendulum. The pendulum bob is released from a position where the string makes an angle of 60° with the vertical. Calculate the distance of nail from the point of suspension such that the bob will just perform revolution with the nail as centre. Assume the length of pendulum to be one meter.*

Ans. Let SP be the initial position with S as point of suspension then,

$$SR = PS\cos 60° = \frac{PS}{2} = \frac{1}{2}$$

and
$$RQ = SQ - SR = SP - SR$$

$$= 1 - \frac{1}{2} = \frac{1}{2} \text{ m}$$

Now, bob comes from P to Q descending vertically by distance RQ.

Using $v^2 = u^2 + 2gd$ for this journey with $u = 0$, $g = 9.8$ m/sec^2, $S = \frac{1}{2}$ m

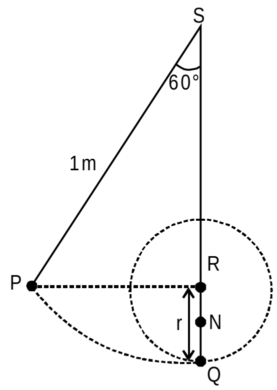


Fig. 30

we get $$v^2 = 0 + 2 \times 9.8 \times \frac{1}{2} = 9.8 \quad ...(1)$$

We know that velocity of the particle at bottom of a vertical circle, so that the particle may successfully describe vertical circle, is given by

$$v = \sqrt{5rg}; \; r \text{ is the radius of circle.}$$

or $$v^2 = 5 \times r \times 9.8 \quad ...(2)$$

Equating Eq. (1) and (2), we get

$$9.8 = 5 \times r \times 9.8$$

or $$r = \frac{1}{5} = 0.2 \text{ meter} = \text{NQ}$$

i.e., $$\text{RN} = \text{RQ} - \text{NQ}$$
$$= 0.5 - 0.2 = 0.3 \text{ m}$$

$\therefore$ $$\text{SN} = \text{SR} + \text{RN}$$
$$= 0.5 + 0.3 = 0.8 \text{ m}$$

$$\boxed{\text{SN} = 0.8 \text{ m}}$$ **Ans.**

Q. 60. *A particle of mass 0.2 kg is moving inside a smooth vertical circle of radius r = 50 cm. If it is projected horizontally with velocity v = 4 m/sec from its lowest position, find the angle θ at which it will loose contact with the circle.*

Ans. Situation is shown in figure 31.

Here $m = 0.2$ kg, $r = 50$ cm $= 0.5$ m, $g = 9.8$ m/sec^2

Let h = height from lowest position where particle loses contact with vertical circle.

θ = angle with vertical and v = velocity at Q then, component of mg along QO = $mg \cos\theta$.

$$\text{centrifugal force} = \frac{mv^2}{r} = \text{T} + mg\cos\theta$$

But $$\text{T} = 0 \text{ at Q.}$$

So, $$mg\cos\theta = \frac{mv^2}{r} \quad ...(1)$$

where $$\cos\theta = \frac{\text{RO}}{\text{OQ}} = \frac{h-r}{r}$$

Again, velocity at Q is, from $v^2 = u^2 - 2gs$, with $u = 4$ m/sec

$$v^2 = 16 - 2 \times 9.8 \times h \quad ...(2)$$

putting value of $\cos\theta$ in (1), we get

$$mg\left(\frac{h-r}{r}\right) = \frac{mv^2}{r}$$

or $g\,(h - r) = v^2$

Substituting the value of v^2 from Eq. (2) we get

$$g\,(h - r) = 16 - 2 \times 9.8 \times h$$

or $9.8\,(h - 0.5) = 16 - 2 \times 9.8 \times h$

or $9.8\,h - 4.9 = 16 - 19.6\,h$

which gives $h = 0.71$ m

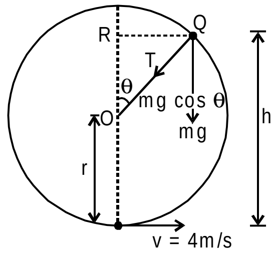


Fig. 31

then $\cos\theta = \dfrac{h-r}{r} = \dfrac{0.71 - 0.50}{0.50} = 0.42$

or $\boxed{\theta = \cos^{-1}(0.42)}$ **Ans.**

Q. 61. *A smooth table is placed horizontally and an ideal spring of constant k = 1000 nt/m and unextended length of 0.5 m has one end fixed to its centre. The other end is attached to a mass of 5 kg which is moving in a circle with constant speed, 10 m/sec. Find*

(1) The tension in the spring.

(2) Extension of spring beyond its normal length.

Ans. Let l = unextended length of spring

= 0.5 m

x = extended length of spring

then, extension in the spring = $(x - l)$

At position P of ball, force on the spring towards centre O, is

F = k × extension = centripetal force

$$= \frac{mv^2}{x} \quad ...(1)$$

But $$\frac{mv^2}{x} = \frac{5 \times 10^2}{x} = \frac{500}{x} \quad ...(2)$$

and Spring force F = $k\,(x - 0.5)$

= $1000 \times (x - 0.5)$

then Eq. (2) gives

$$\frac{500}{x} = 1000\,x - 1000 \times 0.5$$

or $\dfrac{500}{x} = 1000\,x - 500$

or $500 = 1000\,x^2 - 500\,x$

or $2x^2 - x - 1 = 0$

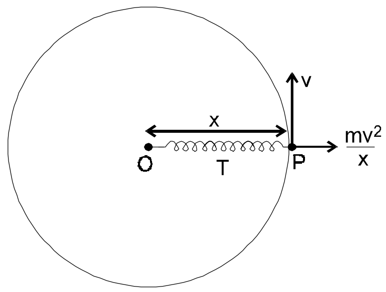


Fig. 32

which gives $x = 1,\ x = -\frac{1}{2}$; but negative elongation is not possible,

So, $x = 1$, Hence tension in string

So, $$F = T = \frac{mv^2}{x} = \frac{5 \times 10 \times 10}{1} = 500 \text{ Newton}$$

and Extension $(x - l) = (1 - 0.5) = 0.5$ meter

F	= 500 Newton
Extension	= 0.5 meter

Ans.

Q. 62. *A string with a ball is held horizontally as shown in the figure 33. A nail located at a distance 'd' vertically below the point of suspension. Show that 'd' must be at least 0.61 l. (l being the length of the string) if the ball is to swing completely around a circle centered on the nail.*

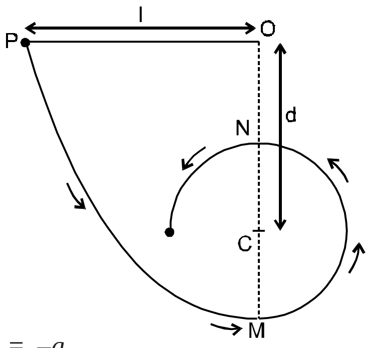


Fig. 33

Ans. Ball falls under gravity from P reaches M (vertical descent = OM = l) then it has to swing completely around the circle with centre at C (nail) of diameter MN = h.

From $v^2 = u^2 + 2gh$, velocity of ball at M is given by

$$v^2 = 0 + 2gl \quad ...(1)$$

Let velocity of ball at N be v_1. Considering the ascent MN and using

$$v^2 = u^2 + 2hg$$

where $u = v,\ v = v_1,\ h = \text{MN},\ g = -g,$

we have $$v_1^2 = v^2 - 2gh \text{ or } v_1^2 = 2gl - 2gh \quad ...(2)$$

For successful completion of circle, condition at N is,

Centrifugal force = weight

i.e., $$\frac{mv_1^2}{h/2} = mg \text{ or } v_1^2 = \frac{gh}{2} \quad ...(3)$$

Equating Eqs. (2) and (3) we get

$$2gl - 2gh = \frac{gh}{2} \text{ or } h = 4l - 4h$$

So $$\text{MN} = h = \frac{4l}{5} = 0.8\ l$$

then $$\text{OC} = d = \text{OM} - \text{CM} = l - 0.4\ l$$

∴ $$\boxed{d = 0.6\ l}$$ **Ans.**

UNSOLVED PROBLEMS

1. Show that a centripetal acceleration acts on a particle moving on a circular path.
2. Explain the statement "Centrifugal force is a pseudo force".
3. Explain the principle of cream separating machine.
4. If a car is suddenly turned to left, a passenger of car strikes with right wall, explain.

5. A passenger in spaceship uses spring clock-give reason.

6. The speed of a body is constant. Can it have a path other than a straight line or circle.

7. Why are tracks banked at sharp turns. An automobile overturns when going too fast around a curve. Which wheel leaves the ground first.

8. The handle of the carpenter's screw driver is much thicker then the handle of a watch makers screw-driver. Explain the reason.

9. A simple pendulum with a bob of mass m swings with an angular amplitude of 40°. When its angular displacement is 20°, then the tension in the string is greater than mg cos 20°. Is the statement correct.

10. As astronaut in spaceship feels weightlessness. Explain why ? If a satellite orbits earth at an altitude of 100 km. Find its time period of revolution in circular orbit.

11. A particle of mass m rotates in a circle of radius a, with a uniform angular speed ω. It is observed from a frame rotating about the z-axis with a uniform angular speed ω_0. Find the centrifugal force on the particle.

12. A smooth circular tube is held firmly in a vertical plane. A particle which can slide inside the tube is slightly displaced from rest at its highest position in the tube. Find the pressure between the tube and the particle in terms of its mass 'm' and the angular displacement θ from its highest position.

13. Obtain expression for the orbiting velocity of artificial satellite.

14. A satellite is moving in a circular orbit round the earth at a distance of 5R from its centre. Calculate its orbital speed and time period.

15. Find an expression for the total energy of a satellite of mass 'm' moving round the earth in circular orbit of radius r.

16. Write short note on

(*a*) Satellite launching (*b*) Escape velocity (*c*) Geo-stationary satellite.

17. In an orbit of radius 1.5×10^{11} meter, earth completes one revolution in 365 days around the sun. Calculate mass of sun ($G = 6.67 \times 10^{-10}$ nt-m^2/kg^2).

18. A motor cyclist with the motor cycle has a mass of 250 kg and travels round a curve in a road of 36 metre radius. Snow and ice on the road reduce the coefficient of friction between the tyres and road to 0.2. If the curved road is banked to 15° in motor cyclist's favour; Find (*a*) maximum velocity attainable without slipping (*b*) the angle, the rider must make with the road surface at this velocity assuming that rider and motorcycle remain in one plane.

19. A train is travelling at 64 km/h and the diameter of one of the wheels of the engine is 1.5 m. Find the velocities of the two points on this wheel. Which are at a height of 1.2 m above the ground.

20. A fighter plane flying in the sky dives with a speed of 360 km/hr in a vertical circle of radius 200 m, weight of the pilot sitting in it is 75 kg. Find the value of force with which pilot presses his seat when aeroplane is (*a*) at highest position (*b*) at lowest position.

21. The moon revolves around the earth in a circle of radius 3.8×10^6 m and requires 27.3 days to make a complete revolution. What is the acceleration of the moon towards the earth.

4

WORK, ENERGY AND MOMENTUM

4.1 WORK

Whenever a force acting on a body produces a change in the position of the body, work is said to be done by the force. If there is no change in the position of the body, work is not done. We may exert a large force on a wall, but if the wall remains intact in its present position then we have not done any work. If a coolie having a heavy box on his head is standing at a fixed place, he is not doing any work. Work is said to be done only when there is a displacement in the direction of the force.

Work is measured by the product of the applied force and the displacement of the body in the direction of the force, that is

Work = force × displacement in the direction of the force.

If a force F acting on a body produces a displacement Δs in the body in the direction of the force (Fig. 1*a*), then the work done by the force is given by

$$W = F \times \Delta s$$

If the force F is making an angle θ with the direction of displacement of the body (Fig. 1*b*), then the work done is

$$W = F \cos\theta \times \Delta s,$$

because $F \cos \theta$ is the component of F in the direction of displacement.

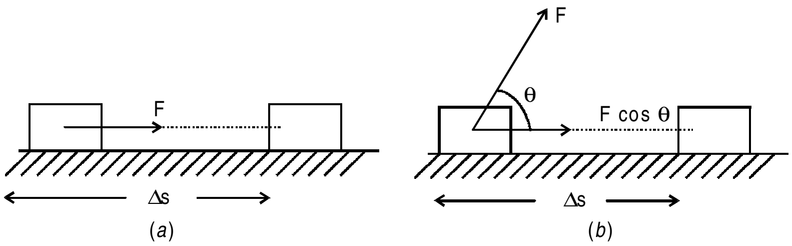


Fig. 1

If $\theta = 90°$, then $\cos \theta = 0$ and so $W = 0$.

This means that if the displacement is perpendicular to the force, no work is done. When a satellite revolves around the earth, the direction of the force applied by the earth is always perpendicular to the direction of motion of the satellite. Hence no work is done on the satellite by the centripetal force.

Force and displacement both are vector quantities but work is a scalar quantity. The unit of work is 'joule'. If a force of 1 newton produces a displacement of 1 meter in the direction of the force, the work done is called 1 joule:

$$1 \text{ joule} = 1 \text{ newton} \times 1 \text{ meter}$$

4.2 POWER

The rate of doing work by an agent or a machine is called power:

$$\text{Power} = \frac{\text{Work}}{\text{time}}$$

If W is the work done by an agent in t second, then his power P is given by

$$P = \frac{W}{t}$$

Since the unit of work is joule, the unit of power will be 'joule/second'. This is called 'watt'. If 1 joule of work is done in 1 second, the power is 1 watt.

$$1 \text{ watt} = 1 \text{ joule/second}$$

Another unit of power is 'horse-power'

$$1 \text{ horse-power} = 746 \text{ watts}$$

The power of a normal person is from 0.05 to 0.1 horse power. If an agent exerts a force F in the direction of motion, then

$$P = \frac{W}{t} = \frac{Fr}{t} = Fv$$

where $v = r/t$

4.3 WORK IN STRETCHING A SPRING

When a spring is stretched slowly, the stretching force increases steadily as the spring elongates, *i.e.*, the force is 'variable'. Let one end of the spring be attached to a wall its length being along the x-axis. Let the origin $x = 0$, coincide with the free end of the spring in its normal, unstretched state. Let the spring be stretched through a distance x by applying a force F_{app} at the free end. The spring, on account of its elasticity, will exert a restoring force F on the stretching agent given by (within elastic limit)

$$F = -kx; \quad \text{(Hooke's law)}$$

Where k is the force-constant or stiffness of the spring. The minus sign indicates that the restoring force is always opposite to the displacement x. Since the restoring force F is equal and opposite to the applied force F_{app}; the latter is given by

$$F_{app} = -F = kx$$

The work done by the (varying) applied force F_{app}. in the displacement from $x = 0$ to $x = x$ is

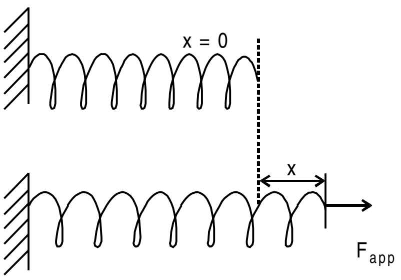


Fig. 2

$$W = \int_{x=0}^{x=x} \vec{F}_{app.} \, d\vec{x} = \int_{x=0}^{x=x} F_{app} \, dx$$

$\therefore$ Angle between $\vec{F}_{app}$ and $d\vec{x}$ is zero.

$$= \int_0^x kx\,dx = k\left[\frac{x^2}{2}\right]_0^x = \frac{1}{2}kx^2$$

Similarly if the spring is stretched so that its free end moves from x_1 to x_2, the work done is

$$W = \int_{x=x_1}^{x=x_2} F_{app}\,dx = k\int_{x_1}^{x_2} x\,dx = k\left[\frac{x^2}{2}\right]_{x_1}^{x_2}$$

$$= \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2 = \frac{1}{2}k\left(x_2^2 - x_1^2\right)$$

4.4 ENERGY

The capacity of doing work is called energy. Energy has various forms such as mechanical energy, heat energy, light energy, magnetic energy, sound energy, chemical energy etc.

Mechanical energy has two forms: Kinetic energy and potential energy.

4.5 KINETIC ENERGY

The kinetic energy of a moving body is measured by the amount of work which has been done in bringing the body from the rest position to its present position, or which the body can do in going from its present position to the rest position.

Let a body of mass m be in the rest position. When we apply a constant force F on the body, it starts moving under an acceleration. If a be the acceleration, then by Newton's second law, we have

$$a = \frac{F}{m}$$

suppose the body acquires a velocity v in moving a distance S. According to equation $v^2 = u^2 + 2as$ (u = 0, since the body was initially at rest) we have

$$v^2 = 2as = 2 \times \frac{F}{m} \times s$$

$$F \times S = ½\,mv^2$$

But F × S is the work which the force F has done on the body in moving it a distance 'S'. It is due to this work that the body has itself acquired the capacity of doing work. This is the measure of the kinetic energy of the body. Hence if we represent kinetic energy of a body by K, then

$$K = F \times S = ½\,mv^2$$

Thus, the kinetic energy of a moving body is equal to half the product of the mass (m) of the body and the square of its speed (v^2). In this formula, v occurs in the second power and so the speed has a larger effect, compared to mass, on the kinetic energy. It is because of this reason that the bullet fired from a gun injures seriously inspite of its very small mass.

Now, suppose a body is initially moving with a uniform speed u. When a force is applied on it then its speed increases from u to v in a distance S. Now, we have

$$v^2 = u^2 + 2as$$

$$v^2 - u^2 = 2 \times (F/m) \times s$$

$$F \times S = ½\ mv^2 - ½\ mu^2$$

But F × S is the work done W on the body by the force. Hence

$$\boxed{W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2}$$

According to definition, $\left(W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2\right)$ is the increase in the kinetic energy of the body. Thus when a force acts upon a moving body, then the kinetic energy of the body increases, and the increase is equal to the work done.

4.6 POTENTIAL ENERGY

Bodies can do work also by virtue of their 'position' or 'state of strain'. The energy in a body due to its position or state of strain is called the 'potential energy' of the body. For example, the water at the top of a water-fall can rotate a turbine when falling on it. The water has this capability by virtue of its position (at a height). Similarly, a wound clock-spring keeps the clock running by virtue of its state of strain. Thus water and wound spring both have potential energy, the former has 'gravitational' potential energy and the later has 'elastic' potential energy.

A system of electric charges also has potential energy which is called 'electrostatic' potential energy. If two opposite charges are taken further away from each other, the work done against their mutual attraction is stored in the form of potential energy, so that the potential energy of the system increases. In case of similar charges, the potential energy increases when they are brought closer.

The potential energy of a body is measured by the amount of work which has been done in bringing the body from its zero-position to the present position or which that body can do in going from its present position to the zero-position. Therefore, if a body is taken, under the action of a force, from one position to the other, then the work done is stored in the body in the form of its potential energy (provided there is no loss of work against friction, etc.).

4.7 GRAVITATIONAL POTENTIAL ENERGY

A mass held stationary above the ground has energy, because, when released, it can raise another object attached to it by a rope passing over a pulley, for example. A coiled spring also has energy, which is released gradually as the spring uncoils. The energy of the weight or spring is called potential energy, because it arises from the position or arrangement of the body and not from its motion. In the case of the weight, the energy given to it is equal to the work done by the person or machine which raises it steadily to that position against the force of attraction of the earth. So this is gravitational potential energy. In the case of the spring, the energy is equal to the work done in displacing the molecules from their normal equilibrium positions against the forces of attraction of the surrounding molecules. So this is molecular potential energy.

Suppose a body of mass m is raised to a height h from the earth's surface. In this process, work is done against the force of gravity (mg). This work is stored in the body in the form of gravitational potential energy U. Thus

$$\begin{aligned} U &= \text{work done against force of gravity} \\ &= \text{weight of the body} \times \text{height} \\ &= mg \times h = mgh \end{aligned}$$

4.8 WORK-ENERGY THEOREM

Let us consider a body of mass m acted upon by a resultant accelerating force F along the x-axis. Suppose as the body moves from a position x_1 to a position x_2 along the x-axis, its velocity increases from v_1 to v_2. The work done by the force in the displacement is

$$W = \int_{x=x_1}^{x=x_2} F\, dx$$

By Newton's second law, we have

$$F = ma = m\frac{dv}{dt} = m\frac{dv}{dx}.\frac{dx}{dt} = mv\frac{dv}{dx}$$

$$\because \qquad v = dx/dt$$

$$\therefore \qquad W = m\int_{x=x_1}^{x=x_2} v\,\frac{dv}{dx}dx = m\int_{v=v_1}^{v=v_2} v\, dv$$

$$= m\left[\frac{v^2}{2}\right]_{v_1}^{v_2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

The quantity ½ mv^2 is defined as the kinetic energy K of the body. Thus the above equation may be written as

$$W = K_2 - K_1$$

Where K_2 and K_1 are the final and initial kinetic energies of the body. Thus, if ΔK represents the change in kinetic energy, $\Delta K = K_2 - K_1$, then we have

$$\boxed{W = \Delta K}$$

Thus we conclude that "whenever a body is acted upon by a number of forces such that the resultant force is not zero, then the work done by the resultant force (whether constant or variable) is equal to the change in the kinetic energy of the body. This is known as the work-energy theorem.

If the kinetic energy of the body decreases, the work done on it by the resultant force is negative, or the work is done by the body against the resultant force. Therefore, a moving body is said to have a store of (kinetic) energy in it, which it loses in doing work. Hence the kinetic energy of a moving body is defined as the work it can do before coming to rest.

The units of kinetic energy and of work are the same. Kinetic energy, like-work, is a scalar quantity.

When several forces act upon the body, the work done by the resultant force is the algebraic sum of the works done by the individual forces *i.e.,*

$$W = W_1 + W_2 + \text{-----------------}$$

Hence the work-energy theorem may also be written as

$$\boxed{W_1 + W_2 + \text{-------------} = \Delta K}$$

4.9 SIGNIFICANCE OF THE WORK-ENERGY THEOREM

The work-energy theorem is useful for solving problems in which the work done by the resultant force is easily computed and in which we are interested in finding the particle's speed at certain positions of greater significance, perhaps is the fact that the work-energy theorem is the starting point for a sweeping generalization in physics. It has been emphasized that the work-energy theorem is valid when W is interpreted as the work done by the resultant force acting on the particle. However, it is helpful in any problems to compute separately the work done by certain types of force and give special names to the work done by each type. This leads to the concepts of different types of energy and the principle of the conservation of energy.

4.10 CONSERVATIVE FORCE: *FIRST DEFINITION*

A force acting on a particle is conservative if the particle, after going through a complete round trip, returns to its initial position with the same kinetic energy as it had initially.

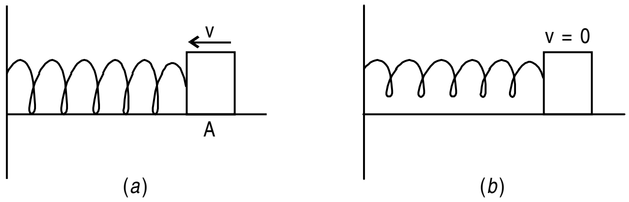


Fig. 3

When we throw a ball upward against gravity, the ball reaches a certain height coming momentaily to rest so that its kinetic energy becomes zero. Then it returns to our hand under gravity with the same kinetic energy with which it was thrown (provided the air-resistance is assumed zero). Thus the force of gravity is conservative. Similarly, when a block is moved from a position A on a horizontal plane with a velocity v so as to compress on a spring (Figure 3*a*). It is first brought to rest by the elastic (restoring) force of the spring and loses all its kinetic energy (Figure *b*). Then the compressed spring re-expands and the block moves back under the elastic force gaining kinetic energy. As the block returns to its initial position A, it has the same velocity v, and hence the same kinetic energy, as it had before (provided the horizontal plane is frictionless and the spring is ideal). Thus the elastic force exerted by an ideal spring is conservative. The electrostatic force is also conservative.

A force is termed to be conservative if the work done by it on a particle in moving it from one point to another in space, depends only on the location or position of the particle and not on the path followed by the particle.

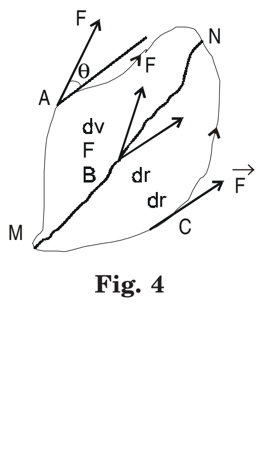


Fig. 4

Imagine a particle moving from point M to N under action of force $\vec{F}$. There may be three alternative paths, Path MAN, Path MBN, Path MCN. If the force $\vec{F}$ is conservative, then

$$W_{MN} = \underset{\text{Path MAN}}{\int_M^N \vec{F}.d\vec{r}} = \underset{\text{Path MBN}}{\int_M^N \vec{F}.d\vec{r}} = \underset{\text{Path MCN}}{\int_M^N \vec{F}.d\vec{r}}$$

Work done by a conservative force around a closed path

Consider the closed path MANBM. under action of a conservative force $\vec{F}$, a particle moves from M to N along path MAN and returns back to M, along the path NBM.

Now, work done by the force $\vec{F}$ in moving the particle from M to N along path 1, is

$$W_{MN} = \int_M^N \vec{F}.d\vec{r} = T_N - T_M$$

Where T_N = K.E. of particle at N and T_M = K.E. of particle at M. Similarly the work done in carrying the particle from N to M, by the force $\vec{F}$ is given by

$$W_{NM} = \int_N^M \vec{F}.d\vec{r} = T_M - T_N$$

So, total work done by the conservative force $\vec{F}$, along closed path MANBM, is

$$\underset{\text{Path MANBM}}{\int \vec{F}.d\vec{r}} = \underset{\text{Path 1(MAN)}}{\int \vec{F}.d\vec{r}} + \underset{\text{Path 2(MBN)}}{\int \vec{F}.d\vec{r}}$$

$$W_{MANBM} = T_N - T_M + T_M - T_N = 0$$

or $$\oint \vec{F}.d\vec{r} = 0$$

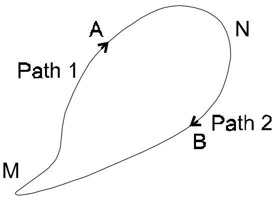


Fig. 5

Thus, work done by a conservative force around a closed path is always zero *i.e.*, the change in kinetic energy of the particle in motion due to a conservative force, over a closed path is zero.

Examples of conservative forces

Position dependent forces depend on the instantaneous position of the particle or body (not on its velocity). Central forces are position dependent and in addition, are directed towards a fixed center. Position dependent forces are (*a*) Gravitational force (*b*) Electrostatic force (*c*) Elastic force. Most of the position dependent forces are conservative in nature.

4.11 A CENTRAL FORCE IS CONSERVATIVE

A force is termed central force if it acts on a particle in such a way that it is directed towards or away from a fixed point and its magnitude depends only on the distance of particle from the fixed point.

It may be observed, that central forces are conservative. Consider a central force $\vec{F}$, by definition:

$$\vec{F} = F_s\hat{S}$$

Where F_s is function of position 's' only and $\hat{S}$ is unit vector along $\vec{S}$. The above force is directed away from the fixed point and acts on a particle. Let the particle move from M to N under action of this force. Now, total work done by the central force in this movement is,

$$W_{MN} = \int_M^N \vec{F}.d\vec{r}$$

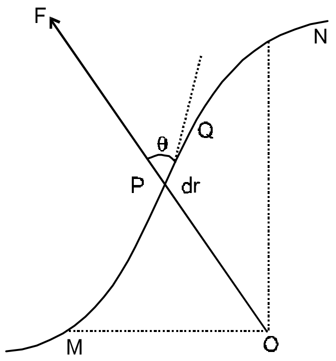


Fig. 6

Putting the value of force $\vec{F}$,

$$W_{MN} = \int_M^N F_s\hat{S}.d\vec{r} = \int_M^N F_s.dr\cos\theta$$

or

$$\boxed{W_{MN} = \int_M^N F_s\, dS}$$

As F_s is function of s only, so its integral (say α) is also a function of S, hence

$$W_{MN} = \int_M^N F_s dS = [\alpha]_M^N = \alpha_N - \alpha_M$$

Thus, the work done by the central force depends only on the position of points M and N (not on path followed). This suggests that central force is a conservative force.

Properties of Work done by conservative force

(*i*) It is independent of path.

(*ii*) It is equal to difference between final and initial values of energy function.

(*iii*) It is completely recoverable.

4.12 NON-CONSERVATIVE FORCE

A force is non-conservative if the work done by it on the particle moving between two points depends on the path followed by the particle. Thus, if a particle moves from M to N under influence of a non-conservative force $\vec{F}$, then work done along three possible paths by the force is not same (*i.e.,* it is path dependent). Thus

$$W_{MN} = \underset{\text{Path 1}}{\int_M^N \vec{F}.d\vec{r}} \neq \underset{\text{Path 2}}{\int_M^N \vec{F}.d\vec{r}} \neq \underset{\text{Path 3}}{\int_M^N \vec{F}.d\vec{r}}$$

Thus, work done by a non-conservative force around a closed path is not zero *i.e.*,

$$W_{MN} = \int_M^N \vec{F}.d\vec{r} \neq 0$$

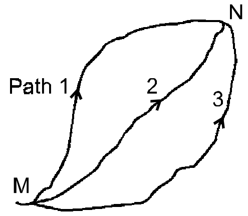


Fig. 7

Again, if a conservative and a non-conservative both, forces act upon the particle and W_C be the work done by conservative force and $W_{M\text{-}C}$ be the work done by non-conservative force, then

$$W_C + W_{N-C} = \text{Change in K.E. of Particle} = \Delta T$$

Example of non-conservative force: Frictional forces, viscous forces etc are examples of path dependent forces.

Potential Energy

Potential energy of a body is defined as the energy stored in body due to its position, configuration or state of strain. Consider a particle lying in a conservative force field due to force F_C. The force F_C that acts on the particle may be balanced by applications of suitable applied force F_{app}. Now the particle may be moved extremely slowly (so that no K.E. develop) by applied force against the conservative force F_C. The work done in moving the particle will appear as potential energy of the particle.

Thus, potential energy is measured by the amount of work that it can do when it moves from referred position to some standard position. Potential energy is generally denoted by U.

Potential Energy Difference

The difference in potential energy between two positions of a particle is defined as the work done by applied force on the particle in moving it from the first position to second position. Thus, if r_2 and r_1 are the final and initial position, then by definition

$$U(r_2) - U(r_1) = \int_{r_1}^{r_2} \vec{F}_{app.}d\vec{r} = -\int_{r_1}^{r_2} \vec{F}_c.d\vec{r} \qquad (\text{as } F_{app} = -F_c)$$

$$U(r_2) - U(r_1) = -\int_{r_1}^{r_2} \vec{F}_c.d\vec{r}$$

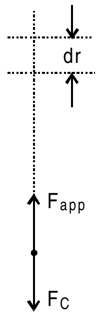


Fig. 8

If first position is at infinity (*i.e.*, $r_1 = \infty$) having potential energy zero there (*i.e.*, $U(r_1) = 0$), then potential energy at position r (*i.e.*, $r_2 = r$) is given by

$$0 - U(r) = \int_{\infty}^{r_2} \vec{F}_c.d\vec{r}$$

or

$$-U(r) = \int_{\infty}^{r_2} \vec{F}_c.d\vec{r}$$

or
$$\boxed{U(r) = \int_{\infty}^{r} \vec{F}_{\text{app}} . d\vec{r}}$$

Thus potential energy of a particle at a point r may be defined as the work done by the externally applied force in moving the particle from infinity to that point.

The position at which the conservative force acting on the particle is zero, is the reference position with respect to potential energy.

In case of gravitational and electrostatic forces the reference position is infinity.

In case of springs the normal unstretched length is considered as reference position.

In case of earth's gravitational force field, earth's surface is the reference position.

4.13 RELATION BETWEEN CONSERVATIVE FORCE AND POTENTIAL ENERGY

By definition of potential energy

$$U(r) = -\int_{\infty}^{r} \vec{F} . d\vec{r}$$

In rectangular Cartesian system one may write

$$\vec{F} = \hat{i} F_x + \hat{j} F_y + \hat{k} F_z$$

and
$$d\vec{r} = \hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz$$

Then the potential energy will be given by

$$U(\vec{r}) = U\ (x,\ y,\ z)$$

i.e.,
$$U(r) = -\int_{\infty}^{r} (\hat{i} F_x + \hat{j} F_y + \hat{k} F_z).(\hat{i}\, dx + \hat{j}\, dy + \hat{k} dz)$$

or
$$U(\vec{r}) = -\int_{\infty}^{x} F_x\, dx - \int_{\infty}^{y} F_y\, dy - \int_{\infty}^{z} F_z\, dz$$

Differentiating the above equation partially

$$\frac{\partial U}{\partial x} = -F_x, \frac{\partial U}{\partial y} = -F_y, \frac{\partial U}{\partial z} = -F_z$$

In view of these relations, the force $\vec{F}$ can be expressed as

$$\vec{F} = -\hat{i}\frac{\partial U}{\partial x} - \hat{j}\frac{\partial U}{\partial y} - \hat{k}\frac{\partial U}{\partial z}$$

or
$$\vec{F} = -\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]U$$

or $$\vec{F} = -\vec{\nabla} U$$

where $$\vec{\nabla} = +\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Vector operator $\vec{\nabla}$ is called 'del' or 'nebla'.

Thus, a conservative force is the negative gradient of potential energy U. Only for one dimension

$$F = -\frac{dU}{dx} \quad \text{or} \quad U = -\int F\,dx$$

4.14 THE CURL OF A CONSERVATIVE FORCE IS ZERO

Now let us see what is the value of curl F for a conservative force:

A conservative force may be expressed as negative gradient of potential energy *i.e.*,

$$\vec{F} = -\vec{\nabla} U$$

The curl of $\vec{F}$ is,

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \vec{\nabla} \times (-\vec{\nabla} U)$$

$$= -\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix}$$

$$= -\left[\hat{i}\left(\frac{\partial^2 U}{\partial y \partial z} - \frac{\partial^2 U}{\partial z \partial y}\right) - \hat{j}\left(\frac{\partial^2 U}{\partial x \partial z} - \frac{\partial^2 U}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 U}{\partial y \partial x}\right)\right]$$

Since, U is perfect differential, so

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x} \text{ and so on}$$

$\therefore$ $$\text{Curl } \vec{F} = -\left[\hat{i}(0) - \hat{j}(0) + \hat{k}(0)\right] = 0$$

i.e., $$\boxed{\vec{\nabla} \times \vec{F} = 0}$$

Thus, the curl of a conservative force is zero.

The Total Mechanical Energy is conserved in a conservative force field

Consider a particle to be displaced by a conservative force $\vec{F}$ from a position M to N. By work-energy theorem, the work done by force is equal to the increase in K.E. of particle, *i.e.*,

$$W_{MN} = \int_M^N \vec{F}.d\vec{r} = T_N - T_M$$

one can express the same work done by decrease in potential energy of the particle, *i.e.*,

$$W_{MN} = -\int_M^N \vec{F}.d\vec{r} = U_M - U_N$$

Equating above,

or $$T_N - T_M = U_M - U_N$$

$$\boxed{T_N + U_N = T_M + U_M}$$

which suggests that the sum of kinetic and potential energy of a particle under action of conservative force remains constant, *i.e.*,

$$T + U = \text{Total Mechanical Energy}$$

(Constant in conservative force field)

Sometimes it is convenient to call the quantity E = T + U as the energy function. This energy function is invariant with respect to change in time.

Motion of a Body near the Surface of the Earth

Let us consider a body of mass m be situated at a height h in the rest position above the earth's surface. Suppose that its potential energy is zero at the earth's surface. Let the x direction be normal to the surface of the earth and directed upwards.

If the body starts to fall down and at any instant its height above the earth's surface is x, then work done by the gravitational force $-mg$ on the body is

$$W = \int_h^x (-mg)dx = -mg\,(x-h) = mg\,(h-x)$$

Where the direction of force is opposite to x.

This amount of work done on the body will increase its kinetic energy

$$W = \tfrac{1}{2} mv^2 \qquad \therefore \text{ initial velocity} = 0$$

$$\therefore \quad \tfrac{1}{2} mv^2 = mg\,(h - x)$$

or $$mgh = \tfrac{1}{2} mv^2 + mgx \qquad ...(i)$$

But a body situated at a height x has the capacity to work $mg \times x$, hence the potential energy of the body at this height is mgx. Thus, initially, the body has no kinetic energy, but only potential energy mgh.

$$\therefore \quad \text{Initial total energy} = \text{initial K.E.} + \text{initial P.E.}$$

$$= 0 + mgh = mgh$$

At any height x, total energy $= \tfrac{1}{2} mv^2 + mgx = mgh$ (from Eq. (i))

At earth surface ($x = 0$), total energy = $mgh + 0 = mgh$

Hence, the total energy of a freely falling body is the same initially, finally and in any middle position.

In other words, the sum of potential and kinetic energies of a freely falling body remains constant throughout the motion.

4.15 LINEAR RESTORING FORCE

When a force acting on a particle is directly proportional to the displacement of the particle from a certain fixed point (called mean position) and is directed oppositely to the displacement, it is called linear restoring force. Thus

$$\vec{F} \alpha - \vec{r}$$

or $$\vec{F} = -k\hat{r}\, r$$

Where $\vec{r}$ = position vector of particle at any instant = $\hat{r}\, r$

Now $$\vec{r} = \hat{r}\, r = \hat{i}x + \hat{j}y + \hat{k}z$$

or $$\vec{F} = -k\,\vec{r}$$

Linear restoring force is a conservative force: For a conservative force

$$\vec{\nabla} \times \vec{F} = 0$$

and $$\vec{F} = -k\,\vec{r} = -k(\hat{i}x + \hat{j}y + \hat{k}z)$$

so $$\vec{\nabla} \times \vec{F} = \vec{\nabla} \times (-k\,\vec{r})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -kx & -ky & -kz \end{vmatrix} = -k \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

Thus, linear restoring force is a conservative force.

Massless Horizontal Spring – One End Fixed

For small displacement, an ideal compressed or stretched spring produces a linear restoring force (due to elastic properties of the spring in accordance to Hook's law). Let there be a massless spring whose one end is rigidly attached to a wall and other end be tied to a block which could slide on a horizontal table having no friction.

Normal Position

No force is produced in the spring. Let us regard the origin at the normal position of block.

Stretched Position: If an external force stretches the spring slowly, then work will be done on the system against the restoring force which constitutes the potential energy of the system. The linear force produced in the spring is in the direction opposite to displacement; *i.e.*,

$$\vec{F} = -kx\hat{i}$$

where k = force constant of spring (or spring factor) and x = displacement.

Now, potential energy, by definition, is

$$U = -\int_0^x \vec{F}.d\vec{r} = -\int_0^x \vec{F}.dx\cos\theta = -\int_0^x (-kx).dx$$

or $$U = \int_0^x kxdx = \frac{1}{2}kx^2$$

∴ $$U = \tfrac{1}{2}kx^2$$

Which gives the potential energy stored in stretched spring suffering extension x. If 'a' is maximum stretching (or elongation) of spring

$$\boxed{U_{max} = \frac{1}{2}ka^2} \quad ...(1)$$

Motion after release: If body be stretched up to maximum stretching and then released it moves under linear restoring force towards origin.

Initial total energy just at the instant of release = $\tfrac{1}{2}\ ka^2$

Total energy at position x during return = $\frac{1}{2}kx^2 + \tfrac{1}{2}\ mv^2$

(where v = velocity in position x)

Using energy conservation principle,

$$\tfrac{1}{2}ka^2 = \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2$$

or $$\tfrac{1}{2}mv^2 = \tfrac{1}{2}k(a^2 - x^2)$$

or $$v_0 = \pm\sqrt{\frac{k}{m}(a^2 - x^2)}$$

the velocity will be maximum in mean position (x = 0), is given by

∴ $$v_{\max} = \pm a\sqrt{\frac{k}{m}} \quad ...(2)$$

Compressed Position: When spring is compressed then the displacement is $-x$ and, thus, the linear restoring force produced is

$$\vec{F} = -k.(-\hat{i}x)$$

$$= \hat{i}\,kx$$

The spring linear force in compressed spring is in $+x$ direction. The maximum energy (U_{max}) and maximum velocity (V_{max}) gained in mean position will be given by Equation (1) and Equation (2) respectively.

Equation of motion: Restoring force at stretching x of the spring is $F = -kx$. By Newton's law

$$F = m\frac{d^2x}{dt^2}$$

So equation of motion is

$$m\frac{d^2x}{dt^2} = -kx$$

or

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

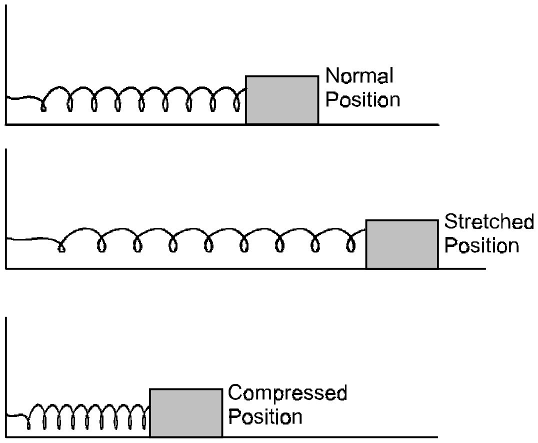


Fig. 9

which is the differential equation of motion of spring having time period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Vertical spring in uniform gravitational field with mass attached to spring: Figure 10 (*a*) shows the natural length of an ideal massless, vertical spring of force constant k. When a weight mg is gently attached to the free end of spring, an elongation x_0 is produced. Figure (*b*) shows the force in the spring (acting upward) balances the weight mg. Thus

$$mg = kx_0$$

or

$$mg - kx_0 = 0$$

If a further, displacement x is given to the mass, figure 10 (*c*) then net or effective restoring force

$$= [mg - k(x_0 + x)]$$
$$= kx_0 - kx_0 - kx$$
$$= -kx$$

From Newton's law

$$\text{Force} = m\frac{d^2x}{dt^2}$$

thus $$m\frac{d^2x}{dt^2} = -kx$$

or $$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

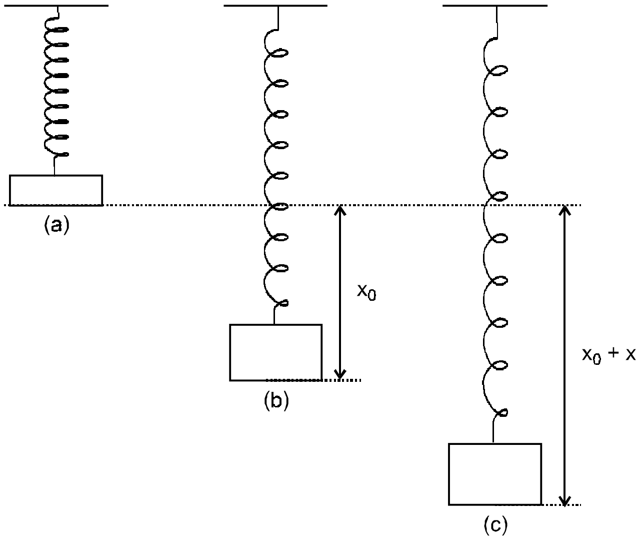


Fig. 10

which is the equation of motion. Note that gravity has no effect in the motion which is simple harmonic of time period given by

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{or} \quad \text{frequency } (n) = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Potential Energy Curve

If the potential energy U of a particle is a function of position (*i.e.,* changes from point to point), then a graph may be plotted to show the variation of potential energy with the position of the particle. Such a graph is known as potential energy curve. In case, if the particle is allowed to move in one dimension only, say along x-axis, the potential energy U will be the function of x-coordinate only, and then the force (conservative in nature) on the particle will be given by

$$F = -\frac{dU_{(x)}}{dx} \qquad ...(1)$$

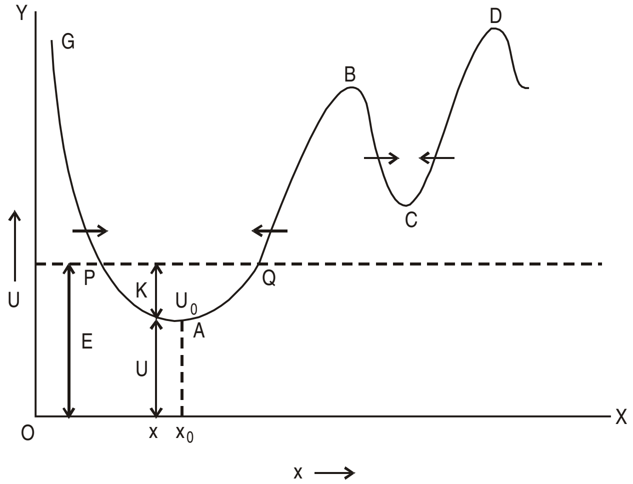


Fig. 10(*a*)

Fig. 10(*a*) shows a possible potential energy curve for one dimensional motion. The slope (dU/dx) of this curve at any point gives the force ($F = -dU/dx$), acting on the particle placed at that point. The slope (dU/dx) is positive, whenever U increases with the increase in x (*e.g.,* for the part AB and CD of the curve) and negative whenever U decreases with increase in x (*e.g.,* for the part GA and BC). Therefore the force F is negative or directed to the left whenever the potential energy is increasing with x and positive or directed towards right whenever the potential energy is decreasing with x. This means that the force acting on the particle at any point tries to bring the particle to the region of lower potential energy.

At points A, B, C and D, the potential energy U is minimum or maximum, the value of dU/dx is zero. Therefore, if a particle is placed at such a point with zero velocity, it will experience no force ($F = -dU/dx = 0$) and so it will remain at rest. These points of the curve are known as positions of equilibrium.

B, and D are the points corresponding to maximum potential energy. A particle at rest at such a point will remain at rest. However, if the particle is displaced even the slighest distance from this point the force $F(x) = -dU/dx$ will tend to push the particle farther away from the equilibrium position. Hence the points B and D are the positions of unstable equilibrium.

A and C are the points corresponding to minimum potential energy. A particle at rest at such a point will remain at rest. If the particle is displaced slightly in either direction by giving a little energy the force will tend to take it back towards the equilibrium position. Hence the points A and C are the positions of stable equilibrium.

In case, if U is constant in a region, then the slope dU/dx and hence the force acting on a particle placed at any point of that region is zero. In such a region, if a particle is displaced from one point to the other, it will remain there without experiencing any restoring force. So we call the region of constant potential energy as the region of neutral equilibrium. For example, a book placed on a table anywhere remains in equilibrium.

Bounded region or potential well. Now let us discuss about the point A (or C) of stable equilibrium and the region near to it, which is of great interest in physics. A particle, placed at the point A at rest, can be displaced from this point by giving some energy so that the total energy E of the particle becomes more than minimum value of the potential energy U_0. The total energy E = K + U of the particle has been indicated.

NUMERICALS

Q.1. *An object of mass 5 kg falls from rest through a vertical distance of 20 m and reaches a velocity of 8 m/s. Calculate the work done by push of air exerted on the object.*

Solution. The forces acting on the body are:

(*i*) the weight, acting downward,

(*ii*) the air push, acting upward.

The body is getting accelerated downward under influence of resultant force (mg – P), P being the push of air.

Now work done by resultant force in 20 m fall

$$= (mg - P) \times 20 \text{ Joules}$$

$$\text{Gain in KE in this fall} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$= \frac{1}{2}m(8)^2 - \frac{1}{2}m \times 0 = 32\,m \text{ Joules}$$

as, work done = change in K.E.

$$\therefore \quad (mg - P) \times 20 = 32\ m$$

or $$mg - P = \frac{32m}{20}; \text{ or } P = 5 \times 9.8 - \frac{32 \times 5}{20}$$

or $$P = 41 \text{ Newton}$$

So, work done by push of air (force)

$$= -41 \times 20 = -820 \text{ Joule}$$

(–*ve* sign refers to expense of work).

Q.2. *A man pulls a block weighing 10 kg upto a distance of 10 meter on a horizontal rough surface (coefficient of kinetic function = 0.20) at a constant speed. Calculate the work done by the man if the pull exerted by him makes an angle of 30° with the horizontal.*

Solution. In the figure various forces are shown

where P = pull exerted by man,

mg = weight of block,

μN = frictional force.

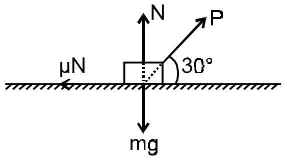


Fig. 11

Then; work done by the man

$$W = \vec{P} \cdot \vec{r} = Pr \cos 30° \qquad (r = \text{displacement})$$

As block moves with uniform speed; net force is zero

$$\therefore \quad P \cos 30° - \mu N = 0 \qquad ...(i)$$

and also, $$N + P \sin 30° - mg = 0$$

which gives, $$N = mg - P \sin 30° \qquad ...(ii)$$

Putting this value in (*i*)

$$P \cos 30° - \mu\, mg + \mu P \sin 30° = 0$$

or $$P (\cos 30° + \mu \sin 30°) = \mu\, mg$$

or $$P = \frac{\mu\, mg}{\cos 30° + \mu \sin 30°} = \frac{0.2 \times 10 \times 9.8}{\sqrt{3}/2 + 0.2 \times \frac{1}{2}}$$

$$\Rightarrow \quad P = \frac{19.60}{0.966} = 20.3 \text{ Newton}$$

So, $$\text{work done} = Pr \cos 30° = 20.3 \times 10\ \sqrt{3}/2$$
$$= 175.803 \text{ Joule}$$

Q.3. *A block initially sliding on a rough horizontal surface with velocity of 10 m/sec stops after travelling a distance of 70 m. Find the coefficient of kinetic friction between surface and the block.*

Solution. $$\text{Initial K.E.} = \frac{1}{2}mv^2 = \frac{1}{2}m \times 10^2 = 50\, m \text{ Joules}$$

Frictional force acting on block = $\mu N = \mu\ mg$

Work done against frictional force in moving 70 m

$$= \mu\ mg \times 70 \text{ Joule}$$

This work done must be equal to the initial K.E. possessed by the block

i.e., $$\mu\, mg \times 70 = 50\ m$$

or $$\mu = \frac{50}{70\, g} = \frac{50}{70 \times 9.8} = 0.073$$

Q.4. *A body of mass 40 kg climbs a vertical rope 10 meters in length in 15 sec with constant velocity. Calculate:*

(*i*) *Work done by the body,*

(*ii*) *Power output.*

Solution. Velocity of the body $= \frac{10}{15} = 0.667$ m/sec

The body has to exert force to support his own weight.

So force exerted = $mg = 40 \times 9.8 = 392$ Newton

Thus, Work = force × displacement

$$= 392 \times 10 = 3920 \text{ J}$$

and $$\text{Power} = \frac{W}{t} = \frac{3920}{15} = 261.33 \text{ Watt}$$

Q.5. *If a net force of 10 Newton acts on a body, initially at rest, of mass 30 kg; then what is the*

(*i*) *Work done by the force in the third second.*

(*ii*) *Instantaneous power exerted by force at the end of third second.*

Solution. (*i*) The acceleration of the body, $a = \frac{\text{Force}}{\text{Mass}}$

i.e., $a = \frac{10}{30} = \frac{1}{3}$ m/sec^2

Displacement in 1 sec.

$$S = ut + \frac{1}{2}at^2$$

$$= 0 + \frac{1}{2} \cdot \frac{1}{3}(1)^2 = \frac{1}{6} \text{ meter}$$

Work done at the end of 1 sec.

$$W = F \times \text{displacement}$$

$$= 10 \times \frac{1}{6} = 1.667 \text{ Joule}$$

Displacement in 2 sec.

$$S = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2} \times \frac{1}{3}(2)^2 = \frac{2}{3} \text{ meter}$$

Work done at the end of 2 sec = W = F × displacement

$$= 10 \times \frac{2}{3} = 6.667 \text{ Joule}$$

Displacement in 3 sec.

$$S = 0 + \frac{1}{2} \times \left(\frac{1}{3}\right) \times 3^2 = \frac{9}{6} m.$$

Work done at the end of 3 sec;

$$W = F \times \text{displacement}$$

$$= 10 \times \frac{9}{6} = 15 \text{ Joule}$$

∴ Work done in the 3rd sec = Work done in 3 sec – Work done in 2 sec.

$$= (15 - 6.667) = 8.33 \text{ Joules}$$

(*ii*) Velocity at the end of 3rd sec.

$$v = u + at$$

or $$v = 0 + \frac{1}{3} \times 3 = 1 \text{ m/sec.}$$

So, instantaneous power,

$$P = F \times v$$

$$= 10 \times 1 = 10 \text{ watt.}$$

Q.6. *When a 500 kg stone is pushed over a level horizontal rough surface, by a horizontal force of 100 Newton, its velocity is found to be uniform. Calculate the work done in following cases, assuming frictional force to be constant:*

(*a*) *Velocity remains constant.*

(*b*) *Velocity increases from 0 to 1 m/sec.*

(*c*) *Velocity decreases from 1 to 0 m/sec. in a distance of 5 meter.*

Solution. (*a*) For uniform velocity over rough surface

$$F_{applied} = F_{frictional}$$

∴ Work done in 5 meter distance against frictional force;

$$W = 100 \times 5 = 500 \text{ Joule}$$

(*b*) Here, the applied force increases the velocity from 0 to 1 m/sec. in addition to overcoming the friction.

Using $$v^2 = u^2 + 2aS$$

$$= 0 + 2a \times 5$$

So $$a = \frac{1}{10} \text{ m/sec}^2$$

Force required for this acceleration,

$$F = ma = 500 \times \frac{1}{10} = 50 \text{ Newton}$$

$$\text{Total force} = 100 + 50 = 150 \text{ Newton}$$

So, Work done in 5 meter distance

$$= 150 \times 5 = 750 \text{ Joule}$$

(*c*) In this case velocity decreases from 0 to 1 m/sec.

So, $$v^2 = u^2 + 2aS$$

$$0 = 1 + 2a \times 5$$

$$a = -\frac{1}{10} \text{ m/sec}^2 \text{ (retardation)}$$

So, $$\text{retarding force} = ma = -500 \times \frac{1}{10} = -50 \text{ Newton}$$

∴ $$\text{Total force acting} = 100 - 50 = 50 \text{ Newton}$$

So, $$\text{Work done} = 50 \times 5 = 250 \text{ Joules}$$

Q.7. *A small block of mass m slides along the frictionless track as shown in the figure 12. Calculate the height h at which the mass must be released on the track to be able to go round the rack of radius R.*

If h = 5R, what is the reaction force exerted by the track on the block when it is at (i) point A (ii) point B (iii) point C.

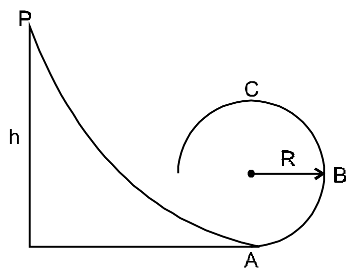


Fig. 12

Solution. Motion is under gravitational force, which is conservative, so mechanical energy is conserved *i.e.,* loss in P.E. in coming from P to C = Gain in K.E.

$$\therefore \qquad mg(h - 2R) = \frac{1}{2}mv^2 \qquad ...(i)$$

[where v = velocity at C]

Now velocity at C should be such that centrifugal force at C = weight of block for successful completion of loop.

i.e.,
$$\frac{mv^2}{R} = mg \text{ or } v^2 = Rg \qquad ...(ii)$$

From (*i*) and (*ii*)

$$Rg = 2g\,(h - 2R)$$
$$R = 2h - 4R$$
$$h = \frac{5R}{2}$$

if h = 5R, velocity at A is given by

$$\text{Loss in P.E.} = \text{gain in K.E.}$$

i.e.,
$$mgh = \frac{1}{2}mv_A^2$$

or
$$v_A^2 = 2gh = 2g \times 5R = 10gR$$

At pt. A, centrifugal force,

$$= \frac{mv_A^2}{R} = \frac{m \times 10g\,R}{R} = 10mg$$

Weight of block at A; acting down wards = mg

So, total outward force (action) = 10 mg + mg = 11 mg

Thus, reaction by track on block at A = 11 mg

Similarly, velocity of block at B will be

$$mg(h - R) = \frac{1}{2}mv_B^2$$

or
$$mg(5R - R) = \frac{1}{2}mv_B^2$$

or
$$v_B^2 = 8\,gR$$

i.e., Centrifugal force at B;

$$= \frac{mv_B^2}{R} = \frac{m \times 8gR}{R} = 8mg$$

Since mg acts at 90° to centrifugal force at B; so action by block on the track = 8mg

Thus, reaction exerted by track on the block at B = 8 mg

The velocity of block at C, is given by

$$mg(h - 2R) = \frac{1}{2}mv_C^2$$

or
$$mg(5R - 2R) = \frac{1}{2}mv_C^2$$

$$v_C^2 = 6gR$$

Then, centrifugal force at C $= \dfrac{mv_C^2}{R} = \dfrac{m6gR}{R} = 6mg$

weight mg of block acts in opposite direction to centrifugal force;

So, Action force of block = 6 mg – mg = 5 mg

Thus, reaction exerted by track on the block at C = 5 mg.

Q.8. *A particle slides along a track with elevated ends and a flat central part, as shown in the figure. The flat part has a length l = 2.0 meter. The curved parts are frictionless while for the flat part the coefficient of kinetic friction is 0.20. The particle is released at a point A, which is at a height h= 1.0 meter above the flat part of the track. Where does the particle finally come to rest ?*

Solution.

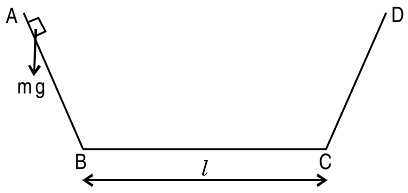


Fig. 12

P.E. of particle at A w.r.t. B = mgh

On reaching B, entire energy is kinetic; so

$$\text{K.E. at B} = \frac{1}{2}mv^2 \text{ so; } \frac{1}{2}mv^2 = mgh \qquad ...(i)$$

Suppose particle travels distance d in flat part before coming to rest. Thus all K.E. is used against friction, so

$$\frac{1}{2}mv^2 = \mu N \times \text{distance} = \mu\ mgd \qquad (\text{as } N = mg)$$

or
$$\frac{1}{2}mv^2 = \mu mgd \qquad ....(ii)$$

(i) and (ii) give

$$mgh = \mu mgd$$

$$d = \frac{h}{\mu} = \frac{1}{0.2} = 5 \text{ meters}$$

Thus particle travels B to C (2 meter) then climbs towards D (no energy loss, as path is frictionless), then descends to C, then comes from C to B (2 meters); again climbs towards A (no energy loss, as track is frictionless); comes back to B then travels half of BC (1 meter) and stops.

Total distance travelled on flat part = 2 + 2 + 1 = 5 meters.

Q.9. *A constant force of 5 Newton acts for 10 sec. on a body whose mass is 2 kg. The body was initially at rest. Calculate*

(a) *Work done by the force,*

(b) *final kinetic energy,*

(c) *average power of the force.*

Solution. (*a*) Acceleration, $a = \frac{\text{Force}}{\text{mass}} = \frac{5}{2} = 2.5 \text{ m/sec}^2$

Now displacement is given by

$$S = ut + \frac{1}{2}at^2$$

$$= 0 + \frac{1}{2} \times 2.5 \times (10)^2$$

$$= 125 \text{ meter}$$

So work done by force = 5 × 125 = 625 J.

(*b*) The final velocity v is given by

$$v = u + at$$

$$v = 0 + 2.5 \times 10 = 25 \text{ m/sec.}$$

So, final K.E. $= \frac{1}{2}mv^2 = \frac{1}{2} \times 2 \times (25)^2 = 625$ J.

(*c*) $$P_{avg} = \frac{\text{Work done}}{\text{time}} = \frac{625}{10} = 62.5 \text{ Watt.}$$

Q.10. *A rod of length 1.0 m and mass 0.5 kg fixed at one end, is initially hanging vertically. The other end is now raised until it makes an angle of 60° with respect to the vertical. How much work is required ?*

Solution. Given OP = 1 m

$\angle$POQ = 60°, m = 0.5 kg

Rod is moved aside against gravity, from position P to Q. This makes C.G. of rod to move from C to D. The vertical displacement of C.G.

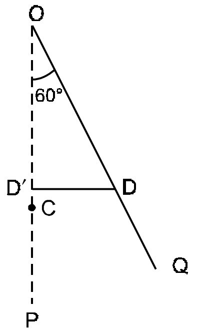


Fig. 13

$$= CD' = OC - OD'$$

$$= 0.5 - 0.5 \cos 60°$$

$$= 0.25 \text{ meter.}$$

So, work done in moving the rod against gravity.

$$= mg\,(CD') = mg \times 0.25$$

$$= 0.5 \times 9.8 \times 0.25 = 1.225 \text{ Joule.}$$

Q.11. *A body of mass 0.5 kg starts from rest and slides vertically down a curved track which is in the shape of one quadrant of a circle of radius 1 meter. At the bottom of track, speed of the body was 3 m/sec. What was the work done by the frictional force ?*

Solution. The motion is under gravity. Frictionless force resist the motion and in doing so it does work.

Supposing that curved track were frictionless, the velocity of body in vertical descent of 1 *m is*,

$$v^2 = u^2 + 2gS$$

$$= 0 + 2g \times 1 \quad \text{or} \quad v = \sqrt{2g} = \sqrt{2 \times 9.8}$$

$$v = 4.427 \text{ m/sec.}$$

and K.E. would have been $= \frac{1}{2}mv^2 = \frac{1}{2} \times 0.5 \times 19.6 = 4.9$ Joules.

but available K.E. at bottom $= \frac{1}{2} \times 0.5 \times (3)^2$

$= 2.25$ Joules.

The difference of K.E. is 4.9 – 2.25 = 2.65 J has been spent in overcoming friction. So work done by frictional force = 2.65 J.

Q.12. *Find the force described by the potential energy* $U = \frac{\vec{P}.\vec{r}}{r^3}$ *where P is constant.*

Solution. As force is negative gradient of potential energy *i.e.*,

$$F = -\nabla U$$

So force,

$$F = -\nabla\left(\frac{\vec{P}.\vec{r}}{r^3}\right) = -\frac{\delta}{\delta r}\left[\frac{\vec{P}.\hat{r}}{r^3}\right]$$

$$= \frac{\delta}{\delta r}\left(\frac{P\cos\theta}{r^3}\right)$$

$$= 2P\cos\theta \times \frac{1}{r^3} = \frac{2\vec{p}.\hat{r}}{r^3} = \frac{2\vec{p}.\vec{r}}{r^4}$$

Q.13. *The potential energy of a particle is given by*

$$U = 40 + 6x^2 - 7xy + 8y^2 + 32z$$

where U is in joule and x, y and z are in metre.

Find the force acting on the particle when it is in position (–2, 0, 5).

Solution. Using the relation $F = -\frac{dU}{dr}$, we can get components of force along *x, y* and *z* directions.

Thus

$$F_x = -\frac{dU}{dx} = -12x + 7y$$

$$F_y = -\frac{dU}{dy} = +7x + 16y$$

$$F_z = -\frac{dU}{dz} = -32$$

In position (–2, 0, 5) we have, $F_x = 24$, $F_y = -14$ and $F_z = -32$. Resultant force on the particle in given position

$$F = \sqrt{(24)^2 + (-14)^2 + (-32)^2}$$

$$= 42.38 \text{ Newton.}$$

Q.14. *The potential energy function for the force between two atoms in a diatomic molecule is approximately given by*

$$U_x = \frac{a}{x^{12}} - \frac{b}{x^6}$$

where a and b are the +ve constants and x is distance between atoms, determine

(a) value of x at which U_x is zero.

(b) Value of x at which U_x is minimum.

(c) Force between atoms.

(d) Dissociation energy of molecules.

Solution. (*a*)

$$U_x = \frac{a}{x^{12}} - \frac{b}{x^6} = 0$$

or

$$\frac{a}{x^{12}} = \frac{b}{x^6}$$

or

$$x = \left(\frac{a}{b}\right)^{1/6}$$

(*b*) For $(U_x)_{min}$; $\frac{d}{dx}(U_x) = 0$

or

$$\frac{d}{dx}\left(\frac{a}{x^{12}} - \frac{b}{x^6}\right) = 0 \quad \text{or} \quad \frac{-12a}{x^{13}} + \frac{6b}{x^7} = 0$$

or

$$\frac{2a}{x^6} = \frac{b}{1} \quad \text{or} \quad x^6 = \frac{2a}{b}, \text{ or } \quad x = \left(\frac{2a}{b}\right)^{1/6}$$

(*c*) Force is –ve gradient of potential energy so;

$$F = -\frac{d}{dx}(U_x) = -\frac{d}{dx}\left(\frac{a}{x^{12}} - \frac{b}{x^6}\right)$$

$$= -\left[\frac{-12a}{x^{13}} + \frac{6b}{x^7}\right]$$

or

$$F = \left[\frac{12a}{x^{13}} - \frac{6b}{x^7}\right]$$

(d) Dissociation energy (D) is changed in potential energy from its minimum value (which occurs at equilibrium separation) to zero (which occurs at x equal to infinity). So,

$$D = (U_x)_{\text{at } x=\infty} - (U_x)_{\min}$$

$$= O - \left[\frac{a}{\left(\frac{2a}{b}\right)^2} - \frac{b}{\frac{2a}{b}}\right]$$

$$= -\frac{b^2}{4a} + \frac{b^2}{2a} = \frac{b^2}{4a}$$

Thus, dissociation energy, $D = \dfrac{b^2}{4a}$.

Q.15. *Show that following forces are conservative;*

(a) $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$

(b) $\vec{F} = (2xy + z^2)\,\hat{i} + x^2\,\hat{j} + 2xz\,\hat{k}$

Solution. (a) We know that for conservative force, $\vec{\nabla} \times \vec{F} = 0$.

Substituting the value of $\vec{F}$,

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \hat{j}\left[\frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy)\right] + \hat{k}\left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right]$$

$$= \hat{i}\,[x - x] + \hat{j}\,[y - y] + \hat{k}\,[z - z]$$

$$= 0$$

So, given force is conservative.

(b)
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy + z^2) & x^2 & 2xz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2xz) - \frac{\partial}{\partial z}(x^2)\right] + \hat{j}\left[\frac{\partial}{\partial z}(2xy + z^2) - \frac{\partial}{\partial x}(2xz)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^2)\right]$$

$$= \hat{i}\,[0 - 0] + \hat{j}\,[2z - 2z] + \hat{k}\,[2x - 2x]$$

$$= 0$$

So, force is conservative.

Q.16. *Calculate the work done in following cases:*

(a) *Force* $\vec{F} = 2\hat{i} + xy\hat{j} + xz^2\hat{k}$ *acts on a particle and displaces it from position (2, 3, 1) to (2, 3, 4) parallel to z-axis.*

(b) *Force* $\vec{F} = (2xy + z^2)\hat{i} + x^2\hat{j} + 2xz\hat{k}$ *makes a particle to move from position (0, 1, 2) to (5, 2, 7).*

(c) *Force* $\vec{F} = Ax + Bx^2$ *acts parallel to x-axis on a particle and moves it from x = 1 to x = 2.*

Solution. (*a*)

$$W = \int_M^N \vec{F}.\vec{dr} = \int_M^N (2\hat{i} + xy\hat{j} + xz^2\hat{k})\,(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_M^N 2dx + xydy + xz^2dz \text{ (since } \hat{i}\,.\,\hat{j} = \hat{i}\,.\hat{k} = 0)$$

Since motion occurs along z-axis, so no change in x and y occurs *i.e.,* $dx = dy = 0$ and we have

$$W = \int_M^N xz^2dz = x\int_M^N z^2dz = x\left[\frac{z^3}{3}\right]_M^N$$

$$= 2\,.\left[\frac{z^3}{3}\right]_1^4 = \frac{2}{3}[4^3 - 1] = 42 \text{ units}.$$

(*b*)

$$W = \int_M^N \vec{F}\,.\,\vec{dr} = \int_M^N (Fx\hat{i} + Fy\,\hat{j} + Fz\,\hat{k}).(dx\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_M^N Fx\,dx + Fy\,dy + Fz\,dz$$

$$= \int_M^N (2xy + z^2)\,dx + x^2dy + 2xz\,dz$$

$$= \int_M^N (2xy\,dx + x^2dy) + (z^2dx + 2xz\,dz)$$

$$= \int_M^N d(x^2y) + d(z^2x) = \int_M^N d(x^2y + z^2x)$$

$$= \left[x^2y + z^2x\right]_M^N = \left[x^2y + z^2x\right]_{(0,\,1,\,2)}^{(5,\,2,\,7)}$$

$$= 295 \text{ units.}$$

(*c*)

$$W = \int_{x=1}^{x=2} F\,dx$$

$$= \int_1^2 (Ax + Bx^2)\,dx$$

$$= \left[\frac{Ax^2}{2} + \frac{Bx^3}{3}\right]_1^2$$

$$= \frac{A}{2}(2^2 - 1^2) + \frac{B}{3}(2^3 - 1^3)$$

$$= \frac{3}{2}A + \frac{7}{3}B$$

Q.17. *An ideal massless spring can be compressed 2 meter by a force of 200 N. The spring is located at the bottom of a frictionless inclined plane which makes angle of 30° with the horizontal. A 20 kg mass is released from rest at the top of inclined plane which comes to rest after compressing the spring by 4 metres. Calculate*

(a) Distance travelled by mass before coming to rest.

(b) Speed of mass before it reaches the spring.

Solution. (*a*) The linear restoring force in the spring is given by,

$$F = -kx$$

here, $k = \frac{200}{2} = 100$ N/m.

Let, l = distance along inclined plane that mass travels before coming to rest.

x = compression of spring = 4 meter.

The initial P.E. of mass will be converted into P.E. of spring during compression of 4 meter.

i.e., $$mgh = \frac{1}{2}kx^2$$

or $$mgl \sin\theta = \frac{1}{2}kx^2$$

or $$20 \times 9.8 \times l \times \sin 30° = \frac{1}{2} \times 100 \times 4^2$$

or $$l = 8.16 \text{ meter.}$$

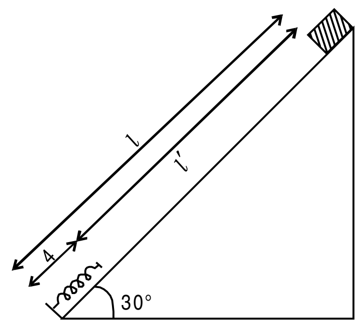


Fig. 14

(b) Let l' be a distance travelled when mass just touches the spring and v be its velocity then,

$$l = l' + 4$$

or $$l' = l - 4 = 8.16 - 4 = 4.16 \text{ meter.}$$

As K.E. of mass at the instant of touching the spring = Initial P.E. of mass

i.e., $$\frac{1}{2}mv^2 = mgl' \sin \theta$$

or $$v^2 = 2gl' \sin \theta = 2 \times 9.8 \times 4.16 \times \frac{1}{2}$$

$$v = 6.385 \text{ m/sec.}$$

Q. 18. *An ideal spring (Force constant = 8 × 10⁴ dynes/cm) hangs vertically and supports 0.8 kg mass at rest. Calculate the distance by which the mass should be pulled down so that it may pass through, on being released, the equilibrium position with a velocity of 1 m/sec.*

Solution. The linear restoring force produced in the stretched spring is

$$F = -kx$$

In stretching the spring, work is done which is stored as P.E. If a vertical downward pull of 'l' cm is made,

the work done against the restoring force = potential energy = $\frac{1}{2}kl^2$

On releasing the mass after stretching, it moves upwards and passes the mean position with velocity v (say). Then

$$\text{K.E. at mean position} = \frac{1}{2}mv^2$$

since entire P.E. at extreme position totally converts into K.E. at mean position,

$$\frac{1}{2}kl^2 = \frac{1}{2}mv^2$$

as $m = 0.8$ kg, $k = 8 \times 10^4$ dynes/cm and $v = 1$ m/sec.

$$l = \sqrt{mv^2/k} = \sqrt{800 \times (100)^2/(8 \times 10^4)} = 10 \text{ cm}$$

Q.19. *A block of mass 2.0 kg is dropped from a height of 0.4 m. onto a spring of force constant K = 1960 N/m. Find the maximum distance through which the spring will be compressed [Neglect Friction]*

Solution. Situation is shown in the diagram

Loss in P.E. of mass = gain in pot. energy by the compressed spring

i.e., $$mg\ (d + y) = \frac{1}{2}ky^2$$

given $m = 2$ kg, $d = 0.4$ m; $k = 1960$ Nt/m, $g = 9.8$ m/sec^2; $y = ?$

So, $$2 \times 9.8 \times (0.4 + y) = \frac{1}{2} \times 1960 \times y^2$$

or $$y^2 - 0.02\ y - 0.008 = 0$$

or $$y = \frac{0.02 \pm \sqrt{(0.02)^2 - 4 \times 0.008}}{2}$$

$$= \frac{0.02 \pm \sqrt{0.0004 - 0.032}}{2}$$

$$= 0.0999 \text{ m (taking +ve value).}$$

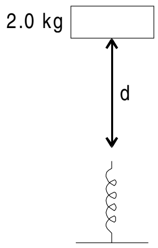


Fig. 15

Q.20. *The scale of spring balance, reading from zero to 100 Newton is 20 cm long. A body suspended from the balance is observed to oscillate vertically at 2 vib/sec. What is weight of the body ?*

Solution. Force constant of spring = $\frac{100}{0.2}$ = 500 Nt/m.

Since frequency of oscillation of vertical spring with mass is given by,

$$n = \frac{1}{2\pi}\sqrt{\frac{K}{m}};\ \text{or } m = \frac{K}{4\pi^2 n^2}$$

with $n = 2$, K = 500 Nt/m, $\pi = 3.14$ kg

$$m = \frac{500}{4 \times (3.14)^2 \times 4} = \frac{500}{157.75} = 3.169 \text{ kg}$$

$$\text{weight } mg = 3.169 \times 9.8 = 31.06 \text{ Newton.}$$

Q.21. *An object is attached to a vertical spring and slowly lowered to its equilibrium position. This stretches the spring by an amount 'd'. If the same object is attached to the same spring but permitted to fall instead, through what distance does it stretch the spring ?*

Solution. Fig. 16(a) shows the normal unextended spring. In Fig. 16(b), by slowly placing the mass, equilibrium state is shown and elongation is d.

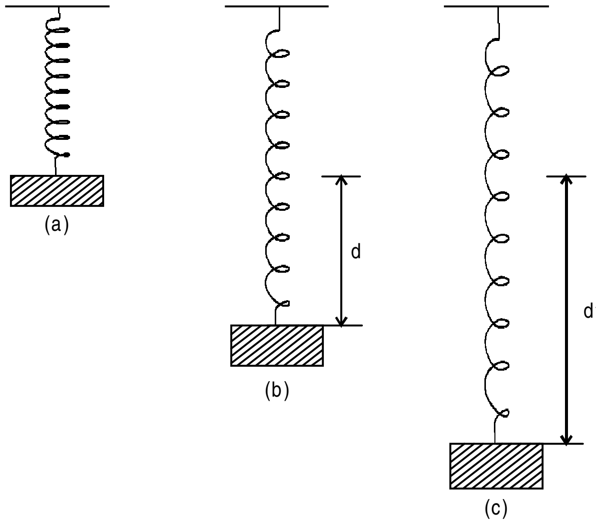


Fig. 16

So, Restoring force = weight

i.e., $$Kd = mg \text{ or } K = \frac{mg}{d} \quad ...(i)$$

In Fig. 16(*c*); If mass is permitted to fall, the mass will go down to an extreme, will rise up and thus will execute S.H.M.

Suppose in extreme position elongation is d then;

Work done by force mg on the spring

$$= mg \times d' \quad ...(ii)$$

P.B. corresponding to extreme position $= \frac{1}{2}Kd'^2$ ...(*iii*)

Equating eqs. (*ii*) and (*iii*)

$$mgd' = \frac{1}{2}Kd'^2$$

Putting for K from (*i*), $$mgd' = \frac{1}{2}\frac{mg}{d}.d'^2$$

or $$d' = 2d$$

i.e., stretching is double of the previous case.

Q.22. *Two block A and B are connected to each other by a string and a spring, the string passes over a frictionless pulley as shown.*

The block B slides over the horizontal top surface of a stationary block C and the block A slides along the vertical side of C, both with the same uniform speed. The coefficient of friction between the surfaces of blocks is 0.2. Force constant of spring is 1960 Nt/m. If mass of A = 2 kg, calculate the mass of block B and energy stored in the spring.

Solution. Let m = mass of B; T = tension in the spring

For motion of B;

$$T = \mu N = \mu mg = 0.2 \ mg \quad ...(i)$$

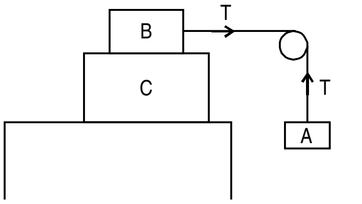


Fig. 17

for motion of A $\quad T = mg = 2g \quad ...(ii)$

from (*i*) and (*ii*) $\quad 2g = 0.2 \ mg$ or $m = 10$ kg

again; $\quad T = 2 \times 9.8 = 19.6$ Newton (from (*ii*))

if x is the elongation of the spring, then magnitude of linear restoring force;

$$F = kx \text{ (upwards } -ve\text{) or } x = \frac{F}{k}$$

The linear restoring force acting in the present case is T itself *i.e.,* F = T

So, $$x = \frac{T}{k} = \frac{19.6}{1960} \quad ...(iii)$$

Energy stored in the spring $= \frac{1}{2}kx^2$

$$= \frac{1}{2} \times 1960 \times \left(\frac{19.6}{1960}\right)^2 = 0.098 \text{ Joule}$$

Q.23. *A body of mass 2 kg is attached to a horizontal spring of force constant 8 Nt/m. A constant force of 6 Nt. is applied along the length of the spring. Find*

(a) The speed of the body when it is displaced through 0.5 m

(b) If force is removed, then; how much farther shall the body move before reaching the state of rest.

Solution. (*a*) Work done by the constant force = 6 × 0.5 = 3 Joule

If v is velocity, K.E. of body in displaced position will be given by

$$\frac{1}{2}mv^2 = \frac{1}{2} \times 2v^2$$

Again, potential energy stored in the spring at displaced position of 0.5 *m*

$$\frac{1}{2}kx^2 = \frac{1}{2} \times 8 \times (0.5)^2$$

So, Work done in 0.5 *m* displacement = K.E. + P.E.

i.e., $$W = \frac{1}{2} \times 2 \times v^2 + \frac{1}{2} \times 8 \times (0.5)^2$$

or $$W = v^2 + 1$$

Since W = 3 Joule, $3 = v^2 + 1$ or $v = 1.4$ m/sec.

(*b*) Total energy of system at the time of removal of force = 3 Joule.

Suppose body moves distance '*l*' still onwards before coming to rest; on removal of force

P.E. at the extreme position = total energy of system

$$\frac{1}{2} \times 8 \times (0.5 + l)^2 = 3$$

or $$(0.5 + l)^2 = \frac{3}{4}$$

So, $$l = 0.36 \text{ m.}$$

Q.24. *A 10 lb block is thrust up a 30° inclined plane with an initial speed 16 ft./sec. If it is found to travel 5 ft. along the plane; stop and then slide back again to the bottom. Calculate*

(a) The force of friction (regarded as constant) on block.

(b) Speed of block when it reaches bottom of the inclined plane.

Solution. (*a*) Upward motion: At the top where block momentarily stops; total energy,

$$E = K.E. + P.E.$$
$$= mg \times 5 \sin 30°$$
$$= 10 \times 5 \times \frac{1}{2} = 25 \text{ ft lb}$$

At foot of incline,

$$E_i = (K.E.)_i + (P.E.)_i$$
$$E_i = \frac{1}{2}\left(\frac{10}{32}\right)(16)^2 + 0$$
$$E_i = 40 \text{ ft-lb}$$

Energy lost = Work done by frictional force in 5 ft. distance

i.e., $25 - 40 = -F \times 5 \quad \text{or} \quad F = 3 \text{ lb}$

(*b*) Downwards motion: Suppose block returns to the foot of incline with speed 'v'.

At bottom where motion ends:

$$E = K.E. + P.E. = \frac{1}{2}\left(\frac{10}{32}\right)v^2 + 0 = \frac{5}{32}v^2 \text{ ft - lb}$$

At top where motion starts: total energy = 25 ft-lb

again energy lost = work done by frictional force

$$\frac{5}{32}v^2 - 25 = -15 \quad \text{or} \quad \frac{v^2}{32} = 2 \quad \text{or} \quad v = 8 \text{ ft/sec.}$$

Q.25. *Figure shows a vertical section of a frictionless track surface. A block of mass 2 kg is released from position A. Compare its K.E. at the positions B, C and D. (g = 9.8 m/sec²).*

Solution. Motion occurs under gravitational force which is conservative, so energy is conserved. In coming down block looses its P.E. and corresponding gain in K.E. occurs;

at point B: $(K.E.)_B = (P.E.)_A - (P.E.)_B = (mgh)_A - (mgh)_B$
$$= mg\ (14 - 5) = 2 \times 9.8 \times 9$$
$$= 176.4 \text{ Joules.}$$

at Point C: $(K.E.)_C = (P.E.)_A - (P.E.)_C = mgh_A - mgh_C$
$$= mg\ (14 - 7) = 0.2 \times 9.8 \times 7 = 137.2 \text{ Joules}$$

at Point D: $(K.E.)_D = (P.E.)_A - (P.E.)_D = mgh_A - mgh_D$
$$= mg\ (14 - 0) = 2 \times 9.8 \times 14 = 274.4 \text{ Joules.}$$

Q.26. *A point mass m starts from rest and slides down the surface of a frictionless sphere of radius r. Measure angles from the vertical and P.E. from the top. Find*

(*a*) *Change in P.E. of the mass with angle.*

(*b*) *K.E. as function of angle.*

(*c*) *Radial and tangential acceleration as function of angle.*

(*d*) *Angle at which mass flies off the sphere.*

(e) *If friction be present between mass and sphere, shall the mass fly off at greater or lesser angle as compared to situation in (d).*

Solution. (a) At top position P (reference point) P.E. = 0

at bottom position at depth, $y = 2r$; $\therefore$ P.E. $= -mgy$

at a general position M;

$$\text{P.E.} = -mg\text{PM}' = -mg\ (\text{PO} - \text{M}'\text{O})$$

$$= -mg\left(r - \frac{\text{M}'\text{O}}{\text{MO}}.\text{MO}\right)$$

$$= -mgr\ (1 - \cos\ \theta)$$

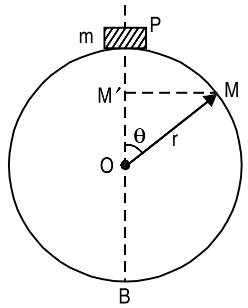


Fig. 18

So change in P.E.; $(\text{P.E.})_\text{M} - (\text{P.E.})_\text{P} = -mgr\ (1 - \cos\ \theta) - 0$

$$\Delta(\text{P.E.}) = -mgr\ (1 - \cos\ \theta) \qquad ...(i)$$

(b) At top total energy is wholly potential = 0. so from energy conservation

At general position M; K.E. + P.E. = 0

i.e., K.E. + [$- mgr\ (1 - \cos\ \theta)$] = 0

or $$\text{K.E.} = mgr\ (1 - \cos\ \theta) \qquad ...(ii)$$

(c) At general position M; v = tangential velocity

$\therefore$ Radial acceleration $= \dfrac{v^2}{r}$

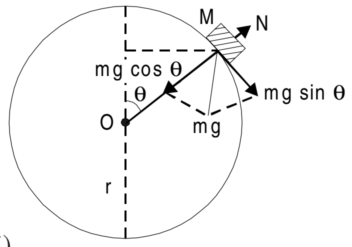


Fig. 19

But K.E. at same position is $\frac{1}{2}mv^2 = mgr\ (1 - \cos\ \theta)$

or $$v^2 = 2gr\ (1 - \cos\ \theta) \qquad ...(iii)$$

So, Radial acceleration $= \dfrac{2gr}{r}(1 - \cos\theta) = 2g\ (1 - \cos\ \theta)$

$$\text{tangential acceleration} = \frac{dv}{dt} = \frac{dv}{d\theta}\cdot\frac{d\theta}{dt}$$

$\therefore$ as $$v = rw = \frac{rd\theta}{dt}\ \therefore\ \frac{d\theta}{dt} = \frac{v}{r}$$

so, $$\frac{dv}{dt} = \frac{v}{r}\frac{dv}{d\theta} \qquad ...(iv)$$

From (iii) $$\therefore\ \frac{dv}{d\theta} = \frac{1}{2}\left[2gr\ (1 - \cos\ \theta)\right]^{-\frac{1}{2}}.\ 2gr\sin\theta$$

then tangential acceleration is

$$\frac{dv}{dt} = \frac{v}{r}\cdot\frac{gr\sin\theta}{v} = g\sin\theta$$

(d) At position where mass flies off the sphere-surface the condition for such event is

Centrifugal force = Component of weight along the radius *i.e.,* $$\frac{mv^2}{r} = mg \cos \theta$$

or $$v^2 = gr \cos \theta \qquad ...(v)$$

equating (*iii*) and (*v*);

$$2gr(1 - \cos \theta) = gr \cos \theta$$

or $$3 \cos \theta = 2 \text{ or } \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

(*e*) If friction is present, then velocity at any general point M shall be less. v is less means $\cos \theta$ is less which suggests θ is large. Therefore mass leaves at greater angle as compared to previous (frictionless) case.

Q.27. *A body of mass 5 kg moves down from state of rest on an inclined plane of length 5 meters having inclination of 60° with the horizontal. If coefficient of friction be 0.2 find the speed of body at the bottom. How far shall it slide on the rough horizontal surface having same coefficient of friction as the inclined plane.*

Solution. At top: P.E. $= mgh = mgd \sin \theta$; and K.E. = O

Now, frictional force $= \mu N = \mu mg \cos \theta$

Work done by frictional force $= -\mu mg \cos \theta \times d$ (*–ve* sign gives, nature or direction)

Since, change in total energy = work done

i.e., change in P.E. + Change in K.E. = work done

So; $$\Delta T + (-mgd \sin \theta) = -\mu\, mgd \cos \theta$$

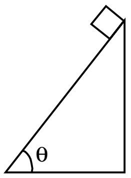


Fig. 20

or $$\Delta T = mgd(\sin \theta - \mu \cos \theta) = \frac{1}{2}mv^2$$

or $$v^2 = 2gd(\sin \theta - \mu \cos \theta)$$

Now, $g = 9.8 \text{ m/sec}^2$; $d = 5$ meters, $\theta = 60°$, $\mu = 0.2$

$$v^2 = 2 \times 9.8 \times (\sin 60° - 0.2 \cos 60°) \times 5$$

which gives, $$v = 8.664 \text{ m/sec}$$

At the start on horizontal rough surface the K.E. possessed by body $= \frac{1}{2}mv^2$,

This becomes zero finally due to expense in doing work against frictional force;

i.e., $$O - \frac{1}{2}mv^2 = \text{Work done against friction}$$

Supposing that l distance is travelled before coming to rest,

then frictional work done $= -\mu mgl$

So, $$-\frac{1}{2}mv^2 = -\mu mgl \quad \text{or} \quad v^2 = 2\mu gl$$

as calculated earlier, $v^2 = 75.06$; $\mu = 0.2$, $g = 9.8$

$$l = \frac{v^2}{2\mu g} = \frac{75.06}{2 \times 0.2 \times 9.8} = 19.15 \text{ meter.}$$

Q.28. *A block of mass 1 kg collides with a horizontal weightless spring of force constant 2Nt/meter. The block compresses the spring 4.0 meter from its normal position. Calculate the speed of block at the instant of collision if kinetic friction between block and surface is 0.25.*

Solution. Suppose v = speed of block at the instant of collision then its kinetic energy

$$= \frac{1}{2}mv^2$$

This energy is spent in overcoming the friction and compressing the spring, if x is compression of spring;

$$\frac{1}{2}mv^2 = \frac{1}{2}kx^2 + \mu mgx$$

given $x = 4$ meters, $k = 2$ Nt/meter, $m = 1$ kg,

$g = 9.8$ m/sec^2, $\mu = 0.25$

$$\therefore \quad \frac{1}{2} \times 1 \times v^2 = \frac{1}{2} \times 2 \times (4)^2 + 0.25 \times 1 \times 9.8 \times 4$$

or $v = \sqrt{51.6} = 7.18$ m/sec.

Q.29. *A horizontal force pushes a 10 kg mass up an inclined plane ($\theta = 30°$) from bottom, by a distance of 3 meters on the inclined plane. If the initial and final speeds of the mass are 1 m/sec and 3 m/s, calculate the work done by the force.*

Solution. Height in the final position,

$$h = \text{AB} \sin 30° = 3 \times \frac{1}{2} = 1.5 \text{ m}$$

$\therefore$ Change in P.E., $\Delta U = mgh = 10 \times 9.8 \times 1.5 = 147$ Joule

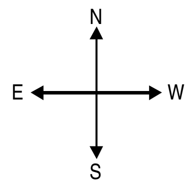
$$\text{Change in K.E.} = \Delta T = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \frac{1}{2}m(v_2^2 - v_1^2)$$

$$= \frac{1}{2} \times 10 \times (3^2 - 1^2) = 40 \text{ J}$$

$\therefore$ Change in total energy = (147 + 40) = 187 Joule

which gives the work done.

Q.30. *A chain of length d and mass m lies on a frictionless horizontal table such that one third of its length hangs over the edge. Calculate the work needed to pull the hanging part back onto table.*



Solution. Mass per unit length of chain = m/d

Mass of infinitesimally small length dy of chain = $\frac{m}{d}.dy$ work done in lifting this segment against force of gravity; by a distance y; is

$$dW = \left(\frac{m}{d}dy\right).g.y$$

So, work done in lifting or raising the $\frac{d}{3}$ length

$$W = \int_0^{d/3}\left(\frac{mg}{d}\right).\,y\,dy$$

$$= \frac{mg}{d}\int_0^{d/3} y\,dy = \frac{mg}{d}\left(\frac{y^2}{2}\right)_0^{d/3}$$

$$= \frac{mg}{2d}\left(\frac{d^2}{9}\right) = \frac{mgd}{18}.$$

Q.31. *A 0.5 kg block slides from point A on a horizontal track with initial speed 3m/s towards a weightless horizontal spring of length 1m and force constant 2 Nt/m. Part AB of track is frictionless while part BC has coefficient of static and kinetic friction 0.22 and 0.2. If AB = 2m and BD = 2.14m, find total distance block moves before coming to rest (g =10m/s²).*

Solution. The block will reach B with initial K.E.

$$V_1 = \frac{1}{2} \times 0.5 \times (3)^2 = 2.25 \text{ J}$$

In going from B to D work will be done against kinetic friction.

$$\therefore \quad W = \mu N\,(BD)$$

$$= 0.2 \times 0.5 \times 10 \times 2.14$$

$$= 2.14 \text{ J}$$

Therefore, the K.E. with which block will hit the spring at D

$$= V_1 - W = 0.11 \text{ J}$$

If block compresses the spring through x, this energy will be partly used up in doing work against friction during the compression x and rest will be stored in spring as its P.E.

i.e.,
$$\frac{1}{2}Kx^2 + \mu N x = 0.11$$

or
$$\frac{1}{2} \times 2 \times x^2 + 0.2 \times 0.5 \times 10x = 0.11$$

i.e.,
$$x^2 + x - 0.11 = 0 \quad \text{which gives } x = \frac{-1 \pm \sqrt{1 + 0.44}}{2} = 0.1 \text{ m}$$

It means total distance converted by block before it comes to rest is

$$D = AB + BD + x$$

$$= 2 + 2.14 + 0.1$$

$$= 4.24 \text{ m.}$$

Q.32. *A body of mass m is accelerated uniformly from rest to a speed V in time. Show that work done on the body as a function of time is*

$$W = \frac{1}{2} m \frac{V^2}{T^2} \cdot t^2.$$

Solution. Let 'a' be the acceleration in body then, since $u = 0$

$$V = 0 + a.T \quad \text{or} \quad a = \frac{V}{T}$$

The velocity gained by body in time t will be

$$v = at = \frac{V}{T} \cdot t$$

Hence K.E. gained in time t, which gives work done

$$W = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{V}{T}.t\right)^2$$

$$= \frac{1}{2}m\,\frac{V^2}{T^2} \cdot t^2$$

Q.33. *The scale of a spring balance reads from 0 to 250 kg and is 25 cm long. What is the potential energy of the spring when a 20 kg weight hang from it ?*

Solution. As given, 25 cm elongation of spring takes place corresponding to 250 kg. So the force constant of spring is given by

$$250 \times 9.8 = K \times 0.25$$

$$K = 9.8 \times 10^3 \text{ N/m}$$

If 20 kg weight hangs from balance, elongation will be obtained by

$$20 \times 9.8 = Kx'$$

i.e.,

$$x' = \frac{20 \times 9.8}{K} \text{ meter}$$

∴ Potential energy stored in spring will be

$$U = \frac{1}{2}K\,(x')^2 = \frac{1}{2} \times 9.8 \times 10^3 \left(\frac{20 \times 9.8}{9.8 \times 10^3}\right)^2$$

$$= \frac{1}{2} \times \frac{(20 \times 9.8)^2}{9.8 \times 10^3}$$

$$= 1.96$$

Q.34. *Calculate the work done by a force $F = kx^2$ acting on a particle at an angle of 60° with x-axis to displace it from x_1 to x_2 along the x-axis.*

Solution. The work done is given by

$$W = \int \vec{F}.\vec{dr}$$

$$= \int_{x_1}^{x_2} F dx \cos 60° = \frac{1}{2}k\int_{x_1}^{x_2} x^2 dx$$

$$= \frac{1}{2}k\left[\frac{x^3}{3}\right]_{x_1}^{x_2} = \frac{1}{6}k\left(x_2^3 - x_1^3\right).$$

Q.35. *A cord is used to lower vertically a block of mass M a distance d at a constant downward acceleration of g/4. Find the work done by the cord on the block.*

Solution. Let T be the tension in the cord acting vertically upward. The net force on the block is T-Mg, acting upward, where Mg is the weight of the block. If a is the downward acceleration of the block, then by Newton's second law, we have

$$T - Mg = -Ma$$

or $$T = M(g - a)$$

Here, $$a = g/4$$

$\therefore$ $$T = M\left(g - \frac{g}{4}\right) = \frac{3}{4}Mg$$

Therefore, work done by the tension T is given by

$$W = \text{tension (T)} \times \text{displacement in the direction of T}$$

$$= T \times (-d) \qquad [\because d \text{ is downward}]$$

$$= -\frac{3}{4}Mgd.$$

Q.36. *A block of mass 10.0 kg is pulled up at constant speed from the bottom to the top of a smooth incline 5.00 meter off the ground at the top. Calculate the work done by the applied force which is parallel to the incline. (g = 9.8 m/s²).*

Solution.

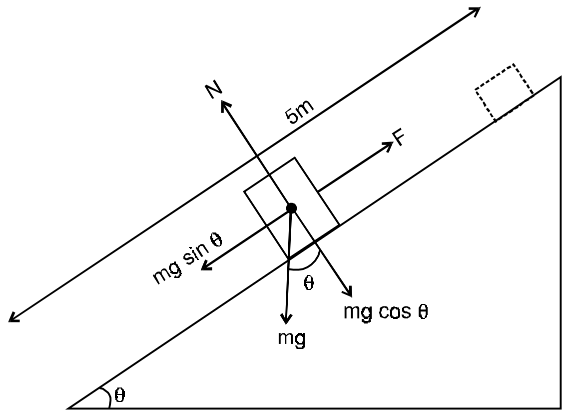


Fig. 21

The force acting on the block at any instant are its weight mg, the normal reaction N and the applied force F. The weight can be resolved into a component mg sin θ to the inclined plane and a component mg cos θ perpendicular to it. The net force parallel to the plane is F mg sin θ. Since the block moves on the plane at constant speed, the net force is zero. Thus

$$F - mg \sin \theta = 0$$

or $$F = mg \sin \theta$$

$$= 10 \times 9.8 \times \frac{3}{5} = 58.8 \text{ Nt}$$

The work done is

$$W = \text{Force} \times \text{displacement along the force}$$
$$= 58.8 \text{ Nt} \times 5 \text{ m} = 294 \text{ Joule}$$

Q.37. *A boy pulls a 5 kg block 10 meter along a horizontal surface at a constant speed with a force directed 45° above the horizontal. If coefficient of kinetic friction is 0.20, how much work does the boy do on block ?*

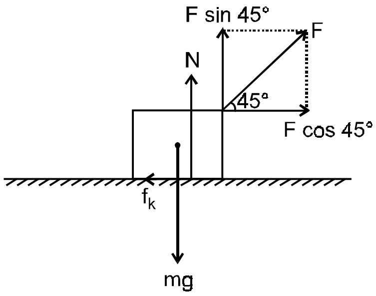


Fig. 22

Solution. The force acting on the block are its weight *mg,* normal reaction N exerted by the surface, sliding frictional force f_K against the motion and the applied force F at 45° above the horizontal. The net horizontal force is F cos 45° – f_K, and net vertical force is N + F sin 45° – *mg*.

The block is unaccelerated, so that from Newton's second law, we obtain

$$F \cos 45° - f_K = 0 \qquad ...(i)$$

and
$$N + F \sin 45° - mg = 0 \qquad ...(ii)$$

Also, we know
$$f_K = \mu_K N \qquad ...(iii)$$

where μ_K is the coefficient of kinetic friction. Substituting the value of N from Eq. (*ii*) in Eq. (*iii*), we get

$$f_K = \mu_K (mg - F \sin 45°)$$

Now putting this value of f_K in Eq. (*i*); we get

$$F \cos 45° - \mu_K (mg - F \sin 45°) = 0 \qquad ...(iv)$$

or
$$F (\cos 45° + \mu_K \sin 45°) = \mu_K mg$$

or
$$F = \frac{\mu_K mg}{\cos 45° + \mu_K \sin 45°}$$

with $\mu_K = 0.20$, $mg = 5 \times 9.8 = 49$ Nt and cos 45° = sin 45° = 0.707,

we obtain

$$F = 11.55 \text{ Nt.}$$

The block is pulled through a horizontal distance *r* = 10 meter. Then the work done is

$$W = Fr \cos 45°$$
$$W = (11.55)(10)(0.707)$$
$$= 81.66 \text{ Joule.}$$

Q.38. *A block of mass m = 3.57 kg is pulled at constant speed through a distance r = 4.06 m along a horizontal surface by a rope exerting a constant force F = 7.68 Nt inclined at θ = 15° to the horizontal. Find (i) the total work done on the block (ii) work done on the block by the rope and by friction, (iii) coefficient of kinetic friction between block and surface.*

Solution. (*i*) Since the block moves at constant speed, the net force on it is zero. Hence the total work done on it is zero.

(*ii*) The force exerted by the rope is inclined at 15° to the horizontal . Hence the work done on the block by the rope through a distance *r* = 4.06 *m* is given by

$$M = Fr \cos 15°´$$
$$= 7.68 \times 4.06 \times 0.966$$
$$= 30.1 \text{ Joule}$$

Since the total work done on the block is zero, the work done on it by friction is –30.1 Joule, *i.e.,* work is done by the block against friction.

(*iii*) From eq. (*iv*) of the last problem, we can write

$$\mu_K = \frac{F \cos 15°}{mg - F \sin 15°}$$
$$= \frac{7.68 \times 0.966}{3.57 \times 9.8 - 7.68 \times 0.259}$$
$$= \frac{7.419}{33.00} = 0.225$$

Q.39. *A running man has half the kinetic energy that a boy of half his mass has. The man speeds up by 1.0 m/s and then has the same kinetic energy as the boy has. Find speed of man and boy.*

Solution. Let m be the mass of man, and $m/2$ that of boy. Let v_1 and v_2 be their original speeds. Then kinetic energy of man is $\frac{1}{2}mv_1^2$ and that of boy is $\frac{1}{2}\left(\frac{m}{2}\right)v_1^2$. Since the energy of man is half that of boy, we have

$$\frac{1}{2}mv_1^2 = \frac{1}{2}\left[\frac{1}{2}\left(\frac{m}{2}\right)v_2^2\right]$$

or
$$v_1^2 = \frac{1}{4}v_2^2 \quad ...(i)$$

When the speed of the man becomes $(v_1 + 1)$, his kinetic energy equals to that of the boy. That is,

$$\frac{1}{2}m\,(v_1 + 1)^2 = \frac{1}{2}\left(\frac{m}{2}\right)v_2^2$$
$$(v_1 + 1)^2 = \frac{1}{2}v_2^2 \quad ...(ii)$$

Solving eqs. (*i*) and (*ii*) we get

$$v_1 = 2.41 \text{ m/s}$$
$$v_2 = 4.82 \text{ m/s}.$$

Q.40. *A 30 gm bullet initially travelling 500 m/s penetrates 12 cm into a wooden block. What average force does it exert ?*

Ans. The kinetic energy of the bullet is given by

$$K = \frac{1}{2}mv^2$$
$$= \frac{1}{2}(30 \times 10^{-3}) \times (500)^2 = 3750 \text{ Joule}$$

It stops after travelling a distance r = 12 cm. = 0.12 m in wooden block. If F Nt be the (retarding) force exerted by the block, then the work done by the bullet against this force is

$$W = Fr$$
$$= F \times 0.12 \text{ Joule}$$

By work-energy theorem, the kinetic energy equals this work done, that is

$$3750 = F \times 0.12$$

$$F = \frac{3750}{0.12} = 31250 \text{ Nt.}$$

Q.41. *Show with the help of work-energy theorem that the minimum stopping distance for a car of mass m moving with speed v along a level road is* $v^2/2\mu_s g$, *where* μ_s *is the coefficient of static friction between tyres and road.*

Solution. The kinetic energy of the car is

$$K = \frac{1}{2}mv^2$$

It is retarded due to the force of static friction f_s (rolling friction neglected), whose maximum value is given by

$$f_s = \mu_s N = \mu_s mg$$

N being the normal reaction which equals weight mg of the car on a level road. If x be the minimum stopping distance, then the work done against the force f_s is given by

$$W = f_s x = \mu_s mgx$$

This, by work-energy theorem, equals the initial kinetic energy K of the car. Thus,

$$\mu_s mgx = \frac{1}{2}mv^2$$

$$x = \frac{v^2}{2\mu_s g}$$

Q.42. *A mass of 0.675 kg is being revolved on a smooth table in a horizontal circle by means of a string which passes through a hole in the table at the centre of the circle. If the radius of circle is 0.50 m and the uniform speed is 10.00 m/s, find the tension in string. If the radius of the circle is reduced to 0.30 m by drawing the string down through the hole, the tension is increased by a factor 4.63. Find the work done by the string on the revolving mass during the reduction of the radius.*

Solution. The tension T of the string supplies the required centripetal force mv^2/R *i.e.,*

$$T = \frac{mv^2}{R} = \frac{0.675 \times (10.0)^2}{(0.5)} = 135 \text{ Nt}$$

On reducing the radius, the new tension is

$$T' = 4.63\ T = 4.63 \times 135 = 625 \text{ Nt}$$

If v' be the new velocity, then we have,

$$T' = \frac{mv'^2}{R'}$$

$$625 = \frac{0.675\, v'^2}{0.30}$$

$$v'^2 = \frac{625 \times 0.3}{0.675} = 278\ (\text{m/s})^2$$

The increase in kinetic energy of the moving mass is therefore

$$\Delta K = \frac{1}{2} m\, (v'^2 - v^2)$$

$$= \frac{1}{2} \times (0.675) \times (278 - 100)$$

$$= 60 \text{ Joule}$$

By work energy theorem, this is equal, to the work done on the moving mass.

Hence, $W = \Delta K = 60$ Joule

Q.43. *A constant force of 5 Nt acts for 10 seconds on a body whose mass is 2 kg. The body was initially at rest. Calculate the work done by the force, the final kinetic energy and the average power of the force.*

Solution. Let a be the (constant) acceleration in the body. Then

$$a = \frac{F}{m} = \frac{5}{2} = 2.5 \text{ m/s}^2$$

The distance x moved in 10 sec is

$$x = \frac{1}{2} a t^2 = \frac{1}{2} \times 2.5 \times (10)^2$$

$$= 125 \text{ m.}$$

The work done by the force F during this distance is

$$W = Fx = 5 \times 125$$

$$= 625 \text{ Joule}$$

The initial kinetic energy of the body is zero, By work-energy theorem the final kinetic energy is the same as the work done, *i.e.,* 625 Joule.

The body covers distance x (= 125 m) in 10 sec. Then its average velocity is

$$\bar{v} = \frac{x}{t} = \frac{125}{10} = 12.5 \text{ m/s}$$

The average power of the force is therefore

$$\vec{P} = F\vec{v}$$

$$= 5 \times 12.5$$

$$= 62.5 \text{ Watt.}$$

Q.44. *A boy whose mass is 51 kg climbs, with constant speed, a vertical rope 6 m long in 10 second. How much work does the boy perform ? What is his power output during the climb ?*

Solution. The body does work against his weight in climbing. This is given by

$$
\begin{aligned}
W &= \text{force} \times \text{distance} \\
&= (51 \times 9.8) \times 6 \\
&= 3000 \text{ Joule}
\end{aligned}
$$

This work is done in 10 sec. Hence the power output is

$$
\begin{aligned}
P &= \frac{W}{t} \\
&= \frac{3000}{10} \\
&= 300 \text{ Watt.}
\end{aligned}
$$

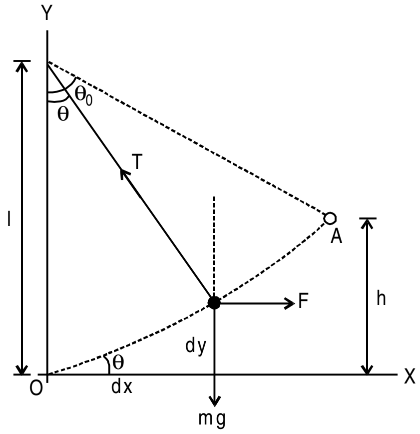


Fig. 23

potential energy when the bob has been raised to a vertical height h in the x-y plane is

$$U_{(x,\ y)} = U_{(x,\ h)} = mgh \qquad ...(i)$$

At the point A, the kinetic energy is zero *i.e.*, the potential energy is equal to the mechanical energy E. Then

$$E = mgh \qquad ....(ii)$$

If the bob be released at the point A, it returns under the gravitational (restoring) force, and the potential energy begins to convert into the kinetic energy. At any point in its path the sum of the two energies remains equal to the mechanical energy. Thus if v be the velocity and y the vertical height of the bob at any point, we have by the law of conservation of mechanical energy.

$$\frac{1}{2}mv^2 + U_{(x,\ y)} = E \qquad ....(iii)$$

But $\quad U(x,\ y) = mgy$

$$\therefore \quad \frac{1}{2}mv^2 + mgy = E \qquad ...(iv)$$

Comparing eq. (*ii*) and (*iv*) we have

$$\frac{1}{2}mv^2 + mgy = mgh$$

or $$\frac{1}{2}mv^2 = mg\,(h - y)$$

or $$v = \sqrt{2g\,(h - y)}$$

Thus at O (y = 0); $v = \sqrt{2gh}$ (maximum); the energy is all kinetic. At A ($y = h$); v = 0 (minimum); the energy is all potential.

Thus v varies between $\sqrt{2gh}$ and O or $\frac{1}{2}mv^2$ varies between mgh and O. This means that from equation (*iii*), U(x, y) can never be greater than E *i.e.*, $U_{(x,\,y)} \ngtr mgh$. In other words the bob cannot rise higher than h, its release point A.

(*iv*) **Magnetic Potential Energy.** Let τ be the (restoring) couple acting upon a magnet of moment M when its axis makes an angle θ with a magnetic field H. Then we have

$$\tau = -\text{MH}\sin\theta$$

Let U(0) be the potential energy in the θ = 0 position. The work in turning the magnet from θ = 0 to θ position is the potential energy further acquired by the magnet. Thus if U(θ) be the potential energy at position θ, then

$$\text{U}(\theta) - \text{U}(0) = -\int_0^\theta \tau\, d\theta$$

$$= \text{MH}\int_0^\theta \sin\theta\, d\theta$$

$$= \text{MH}\,(1 - \cos\theta)$$

If U(0) be assumed to be zero, then

$$\text{U}(\theta) = \text{MH}\,(1 - \cos\theta)$$

This is the required expression.

If the magnet is released from the position θ, its potential energy is converted into the kinetic energy of rotation and in position θ = 0 the entire energy becomes kinetic. It I be the moment of inertia of the magnet about the axis of suspension and ω the angular speed attained at θ = 0 position, then the kinetic energy is $\frac{1}{2}\text{I}\omega^2$. Therefore,

$$\frac{1}{2}\text{I}\omega^2 = \text{MH}\,(1 - \cos\theta)$$

$$\omega = \sqrt{\frac{2\text{MH}\,(1 - \cos\theta)}{\text{I}}}$$

Q.45. *If a force* $\vec{F} = (2xy + z^2)\,\hat{i} + x^2\hat{j} + 2xz\hat{k}$, *then show that it a conservative force. Determine its potential function.*

Solution. Let us find the curl of given force.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^2 & x^2 & 2xz \end{vmatrix}$$

$$= \hat{i}\,(0 - 0) - j(2z - 2z) + \hat{k}\,(2x - 2x)$$

$$= 0$$

As curl $\vec{F}$ is zero, the force $\vec{F}$ is conservative.

The potential energy function of $\vec{F}$ is given by.

$$U = -\int \vec{F}.\vec{dr} = -\int (F_x dx + F_y dy + F_z\, dz)$$

$$= -\int (2xy + z^2)\, dx + x^2\, dy + 2xz\, dz$$

$$= -\int (2xy\, dx + x^2 dy) + (z^2 dx + 2xzdz)$$

$$= -\int d(x^2 y) + d(z^2 x) = -\int d(x^2 y + z^2 x)$$

$$= -(x^2 y + z^2 x)$$

Q.46. *Show that the force represented by* $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ *is conservative.*

Q.47. *The position of a moving particle at an instant is given by* $\vec{r} = A\, cos\theta\, \hat{i} + A\, sin\theta\, \hat{j}$. *Show that the force* acting on the particle is conservative.

Solution. The position vector of the particle at an instant t is given by

$$\vec{r} = A\cos\theta\,\hat{i} + A\sin\theta\,\hat{j} = A\,(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j})$$

where ω is the angular velocity of the particle and $\omega t = \theta$

The linear velocity and acceleration at t are given by

$$\vec{v} = \frac{d\vec{r}}{dt} = A\omega\,(-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j})$$

and

$$\vec{a} = \frac{d\vec{v}}{dt} = -A\omega^2(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}) = -\omega^2\,\vec{r}$$

If m be the mass of the particle, then the force acting upon it is given by

$$\vec{F} = m\vec{a} = -m\omega^2\,\vec{r}$$

Since curl $\vec{r} = 0$, we have

$$\text{curl}\,\vec{F} = 0$$

and so the force is conservative.

Q.48. *What is the potential energy of an 800 kg elevator at the top of a building 380 m above street level ? Assume the potential energy at street level to be zero. What would happen to this energy if the elevator comes down to the street level ?*

Solution. The (gravitational) potential energy of the elevator at a height h when its value at street level is zero, is given by

$$U(h) = mgh$$

Here, $m = 800$ kg and $h = 380$ m

$$\therefore \quad U(h) = 800 \times 9.8 \times 380$$
$$= 2.98 \times 10^6 \text{ Joule.}$$

When the elevator comes down to the street level, the potential energy is first converted into kinetic energy and then into heat when the elevator has come to a stop.

Q.49. *A rod of length 1.0 meter and mass 0.5 kg is fixed at one end and is initially having vertically. The other end is now raised until it makes an angle of 60° with the vertical. How much work is required ?*

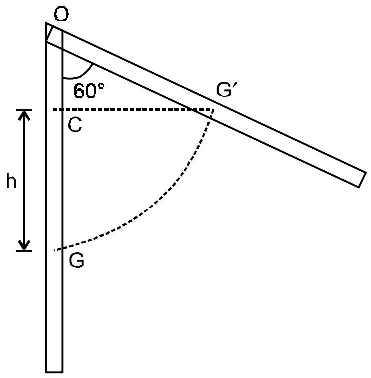


Fig. 24

Solution. The weight mg of the rod acts at its centre of gravity G. As the lower end of the rod is rotated through 60°, G moves to G′, *i.e.,* it is raised through a height h, where,

$$h = CG = OG - OC$$
$$= OG - OG' \cos 60°$$
$$= OG\,(1 - \cos 60°)$$

because $CG = OG'$

Here OG = 0.5 meter and cos 60° = 0.5

$$\therefore \quad h = 0.5\,(1 - 0.5) = 0.25 \text{ meter}$$

Thus; the gain in the potential energy of the rod is

$$U = mgh$$
$$= 0.5 \times 9.8 \times 0.25$$

Since the work done W has been stored as potential energy, we have

$$W = U = 1.225 \text{ Joule}$$

Q.50. *A uniform chain is held on a frictionless table with one-fifth of its length hanging over the edge. If the chain has a length l and a mass m, how much work is required to pull the hanging part back on the table ?*

Solution.

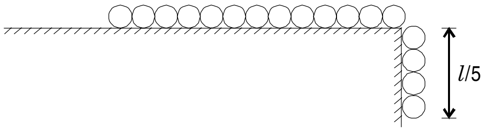


Fig. 25

The mass of the hanging part of the chain is $m/5$. The weight $mg/5$ of this part of the chain acts at its centre of gravity which is distance $l/10$ below the surface of the table. Hence the gain in potential energy in pulling the hanging part on the table is

$$U = \frac{mg}{5}\frac{l}{10} = \frac{mgl}{50}$$

This is also the work done in pulling the chain. Thus

$$W = U = mgl/50$$

Q.51. *The potential energy function for the force between two atoms in a diatomic molecule can be expressed as follows:*

$$U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$$

where a and b are positive constants and x is the distance between the two atoms. Derive an expression for the force between the two atoms and show that the two atoms repel each other for x less than x_0 and attract each other for x greater than x_0. What is the value of x_0?

Solution. $$U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$$

We know that the force is the negative gradient of the potential energy. Therefore the force between the two atoms is given by

$$F = -\frac{dU(x)}{dx}$$

$$= -\frac{d}{dx}\left[\frac{a}{x^{12}} - \frac{b}{x^6}\right]$$

$$= \frac{12a}{x^{13}} - \frac{6b}{x^7}$$

Let the force F be zero, when $x = x_0$, then we have

$$\frac{12a}{x_0^{13}} - \frac{6b}{x_0^7} = 0$$

$$x_0^6 = \frac{2a}{b}$$

$$x_0 = \left(\frac{2a}{b}\right)^{1/6}$$

This is the equilibrium point (a point of stable equilibrium). Now, we may write

$$F = \frac{12a}{x^{13}} - \frac{6b}{x^7}$$

$$= \frac{6b}{x^7}\left[\frac{2a/b}{x^6} - 1\right] = \frac{6b}{x^7}\left\{\left(\frac{x_0}{x}\right)^6 - 1\right\}$$

Clearly, when $x < x_0$ then F is positive and the atoms repel each other. When $x > x_0$ then F is negative and the atoms attract each other.

Q.52. *The potential energy function for the force between two atoms in a diatomic molecule may be expressed as follows:*

$$U(x) = \frac{a}{x^{10}} - \frac{b}{x^5};$$

where a and b are positive constants and x is the distance between the atoms. (a) Calculate the distance x at which P.E. is minimum. (b) Assume that one of the atoms remains at rest and that the other moves along x. Describe the possible motion. (c) The energy needed to break up the molecule into separated atoms (x = ∞) is called the dissociation energy. What is the dissociation energy of the molecule ?

Solution. (*a*) The value of x at which U(x) is a minimum is found from $\frac{d}{dx}U(x) = 0$

so that $$\frac{d}{dx}\left(\frac{a}{x^{10}} - \frac{b}{x^5}\right) = 0$$

or $$-\frac{10a}{x^{11}} + \frac{5b}{x^6} = 0$$

or $$x^5 = \frac{2a}{b}$$

or $$x = \left(\frac{2a}{b}\right)^{1/5}$$

(*b*) The force between the two atom is

$$F = -\frac{d}{dx}U(x) = \frac{10a}{x^{11}} - \frac{5b}{x^6}$$

The force is zero when $x = \left(\frac{2a}{b}\right)^{1/5}$. When x is less than $(2a/b)^{1/5}$, the force is positive and the atoms repel each other. When x is greater than $(2a/b)^{1/5}$, then force is negative and the atoms attract each other. If one of the atoms is fixed, then the other atom would oscillate about the equilibrium separation $(2a/b)^{1/5}$.

(*c*) The dissociation energy D is equal to the change in potential energy from its minimum value at equilibrium separation $x = (2a/b)^{1/5}$ to the zero value at $x = \infty$.

$$\therefore \qquad D = U(x = -\infty) - U\{x = (2a/b)^{1/5}\}$$

$$= D - \left\{\frac{a}{(2a/b)^5} - \frac{b}{2a/b}\right\}$$

$$= b^2/4a$$

If the kinetic energy at the equilibrium position is equal to or greater than its value, the molecule will dissociate.

Q.53. *The potential energy function of the force between two atoms in a diatomic molecule can be expressed approximately as:*

$$U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$$

where a and b are positive constants and x is the distance between atoms (i) At what values of x is U(x) equal to zero and U(x) a minimum ? (ii) Determine the force between atoms (iii) Calculate the dissociation energy of the molecule.

$$\left[\textbf{Ans}.\ (i) \left(\frac{a}{b}\right)^{1/6} \text{ or } \infty, \left(\frac{2a}{b}\right)^{1/6};\ (ii)\ \frac{12a}{x^3} - \frac{6b}{x^7},\ (iii)\ b^2/4a\right]$$

Q.10. *The potential energy between the protons and neutrons inside a nucleus is given by*

$$U(r) = -\frac{r_0}{r} U_0\, e^{-r/r_0}$$

Find the corresponding expression for the force of attraction and compute the ratio of this force at r_0, $2r_0$, $4r_0$ and $10r_0$ to the force at $r = r_0$. What conclusion you draw from the results?

Solution. The force is the negative gradient of potential energy. Thus

$$F(r) = -\frac{d}{dr}U(x) = \frac{d}{dt}\left\{\frac{r_0}{r} U_0\, e^{-r/r_0}\right\}$$

$$= r_0 U_0 \left\{\left(-\frac{1}{r^2}\right)e^{-r/r_0} + \left(\frac{1}{r}\right)\left(e^{-r/r_0}\right)\left(-\frac{1}{r_0}\right)\right\}$$

$$= \frac{-r_0 U_0}{r} e^{-r/r_0} \left(\frac{1}{r} + \frac{1}{r_0}\right)$$

This is the expression for the force of 'attraction'. At $r = r_0$ the force is

$$F_{r_0} = -U_0\, e^{-1}\left(\frac{2}{r_0}\right)$$

Similarly,

$$F_{2r_0} = -U_0\, e^{-2}\, \frac{3}{4r_0}$$

$$F_{4r_0} = -U_0\, e^{-4}\, \frac{5}{16r_0}$$

$$F_{10r_0} = -U_0\, e^{-10}\, \frac{11}{100 r_0}$$

Therefore we have

$$\frac{F_{2r_0}}{F_{r_0}} = \frac{3}{8}e^{-1} \simeq 0.14 \simeq 1.4 \times 10^{-1} \qquad [\because e = 2.718]$$

$$\frac{F_{4r_0}}{F_{r_0}} = \frac{5}{32}\, e^{-3} \simeq 0.0078 \simeq 7.8 \times 10^{-3} \qquad [\because e^3 = 20.09]$$

$$\frac{F_{10r_0}}{F_{r_0}} = \frac{11}{200}e^{-9} \simeq 0.0000067 \simeq 6.7 \times 10^{-6} \qquad [\because e^9 = 8108]$$

The result shows that the force is short range.

Q.54. *The potential energy of a body is given by*

$$U = 40 + 6x^2 - 7xy + 8y^2 + 32z$$

where U is the joule and x, y, z in meter. Deduce the x, y, z components of the force on the body when it is in position (–2, 0, +5).

Solution. $U = 40 + 6x^2 - 7xy + 8y^2 + 32z$

The negative gradient of the potential energy with respect to the potential variables gives the intrinsic force. Therefore, the (x, y, z) components of the force at the position (–2, 0, +5) are given by

$$F_x = \frac{-\partial U}{\partial x} = -12x + 7y = +24 \text{ Nt} \quad [\text{putting } x = -2,\ y = 0]$$

$$F_y = \frac{-\partial U}{\partial y} = 7x - 16y = -14 \text{ Nt} \quad [\text{putting } x = -2,\ y = 0]$$

$$F_z = \frac{-\partial U}{\partial z} = -32 \text{ Nt}$$

Q.55. *If the potential in a plane is given by $U = 4x^2 - 10xy + 2y^2$, deduce the x and y components of field at the point x = 2, y = 4.*

Solution. $U = 4x^2 - 10xy + 2y^2$

The x and y components of the field are the negative gradients of U with respect to x and y respectively. Thus

$$F_x = \frac{-\partial U}{\partial x} = -8x + 10y$$

and

$$F_y = \frac{-\partial U}{\partial y} = 10x - 4y$$

at $x = 2$, $y = 4$ we have

$F_x = 24$ and $F_y = 4$.

Q.56. *The electric potential in a region of space is given by $V = 5x - 7x^2y + 8y^2 + 16yz - 4z$ volt.*

Deduce an expression for the electric field $\vec{E}$. Calculate the y-component of the field at the point (2, 4, –3).

Solution. $$V = 5x - 7x^2y + 8y^2 + 16yz - 4z \text{ volt} \quad ...(i)$$

The electric field $\vec{E}$ may be written in terms of its cartesian components as

$$\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$$

Since the electric field in a direction is the negative potential gradient in that direction, we may write

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k} \quad ...(ii)$$

By partial differential of eq. (*ii*) we get

$$\frac{\partial V}{\partial x} = 5 - 14\ xy,\ \frac{\partial V}{\partial y} = -7x^2 + 16y + 16z \text{ and } \frac{\partial V}{\partial z} = 16y - 4$$

Putting these values in eq. (*ii*), we obtain

$$\vec{E} = (-5x + 14xy)\,\hat{i} + (7x^2 - 16y - 16z)\hat{j} + (-16y + 4)\hat{k}$$

This is the required expression. The y-component at the point (2, 4, –3) is

$$E_y = 7x^2 - 16y - 16z = 12 \text{ volt/meter.}$$

Q.57. *The electric potential in a system is given by*

$$V(x, y, z) = 20 + 6x^2 - 5xy + 4y^2 + 3z^2 \text{ joule/coulomb.}$$

where x, y, z are in meter. Deduce in $\hat{i}, \hat{j}, \hat{k}$ notation the force on a 2×10^{-15} coulomb charge placed at position (2, 0, –3) meter.

Solution. The electric field is the negative derivative of the potential and is given by

$$\vec{E} = \frac{-\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial Z}\hat{k}$$

$$= -(12x - 5y)\,\hat{i} - (-5x + 8y)\,\hat{j} - 6z\hat{k} \quad \text{Nt/Coul.}$$

At position (2, 0, –3) meter, the field is therefore

$$\vec{E} = -(24 - 0)\hat{i} - (-10 + 0)\,\hat{j} - (-18)\,\hat{k}$$

or $$\vec{E} = -24\hat{i} + 10\hat{j} + 18\hat{k} \quad \text{Newton/Coulomb}$$

The force on a charge $q = 2 \times 10^{-15}$ coulomb at this point is

$$\vec{F} = q\vec{E} = (-48\hat{i} + 20\hat{j} - 36\hat{k}) \times 10^{-15} \text{Newton}$$

This is the required expression.

Q.58. *A scalar potential field is given by*

$$V = 6x + 8y - 12xy^2 + 7yz^2 - 5y^2 \text{ joule/coul.}$$

where x, y, z are in meters. Calculate (a) the work done on a 5 coulomb charge in moving it from position (2, 0, 0) to (2, 5, 0). (b) the force on a 4-coulomb charge placed at the origin (0, 0, 0).

Solution. (*a*) The potential at positions (2, 0, 0) and (2, 5, 0) are obtained by the given expression as

$$V_{2,\,0,\,0} = -12 \text{ joule/coulomb and } V_{2,\,5,\,0} = -697 \text{ joule/coulomb}$$

Therefore the work done in carrying a charge $q = 5$ coulomb from a point $V_{2,\,0,\,0}$ to $V_{2,\,5,\,0}$ is given by (work = charge × potential difference)

$$W = q\,(V_{2,\,0,\,0} \sim V_{2,\,5,\,0})$$
$$= 5 \times 685 = 3425 \text{ Joule}$$

(*b*) The expression for the electric field is

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}$$

$$= -(6 - 12y^2)\hat{i} - (8 - 24xy + 7z^2 - 10y)\hat{j} - 14yz\,\hat{k}$$

At (0, 0, 0) we have

$$\vec{E}_0 = -6\hat{i} - 8\hat{j} \text{ Newton/coulomb.}$$

The force on a charge $q = 4$ coulomb is therefore

$$\vec{F} = q\vec{E}_0 = -24\hat{i} - 32\hat{j} \text{ Newton.}$$

The scalar magnitude of this force is

$$F = \sqrt{(24)^2 + (-32)^2}$$
$$= 40 \text{ Newton.}$$

Q.59. *Show that for the same initial speed v_0, the speed v of a projectile will be the same at all points at the same elevation regardless of the angle of projection.*

Solution. In the absence of air resistance, the only force on a projectile is its weight, and the mechanical energy of the projectile remains constant. Fig. 26 shows two trajectories of a projectile with the same initial speed (and hence the same total energy) but with different angles of departure.

Now, at all points at the same elevation the potential energy is the same; hence the kinetic energy is the same and so the speed is the same.

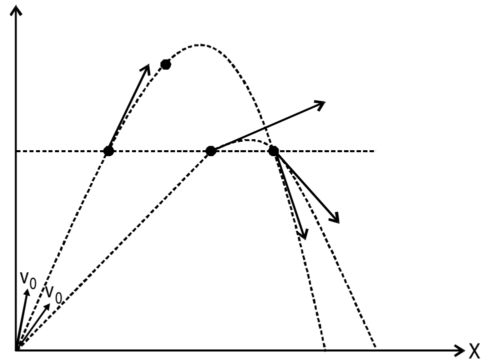


Fig. 26

Q.60. *The bob of a 2m long pendulum has mass m = 0.5 kg. It is pulled to a side to make θ = 30° angle with the vertical. Calculate the change in potential energy, the work done and the speed of the bob when it passes the lowest point after being released.*

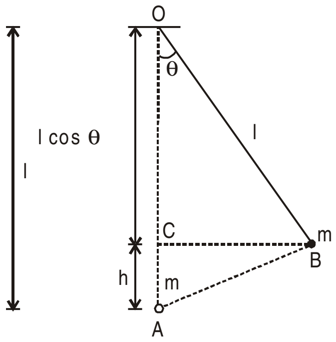


Fig. 27

Solution. As the bob of mass m is moved from A to B, it rises through the height h, where

$$h = CA = OA - OC$$
$$= l - l\cos\theta = l(1 - \cos\theta)$$

Thus the gain in potential energy is

$$U = mgh$$
$$= mgl\,(1 - \cos\theta)$$

Here $m = 0.5$ kg, $l = 2$ meter and $\cos\theta = \cos 30° = 0.866$

$$\therefore \quad U = 0.5 \times 9.8 \times 2\,(1 - 0.866)$$
$$= 1.31 \text{ Joule}$$

All the work done is stored as potential energy. Therefore

$$\text{Work done} = 1.31 \text{ Joule}$$

On being released at B, when bob passes the lowest point A, it has lost all its potential energy which appears as kinetic energy $\frac{1}{2}mv^2$, where v is the velocity at A. No work is done by the tension in the string because it is always perpendicular to the displacement. Hence by the conservation of mechanical energy, we have

$$mgh = \frac{1}{2}mv^2,$$

or

$$v = \sqrt{2gh}$$

Here $h = l(1 - \cos\theta) = 2(1 - 0.866) = 0.268$ meter

$$\therefore \quad v = \sqrt{2 \times 9.8 \times 0.268} = 2.3 \text{ m/s}$$

Q.61. *(a) The bob of a simple pendulum of length l is released from a point in the same horizontal line as the point suspension and at a distance l from it. Calculate the velocity of the bob and the tension in the string at the lowest point of its swing.*

(b) If the string of the pendulum is catched by a nail located vertically below the point of suspension and the bob just swings around a complete circle around the nail, find the distance of the nail from the point of suspension.

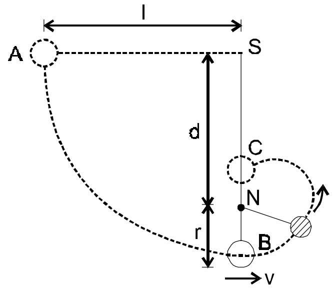


Fig. 28

(c) If the string of the pendulum is made of rubber then show that it will stretch by 3 mg/K (where K is the force constant) on reaching the bob at the lowest point.

Solution. (*a*) Let S be the point of suspension and A the point from which the bob of mass m is released (SA = l). The point A is at a vertical height l above the lowest point B of the swing. Hence is the potential energy mgl. On being released, the bob swings along the dotted arc, reaching the lowest point B where its entire energy is kinetic, $\frac{1}{2}mv^2$ (v being the velocity at B). By the conservation of mechanical energy, we have

$$mgl = \frac{1}{2}mv^2$$

$$\therefore \qquad v = \sqrt{2gl}$$

This is the velocity at the lowest point.

As the bob moves toward B, the tension in the string increases and becomes maximum at B. If T be this (maximum) tension, the net vertically upward force on the bob is T-mg, and this provides the required centripetal force mv^2/l to the swinging bob. Thus

$$T - mg = \frac{mv^2}{l}$$

But at B, $$v^2 = 2gl \text{ (proved above).}$$

$$T = mg + \frac{mv^2}{l}$$

$$T = 3mg$$

(*b*) Now, suppose a nail N is located at distance d vertically below S. As the bob reaches the point B, the swing of the pendulum is caught by the nail and the bob swings around a complete circle of radius r(say). C is the highest point of this circular swing. Clearly the bob will do so provided its velocity at C, say v_c, is such that the required centripetal force $\frac{mv_c^2}{r}$ (downward) is provided by its entire weight mg. That is

$$\frac{mv_c^2}{r} = mg$$

or $$v_c^2 = gr$$

The decrease in kinetic energy as the bob goes from B to C is $\frac{1}{2}m\,(v^2 - v_c^2)$. This appears as gravitational potential energy $mg(2r)$ of the bob at the point C. Thus

$$\frac{1}{2}m\,(v^2 - v_c^2) = mg\,(2r)$$

Substituting the values of v^2 and v_c^2 from above, we get

$$\frac{1}{2}m\,(2gl - gr) = mg\,(2r)$$

$$2gl - gr = 4gr$$

$$2gl = 5gr$$

$$r = \frac{2}{5}l$$

$\therefore$ The distance of the nail from the point of suspension is

$$d = l - r = l - \frac{2}{5}l = \frac{3}{5}l$$

(c) If the string is made of rubber, then on reaching the bob at B the string is stretched by an amount Δl (say). The string would therefore experience an upward elastic force K(Δl). This minus the (downward) weight mg would provide the centripetal force mv^2/l at B. Thus

$$K(\Delta l) - mg = \frac{mv^2}{l}$$

at B; $v^2 = 2gl$

$$\therefore \quad K(\Delta l) - mg = \frac{m(2gl)}{l} = 2mg$$

$$\Delta l = 3mg/K.$$

Q.62. *A light meter stick, pivoted about a horizontal axis through its centre, has a 2kg body attached to one end and a 1 kg body to the other. The system is released from rest with the stick horizontal. What is the velocity of each body as the stick swings through a vertical position ?*

Solution.

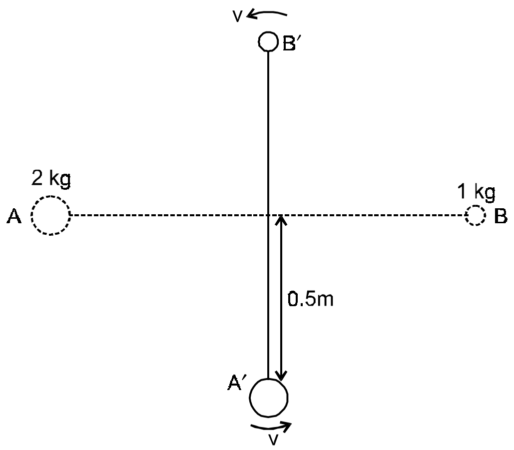


Fig. 29

Initially the bodies are at A and B, having gravitational potential energies mgh only, where h is the vertical height from the lowest point A′(h = 0.5 meter). The mechanical (potential) energy of the system is

$$= 2 \times 9.8 \times 0.5 + 1 \times 9.8 \times 0.5$$

$$= 3 \times 9.8 \times 0.5 \text{ Joules}$$

On being released, the heavier body comes down in the position A′ and the lighter one goes upto B′. Let v be the velocity of each. Now the body at A′ has kinetic energy $\left(\frac{1}{2}mv^2\right)$

alone, but the body at B′ has both the kinetic energy $\left(\frac{1}{2}mv^2\right)$ and the potential energy (mgh); h being now 1 meter. Thus the mechanical energy of the system

$$= \frac{1}{2} \times 2 \times v^2 + \left(\frac{1}{2} \times 1 \times v^2 + 1 \times 9.8 \times 1\right)$$

$$= \frac{3}{2}v^2 + 9.8$$

By the law of conservation of mechanical energy, we have

$$3 \times 9.8 \times 0.5 = \frac{3}{2}v^2 + 9.8$$

or $$\frac{3}{2}v^2 = 3 \times 9.8 \times 0.5 - 9.8 = 0.5 \times 9.8$$

$$\therefore \quad v = \sqrt{\frac{2 \times 0.5 \times 9.8}{3}} = 1.81 \text{ m/s}$$

Q.63. *A light rod of length l and with a mass m attached to its end is suspended vertically. It is turned through 180° and then released. Calculate the velocity of the mass and the tension in the rod when the mass reaches at lowest point.*

If the system be released with the rod horizontal, at what angle from the vertical the tension in the rod would be equal to the weight of the body ?

Solution. Let S be the point of suspension and A the point from which the mass m is released. The point A is at a vertical height $2l$ above the lowest point B of the swing. Hence at A the mass has gravitational potential energy $mg(2l)$.

On being released, the mass swing along the dotted semicircle, reaching the lowest point B where its energy is entirely kinetic, $\frac{1}{2}mv^2$ (v being the velocity at B). By energy conservation, we have mg (2l) = $\frac{1}{2}mv^2$

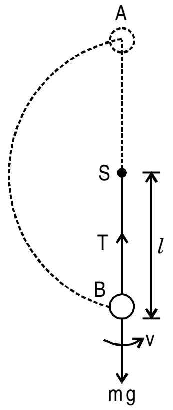


Fig. 30

$$\therefore \quad v = 2\sqrt{gl}$$

As the mass is moving in a circle or radius l, it must be acted upon by a centripetal force. At the point B, the net upward force on the mass is T-mg, where T is the vertical tension in the string. This provides the required centripetal force. Thus

$$\text{T} - \text{mg} = \frac{mv^2}{l}$$

But $$v^2 = 4gl$$

$$\therefore \quad \text{T} = mg + \frac{4mgl}{l} = 5\,mg$$

Now, suppose the mass is released from the point A′. On reaching the point B′, it descends vertically through a distance $l \cos\theta$, thus losing potential energy by $mgl \cos\theta$. It, however, gains kinetic energy $\frac{1}{2}mv'^2$, where v' is the velocity at B′.

Thus

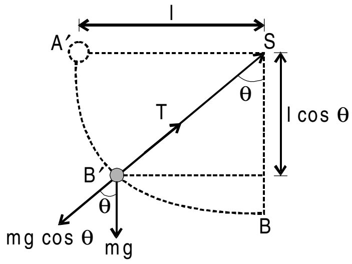


Fig. 31

$$mgl \cos \theta = \frac{1}{2} mv'^2$$

$$\therefore \qquad v'^2 = 2gl \cos \theta$$

At the point B′, the net radically inward force on the mass is T – *mg* cosθ, which provides the required centripetal force mv'^2/l. Thus

$$T - mg \cos\theta = \frac{mv'^2}{l}$$

or

$$T = mg \cos \theta + m \frac{(2gl \cos \theta)}{l} = 3mg \cos \theta$$

Let the angle θ be such that T = *mg*, then we have $mg = 3mg \cos \theta$

or

$$\cos \theta = \frac{1}{3}$$

$$\therefore \qquad \theta = \cos^{-1}\left(\frac{1}{3}\right) = 71°$$

Q.64. *A ball is tied to a cord and set in rotation in a vertical circle prove that the tension in the cord at the lowest point exceed that at the highest point by six times the weight of the ball.*

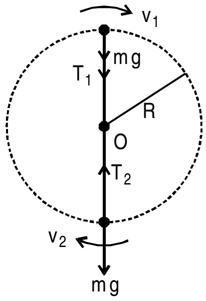


Fig. 32

Solution. Let v_1 be the velocity of the ball as, it passes the highest point. The force acting on it are its weight *mg* and the tension T_1 in the cord both acting downward. The resultant force is thus $T_1 + mg$, which provides the centripetal force $\frac{mv_1^2}{R}$. Thus

$$T_1 + mg = \frac{mv_1^2}{R}$$

Similarly, at the lowest point we shall have

$$T_2 - mg = \frac{mv_2^2}{R}$$

$$T_2 - T_1 = \frac{m}{R}(v_2^2 - v_1^2) + 2\,mg \qquad ...(1)$$

At the highest point, the ball has gravitational potential energy *mg* (2R) and the kinetic energy $\frac{1}{2} mv_1^2$. At the lowest point it has entirely kinetic energy $\frac{1}{2} mv_2^2$. By the law of conservation of energy, we have

$$mg\,(2R) + \frac{1}{2} mv_1^2 = \frac{1}{2} mv_2^2$$

or $$\frac{m}{R}(v_2^2 - v_1^2) = 4\ mg$$

Substituting this value in eq. (1) we have

$$T_2 - T_1 = 4\ mg + 2\ mg = 6\ mg.$$

Q.65. *A body slides down a curved frictionless track which is one quadrant of a circle of radius R. Find its speed at the bottom of the track. Point out the usefulness of the energy method of solving dynamical problems. Will the conclusion regarding the speed at the bottom hold if friction were present ?*

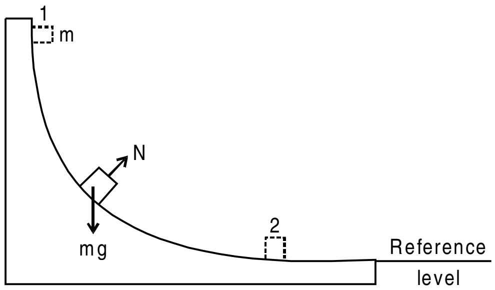


Fig. 33

Ans. Motion on a frictionless inclined plane: Let m be the mass of the body which starting from rest at the point 1, slides down a frictionless curved track to the point 2. As there is no friction the forces acting on the body are (*i*) its weight *mg*, and (*ii*) the normal force N exerted on it by the track. Since the force N is always normal to the direction of motion of the body, it does no work. In other words, only the gravitational force *mg* does work. Since the force is conservative, the mechanical energy of the body is conserved. Thus we can write.

$$E = K_1 + U_1 = K_2 + U_2$$

$$\frac{1}{2}\ m{v_1}^2 + mgh_1 = \frac{1}{2} mv_2^2 + mgh_2$$

Here $v_1 = 0,\ h_1 = R,\ h_2 = 0,\ v_2 = ?$

$$O + mgR = \frac{1}{2} mv_2^2 + 0$$

$$v = \sqrt{2gR}$$

The speed is therefore the same as if the body had fallen vertically through a height R. This proves the fact that the work done by a conservative force in moving a body through a distance is independent of the path chosen.

Here the resultant force acting on the body, and hence the acceleration depends on the slope of the track at each point, *i.e.*, it varies from point to point. Thus the acceleration is variable. As such we cannot use the Newton's equation of motion which holds for constant acceleration. Hence to solve the problem by other methods we shall have to determine the acceleration at each point and then applying integration. All this is avoided in the energy method.

We have seen above that the speed at the bottom is independent of the shape of the surface. This conclusion would not hold if friction were present. This is so because the work of the friction force does depend on the path; the longer the path, the greater the work.

Q.66. *A body of mass 0.5 kg. starts from rest and slides vertically down a curved track which is in the shape of one quadrant of a circle of radius 1 met. At the bottom of the track the speed of the body is 3 m/s. What is the work done by the frictional force?*

Solution. Let *m* be the mass and R the radius of the circle. When the mass is at rest on the top of the track, its energy is entirely potential being equal to *mg* R. On reaching at the bottom the energy is entirely kinetic, being equal to $\frac{1}{2}mv^2$. Let the work done against the frictional force be W_f. Then, by the conservation of total energy. We have

$$mgR = \frac{1}{2}mv^2 + W_f$$

$$W_f = mgR - \frac{1}{2}mv^2$$

$$= (0.5 \times 9.8 \times 1) - \frac{1}{2} \times 0.5 \times (3)^2$$

$$= 2.65 \text{ Joules}$$

Q.67. *A 0.5 kg block is released from rest at a point on a track which is one quadrant of a circle of radius 4m. It slides down the track and reaches its bottom with a velocity of 6 m/s. Then it further slides a distance of 9 m on a level surface and stops. How much work is done against friction in sliding on the circular track and what is the coefficient of sliding friction on the horizontal surface?*

[**Ans.** 10.6 Joule, 0.20]

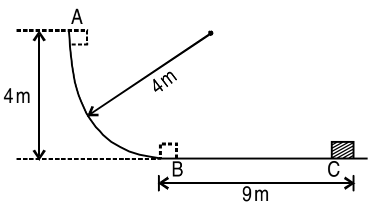


Fig. 34

Q.68. *A body slides along a track with elevated ends and horizontal flat central part. The flat part has a length l = 3met. The curved portions of the track are frictionless. For the flat part the coefficient of kinetic friction is μ_k = 0.20. The body is released at point A which is at a height h = 1.5 meter above the flat part of the track. Where does the particle finally comes to rest.*

Solution. Suppose the body is released form rest at A where it has gravitational potential energy. It reaches B with kinetic energy which then carries it to C against friction. From C it rises up to D (say) where the kinetic energy in it at C is once again converted into potential energy. It then again descends to C, goes to B rises up to a point E and this process continue until finally it comes to rest some where on the horizontal part of the track.

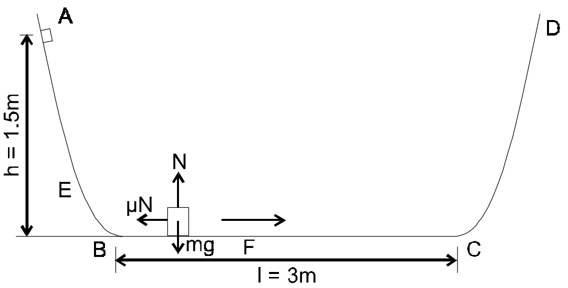


Fig. 35

The initial (potential) energy of the body at the point of release A is *mgh*. Since the curved parts are frictionless energy is not spent up in moving the body on them. Energy is spent up in doing work against the friction only when the body moves on the horizontal part. The friction force against the direction of motion is

$$f_k = \mu N = \mu mg$$

Suppose whole of the initial energy, *mgh*, is consumed in moving a distance *d* on the horizontal part, the work done being.

$$W = f_k \times d = \mu\, mgd$$

By the law of energy conservation, we write

$$mgh = W = \mu mgd$$

$$\therefore \quad d = \frac{h}{\mu}$$

Here $h = 1.5$ meter and $\mu = 0.20$

$$d = \frac{1.5}{0.2} = 7.5 \text{ meter}$$

Thus the body will cover a distance of 7.5 m on the horizontal part which is 3.0 meter long. This means that it will come to rest at a point F such that

$$BC + CB + BF = 7.5$$

$$3 + 3 + BF = 7.5$$

$$BF = 1.5 \text{ meter}$$

Thus the point F is the middle point of the horizontal part where the body would come to rest.

Q.69. *A 3000 kg automobile at rest at the top of an incline 30 m high and 300 m long is released and rolls down the hill. What is its speed at the bottom of the incline if the average retarding force due to friction is $f_k = 200\ g$?*

Solution. Let m be the mass of the automobile, h the height of the incline, x the length and v the velocity at bottom. The potential energy of the automobile when at rest at the top of incline is mgh. The kinetic energy at the bottom is $\frac{1}{2}mv^2$, and the energy dissipated against the friction is $f_k x$.

By conservation of total energy, we have

$$mgh = \frac{1}{2}mv^2 + f_k x$$

$$v^2 = 2gh - \frac{2x}{m} f_k$$

$$= (2 \times 9.8 \times 30) - \left(\frac{2 \times 300}{3000} \times 200 \times 9.8\right)$$

$$= 196$$

$$v = 14 \text{ m/s.}$$

Q.27. *A block of mass m slides down at 30° inclined plane of length l and coefficient of friction μ. With what speed it will reach the bottom. If it further slides on a similar horizontal surface, how far will it go before coming to rest ?*

Solution.

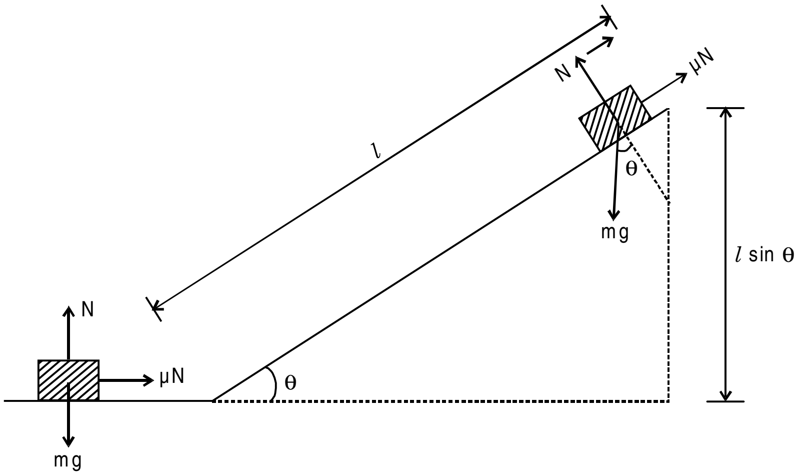


Fig. 36

The weight of the block, *mg*, acts vertically downwards. Let N be the normal reaction exerted by the plane of the block.

Then $$N = mg \cos \theta$$

The friction force opposing its sliding down will be

$$f_s = \mu N = \mu mg \cos \theta$$

Let v be the velocity of the body on reaching the bottom of the plane. At the moment of start the vertical height of the block is l sinθ. Its potential energy is, therefore, mg (l sin θ). This is used up in providing the kinetic energy $\frac{1}{2}mv^2$ to the block and in doing work against the friction f_s. Thus, by the conservation of total energy. We have

$$mgl \sin \theta = \frac{1}{2}mv^2 + f_s = \frac{1}{2}mv^2 + \mu\, mgl \cos \theta$$

or $$2gl \sin \theta - 2\mu gl \cos \theta = v^2$$

or $$v = \sqrt{[2gl\,(\sin \theta - \mu \cos \theta)]}.$$

On reaching the bottom, the block has kinetic energy $\frac{1}{2}mv^2$. Suppose now it moves a distance d on a horizontal surface before coming to rest. Thus the kinetic energy is used up in doing work against this friction. Now the friction force is μN = μ*mg* and the work done is μ*mgd*. Then

$$\frac{1}{2}mv^2 = \mu mgd$$

or $$\frac{1}{2}\, m\,[2gl(\sin \theta - \mu \cos \theta)] = \mu mgd$$

or $$d = \frac{l\,(\sin \theta - \mu \cos \theta)}{\mu}$$

Q.71. *A 0.2 kg block is thrust up a plane inclined at an angle of 30° to the horizontal with an initial velocity of 5 m/sec. It goes up 2m along the plane and then slides back to the bottom. Calculate the force of friction and the velocity of the block with which it reaches the bottom of the plane.*

Solution. Let us first consider the upward motion. Let f be the force of friction. At the bottom where the block starts moving, the energy is wholly kinetic (K = 1/2 mv^2). At the top where the block momentarily stops, the energy is wholly potential (U = mgh). The work done against the friction during this motion is $f \times x$, where x is the distance travelled. By the general law of conservation of energy, we have

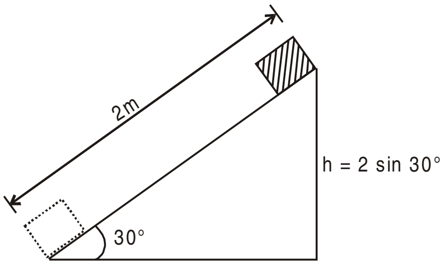


Fig. 37

$$K = U + fx$$

or $$\frac{1}{2}mv^2 = mgh + fx$$

Here, m = 0.2 kg, v = 5 m/s, x = 2 m, h = 2 sin 30° = 1.

$\therefore$ $$\frac{1}{2} \times 0.2\,(5)^2 = 0.2 \times 9.8 \times 1 + f \times 2$$

or $$2.5 = 1.96 + 2f$$

$$f = \frac{2.5 - 1.96}{2} = 0.27 \text{ nt}$$

Let us now consider the downward motion. Let v be the velocity of the block with which it reaches the bottom. Now the potential energy (U = mgh) of the block at the top is used up in giving kinetic energy (K = 1/2 mv'^2) to the block on its reaching the bottom and in doing work against the friction ($f \times x$). Thus

$$U = K + fx$$

$$mgh = \frac{1}{2}mv'^2 + fx$$

$$0.2 \times 9.8 \times 1 = \frac{1}{2} \times 0.2 \times v'^2 + 0.27 \times 2$$

or $$1.96 = 0.1\ v'^2 + 0.54$$

or $$v'^2 = \frac{1.96 - 0.54}{0.1} = 14.2$$

or $$v'^2 = 3.8 \text{ m/sec.}$$

Q.72. *A 12 kg block is pushed 20 m up the sloping surface of a plane inclined at an angle 37 to the horizontal by a constant force of 120 nt acting parallel to the plane. The coefficient of friction between the block and the plane is 0.25. Calculate (i) work done by F (ii) increase in potential energy of the block (iii) work done against the friction and (iv) increase in kinetic energy. What becomes the work done ?*

Solution. (*i*) The force F (= 120 *nt*) pushes to block through a distance x (= 20m). Hence the work done

$W = Fx = 120 \times 20 = 2400$ Joule.

(*ii*) initially, the potential energy is zero, finally it become mg. Here $m = 12$ kg and $h = 20 \sin 37° = 20 \times 0.6 = 12$ m. Hence the increase in P.E.

$$\Delta U = mgh$$
$$= 12 \times 9.8 \times 12$$
$$= 1411 \text{ Joule.}$$

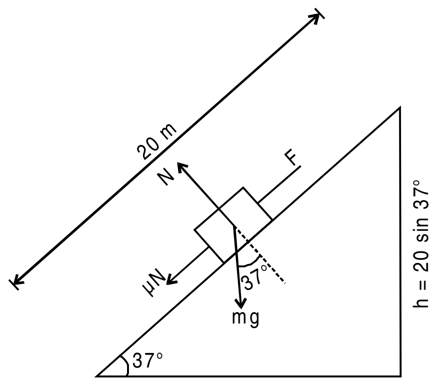


Fig. 38

(*iii*) The force of friction is μN, where μ is the coeff. of friction and N the normal force of reaction exerted by the plane on the block. But $N = mg \cos 37°$ (see Fig.). Thus the force of friction is $\mu mg \cos 37°$. The work done against the force

$$W = \text{force} \times \text{displacement}$$
$$= \mu mg \cos 37° \times x$$
$$= 0.25 \times 12 \times 9.8 \times 0.8 \times 20 = 470.4 \text{ joule}$$

This work is converted into heat.

(*iv*) Initially, the kinetic energy is zero. Let ΔK be the change in kinetic energy. By the law of conservation of energy

$$W = \Delta U + \Delta K + w$$
$$\Delta K = W - \Delta U - w$$
$$= 2400 - 1411 - 470 = 519 \text{ Joule}$$

Q.73. *Figure 39 shows the vertical section of a frictionless surface. A block of mass 2 kg is released from position A. Compute its kinetic energy as it reaches positions B, C, D. (give gravitational field = 9.8 jm⁻¹ kg⁻¹).*

Solution. The (gravitational) force is conservative so that the mechanical energy is conserved. The loss in potential energy as the block comes down to B, C and D from A equals the corresponding gain in the kinetic energy

∴ $$\text{K.E. at B} = \text{loss in P.E. between A and B } (U_A - U_B)$$
$$= mg\,(h_A - h_B)$$
$$= 2\,kg \times 9.8\,jm^{-1}\,\text{kg}^{-1} \times (14 - 5)\,m$$
$$= 176.4 \text{ Joule}$$
$$\text{K.E. at C} = U_A - U_C$$
$$= mg\,(h_A - h_C)$$
$$= 2 \times 9.8 \times 7.0 = 137.2 \text{ Joule}$$
$$\text{K.E. at D} = U_A$$
$$= mgh_A$$
$$= 2.0 \times 9.8 \times 14 = 274.4 \text{ Joule}$$

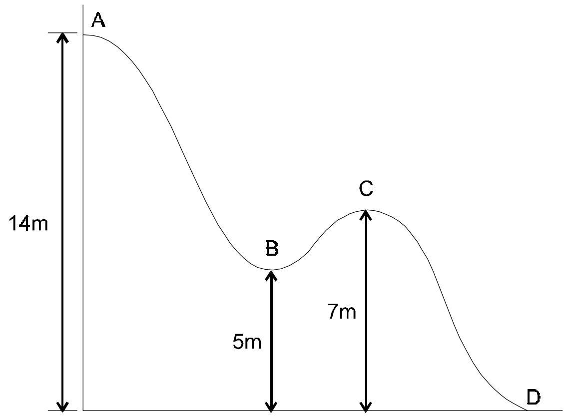


Fig. 39

Q.74. *A small block of mass m slids along a frictionless track as shown in figure 40. If it starts from rest at P, what is the reaction exerted by the track on the block when it is at A, B and C. At what height from the bottom A of the loop should the block be released so that the force it exerts against the track at the top C is equal to its weight.*

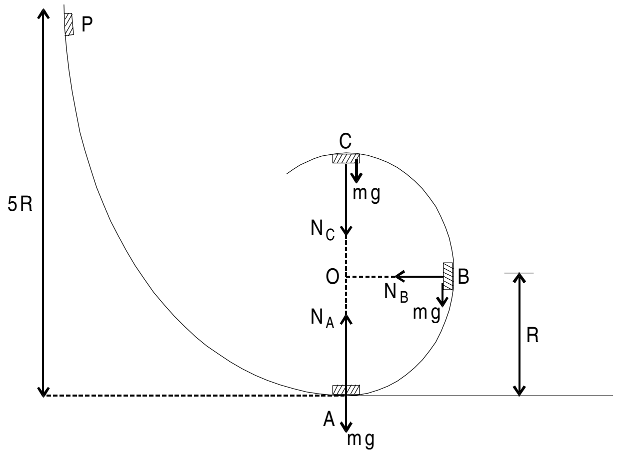


Fig. 40

Solution. Let us first calculate the velocities at A, B and C. At P the entire energy is potential being equal to mg (5R) and at A it is entirely kinetic, being equal to $\frac{1}{2}mv_A^2$. By conservation of mechanical energy, we have

$$mg\ (5R) = \frac{1}{2}mv_A^2$$

$$\therefore \qquad v_A^2 = 10gR \qquad ...(i)$$

At B, the energy is partly kinetic $\frac{1}{2}mv_B^2$ and partly potential mg(R). By energy conservation.

$$mg(5R) = \frac{1}{2}mv_B^2 + mg(R)$$

$\therefore$ $$v_B^2 = 8gR. \qquad ...(ii)$$

Similarly, at C we have

$$mg(5R) = \frac{1}{2}mv_C^2 + mg(2R)$$

$$v_C^2 = 6gR \qquad ...(iii)$$

Now, at any point on the track, the force acting on the block are its weight mg acting vertically downward and the reaction force N exerted by the track acting radially inward. The resultant radial force supplies the required centripetal force at that point.

$$N_A - mg = \frac{mv_A^2}{R}$$

$$= \frac{m}{R}(10gR) \qquad \text{[from eq. } (i)\text{]}$$

$$N_A = 10mg + mg = 11mg.$$

For the point B, we have

$$N_B = \frac{mv^2{}_B}{R}$$

$$= \frac{m}{R}(8gR) = 6mg \qquad \text{[from eq. } (ii)\text{]}$$

$$= 8 \text{ mg}$$

Again for the point C, we write

$$N_C + mg = \frac{mv^2{}_c}{R} \qquad ...(iv)$$

$$= \frac{m}{R}(6gR) = 6\ mg \qquad \text{[from eq. } (iii)\text{]}$$

$\therefore$ $$N_C = 6mg - mg = 5mg.$$

Now, let h be the height from which the block must be released so that N_C equals mg. Then from *eq*. (*iv*), the velocity v_c must be given by

$$mg + mg = \frac{mv_C^2}{R}$$

or $$v_C^2 = 2gR$$

By energy conservation at C, we have

$$mgh = \frac{1}{2}mv_C^2 + mg(2R)$$

$$= \frac{1}{2}m(2gR) + mg(2R)$$

$$= 3mgR$$

$$h = 3R$$

Q.75. *A body is allowed to slide on an inclined frictionless track from rest position under earth's gravity. The track ends in a circular loop of radius R. Show that the minimum height h of the body, so that it may successfully complete the loop is given by* $h = \frac{5}{2}R$.

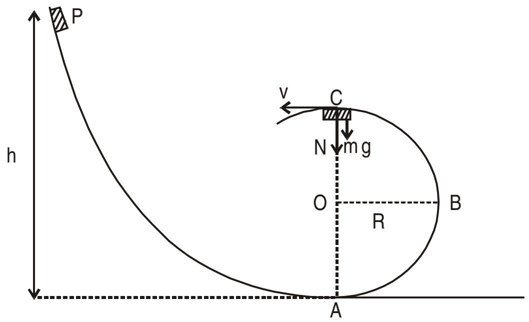


Fig. 41

Solution. Let us first calculate the velocity v of the body which it acquires at the highest point C of the loop, when released from a height h. For this we use conservation of mechanical energy.

At point P, the body is at rest so that its mechanical energy is entirely potential and is equal to mgh At point C it has a velocity v and a height 2R, so that it has kinetic energy equal to $\frac{1}{2}mv^2$ as well as potential energy equal to mg(2R). By energy conservation the mechanical energy at P must be the same as at C because the entire track is frictionless. That is,

$$mgh = \frac{1}{2}mv^2 + mg(2R)$$

or
$$mg(h - 2R) = \frac{1}{2}mv^2$$

or
$$v^2 = 2g(h - 2R) \quad ...(i)$$

Now, the forces acting on the body at C are its weight mg acting vertically downward and the reaction force N exerted by the track which is directed toward the centre O of the circle. In fact, at the point C both are directed toward O. Thus the resultant radial force at C is N + mg, and this supplies the required centripetal force mv^2/R. That is

$$N + mg = \frac{mv^2}{R}$$

Since N cannot be negative, the minimum velocity of the body at C if it is to describe the circle must correspond to N = 0 so that

$$mg = \frac{mv_{min}^2}{R}$$

or
$$v^2_{min} = gR \quad ...(ii)$$

or
$$v_{min} = \sqrt{gR}$$

If the velocity is less than $\sqrt{gR}$, the downward pull of the weight will be larger than the required centripetal force and the body will lose contact from the loop. (The velocity can however be grater than $\sqrt{gR}$).

Now from *eq*. (*i*)

$$v^2_{min} = 2g\,(h_{min} - 2R).$$

Putting this value in eq. (*ii*) we get

$$gR = 2g(h_{min} - 2R)$$

or $$h_{min} = \frac{R}{2} + 2R = \frac{5}{2}R$$

Q.76. *A small body of mass m slides without friction around the loop the loop apparatus, starting at a height 3R above the bottom of the loop, where R, is the radius of the loop. Compute its radial acceleration at the end of a horizontal diameter of the loop. From what minimum height above the bottom of the loop it should start so that it may loop the loop?*

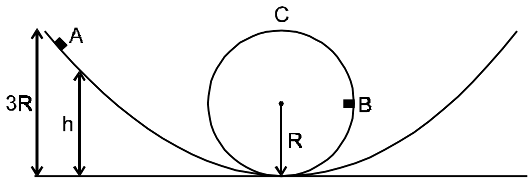


Fig. 42

$$\left[\textbf{Ans. } 4g, \frac{5}{2}R\right]$$

Q.77. *A body is moving on a vertical circular frictionless track. Calculate the minimum velocity which it should have at the lowest point of its path in order to go completely round the track.*

Solution. Let v_1 be the velocity at the highest point of the path, and v_2 the corresponding velocity at the lowest point. In order that the body does not leave the track at the highest point (where it is most likely to do so), the magnitude of V_1 must be atleast such that the centripetal force mV_1^2/R at that point utilises the entire weight mg of the body. That is

$$\frac{mV_1^2}{R} = mg$$

or $$V_1^2 = gR$$

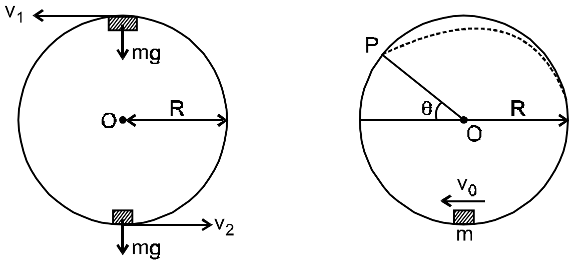


Fig. 43

As the body slides down from the top to the bottom of the track, it loses potential energy by an amount $mg(2R)$ but gains an equivalent amount of kinetic energy $\frac{1}{2}m\,(V_2^2 - V_1^2)$. Thus

$$mg(2R) = \frac{1}{2}m\,(V_2^2 - V_1^2)$$

or $$V_2^2 = V_1^2 + 4gR$$

But $$V_1^2 = gR \text{ (proved above)}$$

$$V_2^2 = gR + 4gR = 5gR$$

or $$V_2 = \sqrt{(5gR)}$$

Q.78. *A particle of mass m = 0.2kg is moving inside a smooth vertical circle of radius R = 50 cm. If it is projected horizontally with velocity v_0 = 4m/s from its lowest position find the angle θ at which it will lose contact with the circle.*

Solution. As the particle moves up the circular track, its velocity decreases. Let v be the velocity at the point P. In this rise the particle gains gravitational potential energy by an amount, mg (R + R sinθ), while it loses an equivalent amount of kinetic energy given by $\frac{1}{2}m$ $(v_0^2 - v^2)$. By energy conservation, we have

$$mg\,(R + R\sin\theta) = \frac{1}{2}m\,(v_0^2 - v^2)$$

or $$v^2 = v_0^2 - 2gR\,(1 + \sin\theta) \qquad ...(i)$$

Let us now consider the forces acting on the particle at the point P. These are particle's weight mg acting vertically downward and the normal reaction N exerted by the track acting radially inward. The net force along the radius is N + mg sinθ which supplies the necessary centripetal force mv^2/R. That is,

$$N + mg\sin\theta = mv^2/R.$$

The particle will loose contact for that value of θ for which N = 0. Thus

$$mg\sin\theta = \frac{mv^2}{R}$$

Putting the value of v^2 from eq. (i) we get

$$g\sin\theta = \frac{v_0^2 - 2gR\,(1 + \sin\theta)}{R}$$

$$= \frac{v_0^2}{R} - 2g - 2g\sin\theta$$

or $$3g\sin\theta = \frac{v_0^2}{R} - 2g$$

or $$\sin\theta = \frac{v_0^2}{3gR} - \frac{2}{3}$$

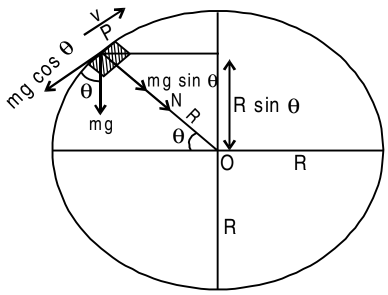


Fig. 44

Here v_0 = 4 m/s and R = 0.5 m

$$\sin\theta = \frac{(4\text{m/s})^2}{3(9.8\text{m/s}^2)\,(0.5\text{m})} - \frac{2}{3}$$

$$= 1.09 - 0.67 = 0.42$$

or $$\theta = \sin^{-1}(0.42) = 25°.$$

Q.79. *A small mass m starts from rest and slides down the smooth surface of a solid sphere of radius R. Assume zero potential energy at the top. Find (a) the change in potential* *energy of the mass with angle (b) the kinetic energy as a function of angle (c) the angle at which the mass flies off the sphere. If there is friction between the mass and the sphere, does the mass fly off at a greater or lesser angle ?*

Solution. (*a*) The potential energy of the mass m at A is zero. As it slides down to P, it descends vertically through. AC. Hence the loss in its potential energy is given by

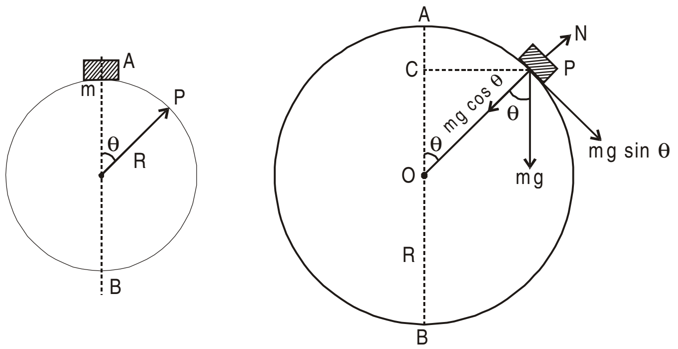


Fig. 45

$$\begin{aligned} U &= -mg\,(AC) \\ &= -mg\,(AO - CO) \\ &= -mg\,(R - R\cos\theta) \\ &= -mg\,R\,(1 - \cos\theta) \end{aligned}$$

(*b*) This lost energy appears as gain in kinetic energy K, as the surface is smooth therefore

$$K = +\,mgR\,(1 - \cos\theta)$$

(*c*) The net radially inward force on the mass *m* at P is mg cos θ – N, where N is the normal reaction of the surface. This supplies the centripetal force, so that

$$mg\cos\theta - N = \frac{mv^2}{R}$$

where v is the velocity at P the mass will fly off the surface at angle θ for which N is zero. *i.e.*,

$$mg\cos\theta = \frac{mv^2}{R}$$

The kinetic energy at P is $\frac{1}{2}mv^2$, which is mgR (1 – cos θ), as shown above. Substituting this in the last expression, we get

$$mg\cos\theta = 2mg\,(1 - \cos\theta)$$

or $$3mg\cos\theta = 2mg$$

or $$\cos\theta = \frac{2}{3}$$

or $$\theta = \cos^{-1}\left(\frac{2}{3}\right).$$

In presence of fiction, the velocity at P will be smaller and hence the mass will fly off the surface at a large angle.

Q.80. *What will be the spring constant if it stretches 10 cm where it has a potential energy of 5600 joule ?*

Solution. The potential energy of a spring stretched through a distance is given by

$$U = \frac{1}{2}Kx^2$$

Hence the spring constant is given by

$$K = \frac{2U}{x^2}$$

Here U = 5600 joule and x = 10 cm = 0.1 m.

$$K = \frac{2 \times 5600}{(0.1)^2} = 1.12 \times 10^6 \text{ nt/m}$$

Q.81. *A spring-gun has a spring constant of 80 Nt/cm. The spring is compressed 12 cm by a ball of mass 15 gm. How much the potential energy of spring ? If the trigger is pulled, what will be the velocity of the ball be ?*

Ans. The spring (elastic) force is conservative and hence the mechanical energy of the system is conserved. Before the trigger is pulled the spring has an elastic potential energy of compression given by

$$U = \frac{1}{2}Kx^2$$

where K is spring-constant and x is the distance of compression.

Here K = 80 nt/cm = 8×10^3 nt/m and x = 12 cm = 0.12 m

$$\therefore \quad U = \frac{1}{2} \times (8 \times 10^3)\,(0.12)^2 = 57.6 \text{ joule}$$

This energy, when the trigger is pulled is converted into the kinetic energy $\frac{1}{2}mv^2$ of the ball, where m is the mass and v the velocity of the ball. Thus

$$\frac{1}{2}mv^2 = 57.6$$

Here m = 15 gm = 0.015 kg

$$\therefore \quad v^2 = \frac{2 \times 57.6}{m} = \frac{2 \times 57.6}{0.015} = 7680$$

$$\therefore \quad v = \sqrt{7680} = 87.6 \text{ m/s}$$

Q.82. *A mass of one kg suspended by a spring of force constant 8×10^4 dyne/cm. Find the distance through which it should be pulled so that on releasing it passes through its equilibrium position with a velocity of 1.0 m/s.* [**Ans.** 11.2 cm]

Q.83. *A block of mass 1 kg is forced against a horizontal spring of force constant K = 100* nt/m. *Which is compressed by x_1 = 0.2 m. When released, the block moves on a level surface a distance x_2 = 1.0 m before coming to rest. Obtain the coefficient of friction μ_k between the block and the surface.*

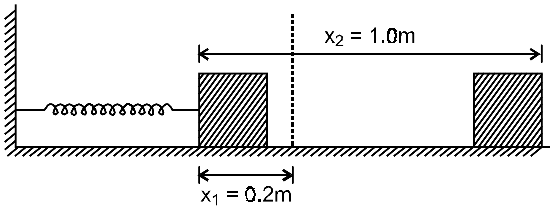


Fig. 46

Solution. When the block is released, it receives the (elastic) potential energy $\frac{1}{2}Kx_1^2$ stored in the compressed spring. This energy is used up in doing work against the friction force $\mu_K N = \mu_K mg$, in moving the body a distance x_2. Thus

$$\frac{1}{2}Kx_1^2 = \mu_K mg \times x_2$$

or $$\frac{1}{2} \times 100 \times (0.2)^2 = \mu_K \times 1 \times 9.8 \times 1.0$$

$\therefore$ $$\mu_K = 0.20$$

Q.84. *A 2kg body moving on a level surface collides and compresses a horizontal spring of force constant k = 2 nt/m through 2 m. Compute the velocity of the body while colliding. (Between block and surface μ_K = 0.25).*

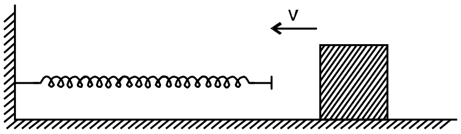


Fig. 47

Solution. The kinetic energy of the block, $\frac{1}{2}mv^2$ supplies the elastic potential energy, $\frac{1}{2}Kx^2$, to the spring and does work against the friction force $\mu_K N = \mu_K mg$, as the block moves the distance x. Thus, by the general law of energy conservation, we have

$$\frac{1}{2}mv^2 = \frac{1}{2}Kx^2 + \mu_K mgx \quad (g = 9.8 \text{ m/s}^2)$$

$$\frac{1}{2} \times 2 \times v^2 = \frac{1}{2} \times 2 \times (2)^2 + 0.25 \times 2 \times 9.8 \times 2$$

$$v^2 = 4 + 9.8 = 13.8$$

$$v = \sqrt{13.8}$$

$$v = 3.7 \text{ meter/sec.}$$

Q.85. *A 10 kg block slides from the top of a 30° inclined smooth plane and compresses a spring placed at the bottom of the plane through 2.0 meter before coming to rest. Calculate* *the distance through which the block has slide before coming to rest and its speed on reaching the bottom. The spring can be compressed 1.0 meter by a force of 100 Newton.*

Ans. When a force F compresses a spring by x, we have

$$F = kx$$

where k is the force constant of the spring.

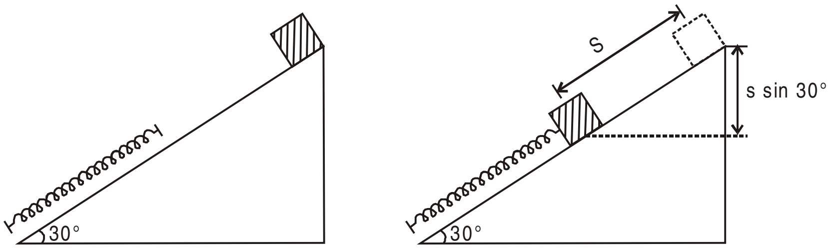


Fig. 48

The given spring can be compressed by 1.0 meter by a force of 100 newton. That is

$$100 = K \times 1$$

$$\therefore \quad K = 100 \text{ Nt/m}$$

Let s the distance through which the block has slide before coming to rest. The vertical distance through which it descends is s sin 30°. Now, at the moment of start, the block has gravitational potential energy, the kinetic energy being zero. At the moment the maximum compression of the spring occurs, again there is no kinetic energy. Hence the loss of gravitational potential energy of the block in sliding down, $mg\ s \sin 30°$, equal to the gain of (elastic) potential energy of the spring, $\frac{1}{2}Kx^2$. That is

$$mg\ s \sin 30° = \frac{1}{2}Kx^2$$

Here $m = 10$ kg, $g = 9.8$ m/s^2, K = 100 Nt/m and $x = 2.0$ m

$$\therefore \quad 10 \times 9.8 \times \frac{s}{2} = \frac{1}{2} \times 100 \times 4$$

$$s = \frac{100 \times 4}{10 \times 9.8} = 4.1 \text{ meter.}$$

This distance of 4.1 meter includes the distance of 2.0 meter travelled while compressing the spring before coming to rest. Thus the distance travelled on reaching the spring (but before compressing) is 2.1 meter. Up to this moment, the loss in potential energy of the block is $mg \times 2.1 \sin 30°$ which appears as kinetic energy of the block $\frac{1}{2}mv^2$. Thus

$$mg \times 2.1 \sin 30° = \frac{1}{2}mv^2$$

$$9.8 \times 2.1 \times \frac{1}{2} = \frac{1}{2}v^2$$

$$v = \sqrt{9.8 \times 2.1}$$

$$v = 4.5 \text{ meter/sec.}$$

Q.86. *A 20 kg block of mass, initially at rest is dropped from a height of 0.40 meter on to a spring whose force constant is 1960 Nt/meter. Find the maximum distance that the spring will be compressed.*

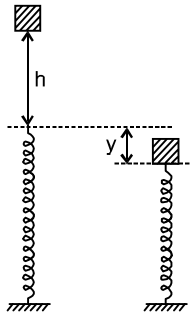


Fig. 49

Ans. At the moment of release, the block has only gravitational potential energy. At the moment of maximum compression of the spring, the block loses some gravitational potential energy which is converted into the elastic potential energy of the spring.

Let m be the mass of the block, and K the force constant of the spring. Let y be the distance through which the spring is compressed. Then the elastic potential energy in the spring is $\frac{1}{2}Ky^2$. The total vertical fall of the block is $(h + y)$, so that the loss in its gravitational potential energy is $mg(h + y)$. By the law of conservation of mechanical energy, we have

$$\frac{1}{2}Ky^2 = mg(h + y) \quad ...(i)$$

or
$$y^2 - \frac{2mg}{K}y - \frac{2mgh}{K} = 0$$

$\therefore$
$$y = \frac{1}{2}\left[\frac{2mg}{K} \pm \sqrt{\left(\frac{2mg}{K}\right)^2 + \frac{8mgh}{K}}\right] \quad ...(ii)$$

The block of mass 20.0 kg and is dropped from a height of 0.40 meter. The force-constant K = 1960 newton/meter. On solving for y, we get

$$y = 0.4 \text{ meter}$$

Q.87. *A 20 kg body is released from rest so as to slide in between vertical rails and compress a vertical spring of force constant K = 1920 Nt/m, placed at a distance h = 1.0 meter from the starting position of the body. The rails offer a friction force of f = 36 Nt opposing the motion of the body. Find (i) the velocity v of the body just before striking with the spring (ii) the distance y through which the spring is compressed, and (iii) the distance h through which the body is rebounded off.*

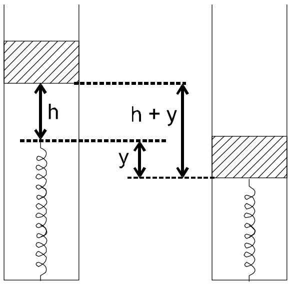


Fig. 50

Ans. (i) The body slides a distance h against the friction force f just before striking the spring. Hence, the loss of gravitational potential energy of the body equals the gain of kinetic energy plus the work done against the friction of f. That is,

$$mgh = \frac{1}{2}mv^2 + fh$$

$$20 \times 9.8 \times 1.0 = \frac{1}{2} \times 20 \times v^2 + 36 \times 1.0$$

$$v^2 = \frac{196 - 36}{10} = 16.0$$

$\therefore$ $$v = 4.0 \text{ m/s}$$

(*ii*) At the moment when maximum compression y of the spring occurs, there is no kinetic energy. Hence, the loss of gravitational potential energy of the body equals the gain of elastic potential energy of the spring plus the work done against the friction force f. Now the total fall of the body is $(h + y)$. Thus

$$mg\,(h + y) = \frac{1}{2}Ky^2 + f\,(h + y)$$

$$20 \times 9.8 \times (1.0 + y) = \frac{1}{2} \times 1920 \times y^2 + 36\,(1.0 + y)$$

$$160\,(1.0 + y) = 960\,y^2 \quad \text{or} \quad 6y^2 - y - 1 = 0$$

or $$y = \frac{1 \pm \sqrt{1 + 24}}{12}$$

or $$y = 0.5 \text{ (– sign is inadmissible).}$$

(*iii*) The compressed spring with elastic energy $\frac{1}{2}Ky^2$ rebounds the body to a height h'. Therefore the loss of elastic energy equals the gain of gravitational energy of the body plus the work done against the friction force f. That is.

$$\frac{1}{2}Ky^2 = mgh' + fh'$$

$$\frac{1}{2} \times 1920 \times (0.5)^2 = 20 \times 9.8 \times h' + 36h'$$

$$240 = 232\,h'$$

$$h' = \frac{240}{232} = 1.03 \text{ m.}$$

Q.88. *A body of mass 4kg slides on a horizontal frictionless table with a speed of 2 m/s. It is brought to rest in compressing a spring in its path. By how much is the spring compressed if the force constant of the spring is 64 kg/m.*

Ans. Kinetic energy of the body

$$= \frac{1}{2}mv^2 = \frac{1}{2} \times 4 \times 2^2 = 8 \text{ Joule.}$$

Let the spring be compressed through a distance x so that the body loses its all kinetic energy and comes to rest. Obviously, the lost kinetic energy will be stored in the form of potential energy $\frac{1}{2}Cx^2$ of the system. Thus,

$$\frac{1}{2}Cx^2 = \frac{1}{2}mv^2 = 8 \text{ Joule}$$

$$x^2 = \frac{16}{C} = \frac{16}{64} = \frac{1}{4}$$

whence, $x = 0.5$ meter.

Q.89. *A body of mass 2 kg is attached to a horizontal spring of force constant 8 N/m and then a constant force of 6 newtons is applied on the body along the length of the spring. Find the speed of the body when it has been displaced through 0.5 met. Now if the force is removed, how much farther the body will move before coming to rest?*

Ans. Work done by constant force 6 Newton in displacing the mass through 0.5 meter, W = 6 × 0.5 = 3 Joule. The potential energy of the system at this displacement

$$U = \frac{1}{2}\ Cx^2 = \frac{1}{2} \times 8 \times (0.5)^2 = 1 \text{ Joule}$$

The work done W on the system increases the potential energy and kinetic energy of the system. If the speed acquired by the body is v m/sec, then

$$W = \frac{1}{2}\ Cx^2 + \frac{1}{2}mv^2$$

$$3 = 1 + \frac{1}{2} \times 2 \times v^2$$

$$v = \sqrt{2} = 1.4 \text{ m/sec.}$$

At the displacement 0.5 meters, the total energy of the system is 3 joule. Now, if the constant force is removed, only 3 joule energy will remain with the system.

Now if the body is further displaced through a distance α against the elastic force of the spring so that it comes to rest, the total energy will become in the form of potential energy *i.e.,*

$$\frac{1}{2}C\ (0.5 + \alpha)^2 = 3 \text{ or } \frac{8}{2}\ (0.5 + \alpha)^2 = 3$$

or $$0.5 + \alpha = \sqrt{3}/2 = 0.866$$

or $$\alpha = 0.366 \text{ meter.}$$

Q.90. *A body of mass m is suspended from a spring. It comes to rest after a downword displacement x_0. If a linear force is acting, then prove that*

(i) the spring (force constant) is mg/x_0.

(ii) the gravitational energy lost is mgx_0.

(iii) the elastic potential energy gained is $\frac{1}{2}mgx_0$.

What happened to the remaining energy ?

Ans. In the equilibrium position, the downward force mg on the spring will be balanced by the restoring force in the spring. So that

$$mg = -\text{Restoring force} = -(-Cx_0) = Cx_0$$

whence $$C = mg/x_0 \qquad ...(i)$$

Loss in gravitational potential energy = mgx_0 ...(*ii*)

$$\text{Elastic potential energy} = \int_0^{x_0} Cx\ dx = \frac{1}{2}Cx_0^2 = \frac{1}{2}mgx_0 \qquad ...(iii)$$

$$\text{Remaining energy} = mgx_0 - \frac{1}{2}mgx_0 = \frac{1}{2}mgx_0 \qquad ...(iv)$$

This energy is first changed to kinetic energy and finally to heat, when initial oscillations stop.

Q.91. *A car carries a framework ABCD shown below, in which a 200 gm mass P is supported between two spring of force constant 5 N/m each. Side AB is kept horizontal and along the length of the car. And the pointer attached to P reads zero when the car is at rest. What will the pointer read when the car has:*

(i) uniform speed 20 m/s on a straight road.

(ii) uniform speed 10 m/s on a circular road of radius 20 m

(iii) uniform acceleration 0.5 m/s² on a straight road.

(iv) uniform acceleration –1.0 m/s² on a straight road.

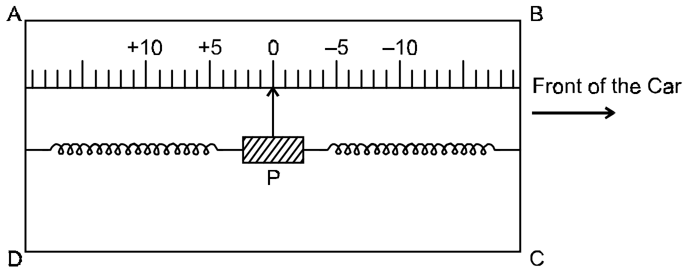


Fig. 51

Ans. (*i*) When the car is moving with constant speed on a straight road, then pointer will read zero.

(*ii*) When the car is moving on a circular path, the force will be perpendicular to the spring and hence again the pointer will read zero.

(*iii*) In the case when car has forward acceleration 0.5 m/s² the mass P will experience a force F = ma = –0.2 × 0.5 = –0.1 newton. As the both spring exerts the force on the mass P, the force at the displacement x is

$$F = -(C_1 + C_2)x \text{ i.e.,}$$

$$0.1 = (5 + 5)x \text{ or } x = \frac{0.1}{10} = 1 \text{ cm} = 10 \text{ mm}$$

Hence, the pointer will read + 10 mm back to the zero position of the scale.

(*iv*) In this case, $F = 0.2 \times 1 = -(5 + 5)x$

or $$x = -\frac{0.2}{10}m = -2\text{cm} = -20 \text{ mm}$$

Hence the pointer will read –20mm forward to the zero of the scale.

Q.92. *A body of mass 1 kg, initially at rest is dropped from a height of 2 meters on to a vertical spring having force constant 490 N/m. Calculate the maximum compression.*

Ans. Loss in gravitational potential energy = Gain in potential energy by the spring

i.e., $$mg(h + x) = \frac{1}{2}Cx^2$$

Here, $m = 1$ kg, $g = 9.8$ m/s^2, $h = 2$m, C = 490 N/m.

$$\therefore \quad 1 \times 9.8\,(2 + x) = \frac{490}{2}x^2$$

or $$100x^2 = 4x + 8$$

or $$25x^2 - x - 2 = 0$$

whence $$x = \frac{1 \pm \sqrt{1 + 200}}{50} = \frac{18.177}{50}$$

$$= 0.3035 \text{ m}$$

(–*ve* value is inadmissible)

SELECTED PROBLEMS

1. Explain clearly the work-energy theorem.

2. A force of 5 Nt acts on a body of 10 kg mass initially at rest. Compute the work done by the force in the third second and also the instantaneous power exerted by the force at the end of zero second.

3. Define conservative force (*a*) show that for a conservative force, $\vec{\nabla} \times \vec{F} = 0$.

(*b*) Central force is conservative.

4. A body of mass 5 kg is released from a position of rest on a frictionless spherical surface. It then moves on a horizontal surface CD whose coefficient of kinetic friction is 0.2. An elastic spring with the force constant K = 900 Nt/m is placed at C. Find the maximum compression of the spring ; (g = 9.8 m/s^2).

5. Prove that the rate of change of K.E. of a body is equal to the power exerted by the force acting on it.

6. An electric field is given by E = $\hat{i}\,(2x + 3y) + \hat{j}\,(5x + 4y)$. Find the scalar point.

7. What is a conservative force ? Show that for a conservative force, the work done around (*a*) closed path is zero.

(*b*) Show that a conservative force can be expressed as negative gradient of potential energy.

(*c*) Prove that curl of a conservative force is zero.

8. (*i*) A light and heavy body have equal momentum, which one has greater kinetic energy ?

(*ii*) A body of mass m is moved to a height h equal to the radius of earth. What is the increase in potential energy ?

(*iii*) Two particles whose masses are in the ratio 1 : 4 have equal momentum. What is ratio of their kinetic energies ?

(*iv*) A 2000 kg motor car was running at a speed of 10 m/s. It was brought to a stop suddenly by breaking. What is the amount of heat produced at the breakers ?

(*v*) A ball is dropped from rest at a height of 12 m, if it loses 25% of its kinetic energy on striking the ground, what is the height to which it bounces ? How do you account for the loss in K.E. ?

9. A ball falls under gravity from a height of 10 m with an initial downward velocity v_0. It collides with the ground and loses 50% of its energy in the collision and then rises back to same height. Find (*i*) initial velocity v_0 (*ii*) the height to which the ball would rise after collision if the initial velocity was directed upward instead of downwards.

10. A bullet of mass m moving with a horizontal velocity 'v' strikes a stationary block of mass M suspended by a string of length L.

5

LINEAR AND ANGULAR MOMENTUM

5.1 CONSERVATION OF LINEAR MOMENTUM

If a force $\vec{F}$ is acting on a particle of mass m, then according to Newton's second law of motion, we have

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v})$$

Where $\vec{p} = m\vec{v}$ is the linear momentum of the particle. Momentum is a vector quantity. If the external force acting on the particle is zero, then

$$\vec{F} = \frac{d\vec{p}}{dt} = 0 \quad \text{or} \quad \vec{p} = m\vec{v} = a \quad \text{constant}$$

Thus, in absence of an external force, the linear momentum of a particle remains constant. The law of conservation of momentum for a system of two particles, which are interacting mutually, in absence of external forces is given by

$$\vec{p}_1 + \vec{p}_2 = \text{constant}$$

or $$m_1\vec{v}_1 + m_2\vec{v}_2 = \text{constant}$$

Now, let us consider a system of n particles whose masses are m_1, m_2 ...., m_n. The system can be a rigid body in which the particles are in fixed positions with respect to one another, or it can be a collection of particles in which there may be all kinds of internal motion.

Suppose that the particles of the system are interacting with each other and are also acted by external forces. If $\vec{p}_1 = m_1\vec{v}_1, \vec{p}_2 = m_2\vec{v}_2, \ldots, \vec{p}_n = m_n\vec{v}_n$ are the momenta of the particles of masses $m_1, m_2, \ldots, m_n$ respectively, then the total momentum $\left(\vec{P}\right)$ of the system is the vector sum of the momenta of individual particles *i.e.*,

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \dots + \vec{p}_n$$

$$\vec{p}_1 = m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \dots + m_n \vec{v}_n$$

Differentiating it with respect to time t, we have

$$\frac{\overrightarrow{dP}}{dt} = \frac{\overrightarrow{dp_1}}{dt} + \frac{\overrightarrow{dp_2}}{dt} + \frac{\overrightarrow{dp_3}}{dt} + \dots + \frac{\overrightarrow{dp_n}}{dt}$$

or
$$\frac{\overrightarrow{dp}}{dt} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n$$

Where $\vec{F}_1, \vec{F}_2, \dots \vec{F}_n$ represent the forces acting on the particles of masses m_1, m_2, m_3 ... m_n respectively.

These forces include external and internal forces both. But according to Newton's third law, the internal forces exist in pairs of equal and opposite forces, they balance each other and so do not contribute any thing to the total force. Hence the right hand side in above equation represents the resultant force $\vec{F}_{ext}$ only due to the external forces acting on all the particles of the system. The internal forces cannot change the total momentum of the system, because being equal and opposite they produce equal and opposite changes in the momentum. Hence, if we want to change the total momentum of the system of the particles, it is necessary to apply external force on that system. Then the sum of external forces is

$$\vec{F}_{ext} = \frac{\overrightarrow{dp}}{dt} = \frac{d}{dt}(\vec{p}_1 + \vec{p}_2 + \dots + \vec{p}_n)$$

If the resultant external force is zero, then

$$\frac{\overrightarrow{dp}}{dt} = 0 \quad \text{or} \quad \vec{P} = a \text{ constant}$$

Thus, if the resultant external force acting on a system of particles is zero, the total linear momentum of the system remains constant. This simple but quite general result is called the law of conservation of linear momentum for a system of particles. It is to be noted that the momenta of individual particles may change, but their sum *i.e.*, the total linear momentum remains unaltered in the absence of external forces.

The law of conservation of momentum is fundamental and exact law of nature. No violation of it has ever been found and it has been thoroughly checked by all kinds of experiments.

5.2 CENTRE OF MASS

Every physical system has associated with it a certain point whose motion characterizes that of the system as a whole. When the system moves under an external force then this point moves in the same way as a single particle would move under the same external force. This point is called the "center of mass" of the system. The motion of the system can be described in terms of the motion of its center of mass.

Let us consider a system of n particles of masses $m_1, m_2, m_3, \ldots m_n$ with position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n$ relative to a fixed origin O. The position vector $\vec{r}_{cm}$ of the center of mass of this system is defined by

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i \qquad \ldots(i)$$

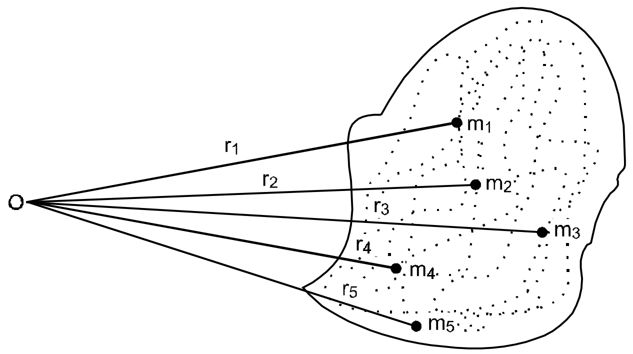


Fig. 1

where M $\left(= \sum_{i=1}^{n} m_i\right)$ is the total mass of the system. This is the expression for the position vector of the center of mass of a system of particles.

Qualitatively, $\vec{r}_{cm}$ represents a geometric point located at the "average" position of the particle weighted in proportion to their masses. From the above definition, we draw two conclusions:

(*i*) If all the n particles have the same mass m, then

$$\vec{r}_{cm} = \frac{m}{M}\sum_{i=1}^{n}\vec{r}_i = \frac{1}{n}\sum_{i=1}^{n}\vec{r}_i$$

That is, the center of mass coincides with the geometric center of the system.

(*ii*) If the origin O is at the center of mass $(\vec{r}_{cm} = 0)$, then

$$\sum_{i=1}^{n} m_i\vec{r}_i = 0$$

That is, the sum of the moments of the masses of the system about the center of mass is zero.

5.3 CARTESIAN COMPONENTS OF THE CENTRE OF MASS

The position vectors $\vec{r}_{cm}$ and $\vec{r}_i$ are related to their Cartesian components by

$$\vec{r}_{cm} = x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$$

and

$$\vec{r}_i = \hat{i}x_i + \hat{j}y_i + \hat{k}z_i$$

Making these substitutions in Equation (i), the Cartesian components of $\vec{r}_{cm}$ are given by

$$x_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \quad y_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i, \quad z_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i,$$

where x_i, y_i, z_i are the Cartesian co-ordinates of the ith particle

5.4 CENTRE OF MASS OF A SOLID BODY

Let us now consider a body with continuous distribution of mass. If the surface may be sub divided into n small elements, the ith element being of mass Δm_i and located approximately at the point (x_i, y_i, z_i). The Co-ordinates of the center of mass are then approximately given by

$$x_{cm} = \frac{\sum_{i=1}^{n} \Delta m_i x_i}{\sum_{i=1}^{n} \Delta m_i}, \quad y_{cm} = \frac{\sum_{i=1}^{n} \Delta m_i y_i}{\sum_{i=1}^{n} \Delta m_i}, \quad z_{cm} = \frac{\sum_{i=1}^{n} \Delta m_i z_i}{\sum_{i=1}^{n} \Delta m_i}$$

An $n \to \infty$, the co-ordinates are defined precisely, by

$$x_{cm} = \frac{\int x\, dm}{\int dm}$$

But $\int dm = M$ (mass of the body).

$$\therefore \quad x_{cm} = \frac{1}{M}\int x\, dm$$

$$y_{cm} = \frac{1}{M}\int y\, dm$$

and

$$z_{cm} = \frac{1}{M}\int z\, dm$$

The vector expression corresponding to these three scalar expressions is

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\, dm$$

The center of mass of a "homogeneous body" (having a uniform distribution of mass) must coincide with the geometric center of the body. Thus if the homogeneous body has a point, a line, or a plane of symmetry, its center of mass must lie at its point, line or plane of symmetry. This fact can be easily understood. From symmetry, the first moment of mass $\int \vec{r}\, dm$ of the "homogeneous" body must be zero with respect to its geometric center:

$$\int \vec{r}\, dm = 0$$

$$\vec{r}_{cm} = 0$$

i.e., the center of mass coincides with the geometric center. From this we see that the center of mass of a body does not necessarily lie within the body. For example, in a homogeneous, ring the center of mass would be at its geometric center which does not lie within the material of the ring.

5.5 POSITION VECTOR OF THE CENTRE OF MASS

Let us consider a system of n particles of masses $m_1, m_2, m_3, \ldots m_n$ with position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n$ relative to a fixed origin. The position vector $\vec{r}_{cm}$ of the center of mass of this system is defined by

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i}$$

$$= \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$

where M $\left(= \sum_{i=1}^{n} m_i\right)$ is the total mass of the system.

5.6 VELOCITY OF THE CENTRE OF MASS

Let us write the expression for the position vector of a system of particles:

$$\vec{r}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n)$$

Differentiating it with respect to time, we obtain

$$\frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left(m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots\ldots + m_n\frac{d\vec{r}_n}{dt}\right)$$

or
$$\vec{v}_{cm} = \frac{1}{M}(m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n)$$

$$\vec{v}_{cm} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{v}_i$$

where $\vec{v}_{cm}$ is the velocity of the center of mass and $\vec{v}_i$ the velocity of the ith particle. This is the required expression.

5.7 CENTER OF MASS FRAME OF REFERENCE

It is a reference frame attached with the center of mass of a system of particles (or a body). It is also known as C-frame of reference. In this frame the velocity of the center of mass is zero by definition ($\vec{v}_{cm} = 0$).

5.8 MOTION OF THE CENTER OF MASS OF A SYSTEM OF PARTICLES SUBJECT TO EXTERNAL FORCES

Let us consider a system of n particles of masses $m_1, m_2, m_3, \ldots m_n$ with position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n$ with respect to a fixed origin, and subjected to external forces $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_n$ exerted by the surroundings.

The position vector of the center of mass of the system is defined by

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots + m_n}$$

$$\vec{r}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n)$$

where M (= $m_1 + m_2 + m_3 + \ldots + m_n$) is the total mass of the system. Differentiating it with respect to time, we obtain

$$\frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left(m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots + m_n\frac{d\vec{r}_n}{dt}\right)$$

or
$$\vec{v}_{cm} = \frac{1}{M}(m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n)$$

Where $\vec{v}_{cm}$ is the velocity of the center of mass and $\vec{v}_i$ is the velocity of the particle m_1, and so on.

Differentiating again, we get

$$\frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\left(m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + \ldots + m_n\frac{d\vec{v}_n}{dt}\right)$$

or
$$M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_n\vec{a}_n$$

where $\vec{a}_{cm}$ is the acceleration of the center of mass, $\vec{a}_1$ is the acceleration of the particle m_1 and so on.

Now, from Newton's second law, the external force $\vec{F}_1$ acting on the first particle is given by $\vec{F}_1 = m_1\vec{a}_1$ and so on. Thus,

$$M\vec{a}_{cm} = \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n$$

or
$$M\vec{a}_{cm} = \vec{F}_{ext}$$

Where $\vec{F}_{ext}$ is the sum of the external force acting on all the particles. Thus the product of the total mass of a system of particles and the acceleration of its center of mass is equal to the sum of the 'external' force acting on the particles. This means that the center of mass of a system of particles moves as if all the mass of the system were concentrated at it and all the 'external' force were applied at it. This result holds whether the system is a rigid body with particles in fixed position or a system of particles with internal motions.

5.9 LINEAR MOMENTUM IN CENTER OF MASS FRAME OF REFERENCE

In the reference frame attached with the center of mass (C-frame) the velocity of the center of mass, $\vec{v}_{cm}$ is zero by definition. Therefore, the total linear momentum of a system of particles is also zero:

$$\vec{P} = M\vec{v}_{cm} = 0$$

Whatever may be the velocities of the constituent particles. Thus the center-of mass coordinate system is also referred to as the center of-momentum coordinate system, to stress the fact that in such a system the total momentum is zero.

The C-frame is important because many experiments that we perform in our laboratory or L-frame of reference can be more simply analyzed in the C-frame of reference (which moves with a velocity $\vec{v}_{cm}$ relative to the L-frame).

5.10 SYSTEM OF VARIABLE MASS

There are certain systems whose mass does not remain constant during their motion, but varies with time. A rocket is propelled by ejecting burnt fuel which causes the total mass of the rocket to decrease as the rocket accelerates. A falling rain drop collapses with smaller drops which increase its mass. Let us see what form Newton's law takes for such system of 'variable mass'.

Let a system of mass M be moving at a velocity $\vec{v}$ at any instant t in a particular reference frame. Let at a later instant $t + \delta t$, a part δM separated from the system be moving with a velocity $\vec{u}$ and the remaining system M-δM with new velocity $\vec{v} + \delta\vec{v}$. If we still treat both parts as forming one and the same system, then for the finite time interval δt, we can write:

$$\vec{F}_{ext} = \frac{\vec{P}_f - \vec{P}_i}{\delta t} = \frac{[(M - \delta M)(\vec{v} + \delta\vec{v}) - \delta M.\vec{u}] - [M\vec{v}]}{\delta t}$$

$$= M\frac{\delta\vec{v}}{\delta t} - \vec{v}\frac{\delta M}{\delta t} - \delta\vec{v}\frac{\delta M}{\delta t} + \vec{u}\frac{\delta M}{\delta t}$$

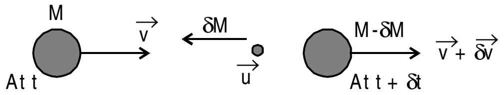


Fig. 2

Now if $\delta t \to 0$ (*i.e.,* the mass of the system is continuously decreasing), then

$$\frac{\delta\vec{v}}{\delta t} = \frac{d\vec{v}}{dt};\ \frac{\delta M}{\delta t} = \frac{-dM}{dt};\ \delta\vec{v} = 0$$

so that

$$\vec{F}_{ext} = M\frac{d\vec{v}}{dt} + \vec{v}\frac{dM}{dt} - \vec{u}\frac{dM}{dt} = \frac{d}{dt}(M\vec{v}) - \vec{u}\frac{dM}{dt}$$

This expresses Newton's second law as applied to a body of variable mass. We note that $\frac{d}{dt}(M\vec{v})$ is not equal to the external force acting on the system (unless the ejected mass comes out with zero speed).

We can write the above equation as

$$M\frac{d\vec{v}}{dt} = \vec{F}_{ext} + (\vec{u} - \vec{v})\frac{dM}{dt}$$

or

$$M\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \vec{v}_{rel}\frac{dM}{dt}$$

Where $\vec{v}_{rel}$ is the velocity of the ejected mass relative to the main body. The last term in the above equation is the rate of change of momentum of the system due to the mass leaving it. It can be taken as the reaction force exerted on the system by the leaving mass. Thus we can write

$$M\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \vec{F}_{reaction}$$

5.11 MOTION OF A ROCKET

The rocket is the most interesting example of a system of variable mass. Its motion can be explained on the basis of Newton's third law of motion and the momentum principle. It consists of a combustion chamber in which liquid or solid fuel is burnt. The heat of combustion raises the pressure very high inside the chamber. Therefore, hot gases (produced by combustion) are expelled from the tail of the rocket in the form of a jet with a very high exhaust velocity. Consequently, the rocket rushes in the forward direction.

The rocket exerts an action force on the gas-jet in the backward direction, while the gas-jet exerts a reaction force on the rocket in the forward direction. These are the internal forces in the (rocket + gas) system. In the absence of external forces, the total momentum of the system (rocket + gas) is constant. The gas-jet acquires momentum in the back ward direction and the rocket acquires an equal momentum in the forward direction.

Let us now derive an expression for the final velocity of the rocket. Let M (a variable) be the mass of the (rocket + unburnt fuel) at an instant t and $\vec{v}$ its velocity in a fixed (laboratory) frame of reference. Suppose in a time-interval dt an amount of Mass dM is ejected from the rocket in the form of gas-jet.

If $\vec{u}$ be the velocity of the gas-jet in the laboratory reference frame, then its velocity relative to the rocket $\vec{v}_{rel}$, would be given by

$$\vec{v}_{rel} = \vec{u} - \vec{v}$$

$\vec{v}_{rel}$ is known as "exhaust velocity". Now, according to the Newton's second law as applied to a system of variable mass, we have

$$M\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \vec{v}_{rel}\frac{dM}{dt}$$

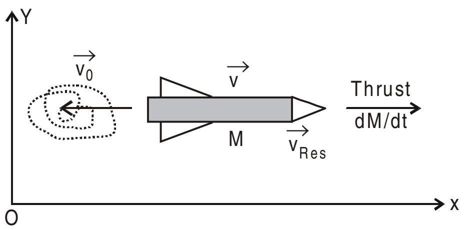


Fig. 3

Where $\vec{F}_{ext}$ is the 'external' force acting on the system and $\vec{v}_{rel}\frac{dM}{dt}$ in the reaction force exerted on the system by the leaving mass. In the case of a rocket, this term is called the 'thrust' exerted on the rocket by the ejecting gas-jet. The external force $\vec{F}_{ext}$ is the force of gravity on the rocket and the air resistance.

To solve the above equation, let us assume that the exhaust velocity $\vec{v}_{rel}$ is constant. Also, neglecting air resistance and the variation of the gravity with altitude, we may write $\vec{F}_{ext} = M\vec{g}$, so that the last equation becomes:

$$M\frac{d\vec{v}}{dt} = M\vec{g} + \vec{v}_{rel}\frac{dM}{dt}$$

Now, suppose the motion of the rocket is vertical. Then $\vec{v}$ is directed upward and $\vec{v}_{rel}$ and $\vec{g}$ downward. The last equation may now be written as

$$M\frac{d\vec{v}}{dt} = -Mg - \vec{v}_{rel}\frac{dM}{dt}$$

or
$$dv = -gdt - v_{rel}\frac{dM}{M}$$

Integrating from the beginning of the motion ($t = 0$), when the velocity is v_0 and the mass of the rocket + fuel is M_0, upto an arbitrary time t, we have

$$\int_{v_0}^{v} dv = -g\int_0^t dt - v_{rel}\int_{M_0}^{M}\frac{dM}{M}$$

or
$$v - v_0 = -gt - v_{rel}\log_e\frac{M}{M_0}$$

or
$$v - v_0 = -gt + v_{rel}\log_e\frac{M_0}{M}$$

or
$$v = v_0 + v_{rel}\log_e\frac{M_0}{M} - gt$$

If t is the time required for burning all the fuel, then M is the final mass and v is the maximum velocity attained by the rocket.

If the force of gravity is ignored, then the above expression reduces to

$$v = v_0 + v_{rel}\log_e\frac{M_0}{M}$$

Further, if the initial velocity v_0 of the rocket is zero, then

$$v = v_{rel}\log_e\frac{M_0}{M}$$

Limitation of One-stage Rocket

In the absence of gravity and air resistance, the ultimate velocity of the rocket at burn out (when all the fuel has been used up) is $v_{rel}\log_e\frac{M_0}{M}$. Thus it can be increased by increasing the exhaust velocity v_{rel} and the ratio $\left(\frac{M_0}{M}\right)$. However, the exhaust velocity cannot be more than 2.5 km/sec with the conventional chemical fuels. Further, the fact that the rocket shell must be strong enough to hold the fuel sets an upper limit on the ratio $\left(\frac{M_0}{M}\right)$ also. In actual practice, this ratio is not greater than 4. Therefore, the maximum velocity of the rocket at burn out is

$$v = 2.5 \log_e 4 = 2.5 \times 1.39 = 3.5 \text{ km/sec}$$

This velocity is much less than the escape velocity (11 km/sec) or the orbital velocity near the surface of the earth (8 km/sec). Thus a single-stage rocket is incapable to put space satellites in orbits or escape the earth's gravitational field.

5.12 MULTI STAGE ROCKET

To attain higher velocities, the rocket is designed in stages. For example, a two-stage rocket means that one rocket is placed on the top of another rocket. When the fuel of the first-stage (lower) is exhausted, its rocket casing is detached and drops off. The velocity attained so far becomes the initial velocity of the second stage which is now ignited . The removal of the surplus mass contained in the first stage considerably helps in attaining still higher velocity.

Suppose that the initial mass of each of the first-and second stage rocket (plus fuel) is $M_0/2$, and the mass of each rocket casing is $M_0/2$. The total initial mass is thus M_0. The velocity v' attained when the first stage is detached is given by

$$v' = v_{\text{rel}} \log_e \frac{M}{M_0/2} = v_{\text{rel}} \log_e 2$$

This is now the initial velocity for the second stage. The final velocity v attained, when the fuel of the second stage is exhausted, is given by

$$v = v' + v_{\text{rel}} \log_e \frac{M_0/2}{M/2}$$

$$v = v_{\text{rel}} \log_e 2 + v_{\text{rel}} \log_e \frac{M_0}{M}$$

Taking $v_{\text{rel}} = 2.5$ km/sec and $M_0/M = 4$, then we have

$$\begin{aligned} v &= 2.5 \text{ km/sec } (\log_e 2 + \log_e 4) \\ &= 2.5 \text{ km/sec} \times \log_e 8 \\ &= 2.5 \times 2.08 \text{ km/sec} \\ &= 5.2 \text{ km/sec} \end{aligned}$$

Thus the final velocity attained by a two-stage rocket is greater than that attained by a single-stage rocket of the same weight and fuel supply. The velocity can be further increased by adding more stages.

The rocket of equal stages is not generally the optimum construction. In fact the first stage should be made much larger than the second in order to obtain a high final speed.

NUMERICALS: SYSTEMS OF VARIABLE MASS: ROCKET

Q. 1. *Solve the last problem is M = 200 kg, m = 10 gm, n = 10/sec and* v_{rel} *= 500 m/s.*

Solution. The magnitude of the acceleration of the trolly at the instant the mass is M, is given by

$$a = v_{rel} \frac{mn}{M}$$

$$= \frac{500 \times 10^{-2} \times 10}{200} = 0.25 \text{ m/s}^2$$

The magnitude of the average thrust on the system is given by

$$F_{reaction} = v_{rel}\, mn$$
$$= 500 \times 10^{-2} \times 10$$
$$= 50 \text{ nt.}$$

Q. 2 (*a*). *A man of mass m is standing on a trolly of mass M which is moving with a velocity* $\vec{v}$ *on frictionless horizontal rails. If the man starts running opposite to the direction of motion of the trolly and his velocity relative to the trolly is* $\vec{v}_{rel}$ *just before he jumps off, find the change in velocity of the trolly.*

(*b*) *Let us now assume that there are n men, each of mass, m, on the trolly. Should they all run and jumps off together or should they do so one by one in order to give a greater velocity to the trolly.*

Solution. (*a*) The initial mass of the (trolly + man) system is (M + m) which is moving with a velocity $\vec{v}$ in a fixed frame of reference. As the man jumps, he acquires a backward momentum and the trolly acquires forward momentum. The Newton's second law, in the absence of external forces, gives

$$(M + m)\frac{d\vec{V}}{dt} = \vec{V}_{rel} \frac{d\,(M + m)\,t}{dt}$$

$$(M + m)\, d\vec{V} = \vec{V}_{rel}\, d\,(M + m) \quad ...(i)$$

$$(M + m)\, \Delta\vec{V} = \vec{V}_{rel}\, \Delta\,(M + m)$$

Here $\Delta(M + m) = -m$ (the man of mass m jumps off). Therefore

$$(M + m)\, \Delta\vec{V} = -\vec{V}_{rel}\, m$$

∴ Change in velocity

$$\Delta\vec{V} = -\vec{V}_{rel} \frac{m}{M + m}$$

This equation shows that the speed of the trolly will increase by $mv_{rel}/(M + m)$ in its initial direction of motion (*i.e.,* opposite to $\vec{v}_{rel}$)

(*b*) Let us now suppose that n men are standing on that trolly. If they all jumps off together, the change in velocity would be

$$\Delta\vec{V} = -\vec{V}_{rel} \frac{mn}{M + mn} \quad ...(ii)$$

But if they jump one by one, the mass of system will go on changing till the last man jumps. In this case we can write eqn. (*i*) as

$$d\vec{V} = \vec{V}_{rel} \frac{d\,(mn + M)}{(mn + M)}$$

$$\Delta\vec{V} = \vec{V}_{rel} \int_{M+mn}^{M} \frac{d\,(M + mn)}{(M + mn)} = \vec{V}_{rel} \left[\log_e (M + mn)\right]_{M+mn}^{M}$$

$$= \vec{V}_{rel} \log_e \frac{M}{M + mn} = -\vec{V}_{rel} \log_e \frac{M + mn}{M}$$

$$= -\vec{V}_{rel} \log_e \left(1 + \frac{mn}{M}\right)$$

$$= -\vec{V}_{rel} \frac{mn}{M} \qquad ...(iii)$$

A comparison of eqns. (*ii*) and (*iii*) shows that the men would give a greater velocity to the trolly by running and jumping off one by one.

Q. 3. *Fine particles of sand are being dropped continuously from a fixed container on to a moving belt. Find out the force necessary to have the belt moving at a constant speed and show that the power supplied by this force is twice the rate of increase of the K.E. of the system.*

Solution. Suppose the belt is moving with a constant velocity $\vec{V}$ in a certain reference frame. The container is fixed in this reference frame. As sand particles are falling on the belt, the mass of the system (belt + material on it) is continuously increasing. Let M be the mass of the system at any instant, dM/dt the rate at which the material is falling on the belt. Then the Newton's second law as applied to a system of varying mass, gives

$$M\frac{d\vec{V}}{dt} = \vec{F}_{ext} + \vec{V}_{rel}\frac{dM}{dt} \qquad ...(i)$$

Where $\vec{F}_{ext}$ is the external force acting on the system and $\vec{V}_{rel}$ is the relative horizontal velocity of the falling mass relative to the belt.

Here $$\frac{d\vec{V}}{dt} = 0 \quad (\text{as } \vec{V} \text{ is constant})$$

$$\vec{V}_{rel} = -\vec{V}$$

and $\frac{dM}{dt}$ is positive (the mass of the system is increasing with time).

Therefore the equation (*i*) becomes

$$0 = \vec{F}_{ext} - \vec{V}\frac{dM}{dt}$$

$$\vec{F}_{ext} = \vec{V}\frac{dM}{dt} \qquad ...(ii)$$

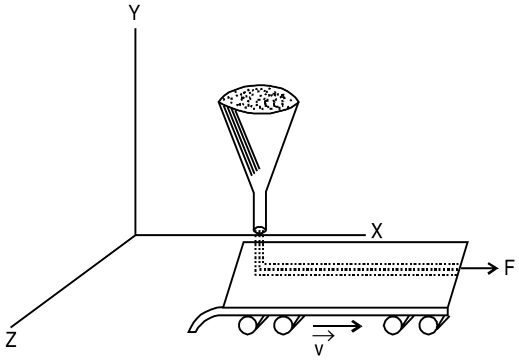


Fig. 4

This is the external force acting on the system under which is moving at a constant rate.

Alternative Method. We can also determine the force by applying momentum principle. Let M′ be the mass of the belt and M that of the material on the belt at any instant. The momentum of the system is

$$\vec{P} = (M' + M)\,\vec{V}$$

Now the rate of change of momentum gives the force. Thus

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(M' + M)\,\vec{V}$$

$$= \vec{V}\frac{dM}{dt}, \text{ (as } \vec{V} \text{ and } M' \text{ are constants)}$$

where $\frac{dM}{dt}$ is the rate at which the material is falling on the belt *i.e.,* the rate of increase of mass of the system.

Now, the power P supplied by the force $\vec{F}_{ext}$ is the rate of doing work. In vector rotation, it is given by the dot product of the force and velocity. That is

$$\vec{P} = \vec{F}_{ext}\,.\,\vec{V}$$

$$= \vec{V}\frac{dM}{dt}\,.\,\vec{V} = V^2\frac{dM}{dt} \qquad \left[\because\ \vec{V}.\vec{V} = V^2\right]$$

$$= \frac{d}{dt}MV^2 \ \ (\because\ V \text{ is constant})$$

$$= 2\frac{d}{dt}\frac{1}{2}(MV^2) = 2\frac{dK}{dt},$$

where $K\left(=\frac{1}{2}MV^2\right)$ is the K.E. Thus the power supplied by the external force acting on the system is twice the rate of increase of K.E. of 'whole' system. This is an example in which the mechanical energy is not conserved.

Q. 4. *A wagon filled with sand has a hole so that sand leaks through the bottom at a constant rate* $-\frac{dm}{dt} = \lambda$. *A force* $\vec{F}$ *acts on the wagon in the direction of its motion. If its instantaneous velocity is* $\vec{V}$, *write the equation of motion.*

Solution. Let m be the instantaneous mass and $\vec{V}$ the instantaneous velocity of the system (wagon + sand). The instantaneous momentum is

$$\vec{P} = m\vec{V}$$

By Newton's law, the rate of change of momentum is the force $\vec{F}$ acting on the system. That is

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(m\vec{V}) = m\frac{d\vec{V}}{dt} + \vec{V}\frac{dm}{dt}$$

The rate of loss of mass is $-\frac{dm}{dt} = \lambda$, so that

$$\vec{F} = m\frac{d\vec{V}}{dt} - \vec{V}\lambda$$

If m_0 be the initial mass, then $m = m_0 - \lambda t$, so that

$$\vec{F} = (m_0 - \lambda t)\frac{d\vec{V}}{dt} - \vec{V}\lambda.$$

This is the required equation.

Q. 5. *A rocket having initial mass of 240 kg ejects fuel at the rate of 6 kg/s with a velocity of 2 km/s vertically downward relative to itself. Calculate its velocity 25 seconds after start, taking initial velocity to be zero and neglecting gravity.*

Solution. Neglecting gravity, the velocity of a rocket at any time t is given by

$$v = v_0 + v_{rel}\log_e\frac{M_0}{M}.$$

where M_0 is the initial (at $t = 0$) mass of the rocket plus fuel and M is the mass remaining at time t. If the initial velocity v_0 is zero, then

$$v = v_{rel}\log_e\frac{M_0}{M}$$

Here v_{rel} = 2 km/s, M_0 = 240 kg. As the fuel is ejected at the rate of 6 kg/sec, the mass consumed in 25 sec is 150 kg. Therefore, the mass remaining after 25 sec is M = 240 – 150 = 90 kg. Thus

$$\begin{aligned} v &= 2 \log_e \frac{240}{90} \\ &= 2 \times 2.3 \log_{10} (2.67) \\ &= 2 \times 2.3 \times 0.4265 \\ &= 1.96 \text{ km/sec.} \end{aligned}$$

Q. 6. *An empty rocket weighs 5000 kg and contains 40,000 kg of fuel. If the exhaust velocity of the fuel is 2.0 km/s, find the maximum velocity gained by the rocket. ($\log_e 10 = 2.3$, $\log_{10} 3 = .4771$).*

Solution. Ignoring gravity effect, the velocity of a rocket at any time t is given by

$$v = v_0 + v_{rel} \log_e \frac{M_0}{M}$$

When M_0 is the initial (at $t = 0$) mass of the rocket plus fuel and M is the mass remaining at time t. The velocity v attains maximum value when all the fuel is burnt. M is then the mass of the empty rocket.

Here, initial velocity $v_0 = 0$, M_0 = 5000 + 40,000 = 45,000 kg, M = 5000 kg and V_{rel} = 2 km/s.

$$\begin{aligned} \therefore \quad v_{max} &= 2 \times \log_e \frac{45000}{5000} \\ &= 2 \times \log_e (3)^2 \\ &= 2 \times 2 \log_e 3 = 2 \times 2 \times 2.3 \times 0.4771 \\ &= 4.4 \text{ km/sec.} \end{aligned}$$

Q. 7. *From the nozzle of a rocket 100 kg of gases are exhausted per second with a velocity of 1000 m/s. What force (thrust) does the gas exert on the rocket.*

Solution. The thrust $(\vec{F}_{reaction})$ exerted by the escaping gas on the rocket is given by.

$$\vec{F}_{reaction} = \vec{V}_{rel} \frac{dM}{dt}$$

Here v_{rel} = 1000 m/s and $\frac{dM}{dt} = 100 \text{ kg/sec}$

$$\begin{aligned} \vec{F}_{reaction} &= 1000 \times 100 \\ &= 10^5 \text{ nt.} \end{aligned}$$

Q. 8. *The first and second stages of a two stage rocket have weights 100 kg and (10 kg and carry 800 kg and 90 kg of fuel supply. The velocity of ejected gases relative to the rocket is 1.5 km/s. Find the final velocity attained by the rocket ($\log_e 10 = 2.3$).*

Solution. Ignoring gravity, the velocity of a rocket at any time t is given by

$$v = v_0 + v_{rel} \log_e \frac{M_0}{M}$$

where v_0 is the initial velocity, M_0 the initial mass of the rocket plus fuel, and M the mass of the rocket plus unburnt fuel at time t.

For the operation of the first stage. We have; $v_0 = 0$, $M_0 = 100 + 10 + 800 + 90 = 1000$ kg and $M = 100 + 10 + 90 = 200$ kg.

$$\therefore \quad v = 0 + 1.5 \log_e 5$$

This is the initial velocity for the second stage. Now, the first stage rocket of mass 100 kg drops off. Thus for the second stage, we have

$$v_0 = 1.5 \log_e 5 \text{ km/s}, M_o = 10 + 90 = 100 \text{ kg and } M = 10 \text{ kg}$$

$$\begin{aligned} v &= 1.5 \log_e 5 + 1.5 \log_e 10 \\ &= 1.5 \log_e 50 \\ &= 1.5 \times 2.3 \times \log_{10} 50 \\ &= 1.5 \times 2.3 \times 1.6990 = 5.86 \text{ km/s.} \end{aligned}$$

If we would have a single stage rocket of the same total mass 110 kg carrying same amount of fuel of 890 kg, then for its operation we would have

$$v_0 = 0, M_o = 110 + 890 = 1000 \text{ kg and } M = 110 \text{ kg}$$

$$\begin{aligned} \therefore \quad v &= 0 + 1.5 \log_e \frac{1000}{110} = 1.5 \log_e 9.1 \\ &= 1.5 \times 2.3 \times \log_{10} 9.1 \\ &= 1.5 \times 2.3 \times 0.959 = 3.31 \text{ km/sec.} \end{aligned}$$

Thus the final velocity attained by a multistage rocket is much greater than that attained by a single stage rocket of the same total mass and fuel supply.

Q. 9. *A 6000 kg rocket is set for vertical firing. If the gas exhaust speed is 1000 m/s, how much gas must be ejected each second to supply the thrust needed (a) to overcome the weight of the rocket, (b) to give the rocket an initial upward acceleration of 20 m/s²*.

Solution. (*a*) The equation for the vertical motion of the rocket is

$$M\frac{dv}{dt} = -Mg - v_{rel}\frac{dM}{dt} \quad ...(i)$$

where M is the mass of the rocket + fuel at any instant t. The term $M\frac{dv}{dt}$ represents the instantaneous net upward force acting on the rocket, and $v_{rel}\frac{dM}{dt}$ is the thrust. In order to just overcome the weight of the rocket, the thrust need not give any net upward force *i.e.*, $M\frac{dv}{dt} = 0.$ Then eqn. (*i*) becomes

$$-Mg - v_{rel}\frac{dM}{dt} = 0$$

$$\frac{dM}{dt} = \frac{-Mg}{v_{rel}} = -\frac{6000 \times 9.8}{1000} = -58.8 \text{ kg/sec}.$$

Hence the gas should be ejected at the rate of 58.8 kg/sec.

(*b*) If the rocket is to be given an initial upward acceleration of 20 m/s^2, then plutting $\frac{dv}{dt} = 20$ in eqn. (*i*) we get

$$20M = -Mg - v_{rel}\frac{dM}{dt}$$

$$v_{rel}\frac{dM}{dt} = -Mg - 20M$$

$$\frac{dM}{dt} = \frac{-M}{v_{rel}}(g + 20) = \frac{-6000}{1000}(9.8 + 20)$$

$$= -178.8 \text{ kg/sec.}$$

Q. 10. *A 8000 kg rocket is set for vertical firing. If the exhaust speed is 800 m/s, how much gas must be ejected/second to supply the thrust needed (i) to overcome the weight of the rocket, (ii) to give the rocket an initial upward acceleration of 3g.*

[**Ans.** (*i*) –98 kg/s, (*ii*) –392 kg/s]

Q. 11. *A rocket of mass 20 kg has 100 kg of fuel. The exhaust velocity of the fuel is 1.6 km/s. Calculate the minimum rate of consumption of fuel so that the rocket may rise from the ground. Also calculate the final vertical velocity gained by the rocket when the rate of consumption of the fuel is (i) 2.0 kg/s, (ii) 20 kg/s.*

Solution. With the minimum rate of fuel consumption, the thrust, $v_{rel}\frac{dM}{dt}$, supplied to the rocket will just overcome its initial weight M_0g and raise it from the ground. Thus

$$v_{rel}\frac{dM}{dt} = M_0 g$$

$$\frac{dM}{dt} = \frac{M_0 g}{v_{rel}} = \frac{200 \times 9.8}{1.6 \times 10^3} = 1.225 \text{ kg/sec.}$$

Now, the velocity attained by the rocket at any time t is

$$v = v_0 + v_{rel}\log_e\frac{M_0}{M} - gt$$

where M is the mass of the rocket plus unburnt fuel at time t. If T is the time in which the entire fuel has burnt, then v is the final velocity attained by the rocket. Thus, if the initial velocity $v_0 = 0$, we have

$$v_{max} = v_{rel}\log_e\frac{M_0}{M} - gT$$

(*i*) $v_{rel} = 1.6 \text{ km/s} = 1.6 \times 10^3 \text{ m/s}$, $\log_e\frac{M_0}{M} = \log_e\frac{200}{20} = \log_e 10 = 2.3$ and $T = \frac{180}{2} = 90$ sec

$$v_{max} = (1.6 \times 10^3 \times 2.3) - (9.8 \times 90) = 3680 - 882 = 2798 \text{ m/sec}$$

$$\simeq 2.8 \text{ km/s}$$

(*ii*) $T = \frac{180}{20} = 9$ sec

$$v_{max} = 3680 - 88.2 = 3591.8 \text{ m/sec}$$

$$\simeq 3.6 \text{ km/sec.}$$

Q. 12. *A 10^3 kg rocket is set vertically on its launching pad. The propellant is expelled at the rate of 2 kg/s. Find the minimum velocity of the exhaust gases so that the rocket just begins to rise. Also find the rocket's velocity 10 sec after ignition, assuming the minimum exhaust velocity.*

Solution. With the minimum velocity of exhaust gases, the thrust, $v_{rel}\,\dfrac{dM}{dt}$, supplied to the rocket will just overcome its initial weight $M_0 g$ and raise it from the ground. Thus

$$v_{rel}\,\frac{dM}{dt} = M_0 g$$

$$v_{rel} = \frac{M_0 g}{dM/dt} = \frac{10^3 \times 9.8}{2}$$

$$= 4.9 \times 10^3 \text{ m/s}$$

Now, the velocity attained by a rocket at any time t is given by

$$v = v_0 + v_{rel}\log_e \frac{M_o}{M} - gt,$$

where M is the mass of the rocket plus unburnt fuel at time t. If the initial velocity v_o is zero, then

$$v = v_{rel}\log_e \frac{M_0}{M} - gt$$

Here $v_{rel} = 4.9 \times 10^3$ m/s, $M_0 = 10^3$ kg, and $t = 10$ sec. As the fuel is expelled at the rate of 2 kg/sec, the mass consumed in 10 sec is 20 kg. Therefore the mass remaining is M = 1000 – 20 = 980 kg. Thus

$$v = (4.9 \times 10^3)\log_e \frac{1000}{980} - (9.8)(10)$$

$$= (4.9 \times 10^3) \times 2.3\left(\log_{10} 1000 - \log_{10} 980\right) - 98$$

$$= 4.9 \times 10^3 \times 2.3 \times (3.0000 - 2.9912) - 98$$

$$= 99 - 98 = 1.0 \text{ m/s}$$

6

COLLISIONS

6.1 COLLISION

It is defined as the phenomenon of the change in the velocities of bodies during the very small time interval of their contact. In the event two bodies collide with one another they are deformed. The kinetic energy before collision of the system transforms completely or partially into potential energy of elastic deformation and into internal energy of the bodies.

In collision process it is not at all assumed that the particles actually come in contact as contrary to every day life's sense of collision. The significance of collision studies lies in the fact that they yield the information regarding the forces which act between particles.

Elastic Collision

If the force of interaction between the colliding bodies are conservative, the kinetic energy remains conserved in the collision and the collision is said to be elastic. Collision between atomic, nuclear and fundamental particles are usually elastic. Collisions between ivory and glass balls are approximately elastic.

Inelastic Collision: When the kinetic energy is changed in the collision, the collision is said to be inelastic (the momentum as well as the total energy is still conserved). Collisions between gross bodies are always inelastic to some extent. When two bodies stick together after collision, the collision is said to be completely inelastic. When a bullet hitting a target remains embedded in the target, the collision is completely inelastic.

6.2 ELASTIC COLLISION IN ONE DIMENSION

Suppose two particles of following specification undergo elastic head-on collision.

Let m_1 = mass of particle 1, m_2 = mass of particle 2, u_1 = velocity of mass 1 before collision, u_2 = velocity of mass 2 before collision, v_1 = velocity of mass 1 after collision, v_2 = velocity of mass 2 after collision.

By the law of conservation of energy

$$\text{K.E. before collision} = \text{K.E. after collision}$$

$$\therefore \qquad \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

or $$m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \quad ...(i)$$

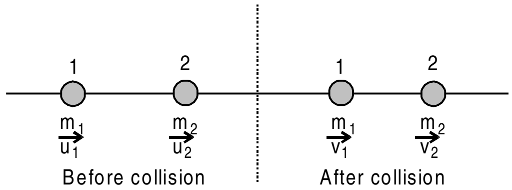


Fig. 1

From law of conservation of Momentum

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

or $$m_1(u_1 - v_1) = m_2(v_2 - u_2) \quad ...(ii)$$

Dividing (i) by (ii),

$$\frac{m_1\left(u_1^2 - v_1^2\right)}{m_1(u_1 - v_1)} = \frac{m_2\left(v_2^2 - u_2^2\right)}{m_2(v_2 - u_2)}$$

or $$(u_1 + v_1) = (v_2 + u_2)$$

or $$(u_1 - u_2) = -(v_1 - v_2) \quad ....(iii)$$

Which suggests that relative velocity of the particles after elastic collision is equal and opposite to the relative velocity before collision.

Putting Equation (iii), in Eq. (ii), we obtain

$$m_1(u_1 - v_1) = m_2(u_1 + v_1 - u_2 - u_2)$$

or $$u_1(m_1 - m_2) + 2m_2u_2 = v_1(m_1 + m_2)$$

or $$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2 \quad ...(iv)$$

Similarly

$$v_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2 + \left(\frac{2m_1}{m_1 + m_2}\right)u_1 \quad ...(v)$$

General Case: Newton experimentally showed that two smooth spherical balls, when approach, collide and separate, then they obey the rule.

$$v_1 - v_2 = -e(u_1 - u_2)$$

$$\boxed{\frac{v_1 - v_2}{u_1 - u_2} = -e}$$

That is, ratio of velocities of separation and approach are in the ratio e:1. Where 'e' has +ve value and is called coefficient of Restitution, which depends on (i) the material of the object (ii) shape and size of colliding objects. Its value lies between 0 and 1. For perfectly inelastic collision, $e = 0$ and for perfectly elastic collision, $e = 1$.

Special cases:

(1) when $m_1 = m_2$ then the equation (iv) and (v)

$$v_1 = u_2 \text{ and } v_2 = u_1$$

i.e., colliding bodies, simply exchange velocities as a result of collision.

(2) when $u_2 = 0$, *i.e.*, second body is initially at rest, then

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) u_1 \quad ...(vi)$$

and

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right) u_1 \quad ...(vii)$$

(3) When $u_2 = 0$ and $m_2 >> m_1$ then from Equations (*vi*) and (*vii*) becomes

$$v_1 = -u_1 \text{ and } v_2 = 0 \text{ (nearly)}$$

Thus, when a lighter body collides with too heavy body at rest, the lighter body reverses the course of motion with no change in the rest state of heavier body.

(4) When $m_2 << m_1$, in addition to $u_2 = 0$, then from Equation (*vi*) and (*vii*)

$$v_1 = u_1 \text{ and } v_2 = 2v_1$$

Thus, when a very heavy body collides with a lighter body at rest, then velocity of heavier body remains nearly unchanged and lighter body moves with nearly twice the velocity of heavy body.

Fraction of K.E. Transferred in Elastic Head on Collision

Suppose a particle 1 collide with particle 2 which is initially at rest. After collision, particle 2 acquire velocity v_2 and has K.E. = ½ $m_2 v^2{}_2$. Before collision, it has K.E. = 0. So kinetic energy transferred = ½ $m_2 v_2^2$.

Initial total K.E. = ½ $m_1 u_1^2$

Thus, fractional K.E. transferred $= \dfrac{½\, m_2 v_2^2}{½\, m_1 u_1^2}$

Putting the value of v_2 from equation (*vii*)

$$v_2 = \frac{2m_1}{m_1 + m_2} u_1$$

We get, Fractional K.E. *transferred* $= \dfrac{\frac{1}{2} m_2 \left(\frac{2m_1}{m_1 + m_2} u_1\right)^2}{\frac{1}{2} m_1 u_1^2} = \dfrac{4 m_1 m_2}{(m_1 + m_2)^2}$

If $m_1 = m_2$, Fractional K.E. transferred = 1.

Thus, when the mass ratio is unity, the whole of the kinetic energy of the moving ball is transferred to the ball initially at rest.

Inelastic Collision (Particle Stick After collision)

(*i*) ***Description in Inertial frame:*** Let a particle of mass m_2 be at rest at origin in an inertial frame. Suppose a particle of mass m_1 moving with velocity u_1 along *x*-direction collides with m_2, both stick and then travel together with velocity v after collision in the same direction.

From law of conservation of momentum

$$m_1 u_1 = (m_1 + m_2)v$$

final velocity of composite particle v, after collision

$$v = \frac{m_1}{(m_1 + m_2)} u_1 \qquad ...(viii)$$

Ratio of K.E. after collision to K.E. before collision

K.E. before collision $K_i = \frac{1}{2} m_1 u_1^2$

K.E. after collision $K_f = \frac{1}{2} (m_1 + m_2)v^2$

$\therefore$ ratio of final to initial K.E. of system

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}(m_1 + m_2)v^2}{\frac{1}{2} m_1 u_1^2}$$

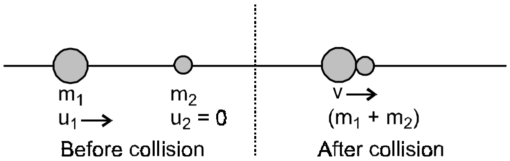


Fig. 2

$$= \frac{\frac{1}{2}(m_1 + m_2)}{\frac{1}{2} m_1 u_1^2} \cdot \left(\frac{m_1 u_1}{m_1 + m_2}\right)^2 \quad \text{using Equation } (viii)$$

or

$$\frac{K_f}{K_i} = \left(\frac{m_1}{m_1 + m_2}\right)$$

It shows $K_f < K_i$ which means during inelastic collision, there is a loss in K.E. of system.

If a moving particle of mass m_1 suffers inelastic collision with a stationary particle of mass m_2, the loss in kinetic energy of system is given by

$$\delta E = \frac{1}{2} m_1 u_1^2 - \frac{1}{2}(m_1 + m_2)\, v^2$$

Since, $$v = \frac{m_1}{(m_1 + m_2)} u_1 \quad \text{from Eq. } (viii)$$

$\therefore$ $$\delta E = \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} u_1^2$$

Description in center of mass frame of reference (C-Frame): velocity of center of mass v_{cm} is given by

$$v_{cm} = \frac{m_1 u_1 + m_2.0}{m_1 + m_2}$$

or

$$v_{cm} = \frac{m_1 u_1}{m_1 + m_2}$$

Velocity of 2nd particle, before collision, in C-frame

$$u'_2 = u_2 - v_{cm} = 0 - \frac{m_1 u_1}{m_1 + m_2}$$

i.e.,

$$u'_2 = -\frac{m_1 u_1}{m_1 + m_2}$$

Velocity of 1st particle, before collision, in C-frame

$$u'_1 = u_1 - v_{cm}$$

$$= u_1 - \frac{m_1 u_1}{m_1 + m_2}$$

$$= \frac{m_1 u_1 + m_2 u_1 - m_1 u_1}{m_1 + m_2}$$

or

$$u'_1 = +\frac{m_2 u_1}{m_1 + m_2}$$

Velocity of combined particle after collision in C-frame

$$v' = v - v_{cm} = \frac{m_1 u_1}{m_1 + m_2} - \frac{m_1 u_1}{m_1 + m_2} = 0$$

$$v' = 0$$

Thus combined particle is at rest in C-frame.

6.3 THE BALLISTIC PENDULUM

It is a device for measuring the speeds of bullets. It consists of a large wooden block mass M hanging vertically by two cords. A bullet of mass m, moving with a speed v, strikes the block and remains embedded in it. If the collision time is very small compared to the time of swing of the pendulum, the supporting cords remain approximately vertical during the collision. Hence no external horizontal force acts on the system during the collision, and the horizontal momentum is conserved.

Let v' be the velocity of the block-bullet combination immediately after the collision. The initial momentum of the system is mv, and the momentum just after the collision is $(m + \text{M})v'$, so that

$$mv = (m + \text{M})v' \qquad ...(i)$$

The kinetic energy of the system immediately after the collision is $\frac{1}{2}(m + M)v'^2$. The block-bullet combination now swings up to a height y at which its kinetic energy is converted into gravitational potential energy. From the conservation of mechanical energy for this part of the motion, we obtain

$$\frac{1}{2}(m + M)\, v'^2 = (m + M)\, gy$$

or
$$v' = \sqrt{2gy}$$

Substituting this of v' in Equation (*i*) and solving for v, we get

$$v = \frac{m + M}{m}\sqrt{2gy}$$

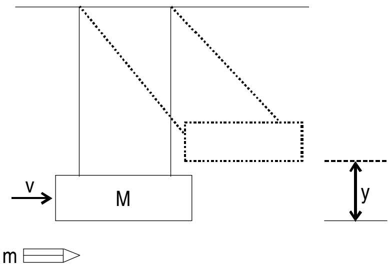


Fig. 3

By measuring m, M and y, we can compute the initial speed v of the bullet. The kinetic energy is not conserved in the collision. The ratio of the kinetic energy of block-bullet combination just after the collision to the initial kinetic energy of the bullet is

$$\frac{\frac{1}{2}(m + M)v'^2}{\frac{1}{2}mv^2} = \frac{m}{m + M}$$

As $m << M$, a very small fraction of the original kinetic energy remains, most of the energy is converted into heat, etc.

6.4 COLLISION IN TWO DIMENSION

(1) *Description in Laboratory Frame:* Suppose a particle of mass m_1 collides with other particle of mass m_2 which is at rest in the laboratory frame of reference. Suppose initial velocity of m_1 is u_1 in this frame and after collision it has velocity v_1 in a direction that makes angle ϕ_1 with the initial direction of travel. Here ϕ_1 is the angle by which colliding particle is deflected and is called scattering angle. Let v_2 be the velocity of m_2, in direction ϕ_2 with initial line. If direction of u_1 is along x-axis and u_1 and v_1 are contained in x-y plane then there is no z-component of v_2.

According to principle of conservative of linear momentum:

x-component

$$m_1u_1 = m_1v_1 \cos \phi_1 + m_2v_2 \cos \phi_2 \quad ...(i)$$

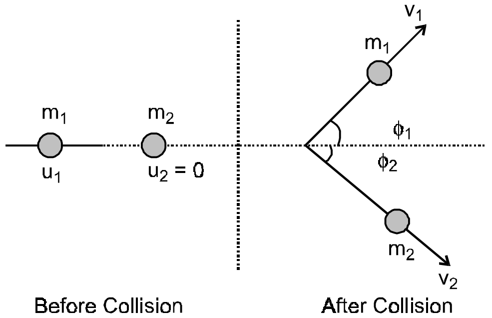


Fig. 4

y-component

$$0 = m_1 v_1 \sin \phi_1 - m_2 v_2 \sin \phi_2 \qquad ...(ii)$$

Since the collision is elastic, K.E. is also conserved

$$\therefore \qquad \frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \qquad ...(iii)$$

If we solve Equations (i), (ii) and (iii), we can know the three unknown quantities of interest. However, the velocity of center of mass in laboratory frame, v_{cm} is given by

$$(m_1 + m_2)\, v_{cm} = m_1\, u_1 + m_2\, .\, 0$$

$$v_{cm} = \frac{m_1 u_1}{m_1 + m_2}$$

(2) ***Description in C-frame:*** Centre of mass frame of reference moves with velocity v_{cm} in lab frame *i.e.*, center of mass remains at rest in c-frame.

In this frame:

Initial velocity of m_1 is, $u'_1 = u_1 - v_{cm}$

Initial velocity of m_2 is, $u'_2 = u_2 - v_{cm} = 0 - v_{cm}$

Final velocity of m_1 is, $v'_1 = v_1 - v_{cm}$

Final velocity of m_2 is, $v'_2 = v_2 - v_{cm}$

v_{cm} is always zero in this frame, hence total momentum before and after collision is zero i.e. momenta of two particles are always equal and opposite as observed in C-frame. So the angle of scattering of both the particles should be same and, thus, they move along a linear path in opposite direction.

Thus $\qquad m_1 u'_1 = m_2 u'_2 \quad$ and $\quad m_1 v'_1 = m_2 v'_2$

Again K.E. conservation yields

$$\frac{1}{2} m_1 u'^2_1 = \frac{1}{2} m_1 v'^2_1 \quad \text{or} \quad u'_1 = v'_1$$

and

$$\frac{1}{2} m_2 u'^2_2 = \frac{1}{2} m_2 v'^2_2$$

or

$$u'_2 = v'_2$$

Thus, in C-frame, the magnitude of velocities of the particles in an elastic collision do not alter.

6.5 VALUE OF THE SCATTERING ANGLE

(i) In the center-of-mass frame of reference

In this frame of reference, there are absolutely no limitations on the value of the scattering angle θ, so that it can have all possible values.

(ii) In the laboratory frame of reference

In this frame of reference, there are some restriction on the value of the scattering angle, θ_1, as will be clear from the following:

we have,

$$\tan \theta_1 = \frac{v_1 \sin \theta_1}{v_1 \cos \theta_1}$$

Now, the y-component of the final velocity of the first particle being the same in the laboratory reference frame as well as the center-of-mass frame, we have $v_1 \sin\theta_1 = v'_1 \sin\theta$. And, since the x-components of this velocity differ by V in the two frames of reference, we have $v_1 \cos\theta_1 = v'_1 \cos\theta + V$. Substituting these values in the relation above, we, therefore, have

$$\tan\theta_1 = \frac{v'_1 \sin\theta}{v'_1 \cos\theta + V} = \frac{\sin\theta}{\cos\theta + \left(\dfrac{V}{v'_1}\right)}$$

As we known,

$$V = \frac{m_1 u_1}{m_1 + m_2} = \frac{m_1 (u'_1 + V)}{m_1 + m_2}$$

$$= \frac{m_1 u'_1}{m_1 + m_2} + \frac{m_1 V}{m_1 + m_2}$$

or $$V\left(1 - \frac{m_1}{m_1 + m_2}\right) = \frac{m_1 u'_1}{m_1 + m_2}$$

or $$V\left(\frac{m_2}{m_1 + m_2}\right) = \frac{m_1 u'_1}{m_1 + m_2}$$

Whence $$V = \frac{m_1}{m_2} u'_1 = \frac{m_1}{m_2} v'_1 \quad \therefore \quad u'_1 = v'_1$$

Substituting this value of V in Equation (i) above, we have

$$\tan\theta_1 = \frac{\sin\theta}{\cos\theta + \left(\dfrac{m_1}{m_2}\right)}$$

This shows that

(a) If $m_1 > m_2$, so that m_1/m_2 a little >1, the denominator can never be zero and hence $\tan\theta_1$ can never be ∞. And therefore, θ_1 must be less than 90°.

(b) If $m_1 = m_2$, so that $m_1/m_2 = 1$, the denominator can be zero for $\cos\theta = -1$ and, therefore, $\tan\theta_1$, can be ∞. Thus, in this case , θ_1 can have any value up to the limiting value of 90°.

(c) If $m_1 < m_2$, so that $m_1/m_2 < 1$, the value of $\tan\theta_1$ can also be negative. In this case alone therefore, all values of θ_1 can be possible.

It follows from case (a) that in an elastic collision, if a massive particle collides against a lighter one at rest, it can never bounce back along its original path. On the other hand, it follows from case (c) that if a lighter particle collides against a massive one at rest, it may well bounce back along its original path. And, it follows from case (b) that in an elastic collision between two particles of equal mass, one of which is initially at rest, the two particles move at right angles to each other after the collision.

6.6 SCATTERING CROSS-SECTION

We can consider three distinct stages in the entire scattering process. We show these three stages of the collision process in below figure. The first stage shown in figure (*a*) corresponds to a time long before the interaction of the colliding particles. At this stage each particle is effectively free, *i.e.*, its energy is positive. As the particles approach each other (figure *b*), interaction forces much larger than any other force acting on them come into play. Finally, long after the interaction (figure C), the emerging particles are again free and move along straight lines with new velocities in new directions. The emerging particles may or may not be the same as the original particles.

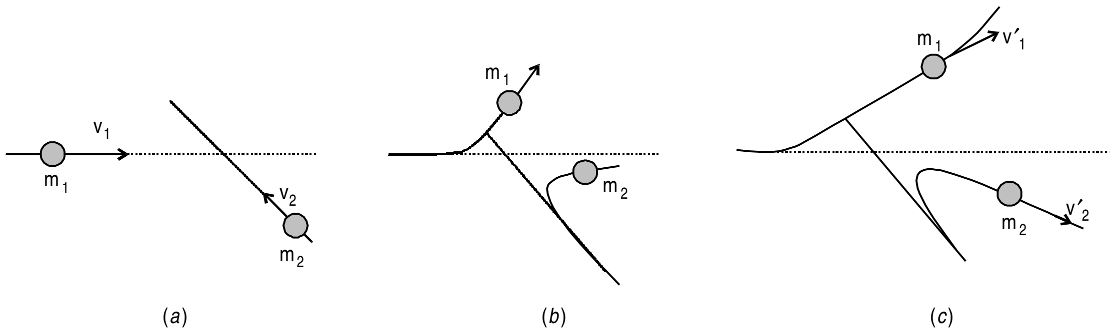


Fig. 5 Scattering of two particles

In a typical scattering experiment, a parallel beam of particles, also called projectiles, of given energy and momentum is incident upon a target (below figure). The particles interact with the target for a short time, which deflects or scatters them in various directions. Eventually these particles are detected at large distances from the target. The scattered particles may or may not have the same energies and momenta.

An experimenter may be interested in knowing the velocities, linear momenta and energies of the particles before and after scattering. Then the changes brought about in these quantities can be determined.

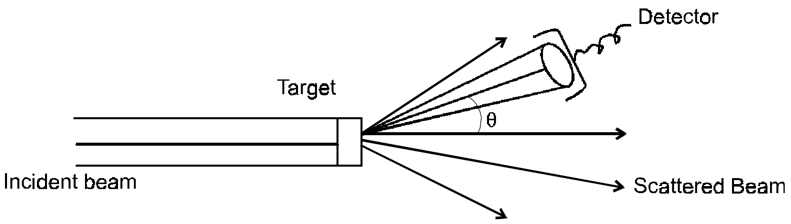


Fig. 6 A typical scattering process

The probability of scattering in a given direction is found by determining the scattering cross-sections. Let us now define the scattering cross-section for a typical scattering process.

Let us suppose that a uniform parallel beam of n particles, all of the same mass and energy, is incident upon a target containing N number of identical particles or scattering centers. Such scattering centers might, for example, be the positive nuclei of atoms in a thin metal foil which could be bombarded by α-particles. Let us assume that the particles in the beam do not interact with each other and the scattering centers in the target are sufficiently far apart. With these assumptions we can regard the incident particles and target particles to be sufficiently far apart. Then we can think of this scattering event as if at a given time only one projectile was being scattered by one target particle, without being affected by the presence of other particles. So, effectively at any instant we deal with a two-body collision process. For convenience, we choose the origin of the coordinate system at the position of the target and one of the axes, say z-axis, in the direction of the incidents beam. The direction of scattering is given by the angles (θ, ϕ) as shown in figure (a). The angle θ, called the angle of scattering, is the angle between the scattered and the incident directions. These two directions define the plane of scattering. The angle ϕ specifies the orientation of this plane with respect to some reference plane containing the z-axis. The shaded plane in figure is a reference plane. The probability of the scattering of a particle in a given direction (θ, ϕ) is measured in terms of the differential cross-section.

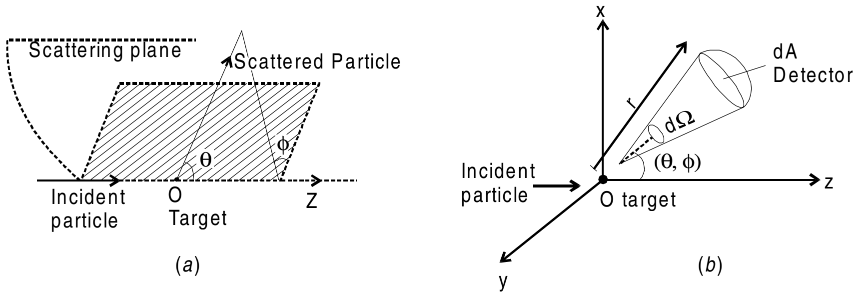


Fig. 7

6.7 DIFFERENTIAL SCATTERING CROSS-SECTION

Let F be the number of projectile incident per unit area per unit time on the target. F represents the incident flux. Let Δn be the number of particles scattered into a small solid angle $d\Omega$ about the angle (θ, ϕ) in time Δt. Then the number of scattered particles by a single target particle in time Δt, must be proportional to the incident flux F, the duration Δt and also the solid angle in which they are scattered, *i.e.*,

$$\Delta n \propto \mathrm{F}\,(d\Omega)\,(\Delta t)$$

The constant of proportionality is defined as the differential scattering cross-section and is denoted by the symbol $\frac{d\sigma}{d\Omega}$. So that

$$\Delta n = \left(\frac{d\sigma}{d\Omega}\right)\mathrm{F}\,(d\Omega)\,(\Delta t)$$

or

$$\frac{d\sigma}{d\Omega} = \frac{\Delta n}{\mathrm{F}\Delta t\; d\Omega}$$

Thus, we can also express the differential scattering cross-section as the following ratio:

$$\frac{d\sigma}{d\Omega} = \frac{\text{The number of particles scattered per unit time in a solid angle d }\Omega\text{ in the direction }(\theta,\phi)}{\text{Incident flux } i.e., \text{ the number of particles incident on the target per unit area per unit time}}$$

The differential scattering cross-section gives a probability. In fact, it is a measure of the probability that an incident particle will be scattered in solid angle $d\Omega$ in the direction

(θ, ϕ). The dimension of $\frac{d\sigma}{d\Omega}$ is area. This explains the use of the term "cross-section". $\frac{d\sigma}{d\Omega}$ is equal to the cross-sectional area of the incident beam that contains the number of particles scattered into the solid angle $d\Omega$ by a single target particle. The unit of $\frac{d\sigma}{d\Omega}$ is $m^2\ sr^{-1}$. The $\frac{d\sigma}{d\Omega}$ depends only on the parameters of the incident particle, nature of the target and the nature of the interaction between the two.

For the N scattering centers the number of particles scattered will be just N times the number scattered by a single scattering center. Thus for N scattering centers, the number of particles scattered is

$$\Delta n' = \frac{d\sigma}{d\Omega} \mathrm{NF}\, d\Omega\, \Delta t$$

Above equation is valid only when the target scattering centers are far enough apart so that the same particle is not scattered by two of them.

6.8 TOTAL CROSS-SECTION

Let us place the detector at all possible values of (θ, ϕ) and count the total number of scattered particles entering all the corresponding solid angles. Then we will get the total scattering cross-section. It is denoted by σ. It can also be calculated from the differential scattering cross-sections by integrating over all possible values of $d\Omega$. Thus

$$\sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega$$

So the total scattering cross-section represents the number of particles scattered in all direction per unit flux of incident particles. It has the dimension of area. So its unit is m^2. Now, we also define the solid angle subtended by an area to be $d\Omega = \sin\theta\, d\theta\, d\phi$, where the limits of θ and ϕ are 0 to π and 0 to 2π, respectively. Using these relations we get,

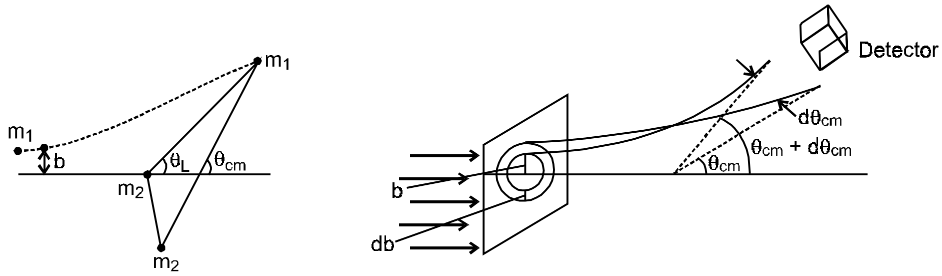
$$\sigma = \int_0^{\pi}\int_0^{2\pi} \left(\frac{d\sigma}{d\Omega}\right) \sin\theta . d\theta . d\phi$$

We can show that for the cases in which the force is central and its magnitude depends only on r, $\frac{d\sigma}{d\Omega}$ is independent of ϕ. We can integrate over ϕ so that

$$\sigma = 2\pi \int_0^{2\pi} \left(\frac{d\sigma}{d\Omega}\right) \sin\theta . d\theta$$

6.9 IMPACT PARAMETERS

Let us suppose that the projectile does not make a head-on collision with the target. Instead, it travels along a path, which if continued in a straight line, would pass target at a distance b (as shown in figure 8). The distance b is known as the impact parameter. b is the perpendicular distance between the projectiles initial path and the target.



(a) The impact parameter b

(b) The particle having impact parameters between b and $b + db$ are scattered into angles between θ_{cm} and $\theta_{cm} + d\theta_{cm}$.

Fig. 8

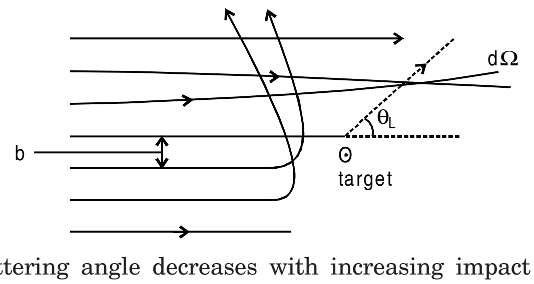
Let us now express the differential scattering cross-sections in terms of the impact parameter. We will study the scattering process in the c.m. frame of reference with θ_{cm} as the angle of scattering (figure b). Let us consider a circular ring having radii between b and $b + db$. The area of the ring is $2\pi bdb$ for infinitesimal values of db. If the incident flux is F then,

The number of incident particles having an impact parameter between b and $(b + db)$

$$= F(\Delta t)\,(2\pi bdb)$$

Let us suppose that these particles are scattered into angles between θ_{cm} and $\theta_{cm} + d\theta_{cm}$. The particles with larger b will be scattered through smaller angles as shown in figure (c). This happens because larger b means lesser interaction, *i.e.*, less scattering. For very large b, scattering will be minimal and the particles will go almost undeflected in a straight line. Now in the c.m frame of reference the number of particles scattered in the solid angle $d\Omega$ in time Δt is given as

$$\Delta n = \left(\frac{d\sigma}{d\Omega}\right)_{cm} F\,(d\Omega)_{cm}\,\Delta t$$



(c) The scattering angle decreases with increasing impact parameter.

Fig. 8

This is the same as the number of incident particles in time Δt having impact parameters between b and $b + db$; given by

$$F(\Delta t)2\pi.b.db = \left(\frac{d\sigma}{d\Omega}\right)_{cm} F(d\Omega)_{cm}\,\Delta t$$

or

$$2\pi.b.db = -\left(\frac{d\sigma}{d\Omega}\right)_{cm} 2\pi\sin\theta_{cm}\,d\theta_{cm}$$

Here we have assumed that $\frac{d\sigma}{d\Omega}$ is independent of ϕ. Taking into account all values of ϕ in $d\Omega$, we have $d\Omega = 2\pi \sin\theta . d\theta$. The negative sign expresses the fact that as b increases, θ_{cm} decreases, *i.e.*, db and $d\theta_{cm}$ have opposite signs. From above Equation we get,

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{b}{\sin\theta_{cm}}\left|\frac{db}{d\theta_{cm}}\right|$$

We have not written the negative sign in above equation because $\left(\frac{d\sigma}{d\Omega}\right)_{cm}$ has the dimension of area and its magnitude has to be positive. So, if we know b as a function of scattering angle θ_{cm}, we can calculate the differential scattering cross-section using above equation.

6.10 RUTHERFORD SCATTERING

The Rutherford scattering experiment was an important milestone in understanding the structure of the atom. Until the early twentieth century Thomson's plum pudding model of the atom was believed to be valid. J.J. Thomson had proposed, in 1898, that atoms were uniform spheres of positively charged matter in which electrons were embedded. It was almost 13 years later that a definite experimental test of this model was made. Now, the most direct way to find out what is inside a plum pudding is to plunge a finger into it. A similar technique was used in the classic experiment performed in 1911, by Geiger and Marsden who were working with Lord Rutherford. They bombarded thin foils of various materials with α-particles (helium nuclei) and recorded the angular distribution of the scattered α-particles.

It was found that most of the α-particles pass through the foil (*i.e.*, scattering angle $\theta < 90°$). However, about 1 in 6.17×10^6 alpha particles was scattered backward, *i.e.*, deflected through an angle greater than 90°. This result was unexpected according to Thomson's model. It was anticipated that the alpha particles would go right through the foil with only slight deflections. This follows from the Thomson model. If this model were correct, only week electric forces would be exerted on alpha particles passing through a thin metal foil. In such a case their initial momenta should be enough to make them go through with only slight deflections. It would indeed need strong forces to cause such considerable deflections in α-particles as were observed.

In order to explain these results Rutherford proposed a nuclear model of the atom. Using this model he calculated the differential scattering cross-section. In doing so, he reasoned that the backward scattering could not be caused by electrons in the atom. The alpha particles are so much more massive than electrons that they would hardly be scattered by them. He assumed that the positive charge in the atom was concentrated in a very small volume, which he termed the nucleus, rather than being spread out over the volume of the atom. So the scattering of alpha particles was due to the atomic nucleus.

Let us consider the scattering of a particle carrying charge q by the atomic nuclei having charge q'. For this scattering process Rutherford derived the relation between the impact parameter b and the angle of scattering θ_{cm} to be

$$b = \frac{r_0}{2}\cot\frac{\theta_{cm}}{2}$$

Where $r_0 = \dfrac{qq'}{4\pi\varepsilon_0 \, E_{cm}}$, E_{cm} is the total mechanical energy of the projectile and the target in the c.m. system.

ε_0 is known as the permittivity of free space. Its value is 8.8×10^{-12} C^2N^{-1} m^{-2}.

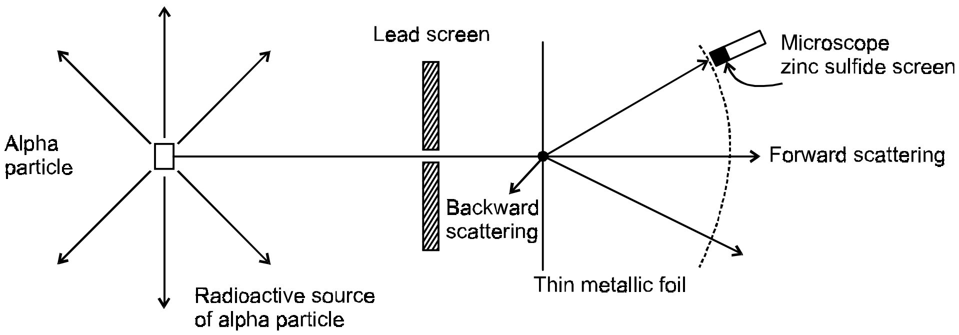


Fig. 9

The differential scattering cross-section in the c.m. system for Rutherford scattering is then given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{b}{\sin\theta_{cm}}\left(\frac{db}{d\theta_{cm}}\right)$$

$$= \frac{r_0 \cot\frac{\theta_{cm}}{2}}{2\sin\theta_{cm}}\,\frac{r_0}{2}.\frac{1}{2}\text{cosec}^2\frac{\theta_{cm}}{2}$$

$$= \frac{r_0^2 \cot\frac{\theta_{cm}}{2}}{16\sin\frac{\theta_{cm}}{2}.\cos\frac{\theta_{cm}}{2}}\text{cosec}^2\frac{\theta_{cm}}{2}$$

or $$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{r_0^2}{16}\text{cosec}^2\frac{\theta_{cm}}{2}$$

where $$r_0 = \frac{qq'}{4\pi\varepsilon_0 E_{cm}}$$

This is the Rutherford scattering cross-section. For scattering of an α-particle by a nucleus of atomic number z, $qq' = (2e)(Ze) = 2Ze^2$ where e is the electronic charge. The Rutherford scattering cross-section is strongly dependent on both the energy of the incoming particle and the scattering angle. The number of particles scattered to increase as Z^2 with increasing atomic number.

NUMERICALS

Q.1. *Assuming that all collisions are completely elastic, find the value of m for which both blocks move with the same velocity after m has collided once with M and once with the wall. (The wall has effectively infinite mass).*

Ans. Suppose after the collision,

$$v_1 = \text{speed of mass M towards right}$$
$$v_2 = \text{speed of mass } m \text{ towards left}$$

Hence, momentum conservation requires;

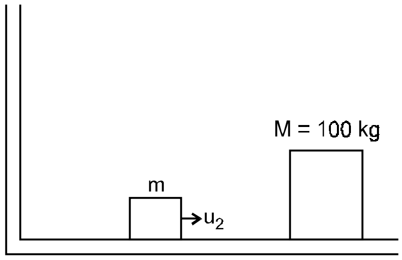


Fig. 10

$$\text{momentum before collision} = \text{momentum after collision}$$

$$mu_2 = Mv_1 - mv_2 \qquad ...(i)$$

The mass m rebounds elastically from the wall and its speed is reversed after the collision with the wall. The mass m has the same speed as that of mass M after its collision with the wall. ($v_2 = v_1$ as given); so (*i*) is

$$mu_2 = (M - m)\, v_1 \qquad ...(ii)$$

The collision is elastic, so,

$$\frac{1}{2}mu_2^2 = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_1^2$$

or,
$$mu_2^2 = (M + m)\, v_1^2 \qquad ...(iii)$$

Substituting the value of v_1 from eqn. (*ii*) in eqn. (*iii*) we get,

$$mu_2^2 = \frac{(M+m)\,(mu_2)^2}{(M-m)^2}$$

or
$$(M - m)^2 = (M + m)\,(m)$$

or
$$M^2 + m^2 - 2Mm = Mm + m^2$$

or
$$M^2 = 3Mm$$

or
$$M = 3m$$

or
$$m = \frac{M}{3} = \frac{100}{3} = 33.33 \text{ kg.}$$

Q. 2. A ball moving with a speed of 8m/sec strikes an identical ball at rest such that after the collision the direction of each ball makes an angle of 30° with the original line of motion. Find the speeds of two balls after the collision. Is the K.E. conserved in this collision?

Solution. Momentum before collision $= m \times 8 + m \times 0 = 8\text{ m}$ ...(*i*)

where m is the mass of each ball.

Suppose after collision their velocities are v_1 and v_2 respectively. Now the final momentum of the balls after the collision along the same line

$$= mv_1 \cos 30° + mv_2 \cos 30°$$

$$= \frac{mv_1\sqrt{3}}{2} + \frac{mv_2\sqrt{3}}{2} \qquad ...(ii)$$

From the law of conservation of momentum

$$8\ m = \frac{mv_1\sqrt{3}}{2} + \frac{mv_2\sqrt{3}}{2}$$

$$\frac{8 \times 2}{\sqrt{3}} = v_1 + v_2 \qquad ...(iii)$$

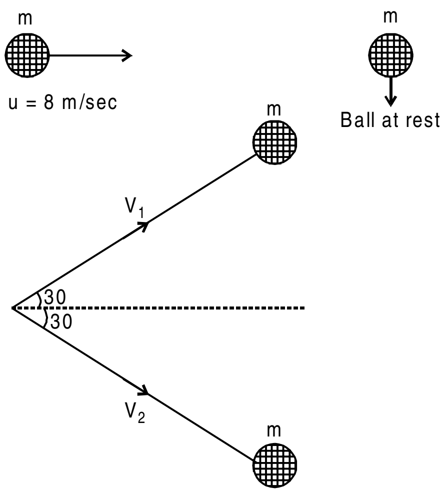


Fig. 11

The initial momentum of the balls along perpendicular direction is zero and final momentum of the balls along perpendicular direction

$$= mv_1 \sin 30° - mv_2 \sin 30° = \left(\frac{m}{2}\right)(v_1 - v_2)$$

Then momentum conservation gives:

$$0 = \left(\frac{m}{2}\right)(v_1 - v_2)$$

$$\therefore \qquad v_1 - v_2 = 0 \qquad ...(iv)$$

from eqns. (*iii*) and (*iv*), we get

$$v_1 = \frac{8}{\sqrt{3}} \text{ m/s and } v_2 = \frac{8}{\sqrt{3}} \text{ m/s}$$

Since, Energy before collision = Energy after collision

$$\therefore \qquad \frac{1}{2}mu_1^2 + \frac{1}{2}mu_2^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

or

$$\frac{1}{2}m\,(8)^2 + 0 = \frac{1}{2}m\left[\left(\frac{8}{\sqrt{3}}\right)^2 + \left(\frac{8}{\sqrt{3}}\right)^2\right]$$

or $$\frac{64m}{2} = \frac{128m}{2 \times 3}$$

Since, L.H.S. $\neq$ R.H.S.

Hence, energy is not conserved in this collision so, collision is inelastic.

Q. 3. *A plastic ball is dropped from a height of 1 metre and rebounds several times from the floor. If 1.3 seconds elapse from the moment it is dropped to the second impact with the floor. What is the coefficient of restitution.*

Ans. Suppose the plastic ball makes first strike on the floor with a velocity V, then ball rebounds with a velocity eV. Now the ball rises to a certain height where its velocity becomes zero and then it retraces its path. The ball strikes the floor with velocity eV making the second impact.

After the second impact, it rebounds with a velocity e^2V. Time taken by the ball before first impact is given by.

$$V = gt_1$$

(Initial velocity $u = 0$, Final velocity V, and acceleration = g)

$$t_1 = \frac{V}{g} \qquad ...(i)$$

Time interval between first and second impact t_2 = 2 × time taken for the velocity to change from eV to O under gravity

$$t_2 = 2 \times \frac{eV}{g} \qquad ...(ii)$$

$$\therefore \quad t_1 + t_2 = \frac{V}{g} + \frac{2eV}{g} = \frac{V}{g}(1+2e) \qquad ...(iii)$$

According to question;

$$\frac{V}{g}(1+2e) = 1.3 \qquad ...(iv)$$

Again, $$V^2 = 0 + 2g.1$$

$$V = \sqrt{2g} \qquad ...(v)$$

Using (v) in (iv)

$$\frac{\sqrt{2g}}{g}(1+2e) = 1.3$$

or $$\sqrt{2/g}\,(1+2e) = 1.3; \text{ or } \sqrt{2/9.8}\,(1+2e) = 1.3$$

or $$(1 + 2e) = 1.3\sqrt{4.9} = 2.8777$$

$$1+2e = 2.8777$$

which gives $$e = 0.94$$

Q. 4. *A meteor of M = 2 kg mass moving with velocity u = 12 km/s explodes above the atmosphere into two pieces which travel along the same path with 15 km/s and 11 km/s. Calculate the mass of each piece and energy set free by the explosion.*

Solution. Due to the absence of external forces;

Linear momentum before explosion = After explosion

i.e., $$2 \times 12 = 15m_1 + 11m_2 \quad ...(i)$$

and $$m_1 + m_2 = 2 \quad ...(ii)$$

From (*i*) and (*ii*);

$$m_1 = \frac{2}{4} = 0.5 \text{ kg}$$

$$m_2 = 2 - 0.5 = 1.5 \text{ kg}$$

$$\text{Energy released } \Delta E = \left(\frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_2^2\right) - \frac{1}{2}mu^2$$

$$= 3 \times 10^6 \text{ J}$$

Q. 5. *A sand bag of mass 10 kg is suspended with a 3 meters long weightless string. A bullet of mass 200 gm is fired with speed 20 m/s into the bag and stays in bag. Calculate*

(i) Speed acquired by the bag.

(ii) Maximum displacement of bag.

(iii) The energy converted to heat in collision.

Solution. (*i*) Collision is inelastic so the momentum is conserved not the K.E. if m = mass of bullet, M = mass of bag, u = velocity of bullet, V = velocity of bag after hit.

then $$mu = (m + M)V$$

$$0.2 \times 20 = (0.2 + 10)V$$

$$V = \frac{0.2 \times 20}{10.2} = \frac{4}{10.2} = 0.39 \text{ m/s}$$

(*ii*) Bag shall oscillate like pendulum after it is hit. If x is the maximum displacement,

Restoring force at maximum displacement; $F = -(M + m)\, g \sin\theta$

i.e., $$F = -(M + m)\, g\frac{x}{l} = -Kx \quad ...(i)$$

or $$K = \frac{10.2 \times 9.8}{3} \quad ...(ii)$$

At maximum displacement, x, K.E. of system will be changed into potential energy

$$\int_0^x Kx\, dx = \frac{1}{2}Kx^2 = \frac{1}{2}mV^2 = \left(\frac{1}{2}\right)10.2\,(0.39)^2$$

or, $$x^2 = \frac{10.2 \times 0.39 \times 0.39}{K} = \frac{10.2 \times 0.39 \times 0.39}{10.2 \times 9.8} \times 3$$

or, $$x = \sqrt{(0.39 \times 0.39 \times 3)/(9.8)} = 0.39\sqrt{3/9.8} = 0.21$$

max. displacement, $x = 0.21$ metre

(*iii*) Energy converted into heat $= \frac{1}{2}mu^2 - \frac{1}{2}(m + \text{M})\text{V}^2$

$$= \frac{1}{2} \times 0.2 \times (20)^2 - \frac{1}{2} \times 10.2 \times (0.39)^2$$

$$= 40 - 0.8 = 38.2 \text{ J}$$

Q. 6. *A steel ball weighing 1 lb is fastened to a cord 27 inches long and is released when the cord is horizontal. At the bottom of its path, the ball strikes a block weighing 5.0 lb which is initially at rest on a frictionless surface as shown.*

The collision is elastic. Find the speed of the ball and the speed of the block just after collision.

Solution. Let, m_1, m_2 = masses of the ball and block respectively u_1 = velocity obtained by the ball after falling through 27″, the velocity by which ball hits the block; V_2 = velocity obtained by the block; V_1 = velocity of ball after impact;

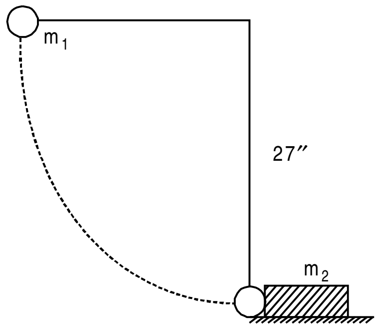


Fig. 12

Conservation of momentum gives.

$$m_1u_1 + 0 = m_1\text{V}_1 + m_2\text{V}_2$$

K.E. obtained in 27"(= 9/4 ft) descent

$$\frac{1}{2}m_1u_1^2 = m_1gh = m \times 32 \times \frac{9}{4}$$

or $$u_1 = 12 \text{ ft/sec.}$$

Since collision is head on, we have

$$\text{V}_2 = \frac{2u_1}{1 + m_2/m_1} = \frac{2 \times 12}{1 + 5/1} = \frac{24}{6} = 4 \text{ ft/sec.}$$

Then from (*i*); $1 \times 12 = 1 \times \text{V}_1 + 5 \times 4$

or $$\text{V}_1 = -8 \text{ ft/sec.}$$

thus, ball recoils in opposite direction with a velocity = 8 ft/sec.

Q. 7. *A moving particle of mass m makes head on collision with a particle of mass 2m which is initially at rest. Show that the colliding particle looses (8/9)th of its energy after collision.*

Ans. Let initial velocity of colliding particle be '*u*' and final velocity ' V ', after collision. If V_1 is velocity of the other particle after collision, from law of conservation of momentum.

$$mu = m\text{V} + 2m\text{V}_1$$

which gives, $$\text{V}_1 = \frac{u - \text{V}}{2}$$

Again from law of conservation of energy

$$\frac{1}{2}mu^2 = \frac{1}{2}m\text{V}^2 + \frac{1}{2}.2m.\text{V}_1^2$$

or $$u^2 = V^2 + 2\left(\frac{u - V}{2}\right)^2$$

or $$u^2 = V^2 + \frac{u^2}{2} + \frac{V^2}{2} - uV$$

or $$3V^2 = u^2 + 2uV$$

or $$3V^2 - 2uV - u^2 = 0$$

which gives final velocity of colliding particle, which is

$$V = \frac{2u \pm \sqrt{4u^2 + 12u^2}}{6}$$

$$= u \text{ or } -\frac{u}{3}$$

Since $+ve$ value of V will not satisfy the law of conservation of momentum.

$$\therefore \quad V = -\frac{u}{3}$$

Therefore, loss in K.E. of colliding particle

$$\Delta U = \frac{1}{2}mu^2 - \frac{1}{2}m\left(-\frac{u}{3}\right)^2$$

$$= \frac{1}{2}mu^2\left(1 - \frac{1}{9}\right) = \frac{8}{9} \text{ initial K.E.}$$

Q. 8. *A bomb moving with velocity* $40\hat{i} + 50\hat{j} - 25\hat{k}$ *m/s exploded into pieces of mass ratio 1 : 4. The small piece goes out with velocity* $200\hat{i} + 70\hat{j} + 15\hat{k}$ *m/s. Deduce the velocity of larger piece after explosion.*

Ans. Force of explosion is the internal force acting on bomb. In absence of external forces, thus, total momentum of the system remains constant.

$\therefore$ Momentum before explosion = momentum after explosion

i.e., $$5M(40\hat{i} + 50\hat{j} - 25\hat{k}) = M(200\hat{i} + 70\hat{j} + 15\hat{k}) + 4\,MV_2$$

or $$4V_2 = 200\hat{i} + 250\hat{j} - 125\hat{k} - 200\hat{i} - 70\hat{j} - 15\hat{k}$$

or $$4V_2 = 180\hat{j} - 140\hat{k}$$

or $$V_2 = 45\hat{j} - 35\hat{k}$$

Q. 9. *A body at rest explodes and breaks up into 3 pieces. Two pieces having equal mass, fly off perpendicular to each other with the same speed of 30 m/sec. The 3rd piece has 3 times the mass of each of the other piece. Find the magnitude and direction of its velocity immediately after the explosion.*

Ans. Mass ratio is 1:1:3 in explosion. Since only internal force acts; so that momentum is conserved;

Momentum before explosion = after explosion

i.e., $5M(O) = M \times 30\hat{i} + M \times 30\hat{j} + 3M\vec{V}_3$

or $3\vec{V}_3 = -30\hat{i} - 30\hat{j}$

or $\vec{V}_3 = -10\hat{i} - 10\hat{j} = -(10\hat{i} + 10\hat{j})$

or $|V_3| = \sqrt{10^2 + 10^2} = 10\sqrt{2}$ m/s

The direction is given by

or $\tan\theta = \dfrac{-10}{10} = -1 \qquad \therefore\ \theta = 90 + 45 = 135°$

Q. 10. *A U^{238} nucleus emits an α-particle and is converted into Th^{234}. If the velocity of the α-particle be 1.4×10^7 m/s and the kinetic energy be 4.1 MeV, calculate the velocity of the recoil and the K.E. of the residual nucleus (Th^{234}).*

Ans. Only internal force act; so total momentum is same before and after the fragmentation (or the nuclear reaction)

i.e., momentum before reaction = after reaction

or $O \times [M_U] = M_{Th} V_{Th} + M_\alpha V_\alpha$

or $O = 234V_{Th} + 4 \times 1.4 \times 10^7$ m/s

or $V_{Th} = -2.4 \times 10^5$ m/s

So, magnitude = 2.4×10^5 m/s and direction will be opposite to motion of α particle.

Kinetic Energy of Th^{234} nucleus:

Again; $$\frac{(K.E.)_{Th}}{(K.E.)_\alpha} = \frac{\frac{1}{2}M_{Th}V_{Th}^2}{\frac{1}{2}M_\alpha V_\alpha^2} = \frac{234}{4}\left[\frac{2.4 \times 10^5}{1.4 \times 10^7}\right]^2$$

$$\frac{(K.E.)_{Th}}{4.1\text{ MeV}} = 58.5 \times \frac{144}{49} \times 10^{-4} = 171.19 \times 10^{-4}$$

$$(K.E.)_{Th} = 171.19 \times 4.1 \times 10^{-4} = .0704$$

K.E. of thorium nucleus = 0.0704 MeV

Q. 11. *A 5 kg object with a speed of 30 m/s strikes a steel plant at an angle 45° and rebounds at the same speed and same angle. What is the change (magnitude and direction) in the linear momentum of the object?*

Ans. Linear momentum before striking; is as below,

$$\vec{P}_1 = mu_x\hat{i} - mu_y\hat{j}$$

Linear momentum after striking is given as;

$$\vec{P}_2 = mu_x\hat{i} - mu_y\hat{j}$$

Change in linear momentum:

$$\Delta\vec{P} = (\vec{P}_2 - \vec{P}_1)$$

$$= mu_x\hat{i} - mu_y\hat{j} - mu_x\hat{i} + mu_y\,\hat{j}$$

$$\Delta\vec{P} = 2mu_x\hat{j}$$

But,
$$u_x = u\,\cos 45° = 30\left(\frac{1}{\sqrt{2}}\right)$$

$$u_y = u\,\sin 45° = 30\left(\frac{1}{\sqrt{2}}\right)$$

So,
$$\Delta\vec{P} = 2\times 5\times\frac{30}{\sqrt{2}}\hat{j} = \frac{300}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\hat{j}$$

or
$$\Delta\vec{P} = 150\sqrt{2}\,\hat{j}$$

So, magnitude, $|\Delta\vec{P}| = 150\sqrt{2}$ Kg m/s, in direction perpendicular to steel plate.

Q. 12. *A particle of mass M, moving with a velocity u, makes a head on collision with a particle of mass m initially at rest so that their final velocities V and v are along the same line. Assuming an elastic collision, prove that* $v = \dfrac{2u}{1+m/M}\cdot$ *If the particle coalesce on colliding, find the final common velocity and the loss in K.E.*

Ans. (*i*) Law of conservation of momentum gives;

$$Mu = MV + mv \quad ...(i)$$

and law of conservation of energy gives;

$$\frac{1}{2}Mu^2 = \frac{1}{2}MV^2 + \frac{1}{2}mv^2$$

$$u^2 = V^2 + (m/M)v^2 \quad ...(ii)$$

Putting for V from (*i*), we obtain,

$$u^2 = \left(\frac{Mu - mv}{M}\right)^2 + \frac{m}{M}v^2$$

$$u^2 = u^2 + \left(\frac{m}{M}\right)^2 v^2 - 2u\frac{m}{M}v + \frac{m}{M}v^2$$

$$2u\,\frac{m}{M}\,v = v^2\left[\left(\frac{m}{M}\right)^2 + \left(\frac{m}{M}\right)\right]$$

$$2u = v\left[\frac{m}{M} + 1\right] \qquad \text{or} \qquad v = \frac{2u}{1 + m/M}$$

(*ii*) When particles stick together (inelastic collision); conservation of momentum gives

$$Mu = (m + M)v$$

$$v = \frac{M}{(m + M)}.u$$

$$\text{Loss of energy} = \frac{1}{2}Mu^2 - \frac{1}{2}(M + m)v^2$$

$$= \frac{1}{2}Mu^2 - \frac{1}{2}(M + m)\frac{M^2u^2}{(m + M)^2}$$

$$= \frac{1}{2}Mu^2 - \frac{1}{2}\frac{M^2u^2}{(m + M)}$$

$$= \frac{1}{2}\left[\frac{Mu^2(m + M) - M^2u^2}{(m + M)}\right] = \frac{1}{2}\left[\frac{Mmu^2 + M^2u^2 - M^2u^2}{m + M}\right]$$

$$\therefore \qquad \text{Loss of energy, } \Delta E = \frac{1}{2}\frac{Mmu^2}{(m + M)}$$

Q. 13. *A radioactive nucleus initially at rest, decays by emitting an electron and a neutrino at right angles to one another. The momentum of the electron is* 1.2×10^{-22} *kg-m/s and that of the neutrino is* 6.4×10^{-23} *kg m/s. Find the direction and magnitude of the recoil nucleus. If its mass is* 5.8×10^{-26} *kg, deduce K.E. of recoil.*

Ans. The nucleus is at rest, before decay. Let P_e = momentum of electron; P_n = momentum of neutrino; P_N = momentum of recoil nucleus. Before decay; total momentum of nucleus is zero. In decay process, no external forces act (only internal one) so momentum remains constant before and after decay.

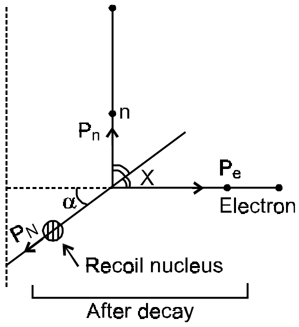


Fig. 13

x-component of momentum

$$O = P_e - P_N \cos\alpha$$

$$P_N \cos\alpha = P_e \qquad ...(i)$$

y-component of momentum

$$O = P_n - P_N \sin\alpha$$

$$P_N \sin\alpha = P_n \qquad ...(ii)$$

(*i*) and (*ii*) give; $$P_N = \sqrt{P_e^2 + P_n^2}$$

$$= \sqrt{(1.2 \times 10^{-22})^2 + (6.4 \times 10^{-23})^2}$$

$$= \sqrt{1.44 \times 10^{-44} + 40.96 \times 10^{-46}}$$

So, magnitude of momentum of recoil nucleus = $\sqrt{1.85} \times 10^{-22}$ Kg-m/s

Its direction will be given by

$$\tan\alpha = \frac{P_n}{P_e} = \frac{6.4 \times 10^{-23}}{1.2 \times 10^{-22}} = \frac{0.64}{1.2} = 0.53$$

$$\alpha = \tan^{-1} 0.53 = 28° \text{ (nearly).}$$

From the given diagram,

$$\theta = 180 - \alpha = 180 - 28 = 152°$$

recoil nucleus will move making angle 152° from electron motion or 242° from neutrino's path.

K.E. of recoil nucleus;

$$\text{K.E.} = \frac{1}{2} M_N V_N^2 = \frac{P_N^2}{2M_N}$$

$$= \frac{(\sqrt{1.85} \times 10^{-22})^2}{2 \times 5.8 \times 10^{-26}} = \frac{1.85 \times 10^{-44}}{11.6 \times 10^{-26}}$$

$$= \frac{1.85}{11.6} \times 10^{-18}\,\text{J} = \frac{18.5}{11.6} \times 10^{-19}\ \text{J}$$

$$= \frac{18.5}{11.6}\, e\text{V}$$

So, K.E. of recoil nucleus = 1.58 eV (nearly)

Q. 14. *A body of 3kg makes elastic collision with another body at rest and afterwards continues to move in the original direction but with one half of its original speed. What is the mass of the struck body?*

Ans. Let, mass of struck body = M

Velocity of struck body = V

Velocity of moving body which collides = v

From law of conservation of momentum,

Total momentum before collision = Total momentum after collision

$$3 \times v = 3 \times \frac{v}{2} + MV \qquad \text{or,} \qquad V = \frac{3v}{2M}$$

Law of conservation of energy gives

$$\frac{1}{2} \times 3v^2 = \frac{1}{2} \times 3\left(\frac{v}{2}\right)^2 + \frac{1}{2} MV^2$$

$$\frac{9}{4} v^2 = MV^2$$

Substituting the value of V we get

$$\frac{9}{4}v^2 = M\left(\frac{3v}{2M}\right)^2 \quad \text{or} \quad M = 1 \text{ kg}.$$

Q. 15. *The empty stages of a two stage, rocket separately weigh 6000 and 500 kg and contain 40000 and 3500 kg fuel respectively. If the exhaust velocity is 1.8 km/s. Find the final velocity. ($\log_e 10 = 2.3$; $\log_{10} 2 = 0.30$).*

Ans. Total mass of rocket = (6000 + 500 + 40000 + 3500) kg = 50,000 kg

Mass at the end of I stage = 6000 + 3500 + 500 = 10,000 kg

exhaust velocity = 1.8 km/s

Thus, velocity at the end of I stage, V_1; is

$$V_1 = V_0 + v\log_e\left(\frac{M_0}{M}\right) = 0 + 1.8\log_e\left(\frac{50,000}{10,000}\right)$$

$$= 1.8\log_e 5 = 1.8\log_e\frac{10}{2} = 1.8\,[\log_e 10 - \log_e 2]$$

$$= 1.8 \times [2.3 - 2.3 \times 0.3] = 1.8 \times [2.3 - 0.69]$$

$$= 1.8 \times 1.61 = 2.898$$

V_1 = 2.898 km/s, which gives initial velocity of second stage

For II stage, V_0 = 2.9 (nearly) M = 500 kg

M_0 = 3500 + 500 = 4000

$$V_2 = V_0 + 1.8\log_e\left(\frac{4000}{500}\right) = 2.9 + 1.8\log_e 2^3$$

$$= 2.9 + 1.8 \times 3 \times 2.3\log_{10} 2 = 2.9 + 1.8 \times 3 \times 2.3 \times 0.3$$

$$= 2.9 + 3.726 = 6.626 \text{ km/sec}$$

∴ Final velocity at the end of II stage = 6.626 km/sec.

Q. 16. *A 5000 kg rocket is set for vertical firing. If the exhaust velocity is 500 m/s, how much gas must be ejected per second to supply the thrust needed to*

(i) *Overcome the weight of rocket.*

(ii) *Give rocket an initial upward acceleration 19.6 m/sec^2*

Ans. The net upward force on rocket at an instant is given by

$$F = M.\frac{dV}{dt} = -V\frac{dM}{dt} - Mg$$

To just overcome the weight of rocket, net force is zero, *i.e.*,

$$0 = -V\frac{dM}{dt} - Mg,$$

or

$$\frac{dM}{dt} = \frac{Mg}{V} = \frac{5000 \times 9.8}{500} = 98 \text{ kg/sec}$$

i.e., rate of gas ejection should be 98 kg/sec.

(*ii*) If '*a*' is the upward acceleration of rocket, we have

$$F = -V\frac{dM}{dt} - Mg = Ma$$

or,

$$\frac{dM}{dt} = -\frac{1}{V}M(g+a) = -\frac{1}{500}.5000\,(9.8+19.6)$$

$$= -294 \text{ kg/sec.}$$

So, gas ejection should be at the rate of 294 kg/sec.

Q. 17. *A rocket starts vertically upwards with speed* u_0, *show that its speed v at a height h is given by* $u_0^2 - V^2 = \dfrac{2gh}{1+\dfrac{h}{R}}$.

where R is the radius of earth, and g is acceleration due to gravity at earth's surface. Hence deduce an expression for maximum height reached by a rocket fired with speed 90% of escape velocity.

Ans. The velocity of rocket at height h is

$$V^2 = u_0^2 - 2\int_o^h g'\,dh \qquad \text{...(i)}$$

[As rocket rises up, h and g change]

At earth surface $\qquad g = \dfrac{GM}{R^2}$

and g' at height h is given by

$$g' = \frac{GM}{(R+h)^2} = \frac{GM}{R^2\,(1+h/R)^2} = \frac{g}{(1+h/R)^2}$$

putting the value of g′ in eqn. (*i*)

$$V^2 = u_0^2 - 2\int_0^h \frac{g}{(1+h/R)^2}.dh$$

$$V^2 = u_0^2 - 2g\left[-\frac{R}{1+h/R}\right]_0^h$$

$$= u_0^2 + 2g\left[\frac{R}{1+h/R} - R\right]$$

$$V^2 = u_0^2 + 2g\left[\frac{R-R-h}{1+h/R}\right] = u_0^2 - \frac{2gh}{1+h/R}$$

$$V^2 - u_0{}^2 = -\frac{2gh}{1 + h/R} \quad \text{or} \quad u_0^2 - V^2 = \frac{2gh}{1 + h/r}$$

Since the escape velocity = $\sqrt{2gR}$, the escape velocity in present case = $0.9\sqrt{2gR}$. At highest point attained by rocket V = 0, hence the above equation gives

$$0 - 0.81 \times 2gR = \frac{2gh}{R + h}.R$$

$$0.81 = \frac{h}{R + h} \quad \text{or} \quad 0.81R + 0.81h = h$$

$$h - 0.81h = 0.81\ R$$

$$h = \frac{81}{19}R = 4.26\ R$$

Q. 18. *The weight of an empty rocket is 5000 kg and it contains 40,000 kg fuel. If the exhaust velocity of the fuel is 2 km/s, find the maximum velocity attained by the rocket. ($\log_e 10 = 2.3$ and $\log_{10} 3 = 0.4171$).*

Ans. When gravity effect is neglected, the velocity of the rocket at any instant is given by;

$$V_1 = V_0 + V \log_e \frac{M_0}{M}$$

where, M_0 = initial mass; M = mass at any instant, V = exhaust velocity; V_0 = initial velocity, V_1 is velocity of the rocket at any instant.

Maximum speed is obtained when all the fuel is exhausted.

Now, $V_0 = 0, \quad m_0 = 5000 + 40{,}000 = 45000$ kg,

$M = 5000$ Kg, $V = 2$ km/sec.

So,
$$V_{max.} = 0 + 2 \log_e (45000/5000) = 2 \log_e 3^2 = 4 \log_e 3$$
$$= 4 \times 2.3 \times \log_{10} 3 = 4 \times 2.3 \times 0.4771$$
$$= 9.2 \times 0.4771 \text{ km/s} = 4.4 \text{ km/s}$$

Q. 19. *A rocket of mass 30 kg has 200 kg of fuel and the exhaust velocity of fuel is 1.6 km/sec. Calculate the minimum rate of consumption of fuel, so that the rocket may rise from the ground. Also calculate the final vertical velocity gained by the rocket when fuel consumption rate is (i) 2 kg/sec. (ii) 20 kg/sec.*

Ans. Rocket rises from the ground when fuel consumption occurs at minimum rate such that the thrust, balances the initial weight of rocket.

$\therefore$ Net force on rocket = 0

$$V\frac{dM}{dt} = Mg \qquad \text{...}(i)$$

Now, $M = 200 + 30 = 230$ kg

$V = 1.6 \text{ km/s} = 1.6 \times 10^3 \text{ m/s}$

from (*i*); $$\frac{dM}{dt} = \frac{Mg}{V} = \frac{230 \times 9.8}{1.6 \times 10^3} = 1409 \text{ kg/s}$$

Final velocity, $$V = V_0 + V \log_e (M_0/M) - gt$$

Case I: $$V = 1.6 \text{ km/s} = 1.6 \times 10^3 \text{ m/s}$$

$$t = \frac{200 \text{ kg}}{2 \text{ kg/sec}} = 100 \text{ sec.}$$

given, $$g = 9.8 \text{ m/s}^2$$

$$v_0 = 0$$

$$M_0 = 230, M = 30$$

$$V_{max} = 1.6 \times 10^3 \times \log_e (230/30) - 9.8 \times 100$$

$$= 1.6 \times 10^3 \times 2.3 \log_{10} (23/3) - 980.0$$

$$= 2.284 \text{ km/s}$$

Case II: All the quantities will be same as in case I:

but, $$t = \frac{200 \text{ kg}}{20 \text{ kg/s}} = 10 \text{ sec}$$

$\therefore$ $$V_{max} = 3.16 \text{ km/s}$$

Q. 20. *A body moving in straight line suddenly explodes into two parts. If one of these is twice as heavy as the other one and the two parts move in opposite direction with same speed, show that mechanical energy released in explosion is at least 8 times the initial kinetic energy.*

Ans. Let, V_1 and V_2 be the speeds of smaller and larger parts. Then using law of conservation of momentum

$$-mV_1 + 2mV_2 = 3mV \quad \text{(V-speed of original body)}$$

But as given, $$V_2 = -V_1 = V' \text{ (say)}$$

$$-mV' + 2mV' = 3\,mV$$

$$V' = 3\,V$$

This means each part moves with thrice the initial speed of original body in opposite direction. So, from law of conservation of energy, the energy released in explosion.

$$\Delta U = \frac{1}{2}(3m)\,V^2 - \left[\frac{1}{2}m(3V)^2 + \frac{1}{2} \times 2m\,(3V)^2\right]$$

$$= \frac{1}{2}(3m)\,V^2\,[1-(3+6)] = -8\,U$$

where U is initial K.E. This proves that released energy is 8 times the initial energy.

Q. 21. *A bullet of mass m moving with horizontal velocity v strikes a stationary block of mass M suspended by a string of length L. The bullet gets embeded in the block. What is the maximum angle made by the string after impact.*

Ans. Suppose block and bullet system moves horizontally with speed V, after bullet gets embeded. Then from law of conservation of momentum:

$$mv = (M + m)\,V$$

$$V = \frac{mv}{M + m}$$

The K.E. gained by block will be converted into P.E. when string reaches final deflected position. Hence if θ = is the maximum angle string makes after impact,

$$\frac{1}{2}(M + m)V^2 = (M + m)\,gh$$

$$\frac{1}{2}V^2 = g\,(L - L\cos\theta)$$

$$V = \sqrt{2gL\,(1 - \cos\theta)} = \sqrt{4gL\sin^2\theta/2}$$

Putting the value of V

$$\frac{mv}{M + m} = \sqrt{4\,g\,L\sin^2\theta/2}$$

$$\sin\frac{\theta}{2} = \frac{mv}{2\,(M + m)\sqrt{gL}}$$

$$\theta = 2\sin^{-1}\left[\frac{mV}{2\,(M + m)}\sqrt{1/gL}\right]$$

Q. 22. *The maximum and minimum distances of a comet from the sun are* 1.4×10^{12} *m and* 7×10^{10} *m. If its velocity nearest to the sun is* 6×10^4 *m/s, what is its velocity when it is in farthest position. Assume in both position that the comet is moving in circular path.*

Ans. From law of conservation of angular momentum;

$$m_1V_1r_1 = m_2V_2r_2 \quad \text{(for circular orbit)}$$

here,

$$m_1 = m_2 = m$$

$$V_1r_1 = V_2r_2$$

$$V_2 = \frac{V_1r_1}{r_2} = \frac{6 \times 10^4 \times 7 \times 10^{10}}{1.4 \times 10^{12}}$$

$$V_2 = 3000 \text{ m/s}$$

Q. 23. *A particle of mass m performs motion along a path which is given by equation* $\vec{r} = \hat{i}\,a\cos wt + \hat{j}\,b\sin wt$.

Calculate the angular momentum and torque about the origin.

Ans.

$$\vec{r} = \hat{i}\,a\cos\omega t + \hat{j}\,b\sin\omega t$$

∴

$$\vec{V} = \frac{d\vec{r}}{dt} = \omega\,(\hat{i}\,a\sin\omega t + \hat{j}\,b\cos\omega t)$$

The angular momentum is given by

$$\vec{J} = \vec{r} \times \vec{P} = \vec{r} \times m\vec{V}$$

$$= (\hat{i}\ a \cos\omega t + \hat{j}\ b \sin\omega t) \times m\ (\hat{i}\ a \sin\ \omega t + \hat{j}\ b \cos\ \omega t)\ \omega$$

$$= ab\ m\ \hat{k}\ \omega\ (\cos^2 \omega t + \sin^2\ \omega t)$$

$$= m\omega a b\ \hat{k}$$

by definition, torque is,

$$\vec{\tau} = \frac{d\vec{J}}{dt}$$

$\therefore$
$$\vec{\tau} = \frac{d}{dt}(m\omega a b\hat{k})$$

$$= 0 \text{ (as all quantities are constant)}$$

Q. 24. *A particle of mass 20 gm moving in a circle of 4 c.m. radius with constant speed of 10 cm/s. What is its angular momentum (1) about the centre of the circle (2) a point on the axis of the circle and at 3 cm distance from its centre.*

Ans. (1) $J = mvr = 20 \times 10 \times 4 = 800$ erg. sec.

indirection perpendicular to plane of circle.

(2) QP = distance of particle from point of reference

$$= \sqrt{3^2 + 4^2} = 5 \text{ cm}$$

So, $J = mvr = 20 \times 10 \times 5 = 1000$ erg. sec.

Its direction will also be perpendicular to plain containing the radius vector (QP) and instantaneous velocity; [Note: The direction of $\vec{J}$ change in this case in First case direction of $\vec{J}$ does not change].

Q. 25. *A particle of mass 1/2 gm starts from the point (1, 0, 2) at a time t = 0. If force acting on it is* $2\hat{i} + (1+6t)\,\hat{j}$ *dynes. Calculate the torque and angular momentum about the point (2, 2, 4) at time t = 1 sec.*

Ans. Force $\vec{F} = m\dfrac{d^2\vec{r}}{dt^2}$; or $\dfrac{d^2\vec{r}}{dt^2} = \dfrac{\vec{F}}{m}$

$$\frac{d^2\vec{r}}{dt^2} = 2\left\{2\hat{i} + (1+6t)\,\hat{j}\right\}$$

Integration yields;

$$\frac{d\vec{r}}{dt} = 2\left\{2t\hat{i} + (t + 6\,t^2/2)\ \hat{j}\right\} + K \qquad ...(1)$$

(K = constant of integration)

as velocity, $V = 0$, at $t = 0$; (1) then yields, $K = 0$

So (1) is $$\frac{d\vec{r}}{dt} = 4t\hat{i} + (2t + 6t^2)\,\hat{j} \qquad ...(2)$$

Further integration gives

$$\vec{r} = 2t^2\,\hat{i} + (t^2 + 2t^3)\hat{j} + k' \qquad ...(3)$$

Given at $t = 0$; particle is at (1, 0, 2)

$$\vec{r}_0 = 1\hat{i} + 0\,\hat{j} + 2\hat{k} = \hat{i} + 2\hat{k} = k'$$

thus, $$\vec{r} = 2.t^2\,\hat{i} + (t^2 + 2t^3)\hat{j} + \hat{i} + 2\hat{k} \qquad ...(4)$$

So, at $t = 1$; $$\vec{r}_1 = 3\hat{i} + 3\,\hat{j} + 2\hat{k} \qquad ...(5)$$

Distance $\vec{R}$ from given point of reference; $(2\hat{i} + 2\,\hat{j} + 4\,\hat{k})$ is

$\therefore$ $$\vec{R} = 3\hat{i} + 3\,\hat{j} + 2\hat{k} - 2\hat{i} - 2\,\hat{j} - 4\,\hat{k} = \hat{i} + \hat{j} - 2\hat{k}$$

Thus at time $t = 1$, (refer eqns. (5), (2) and (1))

$$\left.\begin{aligned} \vec{R} &= \hat{i} + \hat{j} - 2\hat{k} \\ \vec{V} &= 4\hat{i} + 8\hat{j} \\ \vec{F} &= 2\hat{i} + 7\hat{j} \end{aligned}\right] \qquad ...(6)$$

So, torque; $$\vec{\tau} = \vec{R} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 2 & 7 & 0 \end{vmatrix}$$

$$= \hat{i}\,(0 + 14) - \hat{j}\,(0 + 4) + \hat{k}\,(7 - 2)$$

$$\vec{\tau} = 14\hat{i} - 4\hat{j} + 5\hat{k}$$

and the angular momentum at $t = 1$, is = $\vec{R} \times m\vec{V}$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 4 & 8 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left[\hat{i}\,(16) - \hat{j}\,(0 + 8) + \hat{k}\,(8 - 4)\right]$$

$$= 8\hat{i} - 4\hat{j} + 2\hat{k}$$

Q. 26. *A particle of mass m moving in a circular orbit of radius r has angular momentum $\vec{L}$ about the centre. Calculate the K.E. of the particle and the centripetal force acting on it.*

Ans. Angular momentum of a particle moving in circular path

$$L = mr^2\omega; \quad \text{or,} \quad r\omega = \frac{1}{mr}$$

$$K.E. = \frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2$$

So,

$$K.E. = \frac{1}{2}m\left(\frac{1}{mr}\right)^2 = \frac{1}{2}\frac{L^2}{mr^2}$$

$$\text{Centripetal force} = \frac{mv^2}{r} = \frac{m}{r}.(r\omega)^2 = \frac{m}{r}.\left(\frac{L}{mr}\right)^2$$

$$= \frac{L^2}{mr^3}$$

Q. 27. *A meter stick of mass M lies on a smooth horizontal table. A fall of mass m moving with velocity V collides elastically and normally with stick at a distance d as shown in figure. Find the velocity of ball after collision.*

Ans. The linear momentum, angular momentum as well as K.E. will be conserved in elastic collision.

Let, v' = velocity of ball after collision,

V = velocity of centre of mass of stick,

ω = the angular velocity of stick after collision,

and $I = \frac{M \times l^2}{12}$, the moment of inertia of stick about centre of mass.

From law of conservation of linear momentum

$$mV = MV + mV'$$

therefore

$$V = \frac{m\,(V - V')}{M} \quad ...(i)$$

Law of conservation of angular momentum gives

$$mVd = mV'd + Iw$$

$$m(V - V')d = \frac{Mwl^2}{12}$$

$$\omega = \frac{12\,m\,(V - V')d}{Ml^2} \quad ...(ii)$$

Also law of conservation of K.E. yields

$$\frac{1}{2}mV^2 = \frac{1}{2}mV'^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}MV^2$$

$$m\,(V^2 - V'^2) = \frac{Mw^2\, l^2}{12} + MV^2 \qquad \text{...(iii)}$$

Putting the values of V and ω in eqn. (*iii*)

$$m\,(V^2 - V'^2) = \frac{12m^2 d^2\,(V - V')^2\, l^2}{Ml^4} + \frac{m^2\,(V - V')^2}{M}$$

which gives

$$V + V' = \frac{m\,(V - V')}{Ml^2}\,(12d^2 + l^2)$$

$$V' = \frac{m\,(V - V')\,(12d^2 + l^2)}{Ml^2} - V$$

Q. 28. *Determine the impact parameters of 2 MeV, α-particle whose distance of closest approach to a gold nucleus (Z = 79) is 2 × 10⁻¹¹ cm.*

Ans.

$$\text{Initial K.E.} = \frac{1}{2}mu^2 = [2 \times 10^6 \times 1.6 \times 10^{-12}] \text{ ergs}$$

$$= [3.2 \times 10^{-6}]$$

Angular momentum is conserved during scattering

At G; Angular momentum of α^{++} particle about N

$$= m_\alpha Pu \qquad \text{...}(i)$$

Angular momentum of α^{++} particle about N: (when approach is closest)

$$= m_\alpha.S.V \qquad \text{...}(ii)$$

or

$$m_\alpha\, Pu = m_\alpha.S.V$$

$$V = \frac{Pu}{S} \qquad \text{...}(iii)$$

Energy conservation gives, $\frac{1}{2}mu^2 = \frac{1}{2}mV^2 + \frac{Ze.(2e)}{S} = \frac{1}{2}\frac{mu^2P^2}{S^2} + \frac{2\,Ze^2}{S}$

$$\frac{1}{2}mu^2 - \frac{1}{2}\frac{mu^2P^2}{S^2} = \frac{2\,Ze^2}{S}$$

$$\frac{1}{2}mu^2\left(1 - \frac{P^2}{S^2}\right) = \frac{2\,Ze^2}{S}$$

$$1 - \frac{P^2}{s^2} = \frac{2\,Ze^2}{s} \times \frac{2}{mu^2}$$

$$1 - \frac{P^2}{S^2} = \frac{2 \times 79 \times (4.8 \times 10^{-10})^2}{2 \times 10^{-11}} \cdot \frac{1}{3.2 \times 10^{-6}} = 0.57$$

$$\frac{P^2}{S^2} = 1 - 0.57 = 0.43$$

$$P = S\sqrt{0.43} = 2 \times 10^{-11}\sqrt{0.43}$$

$$P = 1.3 \times 10^{-11} \text{ cm}$$

Q. 27. *Calculate the angular momentum of 1 MeV neutron about a nucleus of impact parameter 10^{-10} cm. (m = 1.7 × 10^{-24} gm).*

Ans. Initial K.E. $= \frac{1}{2}mu^2 = 1 \times 10^6$ eV

$$= 1.6 \times 10^6 \times 10^{-12} \text{ ergs}$$

$$= 1.6 \times 10^{-6} \text{ ergs.}$$

So,

$$u = \left[\frac{2 \times 1.6 \times 10^{-6}}{1.7 \times 10^{-24}}\right]^{1/2} \text{cm/s}$$

Neutron being a light neutral particle experiences no force and goes straight way (P = impact parameter). Its angular momentum.

$$= m \times \text{P} \times u$$

$$= 1.7 \times 10^{-24} \times 10^{-10} \times \left[\frac{2 \times 1.6 \times 10^{-6}}{1.7 \times 10^{-24}}\right]^{1/2}$$

$$= 1.93 \times 10^{-25} \text{ gm-cm/sec.}$$

Q. 30. *A light weight man holds heavy dumb-bells in straightened arms when standing on a turn table. The turn table then start rotating at rate of 1 rev/min. The man then pulls the dumb-bells inside toward chest. If initially dumb-bells are 60 cms from his axis of rotation and are pulled to 10 cms from rotation axis, find the frequency of revolution of turn table. Neglect the angular momentum of man in comparison to that of dumb-bells.*

Ans. Initial angular momentum of system (neglecting the angular momentum of man) $= 2\text{M}\omega_1 r_1^2$; where M is mass of dumb-bells, ω is initial rotational frequency, r_1 is distance of dumb-bells from axis.

Final angular momentum when arms are stretched; $= 2\text{M}\omega_2 r_2^2$, where ω_2 and r_2 are changed frequency and distance due to arm stretching. Since no external torque is applied, angular momentum will be conserved.

i.e.,

$$2\text{M}\omega_1 r_1^2 = 2\text{M}\omega_2 r_2^2$$

$$2\pi f_1 r_1^2 = 2\pi f_2 r_2^2$$

$$f_2 = \frac{f_1 r_1^2}{r_2^2} = \frac{1 \times (60)^2}{(10)^2} = \frac{3600}{100}$$

$$f_2 = 36 \text{ rev/min.}$$

Q. 31. *Compute the orbital angular momentum and total energy of the electron in hydrogen atom, assuming the path to be circular.*

Ans. Hydrogen atom is one electron and one proton system. The required centripetal force for circular motion is provided by Coulombian attraction force;

$$\therefore \quad \frac{m\text{V}^2}{r} = \frac{\text{Z}e.e}{r^2} = \frac{e^2}{r^2} \quad ...(i)$$

$$m\text{V}^2 r^2 = e^2 r$$

$$(m\text{V}r)^2 = me^2 r$$

putting the standard values,

$$m = 9 \times 10^{-28} \text{ gm}; \; e = 4.8 \times 10^{-10} \text{ esu}; \; r = 0.5 \times 10^{-8} \text{ cms:}$$

The angular momentum of revolving electron is

$$J = mVr = \sqrt{me^2 r}$$

$$J = \sqrt{9 \times 10^{-28} \times (4.8 \times 10^{-10})^2 \times (0.5 \times 10^{-8})}$$

$$= 4.8 \times 10^{-28} \times 3\sqrt{0.5} = 14.4 \times 10^{-28} \sqrt{0.5}$$

$$= 1.02 \times 10^{-27} \text{ g cm}^2/\text{sec.}$$

from Eqn. (*i*) $$\text{K.E.} = \frac{1}{2} mV^2 = \frac{e^2}{2r}$$

also $$\text{P.E.} = -\frac{e^2}{r}$$

$\therefore$ $$\text{Total energy} = \frac{e^2}{2r} - \frac{e^2}{r} = -\frac{e^2}{2r} = \frac{-(4.8 \times 10^{10})^2}{2 \times 0.5 \times 10^{-8}}$$

$$= -23.05 \times 10^{-12} \text{ ergs}$$

$$= \frac{-23.05 \times 10^{-12}}{1.6 \times 10^{-12}} = -14.4 \text{ eV}$$

Q. 32. *A nucleus of mass m emits a gamma ray photon of frequency ν_o. Show that the loss of internal energy by the nucleus is not $h\nu_o$ but is $h\nu_o \left(1 + h\frac{\nu_o}{2mc^2}\right)$.*

Ans. The initial momentum of nucleus is zero. Let its momentum after emission of photon be P.

$$\text{The momentum of photon} = \frac{h\nu_o}{c}$$

The conservation of linear momentum suggests,

$$0 = P + \frac{h\nu_o}{c}; \quad \text{or} \quad P = \frac{-h\nu_o}{c}$$

Then the internal energy which has converted into K.E. of nucleus;

$$\text{K.E.} = \frac{P^2}{2m} = \frac{(-h\nu_o)^2}{2mc^2}$$

Internal energy which has converted into radiant energy of photon = $h\nu_o$

So total loss in the internal energy of nucleus

$$= \frac{(h\nu_o)^2}{2mc^2} + h\nu_o = h\nu_o \left(1 + \frac{h\nu_o}{2mc^2}\right)$$

Q. 33. *Locate the centre of mass of a system of three particles of masses 1 kg, 2 kg and 3 kg placed at the corners of an equilateral triangle of 1 meter.*

Ans. Choosing the co-ordinate system as shown in figure:

$$\text{Position of } m_1 = 0, \text{ Position of } m_2 = 1\hat{i}$$

$$\text{Position of } m_3 = \frac{1}{2}\hat{i} + \hat{j}\sin 60 = \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$$

Then position of C.M.

$$\vec{R} = \frac{m\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3}}{m_1 + m_2 + m_3}$$

$$= \frac{1 \times 0 + 2 \times (1\hat{i}) + 3 \times \frac{1}{2}(\hat{i} + \sqrt{3}\,\hat{j})}{1 + 2 + 3}$$

$$= \frac{2\hat{i} + \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}}{6}$$

$$= \left(\frac{7}{12}\hat{i} + \frac{\sqrt{3}}{4}\hat{j}\right)$$

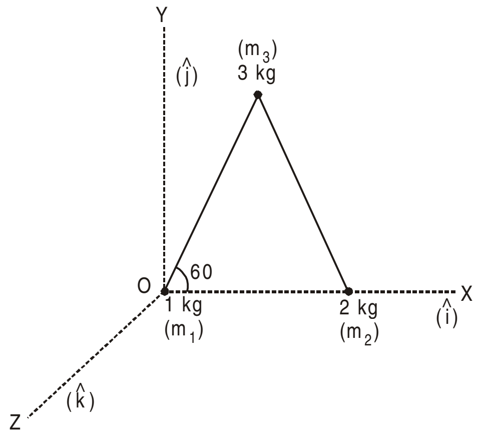


Fig. 14

So co-ordinates of C.M. is, $\left(\frac{7}{12}, \frac{\sqrt{3}}{4}, 0\right)$ meters.

Q. 34. *A particle moves in a force field given by* $\vec{F} = f(r)\,\hat{r}$, *where* $\hat{r}$ *is the unit vector along position vector of particle. Prove that the angular momentum of the particle is conserved.*

Ans. Torque acting on particle is

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times f(r)\,\hat{r} = f(r)\,(\vec{r} \times \hat{r}), \; \vec{\tau} = 0$$

Since external torque on the particle is zero, so according to theorem of conservation of angular momentum, its angular momentum shall be conserved.

Q. 35. *A stream of α-particles is bombarded on a mercury nucleus (z = 80) with velocity 1×10^7 m/s. If an α-particle is approaching the nucleus in head on direction. Calculate the distance of closest approach.*

Ans. Distance of closest approach (d) for α-particles is given, in head on approach, by

$$d = \frac{1}{4\pi \in_o} \frac{4Ze^2}{mV_o^2}$$

$$= \frac{9 \times 10^9 \times 4 \times 80 \times (1.6 \times 10^{-19})^2}{(6.4 \times 10^{-27}) \times (1 \times 10^7)^2}$$

$$= 1.15 \times 10^{-13} \text{ m}$$

Q. 36. *Express in term of angular momentum (J), the kinetic, potential and total energy of a satellite of mass m in a circular orbit of radius r.*

Ans. Angular momentum of the satellite in circular orbit is;

(a) $J = mVr$ if the K.E. be T; then

$$T = \frac{1}{2}mV^2 = \frac{J^2}{2mr^2} \quad ...(1)$$

(b) Potential energy of satellite is: $U = -\frac{GMm}{r}$, where M = mass of the earth. But for a satellite moving in circular orbit we have;

$$\frac{GMm}{r^2} = \frac{mV^2}{r}, \quad \text{or} \quad \frac{GMm}{r} = mV^2$$

then, $\quad U = -mV^2$ and with $J = mVr$

$$U = -\frac{J^2}{mr^2} \quad ...(2)$$

(c) total energy (E) = K.E. + P.E. = $\frac{1}{2}mV^2 - mV^2 = -\frac{1}{2}mV^2$

then in view of Eqn. (2): $\quad E = -\frac{J^2}{2mr^2}$

Q. 37. *"The ratio of maximum to minimum velocity of a planet in its circular orbit equals the inverse of the ratio of the radii of the orbits" – Establish the statement.*

Ans. Let V_1 and V_2 be the maximum and minimum velocity of the planet with mass m and r_1 and r_2 be the radii of the circular orbits. The principle of conservation of angular momentum suggests;

$$mV_1r_1 = mV_2r_2$$

or $\quad \frac{V_1}{V_2} = \frac{r_2}{r_1}$ hence the statement.

Q. 38. *Calculate the momentum of an electron accelerated by a potential of 100 volts.*

Ans. K.E. of electron, $$T = \frac{1}{2}mV^2 = \frac{1}{2m}(mV)^2 = \frac{P^2}{2m}$$

where P = mV = linear momentum of electron.

But $T = eV$; so we have

$$eV = \frac{P^2}{2m}, \text{ or, } P = \sqrt{2meV}$$

here, $m = 9 \times 10^{-28}$ gm, $e = 4.8 \times 10^{-10}$ esu, V = 100 volts = $\frac{100}{300}$ stat volts.

$$P = \sqrt{2 \times 9 \times 10^{-28} \times 4.8 \times 10^{-10} \times 100/300}$$

$$= 5.37 \times 10^{-19} \text{ gm cm/sec.}$$

Q. 39. *A meteorite burns in the atmosphere before it reaches the earth surface. What happens to its momentum.*

Ans. Taking the earth, atmosphere and meteorite to form a closed system, the momentum lost by meteorite is taken up by the combustion products, molecules of the atmosphere (*i.e.,* air) which interact with the meteorite and the earth. The momentum lost and gained remain same and so the total momentum of system is conserved.

Q. 40. *A 20 gm bullet passes through a plate of mass M_1 = 1 kg and then comes to rest inside a second plate of mass M_2 = 2.98 kg. It is found that the two plates initially at rest now move with equal velocities. Find the percentage loss in the initial velocity of the bullet when it is between M_1 and M_2. Neglect any loss of material of the plates due to action of bullet.*

Ans. Let V_1 = initial velocity of bullet; V_2 = bullet velocity after emergence from first plate M_1. Also let m be the mass of the bullet and V be the velocity of plates after impact. Collision is inelastic, so linear momentum is conserved *i.e.,* for first plate.

$$mV_1 = mV_2 + M_1V \quad ...(i)$$

For second plate when bullet enters it with initial velocity V_2, the conservation of momentum gives;

$$mV_2 = (m + M_2)V \quad ...(ii)$$

from (*i*); $$M_1V = m(V_1 - V_2)$$

or $$V = \frac{m}{M_1}(V_1 - V_2) \quad ...(iii)$$

Using this value in eqn. (*ii*)

$$mV_2 = (m + M_2)\frac{m}{M_1}(V_1 - V_2)$$

$$M_1V_2 = (m + M_2)(V_1 - V_2) = mV_1 - mV_2 + M_2V_1 - M_2V_2$$

or $$V_2(m + M_1 + M_2) = (m + M_2)V_1$$

or $$V_2 = \frac{(m + M_2)V_1}{(m + M_1 + M_2)} = \frac{(0.02 + 2.98)V_1}{(0.02 + 1 + 2.98)} = 0.75\ V_1$$

So loss in velocity after emergence from first plate

$$= V_1 - V_2 = V_1 - 0.75V_1 = 0.25V_1$$

$$\text{hence \% loss} = \frac{0.25\ V_1}{V_1} \times 100 = 25\%$$

Q. 41. *A .5 kg block slides down from the point A as shown in figure on a horizontal track with an initial speed of 3 m/s towards weightless horizontal spring of length 1 m and force constant 2 Nt/meter. The part AB of the track is frictionless and the part BC has the coefficient of static and kinetic friction as 0.22 and 0.2 respectively. If the distance AB and BD are 2m and 2.14 m respectively, find total distance through which block moves before it comes to rest completely.*

Ans. Speed of block when it comes at B is same as that at A because path AB is frictionless. Then K.E. of block at position B is

$$= \frac{1}{2}mV^2 = \frac{1}{2} \times 0.5 \times 3^2 = \frac{9}{4} \text{ Joules.}$$

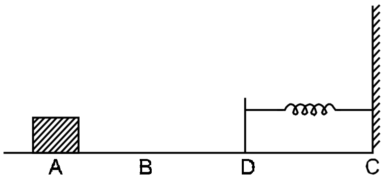


Fig. 15

Suppose the block after travelling frictionful path BD, further travels distance x in which it comes to rest after compressing the spring. Total frictionful path traversed in this journey is (BD + x) *i.e.,* (2.14 + x). Work done in this journey against force of friction is;

$$W(BD + x) = \mu_k.mg \times (2.14 + x) \text{ J}$$
$$= 0.2 \times 0.5 \times 10 \times (2.14 + x) \text{ J}$$
$$= (2.14 + x) \text{ J.}$$

Work done in compressing the spring by x;

$$\frac{1}{2}Kx^2 = \frac{1}{2}2 \times x^2 = x^2 \text{ Joule}$$

$$\text{So total work done} = (2.14 + x + x^2) \text{ J}$$

$$\text{then loss of K.E. of block} = \text{work done}$$

$$\frac{1}{2}mV^2 - 0 = (2.14 + x^2 + x)$$

$$\frac{1}{2} \times 0.5 \times 3^2 = 2.14 + x + x^2 + x)$$

$$x^2 + x - 0.11 = 0; \quad \text{or;} \quad x = \frac{1}{2}\left[-1 \pm \sqrt{1 + 4 \times .11}\right] = 0.1 \text{ m}$$

Now, the compressed spring forces back the block with a force.

$$F = Kx = 2 \times 0.1 = 0.2N$$

But force of friction on the block = $\mu_s mg = 0.22 \times 0.5 \times 10 = 1.1$ N

It is evident that block will not move back as the pushing back force is smaller than the frictional force. Then total distance moved by the block.

$$= AB + BD + x = 2 + 2.14 + 0.1 = 4.24 \text{ m}$$

Q. 42. *Calculate the intrinsic angular momentum of (i) photon, (ii) electron and (iii) π-meson.*

Ans. (*i*) Photon: $P_s = \sqrt{S(S+1)}.\frac{h}{2\pi}$, S = 1, $h = 6.626 \times 10^{-27}$ erg-sec.

So, $$P_s = \sqrt{1(1+1)} \times \frac{6.626 \times 10^{-27}}{2 \times 3.14}$$

$$= 1.414 \times 1.053 \times 10^{-27} = 1.49 \times 10^{-27} \text{ gm cm}^2\text{/sec.}$$

(*ii*) Electron: Again $$P_s = \sqrt{S(S+1)}.\frac{h}{2\pi}; \text{ here } S = \frac{1}{2}$$

So, $$P_s = \sqrt{\frac{1}{2}(1/2+1)} \times \frac{6.62 \times 10^{-27}}{2 \times 3.14}$$

$$= \frac{1.732}{2} \times 1.054 \times 10^{-27} = 9.127 \times 10^{-28} \text{ gm cm}^2\text{/s}$$

(*iii*) π-meson $$P_s = \sqrt{S(S+1)} \times \frac{h}{2\pi}; \text{ here } S = 0$$

So $$P_S = 0$$

Q. 43. *A cart of mass is M moving with a constant velocity v on a horizontal track. A monkey of mass m jumps from a tree on to cart just from above. Find the velocity of cart after the event.*

Ans. Considering cart and monkey as a single system, it is noted that there is no external horizontal force on the system.

Supposing that cart is moving in the $+x$ distance towards right, it is found, initial horizontal momentum of monkey = 0.

Initial horizontal momentum of cart = Mv

The final horizontal momentum of monkey–cart system if after the event system moves with velocity V.

$$= (M + m)V$$

The horizontal momentum of the system shall be conserved in the absence of external force, thus

$$O + Mv = (M + m)V$$

So; $$V = \frac{Mv}{M+m}$$

SELECTED PROBLEMS

1. State and explain the law of conservation of linear momentum, Explain the following

(*a*) A meteorite burns in the atmosphere before it reaches earth. What happens to its linear momentum ?

(*b*) When a ball is thrown up, its momentum first decreases, then increases. Does this violate the principle of conservation of momentum.

(*c*) Show that conservation of linear momentum is equivalent to Newton's third law.

2. What do you mean by "Centre of mass" of a system of particles/show that in absence of any external force the velocity of the centre of mass remains constant.

3. Show that in head on elastic collision between two particles, the transfer of energy is maximum when their mass ratio is unity.

4. Two bodies of mass M and m (at rest) collide and stick. Describe the collision in centre of mass frame. What is the ratio of kinetic energy transferred.

5. Describe principle of rocket. Establish the relation for the motion of rocket in horizontal motion, $V = V_0 + \log_e \frac{M_0}{M}$, where the symbols have their usual meanings.

6. (*a*) What is multistage rocket. Discuss its motion when it is moving in a field fee space with frictional forces are present as also when it moves in a region where gravitational force is present.

(*b*) What is multistage rocket? What is the advantage of a two stage rocket on a single stage rocket.

7. Show that the rocket speed is equal to the exhaust speed when the ratio $\frac{M_0}{M} = e^2$.

Show that the rocket speed is twice the exhaust speed when $\frac{M_0}{M} = e^2$.

CENTRE OF MASS : CONSERVATION OF LINEAR MOMENTUM

Q. 1. *Show that centre of mass of two particles must lie on the line joining them, and the ratio of the distances of the two particles from the centre of mass is the inverse ratio of their masses.*

Solution. Let there be two particles of masses m_1 and m_2 whose position vectors are $\vec{r}_1$ and $\vec{r}_2$ with respect to *a* fixed origin O. Let *c* be the centre of mass whose position vectors with respect to m_1 and m_2 are $\vec{d}_1$ and $\vec{d}_2$.

The position vector of C with respect to the origin O is defined by

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

Now by the law of vector addition, we have

$$\vec{r}_1 + \vec{d}_1 = \vec{r}_{cm} \quad \text{and} \quad \vec{r}_{cm} + \vec{d}_2 = \vec{r}_2$$

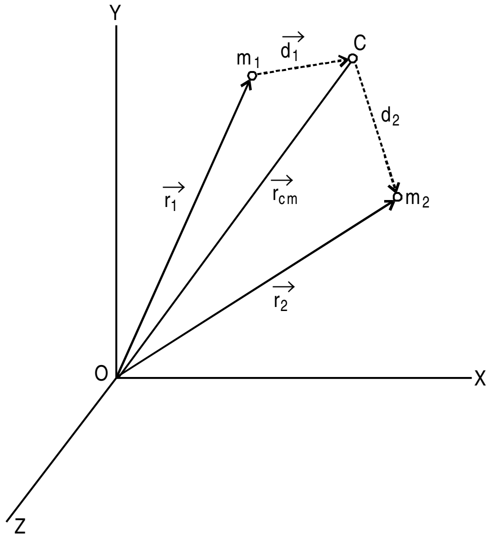


Fig. 16

These give

$$\vec{d_1} = \vec{r}_{cm} - \vec{r_1} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2} - \vec{r_1} = \frac{m_2(\vec{r_2} - \vec{r_1})}{m_1 + m_2}$$

$$\vec{d_2} = \vec{r_2} - \vec{r}_{cm} = \vec{r_2} - \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}(\vec{r_2} - \vec{r_1})$$

$\therefore$

$$\vec{d_1} = \frac{m_2}{m_1}\vec{d_2}$$

This shows that the vectors $\vec{d_1}$ and $\vec{d_2}$ are collinear *i.e.,* the centre of mass O lies on the line joining m_1 and m_2.

Further, from the last equation, we have

$$\frac{\vec{d_1}}{\vec{d_2}} = \frac{m_2}{m_1}$$

Hence the ratio of the distance of the particles from the centre of mass is the inverse ratio of their masses. This also shows that the position of the centre of mass is independent of the origin or the reference of frame chosen.

Q. 2. *The mass of the moon is about 0.013 times the mass of the earth and the distance from the centre of the moon to the centre of the earth is about 60 times the radius of the earth. Find the distance of the centre of the mass of the earth moon system from the centre of the earth. Take the radius of the earth 6400 km.*

Solution. Let d be the distance between earth and moon. Then if x be the distance of the centre of mass from earth, its distance from the moon will be $d - x$. The ratio of x to $d - x$ will be equal to the inverse ratio of the masses of earth and moon. That is,

$$\frac{x}{d-x} = \frac{\text{mass of moon}}{\text{mass of earth}} = 0.013$$

or $$x = (d - x)\ 0.013$$

or $$x\ (1 + 0.013) = 0.013\ d$$

or $$x = \frac{0.013}{1+0.013} d = 0.0128\ d.$$

Here $$d = 60 \times 6400 \text{ km}$$

$\therefore$ $$x = 0.0128 \times (60 \times 6400 \text{ km})$$

$$= \mathbf{4928\ km.}$$

Q. 3. *The distance between the centres of carbon and oxygen atoms in the carbon monoxide gas molecule is 1.130 × 10^{-10} meter. Locate the centre of mass of the molecule relative to the carbon atom.*

Solution. Let the molecule be along the x-axis, the carbon atom being at the origin. The centre of mass relative to the carbon atom is given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

where x_1 and x_2 are the distance relative to the carbon atom.

$$x_{cm} = \frac{(12 \times 0) + (16 \times 1.130 \text{ Å})}{12 + 16}$$

$$= \mathbf{0.6457\ Å\ Along\ the\ line\ of\ symmetry.}$$

Q. 4. *Locate the centre of mass of a system of three particles of masses 1.0 kg, 2.0 kg, and 3.0 kg placed at the corners of an equilateral triangle of 1 meter side.*

Solution. Let the triangle lie in the $x - y$ plane with its corner m_1 (1.0 kg) at the origin and side m_1, m_2 along the x-axis. Let (x_{cm}, y_{cm}) be the coordinates of the centre of mass C. By the definition of the centre of mass, we have

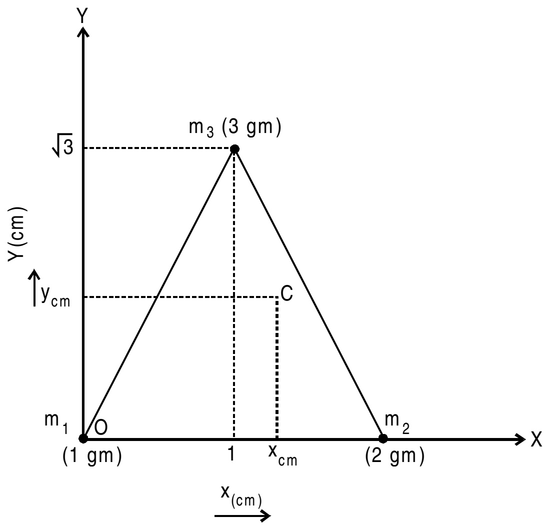


Fig. 17

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0 \text{ kg})\,(0) + (2.0 \text{ kg})\,(1 \text{ meter}) + (3.0 \text{ kg})\,(0.5 \text{ meter})}{(1.0 + 2.0 + 3.0) \text{ kg}}$$

$$= \frac{7}{12} \text{ meter}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$= \frac{(1.0 \text{ kg})\,(0) + (2.0 \text{ kg})\,(0) + (3.0 \text{ kg})\left(\frac{\sqrt{3}}{2} \text{ meter}\right)}{(1.0 + 2.0 + 3.0) \text{ kg}}$$

$$= \frac{\sqrt{3}}{4} \text{ meter}$$

The co-ordinates of the centre of mass are $\left(\frac{7}{12}, \frac{\sqrt{3}}{4}\right)$ meter.

Q. 5. *Two masses 6 and 2 units are at positions* $6\hat{i} - 7\hat{j}$ *and* $2\hat{i} + 10\hat{j} - 8\hat{k}$ *respectively. Deduce the position of their centre of mass.*

Solution. The position vectors of the masses $m_1 = 6$ and $m_2 = 2$ are $\vec{r}_1 = 6\hat{i} - 7\hat{j}$ and $\vec{r}_2 = 2\hat{i} + 10\hat{j} - 8\hat{k}$.

The position vector of the centre of mass is defined by

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$= \frac{1}{8}\left[6(6\hat{i} - 7\hat{j}) + 2(2\hat{i} + 10\hat{j} - 8\hat{k})\right]$$

$$= 5\hat{i} - 2.75\hat{j} - 2\hat{k}$$

The coordinates of the centre of mass are (5, –2.75, –2).

Q. 6. *Calculate the position of the C.M. of a uniform semi-circular plane lamina.*

Ans. ACB is a semi-circular lamina of radius a.

From symmetry we can conclude that the C.M. lies on OC, the perpendicular bisector of the base AB of the lamina.

Let mass of the lamina be M. Let the C.M. be at a distance R from O on the straight line OC.

$$|R| = OR = \frac{1}{M}\int_{o}^{a} r\, dm \qquad ...(1)$$

Here r is the radius of a semicircular element xy with centre at O and breadth 'dr' and mass 'dm'.

$$\text{Area of entire lamina} = \frac{\pi a^2}{2} \text{ (semicircular)}$$

$$\therefore \qquad \text{Mass perunit area} = \frac{M}{\frac{\pi a^2}{2}} = \frac{2M}{\pi a^2} \qquad ...(2)$$

Area of semicircular element, xy = length × breadth dt

$$= \pi r\, dr$$

$$\therefore \text{ Mass of } xy \text{ element, } dm = (\pi r\, dr)\frac{2M}{\pi a^2} = \frac{2M}{a^2}.rdr \qquad ...(3)$$

Substituting in (1), we get,

$$OR = \frac{1}{M}\int_{o}^{a} r\left[\frac{2M}{a^2} r\, dr\right] = \frac{1}{M}\frac{2M}{a^2}\int_{o}^{a} r^2\, dr$$

$$= \frac{2}{a^2}\left[\frac{r^3}{3}\right]_{o}^{a} = \frac{2a^3}{3a^2} = \frac{2a}{3}$$

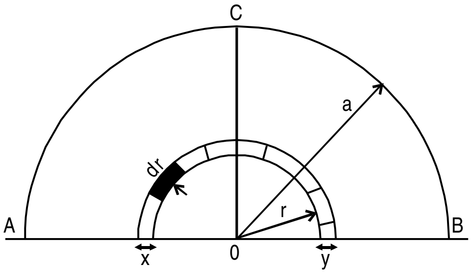


Fig. 18

The centre of mass of a plane semicircular lamina of radius 'a' will be at a height $2a/3$ from the centre and lies on the perpendicular bisector of the base of the lamina.

Q. 7. *A system consists of masses 7, 4 and 10 gm located at (1, 5, –3), (2, 5, 7) (3, 3, –1) respectively. Find the position of its centre of mass.*

Solution. The coordinates of the centre of mass (x_{cm}, y_{cm}, z_{cm}) are given by

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{7\,(1) + 4\,(2) + 10\,(3)}{7 + 4 + 10} = \frac{45}{21}$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{7\,(5) + 4\,(5) + 10\,(3)}{7 + 4 + 10} = \frac{85}{21}$$

$$z_{cm} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3}{m_1 + m_2 + m_3} = \frac{7\,(-3) + 4\,(7) + 10\,(-1)}{7 + 4 + 10} = \frac{-3}{21}$$

The centre of mass is located at $\left(\frac{45}{21}, \frac{85}{21}, \frac{-3}{21}\right)$.

Q. 8. *If the centre of mass of three particles of masses 2, 4 and 6 gm be at the point (1, 1, 1) then where should the fourth particle of mass 8 gm be placed so that the position of the centre of the new system is (3, 3, 3).*

Solution. Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the positions of the particles of masses 2, 4, and 6 gm respectively then

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{2x_1 + 4x_2 + 6x_3}{2 + 4 + 6}$$

$$= \frac{1}{6}(x_1 + 2x_2 + 3x_3)$$

It is given that (x_{cm}, y_{cm}, z_{cm}) is (1, 1, 1). Thus

$$1 = \frac{1}{6}(x_1 + 2x_2 + 3x_3)$$

or $$x_1 + 2x_2 + 3x_3 = 6 \qquad ...(i)$$

Let (x_4, y_4, z_4) be the position of the fourth particle of mass 8 gm, so that the new centre of mass is (3, 3, 3). Then we have

$$3 = \frac{2x_1 + 4x_2 + 6x_3 + 8x_4}{2 + 4 + 6 + 8}$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 30 \qquad ...(ii)$$

Subtracting eqn. (*i*) from (*ii*), we get $x_4 = 6$

In the same way we get $y_4 = 6$, $z_4 = 6$

Hence 8 gm mass should be placed at (6, 6, 6).

Q. 9. *Three particles of masses m_1, m_2, m_3 are at positions $\vec{r}_1, \vec{r}_2, \vec{r}_3$ and are moving with velocities $\vec{V}_1, \vec{V}_2, \vec{V}_3$ respectively (i) what is position vector of the centre of mass? (ii) what is the velocity of the centre of mass ?*

Solution. (*i*) The position vector of the centre of mass is given by

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3}$$

(*ii*) The velocity of the centre of mass is given by $\vec{v}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3}{m_1 + m_2 + m_3}$

Q. 10. *The velocity of two particles of masses m_1 and m_2 relative to an inertial observer are $\vec{v}_1$ and $\vec{v}_2$. Determine the velocity of the centre of mass relative to the observer and the velocity of each particle relative of the centre of mass.*

Solution. The velocity of the centre of mass relative to the observer is given by

$$\vec{v}_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

The velocity of each particle relative to the centre of mass is, by Galilean transformation of velocities, given by

$$\vec{v}_1' = \vec{v}_1 - \vec{v}_{cm} = \vec{v}_1 - \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

$$= \frac{m_2(\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

and

$$\vec{v}_2' = \vec{v}_2 - \vec{v}_{cm} = \vec{v}_2 - \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

$$= \frac{m_1(\vec{v}_2 - \vec{v}_1)}{m_1 + m_2}$$

Thus, in *c*-frame, the two particles appear to be moving in opposite directions.

Q. 11. *Two bodies of masses 10 kg and 2 kg are moving with velocities* $2\hat{i} - 7\hat{j} + 3\hat{k}$ *and* $-10\hat{i} + 35\hat{j} - 3\hat{k}$ *meter/sec respectively. Find the velocity of the centre of mass.*

Solution. The velocity of the centre of mass in the laboratory frame of reference (L-frame) is defined by

$$\vec{v}_{cm} = \frac{1}{m}\sum_{i=1}^{n} m_i \vec{v}_i$$

in the case of two bodies only, we have

$$\vec{v}_{cm} = \frac{1}{m_1 + m_2}\left[m_1 \vec{v}_1 + m_2 \vec{v}_2\right]$$

$$= \frac{1}{10+2}\left[10\,(2\hat{i} - 7\hat{j} + 3\hat{k}) + 2(-10\hat{i} + 35\hat{j} - 3\hat{k})\right]$$

$$= \frac{1}{12}\left[20\hat{i} - 70\hat{j} + 30\hat{k} - 20\hat{i} + 70\hat{j} - 6\hat{k}\right]$$

$$= 2\hat{k} \text{ meter/sec.}$$

Q. 12. *In a system comprising two particles of masses 2.0 kg and 5.0 kg the positions of the particles at t = 0 are* $4\hat{i} + 3\hat{j}$ *and* $6\hat{i} - 7\hat{j} + 7\hat{k}$ *respectively, and the velocities are* $10\hat{i} - 6\hat{k}$ *and* $3\hat{i} + 6\hat{j}$ *respectively. Deduce the velocity of the centre of mass and the position of the centre of mass at t = 0 and t = 4 units.*

Solution.

$$\vec{v}_{cm} = \frac{1}{m_1 + m_2}\left[m_1 \vec{v}_1 + m_2 \vec{v}_2\right]$$

$$= \frac{1}{2.0 + 5.0}\left[2.0\,(10\hat{i} - 6\hat{k}) + 5.0\,(3\hat{i} + 6\hat{j})\right]$$

$$= \frac{1}{7.0}\left[(20\hat{i} - 12\hat{k} + 15\hat{i} + 30\hat{j})\right]$$

$$= \frac{1}{7.0}\,[35\hat{i} + 30\hat{j} - 12\hat{k}]$$

The position vector of the centre of mass at $t = 0$.

$$\vec{r}_{cm} = \frac{1}{m_1 + m_2}\left[m_1\vec{r_1} + m_2\vec{r_2}\right]$$

$$= \frac{1}{2.0 + 5.0}\left[2.0\,(4\hat{i} + 3\hat{j}) + 5.0\,(6\hat{i} - 7\hat{j} + 7\hat{k})\right]$$

$$= \frac{1}{7.0}\left[8.0\hat{i} + 6.0\hat{j} + 30\hat{i} - 35\hat{j} + 35\hat{k}\right]$$

$$= \frac{1}{7.0}\left[38\hat{i} - 29\hat{j} + 35\hat{k}\right]$$

The displacement of the centre of mass in 4 sec is

$$\Delta\vec{S}_{cm} = 4\vec{v} = \frac{4}{7.0}\left[35\hat{i} + 30\hat{j} - 12\hat{k}\right]$$

∴ position vector at $t = 4$ sec.

$$= \frac{1}{7.0}\left[38\hat{i} - 29\hat{j} + 35\hat{k}\right] + \frac{4}{7.0}\left[35\hat{i} + 30\hat{j} - 12\hat{k}\right]$$

$$= \frac{1}{7.0}\left[178\hat{i} + 91\hat{j} - 13\hat{k}\right]$$

Q. 13. *Two particles of masses 100 and 300 gm have position vectors* $2\hat{i} + 5\hat{j} + 13\hat{k}$ *and* $-6\hat{i} + 4\hat{j} - 2\hat{k}$ *and velocity* $(10\hat{i} - 7\hat{j} - 3\hat{k})$ *and* $(7\hat{i} - 9\hat{j} + 6\hat{k})$ *cm/sec respectively. Deduce the instantaneous position of the centre of mass, and the velocity of the second particle in a frame of reference travelling with the centre of mass.*

Solution. The position of the centre of mass in the laboratory frame (L-frame) is given by

$$\vec{r}_{cm} = \frac{1}{m_1 + m_2}\left[m_1\vec{r_1} + m_2\vec{r_2}\right]$$

$$= \frac{1}{100 + 300}\left[100(2\hat{i} + 5\hat{j} + 13\hat{k}) + 300\,(-6\hat{i} + 4\hat{j} - 2\hat{k})\right]$$

$$= \frac{1}{4}\,(-16\hat{i} + 17\hat{j} + 7\hat{k})\ \text{cm.}$$

The velocity of the centre of mass in the L-frame is

$$\vec{v}_{cm} = \frac{1}{m_1 + m_2}\left[m_1 \vec{v}_1 + m_2 \vec{v}_2\right]$$

$$= \frac{1}{100 + 300}\left[100(10\hat{i} - 7\hat{j} - 3\hat{k}) + 300\,(7\hat{i} - 9\hat{j} + 6\hat{k})\right]$$

$$= \frac{1}{4}\,(31\hat{i} - 34\hat{j} + 15\hat{k}) \text{ cm/sec}$$

In the centre of mass reference frame (c-frame) the velocity of centre of mass, $\vec{v}_{cm}$ is zero by definition. Therefore, the velocity of the second particle in the c-frame is

$$\vec{v}_z - \vec{v}_{cm} = (7\hat{i} - 9\hat{j} + 6\hat{k}) - \frac{1}{4}\,(31\hat{i} - 34\hat{j} + 15\hat{k})$$

$$= \frac{1}{4}\,(-3\hat{i} - 2\hat{j} + 9\hat{k}) \text{ cm/sec}$$

Q. 14. *The Fig. 19 shows two blocks of masses 5 kg and 2 kg placed on a frictionless surface and connected by a spring. A external kick gives a velocity 14 meter/sec to the heavier block in the direction of the lighter one. Deduce (i) the velocity gained by the centre of mass, and (ii) the separate velocities of the two blocks in the centre-of-mass coordinates just after the kick.*

Solution. (*i*) The velocity of the centre of mass in the laboratory frame of reference (L-frame) is given by

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

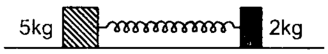


Fig. 19

$$= \frac{5 \text{ kg} \times 14 \text{ m/sec} + 0}{5 \text{ kg} + 2 \text{ kg}} = 10 \text{ ms}^{-1}$$

(*ii*) The centre of mass coordinates reference frame (c-frame) is one which is attached to the centre of mass and hence moving with velocity v_{cm} = 10 m/sec with respect 0 the L-frame (By definition the velocity of the centre of mass in the c-frame is zero), therefore the velocity of the heavier block in the c-frame just after the kick is

$$v_1 - v_{cm} = 14 \text{ ms}^{-1} - 10 \text{ ms}^{-1} = 4 \text{ ms}^{-1},$$

and that of the lighter block is

$$0 - v_{cm} = -10 \text{ ms}^{-1}$$

Q. 15. *A and B are two balls of mass M each. Ball B is at rest and ball A is approaching B with velocity v. Show that to an observer in the centre-of-mass frame of reference both balls appear to approach him with equal speeds v/2 from opposite directions.*

Q. 16. *Three masses of 3 kg, 4 kg and 5 kg are being acted upon by forces $-6\hat{i}$ newton, $8\hat{j}$ newton and $12\hat{i}$ newton respectively. Find the acceleration of the centre of mass of the system.*

Solution.

$$\vec{F} = -6\hat{i} + 8\hat{j} + 12\hat{i}$$
$$= 6\hat{i} + 8\hat{j} \text{ newton}$$

We know that the product of the mass of the system M, and acceleration of its centre of mass $\vec{a}_{cm}$ is equal to the resultant (external) force acting on the system. Thus

$$\vec{a}_{cm} = \frac{\vec{F}}{M} = \frac{(6\hat{i} + 8\hat{j}) \text{ newton}}{(3+4+5) \text{ kg}}$$
$$= \frac{1}{2}\hat{i} + \frac{2}{3}\hat{j} \text{ meter/sec}^2$$

its magnitude is

$$a = \sqrt{\left[\left(\frac{1}{2}\right)^2 + \left(\frac{2}{3}\right)^2\right]} = 0.83 \text{ ms}^{-2}$$

its direction with the x-axis is

$$\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right)$$
$$= \tan^{-1}\left(\frac{2/3}{1/2}\right)$$
$$= \tan^{-1}\left(\frac{4}{3}\right) = 53°6'$$

Q. 17. *Two particles P and Q of masses m_1 = 0.10 kg and m_2 = 0.30 kg respectively are initially at rest 1.0 meter apart they attract each other with a constant force of 1.0 × 10^{-2} nt. No external forces act on the system. Describe the motion of the centre of mass. At what distance from the original position of P do the particles collide?*

Solution. Initially the system is at rest so that its centre of mass is also at rest (zero velocity). If M be the total mass of the system and $\vec{a}_{cm}$ the acceleration of the centre of mass, then

$$M\vec{a}_{cm} = \vec{F}_{ext}$$

Since no external forces act on the system $(\vec{F}_{ext} = 0)$, we have

$$\vec{a}_{cm} = \frac{d\vec{V}_{cm}}{dt} = 0$$

$$\vec{V}_{cm} = \text{constant} \quad ...(i)$$

Hence the centre of mass remains at rest.

Let the particle m_1 lie at the origin of co-ordinates, and the particle m_2 at a distance r from it along the x-axis. The distance of the centre of mass of the system from the origin would be given by

$$x_{cm} = \frac{(m_1 \times o) + (m_2 \times r)}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} r \qquad ...(ii)$$

Each particle is acted upon by a constant force F. Therefore the accelerations of these particles would be

$$a_1 = \frac{F}{m_1} \quad \text{and} \quad a_2 = \frac{F}{m_2}$$

Suppose the particles collide after time t at a distance x from the origin. This means that in time t, the particle m_1 travels a distance x and particle m_2 travels a distance $r - x$. Thus, using the formula $x = \frac{1}{2}at^2$, we have

$$x = \frac{1}{2}a_1 t^2$$

$$r - x = \frac{1}{2}a_2 t^2$$

$$\frac{r - x}{x} = \frac{a_2}{a_1} = \frac{m_1}{m_2} \text{ (From above)}$$

$$\frac{r}{x} - 1 = \frac{m_1}{m_2}$$

$$\frac{r}{x} = \frac{m_1}{m_2} + 1 = \frac{m_1 + m_2}{m_2}$$

$$x = \frac{m_2}{m_1 + m_2} r$$

Here $m_1 = 0.10$ kg, $m_2 = 0.30$ kg, $r = 1.0$ m

$$x = \frac{0.30}{0.10 + 0.30}(1.0) = 0.75 \text{ meter.}$$

Infact, the particle collides at the centre of mass.

Q. 18. *A bullet of mass m and velocity v passes through a pendulum bob of mass M and emerges with a velocity* $\frac{v}{2}$. *The pendulum bob is at the end of a string length l. What is the minimum value of v such that the pendulum bob will swing through a complete circle.*

Solution. In order that the pendulum bob may describes a complete circle, its velocity at the lowest point must be $\sqrt{5\,gl}$. This must be the velocity after collision.

From the law of conservation of momentum.

$$mv = Mx + \frac{mv}{2} \text{ where } x \text{ is the velocity of the bob.}$$

$$Mx = \frac{mv}{2}$$

$$x = \frac{mv}{2M} = \sqrt{5\,gl}$$

$$v = \frac{2M}{m}\sqrt{5\,gl}.$$

Q. 19. *A weightless string is passing over a light frictionless pulley of diameter 5 cm, and a tray of sand weiging 50 gm is tied at each of its two ends. The system is in equilibrium and at rest. Find its centre of mass. Now, 5 gm of sand is transferred from one tray to another but the system is not allowed to move. Find the new centre of mass. If the system is now set free to move, what would be acceleration of the centre of mass.*

Solution. In the first case, by symmetry the centre of mass C will be at the mid point of the line joining the centres of the two trays. In the second case, let x be the distance of the first tray (45 gm) from some arbitrary origin on the left and on the same line as joining the centres of the two trays. The distance of the second tray (55 gm) from the same origin would be x + 5. Then the distance of the centre of mass C from the same origin

$$= \frac{45 \times x + 55\,(x + 5)}{45 + 55} = x + 2.75$$

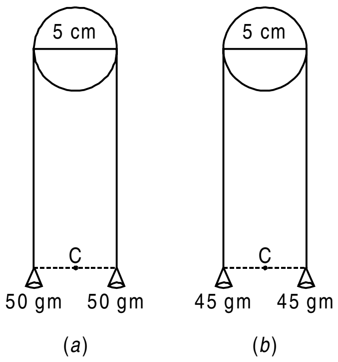


Fig. 20

Thus the centre of mass lies at a distance of 2.75 cm from the lighter tray on the line joining the centres.

When the system is released, the heavier tray goes down and the lighter one goes up. We know that the product of the total mass of the system and the acceleration of its centre of mass is equal to the vector sum of the external forces acting on all parts of the system. That is

$$M\,\vec{a}_{cm} = \vec{F}_1 + \vec{F}_2$$

$$(45 + 55)\,a_{cm} = -45g + 55\,g = 10\,g$$

$$a_{cm} = \frac{10\,g}{100} = \frac{g}{10} \text{ (down)}$$

Q. 20. *A 10 kg boy standing in a 40 kg boat floating on water is 20 meter from the shore of the river. If he moves 8 meter on the boat toward the shore then how far is he now from the shore ? Assume no friction between boat and water.*

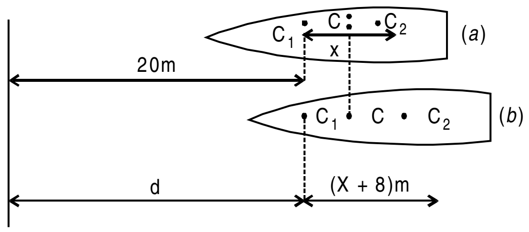


Fig. 21

Solution. Let C_1 be the centre of mass of the boy, C_2 that of the boat and C that of the system (boy + boat), as shown in (*a*). Let x be the distance between C_1 and C_2. The distance of C_1 (10 kg) from the shore is 20 meter and that of C_2 (40 kg) is $20 + x$ meter. Therefore, the distance of the centre of mass C of the system is

$$r_{cm} = \frac{(10 \times 20) + \{40 \times (20 + x)\}}{10 + 40}$$

$$= \frac{1000 + 40x}{50} \text{ meter}$$

when the boy moves 8 meter towards the shore, the distance between C_1 and C_2 becomes $(x + 8)$ meter. Let d be the new distance of the boy from the shore. The distance of centre of mass from the shore is given by

$$r'_{cm} = \frac{(10 \times d) + \{40 \times (d + x + 8)\}}{10 + 40}$$

$$= \frac{50d + 40x + 320}{50} \text{ meter}$$

As no external forces are acting on the system, the velocity of the centre of mass C of the system remains constant. Initially the centre of mass was at rest, So it must continue to remain at rest *i.e.*, its position will still remain unchanged with respect to shore. Thus

$$r_{cm} = r'_{cm}$$

$$1000 + 40x = 50d + 40x + 320$$

$$d = 13.6 \text{ meter}$$

Q. 21. *(a) What is the momentum of a 10000 kg truck whose velocity is 20 m/s? (b) What velocity must a 5000 kg-truck attain in order to have the same momentum (c) the same K.E.*

Solution. (*a*) The momentum of the truck of mass m, moving with velocity v is given by

$$P = mv = 10000 \times 20$$

$$= 2 \times 10^5 \text{ kg m/s}$$

(*b*)

$$m_1 v_1 = m_2 v_2$$

$$10000 \times 20 = 5000 \times v_2$$

$$v_2 = 40 \text{ m/s}$$