CHAPTER 4

MOTION IN A PLANE

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4.1 Introduction

Dear students, we have learnt about the concepts of displacement, velocity and acceleration which are necessary to describe the motion of a body on a straight line path (one dimension). We have seen that in one dimension there are only two possible directions and hence the directions are automatically taken care of by using Ace positive (+) and negative (–) signs, But to describe the motion of the body in two dimensions (in a plane) or in three dimensions (in space), a vector is needed. For this, what a vector is how should addition, subtraction and multiplication of vectors be carried out, what is the result on multiplying a vector with a real number, need to understand etc. We will use vector to define velocity and acceleration in a plane. Then we shall discuss the motion of the body in a plane. We will discuss the motion with constant acceleration as a simple case of motion in a plane and the projectile motion in detail. The circular motion being very important in our day to day life, we will study uniform circular motion also in detail.

The equations obtained for the motion in a plane can be easily transformed into those of motion in three dimensions.

4.2 Scalar and Vector Quantities

In physics, quantities are classified as (1) Scalar quantities and (2) Vector quantities. The basic difference between the scalar and vector quantities is that with scalar quantities, direction is not involved while the direction is involved with vector quantities.

The quantities for which, complete information is obtained by knowing their values only are called scalars. e.g. temperature, time, mass, density, volume, work etc. A scalar is represented by a number showing its magnitude in a proper unit. The combination or associations ofr scalar quantities follow the laws of ordinary algebra. Addition, subtraction, multiplication and division can be done like those of usual numbers.

The quantities, which need the direction as well as their values (magnitudes), to be completely known, are called vectors. e.g. velocity, acceleration, force, torque, area, displacement etc.

A vector quantity is represented by putting an arrow on the symbol of that quantity or as a bold letter. For example, the force vector is shown as $\vec{F}$ or **F** the velocity vector is represented as $\vec{v}$ or $\boldsymbol{v}$. The value of the vector quantity is shown by putting the symbol of that quantity in modulas (i.e. between two vertical bars) or by writing that symbol without the arrow. e.g., The value of $\vec{A}$ is shown by $|\vec{A}|$ or A. The vector quantities obey specific laws of combination.

4.3 Presentation of vector by graphical or geometrical method

To represent a vector quantity geometrically an arrow is drawn such that the length of this arrow is equal to the value of this vector quantity on a proper scale. The head of the arrow is put in the direction in which the effect of this vector quantity is prevailing. This arrow can be drawn from any point. Such vectors are called free vectors. e.g. A train moves with a velocity of 40 km/h South to North direction. To represent this velocity vector, as shown in Figure 4.0, draw an arrow from South to North. Keep the length of the arrow proportional to the value of velocity 4.0cm (taking the scale of 10km/h = 1cm). Since the motion is in North direction, put the head of the arrow in North direction. Point O is called the tail of the arrow. Thus this velocity vector is represented as $\vec{v} = \vec{OP}$.

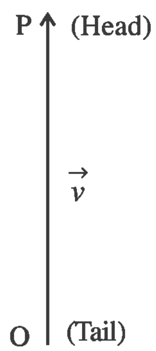


Figure 4.0

4.4 Position and Displacement Vectors

To represent the position of a body we have to mention the reference point which is usually taken as the origin of coordinate axes. Suppose a body moves along the path PQRS as shown in the Figure (4.1). At time t_1 it is at point Q. The vector $\vec{OQ} = \vec{r}_1$ formed by joining the origin O with the point Q is called the position vector of the body at time t_1. Suppose at time t_2 the body reaches the point R. Then, the vector $\vec{OR} = \vec{r}_2$ formed by joining the origin O with the point R is called the position vector of the body at time t_2. During time $t_2 - t_1$ it reaches from Q to R. Hence its displacement vector is shown by $\vec{QR}$.

Here a noteworthy point is that the value of the displacement vector is the minimum distance between the initial position and the final position.

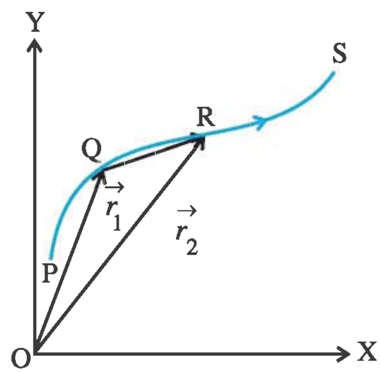


Figure 4.1

4.5 Equality of Vectors

Equal Vectors : If the values and the directions of two vectors are equal, then they are called equal vectors. (Fig. 4.2 *a*)

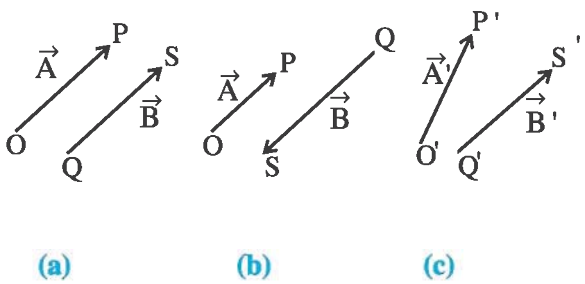


Figure 4.2

Parallel Vectors : The vectors with the same direction are called parallel vectors. (The magnitudes of such vectors can be equal or different). See Fig. 4.2 (a).

Antiparallel Vectors : The vectors having mutually opposite directions are called antiparallel vectors. Fig. 4.2 (b)

Aparallel Vectors : The vectors which are not parallel or antiparallel to each other are called aparallel vectors. Fig. 4.2 (c)

4.6 Vector Algebra

4.6.1 Multiplication of Vectors by real numbers

Multiplying a vector quantity by a real number, the results also in a vector only. Multiplying a vector $\vec{A}$ with a positive number k, the result is vector $k\vec{A}$ whose value is k times that of vector $\vec{A}$.

$$|k\vec{A}| = k|\vec{A}| \qquad \text{if } k > 0$$

When we multiply a vector $\vec{A}$ with a negative number $-k$, the result is $-k\vec{A}$; the direction of which is opposite to that of vector $\vec{A}$ and its magnitude is $|k\vec{A}|$.

The coefficient k which is multiplied with vector $\vec{A}$ can also be a scalar with its physical dimensions. Hence the dimensions of the resultant vector $k\vec{A}$, are the product of dimensions of k and the dimensions of $\vec{A}$. e.g. The product of a constant velocity with a time interval gives displacement vector.

4.6.2 Addition and Subtraction of Vectors Graphical or Geometrical Method

Two vectors are geometrically added as shown below

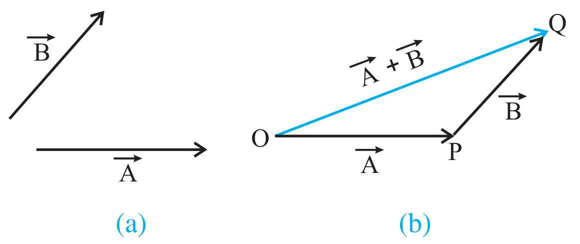


Figure 4.3

Suppose vector $\vec{A}$ and $\vec{B}$ shown in (Fig. 4.3 *a*) are to be added.

For this, as shown in Figure (4.3 *b*) starting from any point O, draw a vector $\vec{OP}$ which has the same magnitude as that of $\vec{A}$ and is in the direction of $\vec{A}$. Thus $\vec{OP} = \vec{A}$. Now draw $\vec{PQ} = \vec{B}$ by putting the tail of vector $\vec{B}$ on the head P of vector $\vec{OP}$. Then, the vector formed by joining the tail O of vector $\vec{A}$ to the head Q of vector $\vec{B}$ is the resultant vector $\vec{R}$ showing the addition of $\vec{A}$ and $\vec{B}$. i.e. $\vec{A} + \vec{B} = \vec{OQ} = \vec{R}$.

In this method of addition of vectors, the two given vectors and their resultant form the three sides of a triangle, hence it is also called the method of triangle for addition of vectors.

From some point O, draw two vectors $\vec{OP}$ and $\vec{OR}$ representing vectors $\vec{A}$ and $\vec{B}$ respectively. Now considering OP and OR as adjacent sides of a parallelogram, complete the parallelogram OPQR, as shown in the Fig. 4.4.

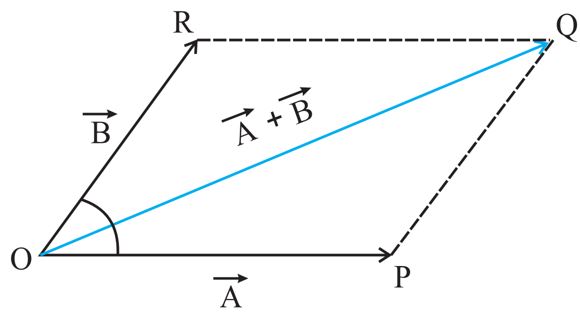


Figure 4.4

Here it is clear that $\vec{OR} = \vec{PQ} = \vec{B}$. The diagonal OQ of this parallelogram drawn from the point O becomes the resultant vector $\vec{R}$ of the addition of vectors $\vec{A}$ and $\vec{B}$ i.e. $\vec{OQ} = \vec{A} + \vec{B}$.

This method is also called the method of parallelogram for addition of vectors. (We will discuss this method in detail in article 4.9.4)

4.6.3 Subtraction of Vectors

Suppose we want to get subtraction of $\vec{A}$ and $\vec{B}$, which are shown in Figure 4.3(*a*).

Since $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$, the meaning of subtracting vector $\vec{B}$ from vector $\vec{A}$, is to add $-\vec{B}$ (a vector with the same magnitude as that of $\vec{B}$ but in opposite direction) into $\vec{A}$. See Fig. 4.5.

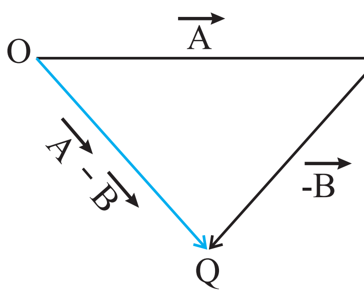
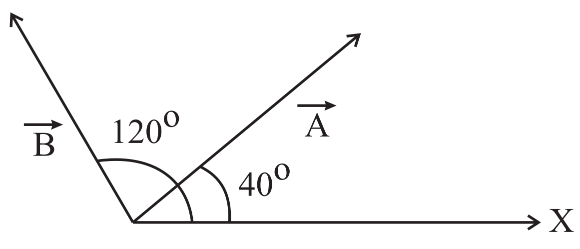


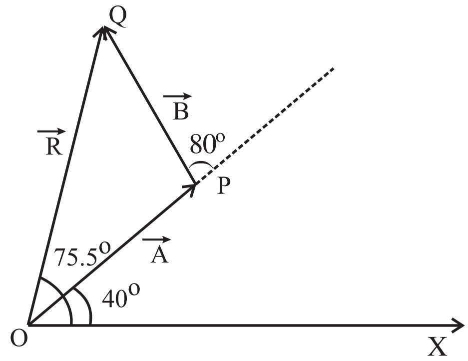
Figure 4.5

In the method of parallelogram, the diagonal formed by joining the points P and Q in Fig. 4.5 shows $\vec{PQ} = \vec{A} - \vec{B}$ (verify this by yourself). Also verify that $\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$.

Illustration 1 : Two vectors $\vec{A}$ and $\vec{B}$, make angles of 40° and 120° respectively with the X–axis. If $|\vec{A}| = 6$ and $|\vec{B}| = 5$ unit, find the resultant vector of these two vectors.



(a)



(b)

Figure 4.6

Solution : The two given vectors $\vec{A}$ and $\vec{B}$ are shown in the Figure 4.6(a). To obtain their addition, draw X–axis from some point O on a graph paper. (See Fig. 4.6 (b)) Taking a proper scale, draw vector $\vec{OP}$ representing $\vec{A}$, and on the point P of this vector put the tail of vector $\vec{B}$ and draw $\vec{PQ} = \vec{B}$.

Joining O and Q, we get $\vec{OQ}$, which in the resultant of $\vec{A}$ and $\vec{B}$, i.e. $\vec{OQ} = \vec{R}$. Measuring the value of the resultant vector $\vec{OQ}$, with a scale, it is found as 8.4 unit. This resultant vector makes an angle of 75.5° with X–axis.

Illustration 2 : River water flows at 40 km/h. In this river a fisherman tries to drive a motorboat at 30 km/h perpendicular to the bank of the river. Find the resultant velocity of the motorboat and its direction with respect to the bank.

Solution : In Figure 4.7, the velocity of flow of water is shown as $\vec{v_r}$ and the velocity of motorboat as $\vec{v_b}$.

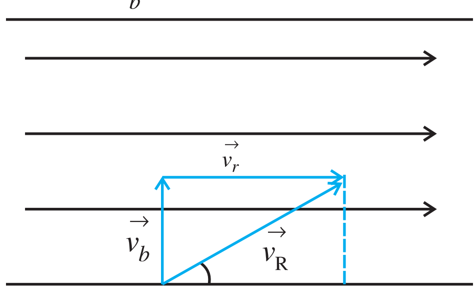


Figure 4.7

Using the method of triangle for the addition of vectors; the resultant velocity of these two velocities is shown as $\vec{v}_R$.

It is clear from the figure that

$$\left| \vec{v}_R \right| = \sqrt{\left| \vec{v}_r \right|^2 + \left| \vec{v}_b \right|^2}$$

$$= \sqrt{(40)^2 + (30)^2}$$

$$= 50 \text{ km/h}$$

If $\vec{v}_R$ makes an angle θ with the bank, it is clear from the figure that,

$$\tan\theta = \frac{v_b}{v_r} = \frac{30}{40} = 0.75$$

$$\therefore \theta = \tan^{-1} 0.75 \approx 37^{o}$$

Thus the resultant velocity of boat is 50km/hr in the direction making an angle of 37^o with the river flow.

4.6.4 Properties of vector addition

(1) Addition of vectors is commutative but Subtraction of vectors is not.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}.$$

$$\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$$

(2) Addition of vectors follows associative law.

i.e., $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

4.7 Null or Zero Vectors

The vector obtained by adding two vectors of equal magnitude (value) and but of opposite directions is called null or a zero vector and it is shown as $\vec{0}$. Thus $\vec{A} - \vec{A} = \vec{0}$. As the value of zero vector is zero its direction cannot be shown. e.g. the acceleration of a train moving with constant valocity is zero vector.

4.8 Unit Vector

A vector of unit magnitude is called a unit vector. A unit vector is symbolically expressed as $\hat{n}$ (Read : *n* hat or *n* carat). By dividing any vector by its value, we get a unit vector in the direction of that vector. e.g. in the Fig. 4.8, vector $\vec{A}$ is shown and suppose $|\vec{A}| = 6$.

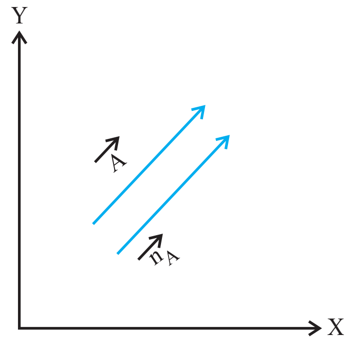


Figure 4.8

If we show the unit vector in the direction of this vector as $\hat{n}_A$, then

$$\hat{n}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{6} \qquad (4.8.1)$$

Thus any vector can be expressed as a product of its value (magnitude) and the unit vector in the direction of that vector.

$$\vec{A} = |\vec{A}|\,\hat{n}_A = A\hat{n}_A \qquad (4.8.2)$$

In the Cartesian co–ordinate system, the unit vectors in the directions of X, Y and Z-axes are expressed respectively as $\hat{i}$, $\hat{j}$ and $\hat{k}$. The vectors shown in the Figure 4.9 can be written as :

$$\vec{B} = 4\hat{i}, \ \vec{C} = 2\hat{j}$$

$$\vec{A} = 4\hat{i} + 2\hat{j} \qquad (4.8.3)$$

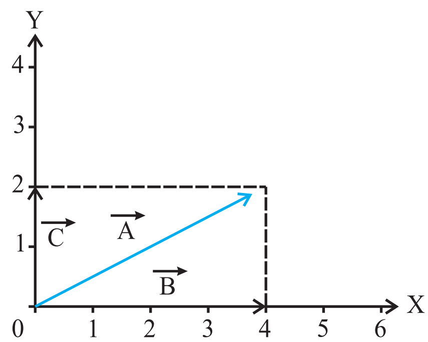


Figure 4.9

4.9 Resolution of a Vector in a Plane

As shown in the Figure 4.10 *a*, consider two non–zero vectors $\vec{a}$ and $\vec{b}$; besides another vector $\vec{A}$ in the same plane. The vector $\vec{A}$ can be represented as combination (addition) of two vectors, one obtained by multiplying the vector $\vec{a}$ with a real number λ and the other by multiplying vector $\vec{b}$ with a real number μ. To verify the above statement, draw a line parallel to $\vec{a}$ and passing through the tail O of vector $\vec{A}$. Similarly draw another line parallel to b and passing through the head P of $\vec{A}$. If these two lines interect in Q, then from Fig. 4.10 *b*.

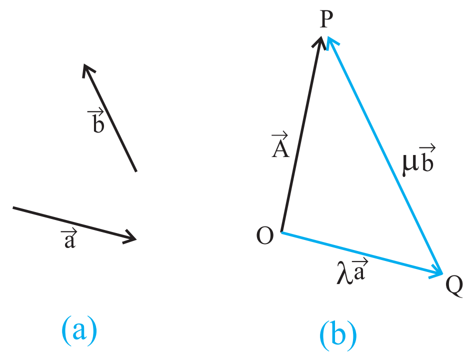


Figure 4.10

$$\vec{A} = \vec{OP} = \vec{OQ} + \vec{QP} \qquad (4.9.1)$$

But $\vec{OQ}$ is parallel to $\vec{a}$ and $\vec{QP}$ is parallel to $\vec{b}$, hence we can write,

$$\vec{OQ} = \lambda\vec{a} \text{ and } \vec{QP} = \mu\vec{b} \qquad (4.9.2)$$

This is called the resolution of vector $\vec{A}$ in the directions of $\vec{a}$ and $\vec{b}$, in the form of vector components $\lambda\vec{a}$ and $\mu\vec{b}$.

Where λ and μ are real numbers.

$$\therefore \vec{A} = \lambda\vec{a} + \mu\vec{b} \qquad (4.9.3)$$

Thus a given vector can be resolved in such a way that its two vector components remain in the direction of two given vectors and all these three vectors remain in a plane.

In the orthogonal co-ordinate system, a given vector can be easily resolved in the directions of the axes using unit vectors.

4.9.1 Perpendicular components of a vector

In Fig. 4.11, a vector $\vec{A}$ is shown in two dimensions. From the head and the tail of this vector perpendiculars are drawn on X– and Y– axes. By doing so, we get PQ = projection of $\vec{A}$ on X–axis or the scalar component (A_x) of vector $\vec{A}$ in the direction of X–axis, MN is the projection of $\vec{A}$ on Y–axis or the scalar component (A_y) of $\vec{A}$ in the direction of Y–axis.

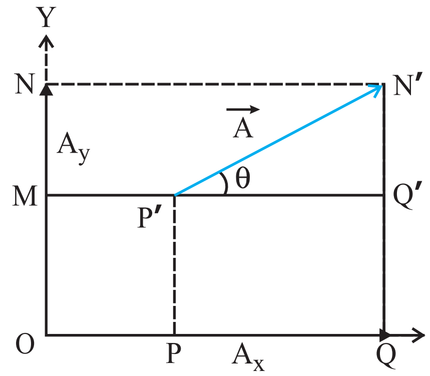


Figure 4.11

Now, from the law of addition of vectors,

$$\vec{A} = \vec{P'Q'} + \vec{Q'N'} = \vec{PQ} + \vec{MN} \qquad (4.9.4)$$

$$\therefore \vec{A} = A_x\hat{i} + A_y\hat{j} \qquad (4.9.5)$$

Here, $A_x\hat{i} = \vec{A}_x$ = vector component of vector $\vec{A}$ in X–direction. $A_y\hat{j} = \vec{A}_y$ = vector component of vector $\vec{A}$ in Y–direction.

From Figure 4.11 in ΔP'Q'N'

$$\cos\theta = \frac{P'Q'}{P'N'} = \frac{A_x}{A} \qquad (4.9.6)$$

$\therefore A_x = A \cos\theta$

Similarly $A_y = A \cos(90^\circ - \theta)$

$$\therefore A_y = A \sin\theta \qquad (4.9.7)$$

From this, we can say that the component of a vector in any given direction is equal to the product of the value of that vector and the cosine of the angle made by that vector with that given direction.

Thus any vector can be resolved in two mutually perpendicular components.

A vector can be described in two ways :

(1) by the magnitude (value) of that vector and the angle made by it with a definite direction or

(2) by the components of that vector.

In the $\Delta P'Q'N'$, in Fig. 4.11,

$$|\vec{A}| = P'N' = \sqrt{(P'Q')^2 + (Q'N')^2}$$

$$= \sqrt{A_x^2 + A_y^2} \qquad (4.9.8)$$

Thus the magnitude (value) of any given vector is equal to the square root of the addition of the square of its matually perpendicular components. For the direction,

$$\tan\theta = \frac{N'Q'}{P'Q'} = \frac{A_y}{A_x} \qquad (4.9.9)$$

$$\therefore \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right), \qquad (4.9.10)$$

where θ = angle between the vector $\vec{A}$ and the x–axis.

In the discussion so far, we have only considered the vector lying in the XY plane. In a similar way a vector in three dimensions can be resolved in three mutually perpendicular components (in directions of X, Y, Z–axes)

The component of a vector representing a physical quantity, in any direction shows the effectiveness of that physical quantity in that direction. e.g. If as shown in Fig 4.12 *a* body makes a displacement of 5 m from A to B. Then it is clear that the distance travelled by it in horizontal direction is (A to C) 4m and the distance travelled by it in the verticale direction (C to B) 3m. From Fig 4.11, in $\Delta P'Q'N'$,

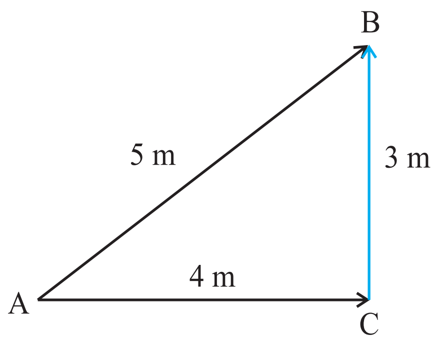


Figure 4.12

In the Fig. 4.13 a vector $\vec{A}$ in three dimensions is shown. The projection of this vector on the XY plane is OQ. By drawing the perpendiculars from the point Q on the X and Y–axes, we get the *x* and *y* components of vector $\vec{A}$ on those axes, as ON = A_x and OM = A_y respectively. Looking three dimensionally we find that PQ = RO = A_z.

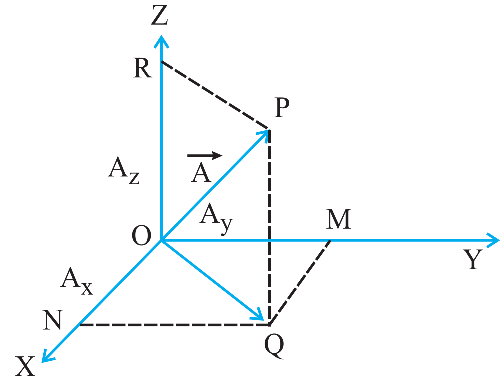


Figure 4.13

Now according to Pythagoras theorem,

$$OQ^2 = MQ^2 + OM^2 = A_x^2 + A_y^2 \qquad (4.9.11)$$

and $OP^2 = OQ^2 + PQ^2$

$$\therefore OP^2 = A_x^2 + A_y^2 + A_z^2 \qquad (4.9.12)$$

$$|\vec{A}|^2 = A_x^2 + A_y^2 + A_z^2$$

$$\therefore |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad (4.9.13)$$

In three dimensions vector $\vec{A}$ can be written as $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

Another way of writing the very same vector $\vec{A}$, is

$$\vec{A} = (A_x, A_y, A_z)$$

If the co-ordinates of a point are (x, y, z) then, its position vector can be written as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = (x, y, z) \qquad (4.9.14)$$

Here x, y and z are the components of $\vec{r}$ in the directions of X, Y, Z axes respectively. The value of this position vector is

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2} \qquad (4.9.15)$$

4.9.2 Addition and subtraction of vectors in Algebraic or analytical method :

We have learned the geometrical method for addition of vectors. This method is convenient for addition of two or three vectors, but when large number of vectors are to be added this method is tedious and has limited accuracy. In such circumstances the algebraic method of vector addition is more convenient.

We have already seen that a given vector can be resolved in mutually perpendicular components. Vectors can be easily added by combinity the components of vectors. Suppose $\vec{A}$ and $\vec{B}$ are in the XY–plane, and their components are A_x, A_y and B_x, B_y respectively.

$$\therefore \ \vec{A} = A_x\hat{i} + A_y\hat{j} \quad (4.9.16)$$

$$\text{and} \ \vec{B} = B_x\hat{i} + B_y\hat{j} \quad (4.9.17)$$

If the resultant vector of the addition of these two vectors is represented as $\vec{R}$ then,

$$\vec{R} = \vec{A} + \vec{B}$$

$$= (A_x\hat{i} + A_y\hat{j}) + (B_x\hat{i} + B_y\hat{j}) \quad (4.9.18)$$

Addition of vectors is commutative and it also follows the associative law, Therefore,

$$\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} \quad (4.9.19)$$

Moreover, $\vec{R} = R_x\hat{i} + R_y\hat{j}$

hence $R_x = A_x + B_x$ and $R_y = A_y + B_y$

Thus every component of the resultant vector is equal to the sum of the corresponding components of $\vec{A}$ and $\vec{B}$.

In three dimensions,

$$R = \vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k} \quad (4.9.20)$$

We will now illustrate, how the algebraic method for addition of vectors is easier, then the geomatric mathod.

Ilustration 3 : If $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{B} = 4\hat{i} + 5\hat{j} + 3\hat{k}$; find the magnitudes of $\vec{A} + \vec{B}$ and $\vec{A} - \vec{B}$

Solution : $\vec{A} + \vec{B} = 6\hat{i} + 8\hat{j} + 7\hat{k}$

$$\therefore \left|\vec{A} + \vec{B}\right| = \sqrt{(6)^2 + (8)^2 + (7)^2}$$

$$= 12.2 \text{ unit}$$

$$\vec{A} - \vec{B} = -2\hat{i} - 2\hat{j} + \hat{k}$$

$$\therefore \left|\vec{A} - \vec{B}\right|$$

$$= \sqrt{(-2)^2 + (-2)^2 + (1)^2}$$

$$= 3 \text{ unit}$$

Illustration 4 : Add vectors $\vec{A}$ and $\vec{B}$ shown in the Fig. 4.14 by the algebraic method. $|\vec{A}| = 10$ unit; $|\vec{B}| = 8$ unit.

Solution : For this we will determine the X and Y components of both the vectors.

$A_x = A \cos 30^o = 10 \cos 30^o$

$= 10 \times 0.8660 = 8.66$ unit

$B_x = B \cos 60^o = 8\cos 60^o = 8 \times 0.5$

$= 4.0$ unit

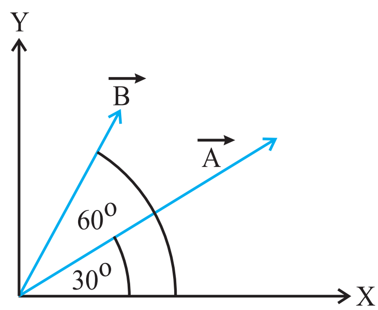


Figure 4.14

$A_y = A \sin 30^o = 10 \sin 30^o$

$= (10)(0.5) = 5.0$

$B_y = B \sin 60^o = 8 \sin 60^o$

$= (8)(0.8660) = 6.928 \approx 6.93$

If the resultant of these two vectors is $\vec{R}$, then

$R_x = A_x + B_x = 8.66 + 4.0 = 12.66$

$R_y = A_y + B_y = 5 + 6.93 = 11.93$

$\therefore$ The magnitude of the resultant vector $\vec{R}$ is

$$|\vec{R}| = \sqrt{R_x^{\,2} + R_y^{\,2}}$$

$$= \sqrt{(12.66)^2 + (11.93)^2}$$

$$= 17.4 \text{ unit}$$

Suppose the resultant vector makes an angle θ with the X–axis, then

$$\tan\theta = \frac{R_y}{R_x} = \frac{11.93}{12.66} = 0.9423$$

$$\therefore \theta = \tan^{-1} 0.9423 \approx 43^{\circ}\, 8'$$

This process can be made easier in the following way. As we can take the X and Y-axes in any way convenient to us,.

Let us take X–axis in the direction of $\vec{A}$. Hence the angle between $\vec{A}$ and X–axis becomes zero and the angle between $\vec{B}$ and X–axis becomes 30° Fig. 4.15.

In this condition,

$$A_x = A\cos 0^{\circ} = 10\cos 0^{\circ} = 10$$

$$B_x = B\cos 30^{\circ} = 8(0.8660) = 6.93$$

$$A_y = A\sin 0^{\circ} = 10\sin 0^{\circ} = 0$$

$$B_y = B\sin 30^{\circ} = (8)\,(0.5) = 4.0$$

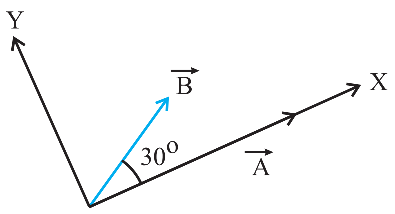


Figure 4.15

$$\therefore R_x = A_x + B_x = 10 + 6.93 = 16.93$$

$$R_y = A_y + B_y = 0 + 4 = 4$$

$$\therefore |\vec{R}| = \sqrt{R_x^{\,2} + R_y^{\,2}}$$

$$= \sqrt{(16.93)^2 + (4)^2}$$

$$\approx 17.4 \text{ unit}$$

Varify the direction of the resultant vector yourself.

Illustration 5 : Find the resultant vector of the three vectors shown in the Figure (4.16).

Solution : We will first find the x and y components of all these three vectors and then obtain the resultant vector by addition of the corresponding components.

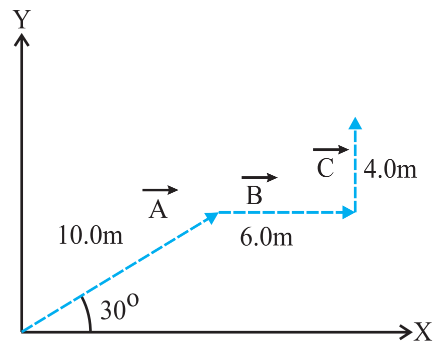


Figure 4.16

Taking x–components of $\vec{A}$, $\vec{B}$ and $\vec{C}$ we get,

$$A_x = A\cos 30^{\circ} = 10\cos 30^{\circ}$$
$$= (10)(0.8660) = 8.66$$
$$B_x = B\cos 0^{\circ} = 6\cos 0^{\circ} = 6$$
$$C_x = C\cos 90^{\circ} = 4\cos 90^{\circ} = 0$$

Taking y components of $\vec{A}$, $\vec{B}$ and $\vec{C}$, we get,

$$A_y = A\cos 60^{\circ} = (10)(0.5) = 5$$
$$B_y = B\cos 90^{\circ} = (6)(0) = 0$$
$$C_y = C\cos 0^{\circ} = (4)(1) = 4$$

If we write the resultant vector as $\vec{R}$, then,

$$R_x = A_x + B_x + C_x$$
$$= 8.66 + 6 + 0 = 14.66$$
$$R_y = A_y + B_y + C_y$$
$$= 5 + 0 + 4 = 9$$

The value of $\vec{R}$ is,

$$|\vec{R}| = \sqrt{R_x^{\,2} + R_y^{\,2}}$$

$$= \sqrt{(14.66)^2 + (9)^2} = 17.2\text{m}$$

If $\vec{R}$ makes an angle θ with the X–axis then,

$$\tan\theta = \frac{R_y}{R_x} = \frac{9}{14.66} = 0.6139$$

$$\therefore \theta = \tan^{-1}(0.6139)$$

$$= 31^{\circ}\, 27'$$

Illustration 6 : If the summation of the three vectors shown in Fig. 4.17 is zero, find magnitudes of the vectors $\vec{B}$ and $\vec{C}$.

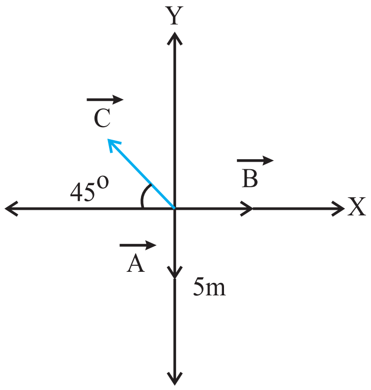


Figure 4.17

Solution : Taking x components of these three vectors

$A_x = A \cos 270^{\circ} = 0$

$B_x = B \cos 0^{\circ} = B$

$C_x = C \cos 135^{\circ} = -\frac{1}{\sqrt{2}} C$

Now, taking the y components of these vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$

$A_y = A \cos 180^{\circ} = -A$

$B_y = B \cos 90^{\circ} = 0$

$C_y = C \cos 45^{\circ} = \frac{1}{\sqrt{2}} C$

If the resultant vector is denoted by $\vec{R}$

and since $\vec{R} = \vec{A} + \vec{B} + \vec{C}$,

$R_x = A_x + B_x + C_x = 0 + B - \frac{1}{\sqrt{2}} C$

$R_y = A_y + B_y + C_y = -A + 0 + \frac{1}{\sqrt{2}} C$

It is given that the magnitude of the resultant vector $\vec{R}$ is zero. Hence the magnitudes of its components should also be zero. So,

$R_x = 0 + B - \frac{1}{\sqrt{2}} C = 0 \Rightarrow B = \frac{1}{\sqrt{2}} C$

$R_y = -A + 0 + \frac{1}{\sqrt{2}} C = 0 \Rightarrow A = \frac{1}{\sqrt{2}} C$

$\therefore A = B$

As shown in the figure $|\vec{A}| = A = 5m$

$\therefore C = A\sqrt{2} = 5\sqrt{2}\ m$

and $B = A = 5m$

Illustration 7 : As shown in Fig. 4.18 six vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, $\vec{E}$, and $\vec{F}$ form a regular haxagon. Using the algebraic method of addition of vectors shows that their resultant is zero.

Solution : Since these vectors form a regular hexagon, their lengths are the same. Suppose, this length is P. Hence, A = B = C = D = E = F = P.

Taking X and Y axes as shown in the figure and taking x and y component of these vectors,

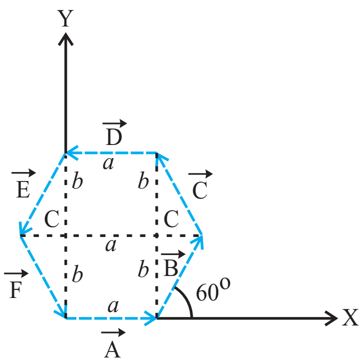


Figure 4.18

From the Fig. 4.18,

$\vec{A} = a\hat{i}$

$\vec{B} = c\hat{i} + b\hat{j}$

$\vec{C} = -c\hat{i} + b\hat{j}$

$\vec{D} = -a\hat{i}$

$\vec{E} = -c\hat{i} - b\hat{j}$

$$\vec{F} = c\hat{i} - b\hat{j}$$

$$\vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E} + \vec{F}$$

$$= (a\hat{i}) + (c\hat{i} + b\hat{j}) + (-c\hat{i} + b\hat{j}) +$$

$$(-a\hat{i}) + (-c\hat{i} + b\hat{j}) + (c\hat{i} - b\hat{j}) = 0$$

When the vectors form a close loop then their vector addition is zero.

4.9.3 Law of parallelogram for addition of two vectors

"If a parallelogram is completed by taking the two given vectors as adjacent sides, then the diagonal of the parallelogram drawn from the common point, gives the addition of the two given vectors." And the other diagonal shows subtraction of the two vectors. As shown in the Fig. 4.19 the given two vectors $\vec{A}$ and $\vec{B}$ are taken as adjacent sides of the parallalogram OP and OQ respectively and parallalogram OQRP is completed. Here θ is the angle between vectors $\vec{A}$ and $\vec{B}$. Diagonal OR is a resultant vector $\vec{R} = \vec{A} + \vec{B} = \vec{OR}$. This can be seen in the following manner.

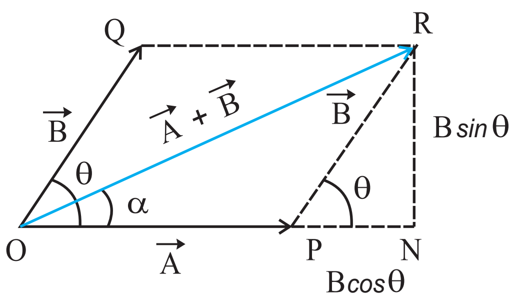


Figure 4.19

Suppose $\vec{A}$ is in the X–direction

$\therefore$ $\vec{A} = A_x\hat{i}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j}$

$\therefore$ Algebraic method gives

$$\vec{R} = A_x\hat{i} + B_x\hat{i} + B_y\hat{j}$$

$$= (A_x + B_x)\hat{i} + B_y\hat{j} \qquad (4.9.20)$$

$$\therefore \left|\vec{R}\right| = \left[(A_x + B_x)^2 + {B_y}^2\right]^{\frac{1}{2}}.$$

$$= \left[{A_x}^2 + 2A_xB_x + {B_x}^2 + {B_y}^2\right]^{\frac{1}{2}}$$

If resultant $\vec{R}$ is subtending angle α with vector $\vec{A}$ then $\tan\alpha = \dfrac{B_y}{A_x + B_x}$

$$\therefore \alpha = \tan^{-1}\frac{B_y}{A_x + B_x} \qquad (4.9.21)$$

From the geometry of the Figure.

$B_x = PN = B\cos\theta$ and $B_y = NR = B\sin\theta$ (4.9.22)

But $A_x = A$ and ${B_x}^2 + {B_y}^2 = B^2$

$$\therefore \left|\vec{R}\right| = \left[A^2 + B^2 + 2AB_x\right]^{\frac{1}{2}}$$

Now $B_x = B\cos\theta$

$$\therefore \left|\vec{R}\right| = \left[A^2 + B^2 + 2AB\cos\theta\right]^{\frac{1}{2}} \qquad (4.9.23)$$

If resultant $\vec{R}$ is makes angle α with $\vec{A}$ i.e. with X–axis then from Fig. 4.19.

$$\tan\alpha = \frac{RN}{OP + PN} = \frac{B_y}{A_x + B_x}$$

$$= \frac{B\sin\theta}{A + B\cos\theta}$$

$$\therefore \alpha = \tan^{-1}\frac{B\sin\theta}{A + B\cos\theta} \qquad (4.9.24)$$

Thus from equations (4.9.23) and (4.9.24) (using Law of Parallelogram) magnitude and direction of $\vec{A} + \vec{B}$ can be obtained respectively.

Try yourself for $\vec{A} - \vec{B}$ and obtain

$$\left|\vec{R}\right| = \left[A^2 + B^2 - 2AB\cos\theta\right]^{\frac{1}{2}}$$

and $\alpha = \tan^{-1}\dfrac{B\sin\theta}{A - B\cos\theta}$

Where θ is the angle between $\vec{A}$ and $\vec{B}$.

4.10 Multiplication of Two Vectors

Vector quantities have both magnitude and direction. Hence their products do not obey ordinary laws of algebra. By taking product of two vector quantities in a specific way a new physical quantity can be derived. The quantity derived this way may be a vector quantity or a scalar quantity. If the product of two vector quantities results into a vector then the product is called a vector product and if it results in to a scalar then the product called a scalar product. Here one may understand in general that of product of two vectors means specific type of combination of two vectors which looks like a product. Thus vector product can be carried out in two ways (1) scalar product (2) Vector product.

4.10.1 Scalar products of two vectors

The scalar products of the two vectors $\vec{A}$ and $\vec{B}$ is defined as follows :

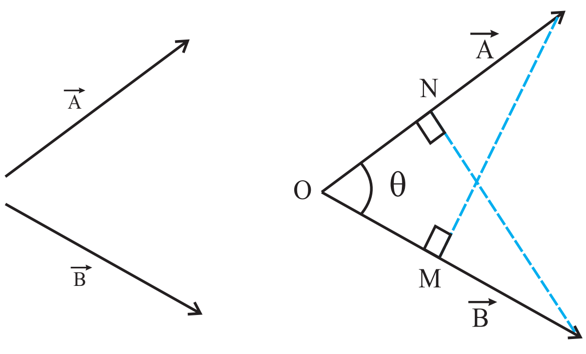
$$\vec{A} \cdot \vec{B} = |\vec{A}|\ |\vec{B}| \cos\theta$$

$$\therefore \vec{A} \cdot \vec{B} = AB \cos\theta \qquad (4.10.1)$$

Where θ is the angle between $\vec{A}$ and $\vec{B}$.

Such product is represented by keeping a dot ($\cdot$) between two vectors, it is also called dot product.

To obtain scalar products of two vectors shown in Fig 4.20.(a), draw these vectors from the common point O as shown in Fig 4.20(b). Now draw perpendicular from the head of the $\vec{A}$ on $\vec{B}$. Hence OM is called projection of $\vec{A}$ on $\vec{B}$.



(*a*) (*b*)

Figure 4.20

From equation (4.10.1)

$$\vec{A} \cdot \vec{B} = AB \cos\theta$$

$$\therefore \vec{A} \cdot \vec{B} = B(A \cos\theta) \qquad (4.10.2)$$

From Fig. 4.20.(b)

$$\cos\theta = \frac{OM}{A}$$

$$\therefore OM = (A)(\cos\theta) \qquad (4.10.3)$$

$$\therefore \vec{A} \cdot \vec{B} = B(OM)$$

$$= (\text{magnitude of } \vec{B})\ (\text{Projection of } \vec{A} \text{ on } \vec{B}) \qquad (4.10.4)$$

$$\text{or } \vec{A} \cdot \vec{B} = A(B \cos\theta) = A(ON)$$

$$= (\text{magnitude of } \vec{A}) \times (\text{projection of } \vec{B} \text{ on } \vec{A}) \qquad (4.10.5)$$

Thus, the scalar product of two vectors is equal to the product of the magnitude of first vector with the projection of second vector on the first vector.

4.10.2 Properties of scalar product

(1) Commutative Law :

$$\vec{A} \cdot \vec{B} = AB \cos\theta = BA \cos\theta = \vec{B} \cdot \vec{A} \qquad (4.10.6)$$

Thus scalar product of two vectors is commutative

(2) Distributive Law :

As shown in Fig. 4.21

$\vec{OP} = \vec{A}$, $\vec{OQ} = \vec{B}$ and $\vec{QR} = \vec{C}$ Now,

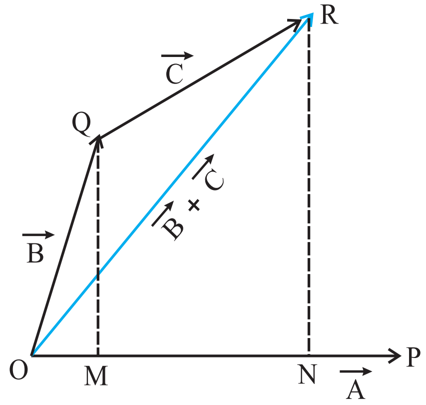


Figure 4.21

$$\vec{A} \cdot (\vec{B} + \vec{C}) = (\text{magnitude of } \vec{A})\ [\text{projection of } (\vec{B} + \vec{C}) \text{ on } \vec{A}]$$

$$= |\vec{A}|(ON)$$

$$= |\vec{A}|(OM + MN)$$

$$= |\vec{A}|(OM) + |\vec{A}|(MN) \qquad (4.10.7)$$

$\therefore \vec{A} \cdot (\vec{B} + \vec{C}) = |\vec{A}|$ (projection of $\vec{B}$ on $\vec{A}$) + $|\vec{A}|$ (projection of $\vec{C}$ on $\vec{A}$)

$\therefore \vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$ (4.10.8)

Thus scalar product of vectors is distributive with respect to summation.

(3) If $\vec{A} \parallel \vec{B}$, $\theta = 0^o$

$\therefore \vec{A} \cdot \vec{B} = AB \cos 0^o = AB$ (4.10.9)

Also $\vec{A} \cdot \vec{A} = |\vec{A}|\ |\vec{A}| = A^2$

$\therefore |\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$ (4.10.10)

Thus the magnitude a vector is equal to the square root of the scalar product of the vector with itself.

(4) If $\vec{A} \perp \vec{B}$ $\theta = 90^o$:

$\therefore \vec{A} \cdot \vec{B} = AB \cos 90^o = 0$

Thus the scalar product of two mutually perpendicular vectors is zero.

(5) Scalar products of unit vectors in Cartesian co–ordinate system :

$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$ (4.10.11)

(6) Scalar product in terms of Cartesian Component of vectors :

If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$\vec{A} \cdot \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \cdot (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

$\therefore \vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$ (4.10.12)

(7) Angle between two vectors

$\vec{A} \cdot \vec{B} = |\vec{A}|\ |\vec{B}| \cos\theta$

$\therefore \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|\ |\vec{B}|}$

$= \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$ (4.10.13)

The angle between two vectors can be found out using this formula.

Illustration 8 : Find the scalar products of two vectors $\vec{A} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{B} = \hat{i} + \hat{j} - 3\hat{k}$

Solution : $\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$

$= 2 + 3 + 12$

$= 17$ units

Illustration 9 : Find the angle between two vectors $\vec{A} = -2\hat{i} + 2\hat{j} - 4\hat{k}$ and $\vec{B} = 2\hat{i} + 4\hat{j} - 2\hat{k}$

Solution :

$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$

$= \frac{-4 + 8 + 8}{\sqrt{24}\sqrt{24}} = \frac{12}{24} = \frac{1}{2}$

$\therefore \theta = 60^o$

Illustration 10 : If vector $\vec{A} = 4\hat{i} - 6\hat{j} + 2\hat{k}$ and $\vec{B} = 6\hat{i} + 8\hat{j} + m\hat{k}$ are mutually perpendicular, find the value of m

Solution : As $\vec{A}$ and $\vec{B}$ are perpendicular to each other $\vec{A} \cdot \vec{B} = 0$

$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z = 0$

$= 24 - 48 + 2m = 0$

$\therefore 2m = 24$

$m = 12$

Illustration 11 : The co-ordinates of a point P in (x, y) plane are x and y. The position vector $\vec{r}$, of this point, makes an angle θ with the X–axis. Find the unit vectors $\hat{n}_r$ and $\hat{n}_\theta$ (in XY plane) which are parallel and perpendicular to $\vec{r}$ respectively.

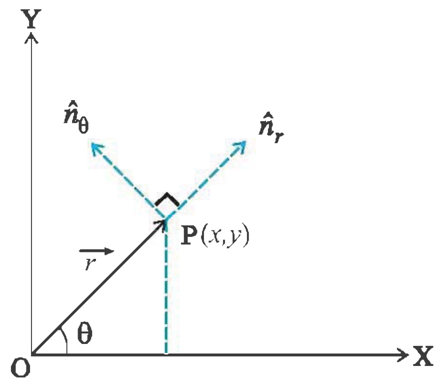


Figure 4.22

Solution : By definition,

$$\hat{n}_r = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j}}{r}$$

$$\therefore \hat{n}_r = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j}$$

or $\hat{n}_r = \cos\theta\,\hat{i} + \sin\theta,\ \hat{j}$ (From Fig. 4.22)

The vector, obtained by rotating $\hat{n}_r$ by $\frac{\pi}{2}$, would be perpendicular to $\hat{n}_r$. We denote this new vector by $\hat{n}_\theta$.

$$\hat{n}_\theta = \cos\left(\theta + \frac{\pi}{2}\right)\hat{i} + \sin\left(\theta + \frac{\pi}{2}\right)\hat{j}$$

$$\therefore \hat{n}_\theta = -\sin\theta\ \hat{i} + \cos\theta\ \hat{j}$$

Note : Here, θ is increased by $\frac{\pi}{2}$ in anti-clockwise sense.

4.10.3 Vector product of two vectors

The vector product of two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{A} \times \vec{B} = |\vec{A}|\ |\vec{B}|\ \sin\theta\ \hat{n}$$

Where θ is the angle between $\vec{A}$ and $\vec{B}$. $\hat{n}$ is the unit vector in the direction perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$. The direction of $\hat{n}$ can be given by right handed screw rule.

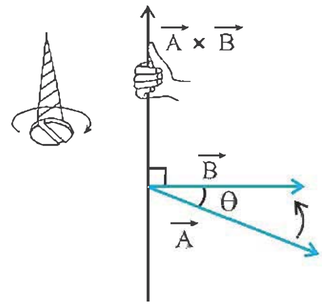


Figure 4.23

As shown in the Fig. 4.23 keep the right handed screw perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ and rotate it from $\vec{A}$ towards $\vec{B}$. The direction of advancement of the screw is taken as the direction of $\hat{n}$. The direction of $\vec{A} \times \vec{B}$ can be determined using Right hand rule : **Open up your right hand palm and wrap the fingers sweeping from $\vec{A}$ to $\vec{B}$ Your streached thumb points in the direction of $\vec{A} \times \vec{B}$**

A vector product is represented by keeping cross sign (×) between two vectors hence it is also called cross product of vectors.

4.10.4 Properties of vector product of two vectors

(1) $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$, The vector product of two vectors is not commutative

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \qquad (4.10.14)$$

This can be understood from the right handed screw rule.

(2) Distributive law

$$\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C}) \qquad (4.10.15)$$

holds for vector product too.

(3) If two vectors are parallel (θ = 0°) or antiparallel (θ = 180°), their vector product is zero because $\sin(0^\circ) = \sin(180^\circ) = 0$.

(4) If $\vec{A} \perp \vec{B}$, θ = 90°

$\therefore \sin\theta = \sin 90^\circ = 1$

$$\therefore \vec{A} \times \vec{B} = AB\sin 90^\circ = AB\,\hat{n} \qquad (4.10.16)$$

(5) Vector products of unit vectors of Cartesian co-ordinate system :

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \quad (4.10.17)$$

and $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

$$\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \hat{i} \times \hat{k} = -\hat{j} \quad (4.10.18)$$

(6) Vector products of two vectors,

$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$\vec{A} \times \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$ is

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k} \quad (4.10.19)$$

$$\therefore \text{Now,} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k} \quad (4.10.20)$$

From equations (4.10.19) and (4.10.20)

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (4.10.21)$$

Illustration 12 : Find the vector product of vectors $\vec{A} = 4\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{B} = \hat{i} + 3\hat{j} + 4\hat{k}$.

Solution : $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & -1 \\ 1 & 3 & 4 \end{vmatrix}$

$= (8 + 3)\hat{i} + (-1 - 16)\hat{j} + (12 - 2)\hat{k}$

$= 11\hat{i} - 17\hat{j} + 10\hat{k}$

Illustration 13 : If vector $\vec{A} = 2\hat{i} - 10\hat{j}$ and vector $\vec{B} = 4\hat{i} - 20\hat{j}$ then show that they are parallel to each other.

Solution : If the two vectors are parallel to each other then their vector product is zero

$$\therefore \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -10 & 0 \\ 4 & -20 & 0 \end{vmatrix} = 8\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -5 & 0 \\ 1 & -5 & 0 \end{vmatrix} = 0$$

Hence $\vec{A}$ and $\vec{B}$ are parallel to each other.

Illustration 14 : Show that the magnitude of cross product of $\vec{A}$ and $\vec{B}$ is equal to twice the area of the triangle of which $\vec{A}$ and $\vec{B}$ are the adjacent sides.

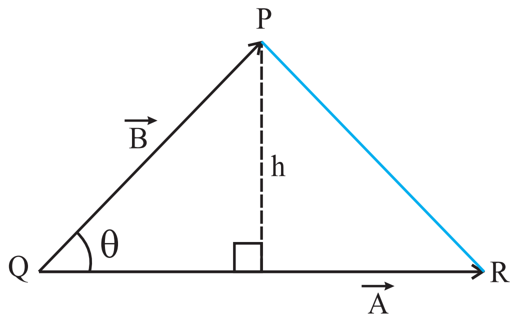


Figure 4.24

Solution : In Fig 4.24, the area of ΔPQR

$= \frac{1}{2}|\vec{A}|h$

$= \frac{1}{2}|\vec{A}||\vec{B}|\sin\theta$

$= \frac{1}{2}\left|\vec{A} \times \vec{B}\right|$

$\therefore \left|\vec{A} \times \vec{B}\right| = 2$ (Area of ΔPQR)

Illustration 15 : Using vector products show that for a plane triangle.

$$\frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C}$$

Where α, β, and γ are the angles and A, B and C are the lengths of the sides opposite to α, β, and γ respectively.

Solution : We know that the magnitude of a cross product of two vectors is twice the area of the triangle of which the vectors form two adjacent sides. In view of this, we have from Fig. 4.25.

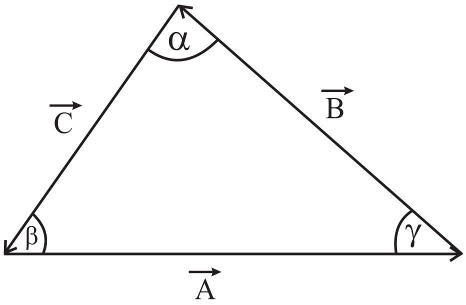


Figure 4.25

$$\left|\vec{A}\times\vec{B}\right| = \left|\vec{B}\times\vec{C}\right| = \left|\vec{C}\times\vec{A}\right|$$

∴ AB sin (π − γ) = BC sin (π − α) = CA sin (π − β)

∴ AB sin γ = BC sin α = CA sin β

Dividing each term by ABC, we have,

$$\therefore \frac{\sin\gamma}{C} = \frac{\sin\alpha}{A} = \frac{\sin\beta}{B}$$

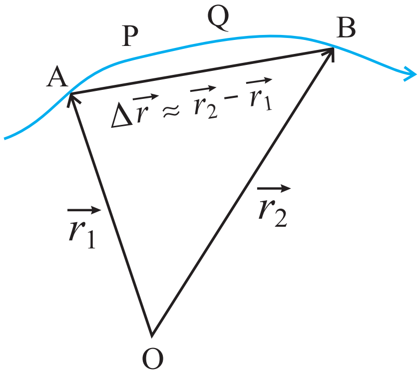
4.11 Instantaneous Velocity

Fig. 4.26(a) shows curved path APQB of a particle moving in the XY plane. Suppose the particle is at point A at time t and reaches point B at time $t + \Delta t$. With respect to certain reference points, the position vectors of these two points are $\vec{r}_1 = \vec{OA}$ and $\vec{r}_2 = \vec{OB}$ respectively.

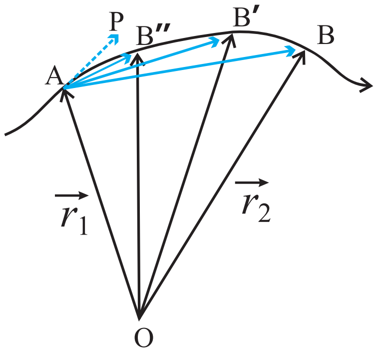
During the motion of the particle from point A to point B, the change in its position is represented by displacement vector $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$. Δt is the time taken for this displacement. By definition the average velocity of the particle in the given time interval Δt is,

$$\text{Average velocity} = \frac{\text{displacement(vector)}}{\text{time(scalar)}}$$

$$\therefore \langle \vec{v} \rangle = \frac{\Delta\vec{r}}{\Delta t} \quad (4.11.1)$$



(a)



(b)

Figure 4.26

The average velocity is a vector quantity and its direction is in the direction of $\Delta\vec{r} = \vec{AB}$. If the average velocity of the particle is same during different time intervals then the motion of the particle is said to be a motion with uniform velocity. The displacement vector $\Delta\vec{r}$ in time interval Δt is the vector joining the initial and final positions of the particle in the time interval Δt. Hence it may not show the actual distance covered by the particle. In reality the particle has moved along the path APQB and reached from A to B. Moreover during time interval Δt changes in the velocity might have taken place. **Hence from average velocity of the particle, we do not get the actual path of its motion and information of velocity at various points on the path of motion.**

As shown in the Fig. 4.26(b) if we keep on decreasing the time interval Δt then the particle which is at point A at time t, after time interval

Δt will be at B' instead of B, will be at B" instead of B' .... and so on. In this manner if we continue to make Δt smaller and smaller, that is we give lesser and lesser time ($\Delta t \to 0$) for change in velocity. Now taking $\lim_{\Delta t \to 0}$ it can be seen from the Fig. 4.26(b) that the displacement vector becomes tangent at point A in the direction AP on the path of the motion of the particle. In these circumstances velocity of the particle has a definite value and direction. This velocity of the particle is called instantaneous velocity ($\vec{v}$) at time t at point A. Symbolically it is respresented as follows.

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \quad (4.11.2)$$

Here $\frac{d\vec{r}}{dt}$ is called derivative of $\vec{r}$ with respect to time t and $\frac{d\vec{r}}{dt}$ is represented symbolically as $\dot{\vec{r}}$. In general instantaneous velocity is termes as velocity. SI unit of velocity is m s^{-1}.

Velocity of a particle at any point on the path of its motion is along the tangent drawn at that point.

To represent the velocity in its components suppose the co-ordinates of points A and B in the Fig. 4.26(a) are (x_1, y_1) and (x_2, y_2) respectively.

$$\therefore \quad \vec{r_1} = x_1\hat{i} + y_1\hat{j} \text{ and } \vec{r_2} = x_2\hat{i} + y_2\hat{j}$$

$$\therefore \Delta\vec{r} = \vec{r_2} - \vec{r_1}$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

$$= \Delta x\hat{i} + \Delta y\hat{j} \quad (4.11.3)$$

Where $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ using equation (4.11.3) in equation (4.11.2)

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta x\hat{i} + \Delta y\hat{j}}{\Delta t}$$

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j} \quad (4.11.4)$$

Where $v_x = \frac{dx}{dt} = \dot{x}$ is the X component of velocity $\vec{v}$. (4.11.5)

and $v_y = \frac{dy}{dt} = \dot{y}$ is the Y component of velocity $\vec{v}$. (4.11.6)

If x and y co-ordinates of the particle in motion are functions of time. x and y components (v_x and v_y) of the velocity $\vec{v}$ of the particle can be obtained, by using above formulae and they can be used to obtain the magnitude and direction of the velocity $\vec{v}$ from equations

$$v = \sqrt{v_x^2 + v_y^2} \text{ and } \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

respectively. Here θ is the angle between the X–axis and direction of velocity.

Illustration 16 : Position vector of a particle is given by the formula $\vec{r}(t) = t^2\hat{i} + 3t\hat{j} + 24\hat{k}$.

(i) Obtain formula for the velocity of the particle.

(ii) Find magnitude and direction of its velocity at $t = 2$ s.

Remember,

$$\left[\frac{d(x^n)}{dx} = nx^{n-1}\right]$$

Solution :

(i) Velocity at any instant of time

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(t^2\hat{i} + 3t\hat{j} + 24\hat{k})$$

$$\therefore \quad \vec{v}(t) = 2t\hat{i} + 3\hat{j}$$

(ii) To obtain velocity at $t = 2$ s substitute $t = 2$ in the above expression

$$\vec{v}_2 = 2(2)\hat{i} + 3\hat{j}$$

$$= 4\hat{i} + 3\hat{j}$$

$\therefore$ $v_x = 4\text{m s}^{-1}$ and $v_y = 3\text{m s}^{-1}$

$\therefore$ Magnitude of velocity

$$\left|\vec{v}_2\right| = \sqrt{(4)^2 + (3)^2} = 5\text{m s}^{-1}$$

If direction of velocity is in the direction making angle θ with the X–axis then,

$$\theta = tan^{-1}\left(\frac{v_y}{v_x}\right) = tan^{-1}\left(\frac{3}{4}\right) = tan^{-1} 0.75 \approx 37^{o}$$

4.12 Acceleration

Time rate of change of velocity is called acceleration.

Suppose a particle is at point P on its path of motion (as shown in Fig. 4.27) at time t and its velocity is v at this point. Now it reaches at point P_1 at time $t + \Delta t$ and its velocity is v' at P_1. Thus the change in velocity of the particle in time

Δt, $\Delta\vec{v} = \vec{v}' - \vec{v}$

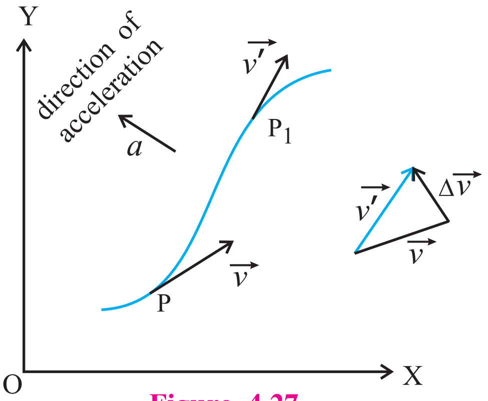


Figure 4.27

As per definition average acceleration

$$= \frac{\text{Change in velocity}}{\text{time}}$$

$$\therefore \langle a \rangle = \frac{\Delta\vec{v}}{\Delta t} \qquad (4.12.1)$$

Average acceleration $\langle \vec{a} \rangle$ is vector quantity and its direction is in the direction of the vector representing the change in velocity $\Delta\vec{v}$.

The information regarding how the velocity of the particle change at each moment on its actual path between points P and P_1 cannot be obtained from average acceleration. Taking $\Delta t \to 0$ in equation (4.12.1), instantaneous acceleration $(\vec{a})$ is obtained. Generally instantaneous acceleration is called acceleration. SI unit of acceleration is m s^{-2}.

Instantaneous acceleration

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \qquad (4.12.2)$$

Now, $\vec{v} = \dfrac{d\vec{r}}{dt}$

$$\therefore \vec{a} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} \qquad (4.12.3)$$

Substituting $\vec{v} = v_x\hat{i} + v_y\hat{j}$ in equation. (4.12.2)

$$\vec{a} = \frac{d}{dt}(v_x\hat{i} + v_y\hat{j}) = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$$

$$\therefore \vec{a} = a_x\hat{i} + a_y\hat{j}$$

Where, $a_x = \dfrac{dv_x}{dt} = \dot{v}_x$ = X component of the acceraletion $\vec{a}$ of the particle (4.12.4)

$a_y = \dfrac{dv_y}{dt} = \dot{v}_y$ = Y component of the accelaration $\vec{a}$ of the particle. (4.12.5)

If the co-ordinates x and y of the particle in motion are functions of time, then using equations (4.11.5) and (4.11.6) the X and Y components of velocity of the particle (v_x and v_y) can be obtained. Substituting them in equations (4.12.4) and (4.12.5), X and Y components of the acceleration (a_x and a_y) of the particle can be obtained.

Velocity is a vector quantity, hence, it can be changed in three ways :

(i) by changing its magnitude only (ii) by changing its direction only (iii) by changing both its magnitude and direction.

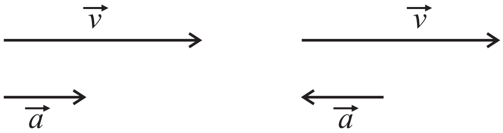


Figure 4.27 (a)

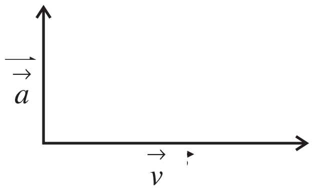


Figure 4.27 (b)

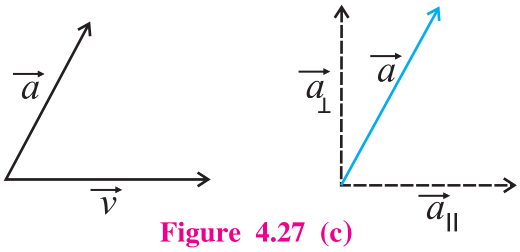


Figure 4.27 (c)

As shown in the Fig. 4.27(*a*) if acceleration $\vec{a}$ is in the direction of velocity ($\vec{v}$) the magnitude of the velocity increases or when $\vec{a}$ is in the direction opposite to velocity ($\vec{v}$) then the magnitude of the velocity decreases respectively.

As shown in Fig 4.27(*b*) if acceleration $\vec{a}$ is in direction perpendicular to the direction of the velocity ($\vec{v}$) then only the direction of the velocity changes.

As shown in the Fig. 4.27(*c*) for some angle between the directions of acceleration ($\vec{a}$) and velocity ($\vec{v}$) of other than 0º, 90º or 180º.Consider the two components of acceleration (i) parallel to the velocity (a_{11}) and (ii) perpendicular to the velocity ($a_\perp$). It can be seen that due to component a_{11} the magnitude of the velocity changes and due to component $a_\perp$ the direction of the velocity changes.

Illustration 17 : Velocity of a particle at time t is $\vec{v}(t) = 7t\hat{i} + 16\hat{k}$. Find acceleration of the particle.

Solution : Acceleration $\vec{a} = \dfrac{d\vec{v}}{dt}$

$$\therefore \ \vec{a} = \frac{d}{dt}(7t\hat{i} + 16\hat{k})$$

$$= 7\hat{i} \text{ m s}^{-2}$$

Illustration 18 : The position vector of a moving particle changes with time according to the formula $\vec{r} = \alpha t\hat{i} - \beta t^2\hat{j}$, where α and β are positive constants. Then (*a*) determine the path of motion of the particle, (*b*) obtain the formula for velocity and acceleration as functions of time and also obtain their magnitudes.

Solution :

(a) $\vec{r} = \alpha t\hat{i} - \beta t^2\hat{j}$ is given and

$$\vec{r} = x\hat{i} - y\hat{j}$$

$\therefore$ $x = \alpha t$ and $y = -\beta t^2$. Eliminating t from these equations we get,

$y = \dfrac{\beta x^2}{\alpha^2}$ which is similar to the equation of parabola viz, $y = ax - bx^2$ (where $a = 0$ and $b = \dfrac{\beta}{\alpha^2}$). Hence, the path of the said particle is a parabola.

(b) The velocity of the particle $\vec{v} = \dfrac{d\vec{r}}{dt}$

$$\therefore \ \vec{v} = \frac{d}{dt}(\alpha t\hat{i} - \beta t^2\hat{j}) = \alpha\hat{i} - 2\beta t\hat{j}$$

This equation gives the velocity of the particle as a function of time.

The magnitude of velocity

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{\alpha^2 + (-2\beta t)^2}$$

$$= \sqrt{\alpha^2 + 4\beta^2 t^2}$$

Now, the acceleration of the particle $\vec{a} = \dfrac{d\vec{v}}{dt}$

$$\therefore \ \vec{a} = \frac{d}{dt}(\alpha\hat{i} - 2\beta t\hat{j}) = -2\beta\hat{j}$$

Since the expression for acceleration does not contain t we can say that the acceleration of the particle is constant; and it is in the direction of negative Y axis.

The magnitude of acceleration

$$\left|\vec{a}\right| = \sqrt{a_x^2 + a_y^2} = \sqrt{(0)^2 + (-2\beta)^2}$$

$$= 2\beta$$

Illustration 19 : The position vector of on particle, as a function of time, is given by :

$\vec{r} = \vec{b}\, t\,(1 - \alpha t)$; where $\vec{b}$ is a constant vector and α is some positive constant, (i) Obtain the velocity and acceleration of the particle as function of time and (ii) find the time taken by the particle to come back to the same point from where it had started.

Solution : (i) $\vec{r} = \vec{b}\, t\,(1 - \alpha t)$ (1)

$$\therefore \quad \vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}\{\vec{b}\, t\,(1 - \alpha t)\}$$

$$= \frac{d}{dt}\{(\vec{b}\, t - \vec{b}\, \alpha t^2)\}$$

$$\therefore \quad \vec{v}(t) = \vec{b} - 2\vec{b}\, \alpha\, t$$

$$= \vec{b}(1 - 2\alpha t) \quad (2)$$

Similarly, acceleration $\vec{a}(t) = \frac{d\vec{v}}{dt} =$

$$\frac{d}{dt}\{\vec{b} - 2\vec{b}\,\alpha t\} = 0 - 2\vec{b}\,\alpha$$

$$\therefore \quad \vec{a}(t) = -2\vec{b}\,\alpha$$

(ii) This particle starts (i.e. at time $t = 0$) its motion from $\vec{r} = \vec{0}$. It can be seen from eqn. (1) that again at time $t = \frac{1}{\alpha}$ it will return back to $\vec{r} = \vec{0}$. Thus, in time interval $\Delta t = \frac{1}{\alpha}$, the particle will return back to the point from where it started its motion.

4.13 Relative Velocity

Uptill now we have discussed the motion of a particle with respect to some given frame of reference. We also noticed that the choice of frame of reference is quite arbitrary. The position vector $\vec{r}$, velocity $\vec{v}$ and acceleration $\vec{a}$ depend upon the frame of reference chosen. Now we shall obtain the relations between such quantities in different frames of reference.

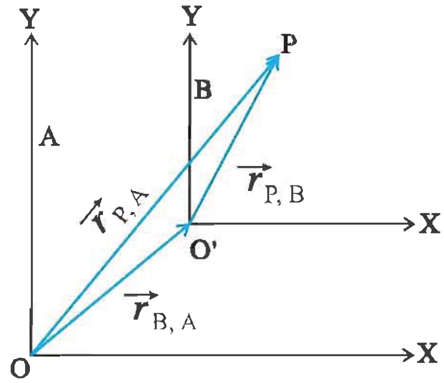


Figure 4.28

In Fig. 4.28 two frames of reference A and B, moving with uniform velocity with respect to (w.r.t.) each other are shown. Such frames of references are called inertial frames of reference and they are discussed in detail in article 5.11. Suppose two observers, one from A and the other form B study the motion of a particle P.

Let the position vectors of particle P at some instant of time with respect to the origin O of frame A be $\vec{r}_{P,A} = \vec{OP}$ and that with respect to the origin O' of frame B be $\vec{r}_{P,B} = \vec{O'P}$.

The position vector of O' w.r.t. O is $\vec{r}_{B,A} = \vec{OO'}$. From Figure 4.28 it is clear that

$$\vec{OP} = \vec{OO'} + \vec{O'P} = \vec{O'P} + \vec{OO'}$$

$$\therefore \quad \vec{r}_{P,A} = \vec{r}_{P,B} + \vec{r}_{B,A} \quad (4.13.1)$$

Differentiating this equation with respect to time we get

$$\frac{d}{dt}(\vec{r}_{P,A}) = \frac{d}{dt}(\vec{r}_{P,B}) + \frac{d}{dt}(\vec{r}_{B,A})$$

$$\therefore \quad \vec{v}_{P,A} = \vec{v}_{P,B} + \vec{v}_{B,A} \quad (4.13.2)$$

Here $\vec{v}_{P,A}$ is the velocity of the particle w.r.t. frame of reference A, $\vec{v}_{P,B}$ is the velocity of the particle w.r.t reference frame B and

$\vec{v}_{B,A}$ is the velocity of frame of reference B with respect to frame A.

Suppose velocities of two particles A and B are respectively $\vec{v}_A$ and $\vec{v}_B$ relative to a frame of reference (suppose earth) then velocity ($\vec{v}_{AB}$) of A relative to B is

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B \quad (4.13.3)$$

and velocity $\vec{v}_{BA}$ of B relative to A is

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A \quad (4.13.4)$$

Thus $\vec{v}_{AB} = -\vec{v}_{BA}$

and $|\vec{v}_{AB}| = |\vec{v}_{BA}|$

For example a car is moving with velocity 80 km/h on a highway towards East. A truck is also moving toward East with velocity 60 km/h and a motorbike is moving towards West with velocity 40 km/h . All the velocities are relative to the earth and written as follows.

$\vec{v}_{CG} = 80\hat{i}$ km/h, $\vec{v}_{TG} = 60\hat{i}$ km/h and $\vec{v}_{BG} = -40\hat{i}$ km/h

Now velocity of car relative to motorbike

$\vec{v}_{CB} = \vec{v}_{CG} - \vec{v}_{BG} = 80\hat{i} - (-40\hat{i}) = 120\hat{i}$,

velocity of car relative to truck $\vec{v}_{CT} = \vec{v}_{CG} - \vec{v}_{TG} = 80\hat{i} - 60\hat{i} = 20\hat{i}$ and velocity of motorbike relative to truck $\vec{v}_{BT} = \vec{v}_{BG} - \vec{v}_{TG} = -40\hat{i} - 60\hat{i}$

$= -100\hat{i}$

Generally if we know the velocities of two objects P and Q w.r.t. third X then

$$\vec{v}_{PQ} = \vec{v}_{PX} + \vec{v}_{XQ} = \vec{v}_{PX} - \vec{v}_{QX} \quad (4.13.5)$$

This formula holds true for (a) when the velocities are not very large, (b) if the object is not performing rotational motion and (c) the time interval are the same for all the frames of referance.

Illustration 20 : A boat can move in river water with speed of 8km/h. This boat has to reach to a place from one bank of the river to a place which is in perpendicular direction onthe other bank of the river. Then (i) in which direction should the boat has to be moved ? (ii) If the width of the river is 600 m; then what will be the time taken by the boat to cross the river ? The river is flows with velocity 4km/h.

Solution : Suppose the river is flowing in positive X direction as shown in Fig. 4.29. To reach to a place in the perpendicular direction on the other bank, the boat has to move in the direction making angle θ with Y direction as shown in the Fig. 4.29. This angle should be such that the velocity of the boat relative to the opposite bank is in the direction perpendicular to the bank.

Note : When we say boat can move in water velocity 8 km/h it means that the velocity of boat is 8 km/h relative to water. When air hostess anounces in aeroplance, that the velocity of the plane is 700 km/h then it means that it is relative to atmosphere.

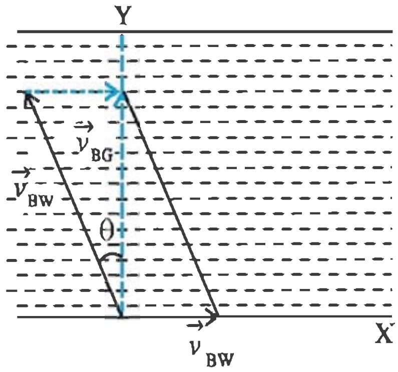


Figure 4.29

Suppose $\vec{v}_{BW}$ = velocity of boat relative to water is 8 km/h in the direction making angle θ with Y–axis.

$\vec{v}_{WG}$ = velocity of water–relative to bank which is 4 km/h in the positive X direction and

$\vec{v}_{BG}$ = velocity of the boat relative to bank which is to be found.

It is clear from the Fig. 4.29

$$\vec{v}_{BG} = \vec{v}_{BW} + \vec{v}_{WG} \quad (a)$$

Taking x components in this equation

$0 = 8\ cos\ (90 + \theta) + 4\ cos\ 0 = -8\ sin\ \theta + 4$

$$\therefore\ sin\ \theta = \frac{4}{8} = \frac{1}{2}\ \therefore\ \theta = 30^{\circ}$$

(ii) Taking Y–components in the equation (a)

$v_{BG} = 8\ cos\ 30^{\circ} + 0 = 8 \times 0.866 = 6.928$ $\approx$ 6.93 km/h

Thus the velocity of boat relative to bank v_{BG} = 6.93 km/h The time taken with this velocity to cover distance of 600 m.

$$= \frac{\text{displacement in Y-direction}}{\text{velocity Y-direction}}$$

$$= \frac{600\text{km}}{6.93\text{km/h}}$$

= 0.8658 hr $\approx$ 5.2 minute

4.14 Equations of motion in a plane (two dimensions) with uniform acceleration :

Suppose a particle moves in the XY plane with uniform acceleration $\vec{a}$. Its velocities at time $t = 0$ and $t = t$ are v_0 and v respectively. As it is moving with uniform acceleration in any time interval its average acceleration and instantaneous acceleration will be the same.

Now change in velocity in time interval

$\Delta t = t - 0$ is $\Delta \vec{v} = \vec{v} - \vec{v_0}$

$$\text{Using } \vec{a} = \frac{\Delta \vec{v}}{\Delta t}$$

$$\vec{a} = \frac{\vec{v} - \vec{v_0}}{t - 0} = \frac{\vec{v} - \vec{v_0}}{t} \quad (4.14.1)$$

$$\therefore\ \vec{v} = \vec{v_0} + \vec{a}\,t \quad (4.14.1a)$$

Writing this equation in terms of components (x and y components)

$$v_x = v_{0x} + a_x t \quad (4.14.2)$$

$$v_y = v_{0y} + a_y t \quad (4.14.3)$$

Suppose the positions of the object at time t = 0 and $t = t$ are represented by position vectors $\vec{r_0}$ and $\vec{r}$ respectively. During this time interval $(t - 0)$

$$\text{Average Velocity} = \frac{\vec{v_0} + \vec{v}}{2}$$

$\therefore$ Displacement taking place in time t = average velocity X time

$$\therefore\ \vec{r} - \vec{r_0} = \left(\frac{\vec{v_0} + \vec{v}}{2}\right)t \quad (4.14.4)$$

Substituting the value of $\vec{v}$ from equation (4.14.1)

$$\vec{r} - \vec{r_0} = \left(\frac{\vec{v_0} + \vec{v_0} + \vec{a}\,t}{2}\right)t$$

$$= \vec{v_0}\,t + \frac{1}{2}\vec{a}\,t^2$$

$$\therefore\ r = \vec{r_0} + \vec{v_0}\,t + \frac{1}{2}\vec{a}\,t^2 \quad (14.14.5)$$

Presenting this equation in the form of components (x and y components)

$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \quad (14.14.6)$$

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \quad (14.14.7)$$

It is clear from the equations (4.14.6) and (4.14.7) that motions in X and Y direction can be described independently.

Thus the motion in a plane (two dimension) with uniform acceleration can be considered as a combination of two simultaneous one dimentional motions in mutually perpendicular directions, with different uniform acceleration. This is an important result. (This type of equations can also be used for motion in three dimensions). Selection of two perpendicular directions is arbitrary.

Thus the equations of motion in plane (two dimensions) with uniform acceleration $\vec{a}$ can be written as follows.

$$\vec{v} = \vec{v}_0 + \vec{a}\,t$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0\,t + \frac{1}{2}\vec{a}\,t^2$$

Taking dot product of the equations (4.14.1) and (4.14.4)

$$(\vec{a})\cdot(\vec{r} - \vec{r}_0) = (\vec{v} - \vec{v}_0)\cdot\left(\frac{\vec{v}_0 + \vec{v}}{2}\right)$$

$$v^2 - v_0^2 = 2\vec{a}\cdot(\vec{r} - \vec{r}_0)$$

From these equations, the equations for the motion in one dimensions with uniform acceleration a can be written as

$$v = v_0 + at$$

$$d = v_0 t + \frac{1}{2}at^2 \quad \text{Here } d = r - r_0$$

$$v^2 - v_0^2 = 2ad$$

Here, d is the displacement in time t.

Illustration 21 : A particle starts its motion from the origin with velocity $2\hat{i}$ m s^{-1} and moves in the XY plane with uniform acceleration $\hat{i} + 3\hat{j}$, (i) what will be the value of its y co-ordinate when the value of its x co-ordinate is 30 m (ii) at this time what will be its speed ?

Solution : (i) Formula for the displacement of a particle in two dimension is

$$\vec{r} = \vec{r}_0 + \vec{v}_0\,t + \frac{1}{2}\vec{a}\,t^2$$

Here, $\vec{r}_0 = 0$

$$\therefore \vec{r}(t) = \vec{v}_0\,t + \frac{1}{2}\vec{a}\,t^2$$

$$\vec{v}_0 = 2\hat{i}\text{ m s}^{-1} \text{ and } \vec{a} = \hat{i} + 3\hat{j}\text{ m s}^{-1}$$

$$\therefore \vec{r}(t) = (2\hat{i})t + \frac{1}{2}\hat{i}(+3\hat{j})t^2$$

$$= (2t + \frac{1}{2}t^2)\hat{i} + \frac{3}{2}\hat{j}t^2$$

$$\therefore x(t) = 2t + \frac{1}{2}t^2 \text{ and } y(t) = \frac{3}{2}t^2$$

At the instant of time t, $x(t) = 30$m is given

$$\therefore 30 = 2t + \frac{1}{2}t^2$$

$$\therefore t^2 + 4t - 60 = 0$$

$$\therefore (t + 10)(t - 6) = 0$$

$\therefore t = -10$s or $t = 6$s but $t = -10$s is not possible.

$\therefore t = 6$ sec. Substituting $t = 6$ in equation.

$$y(t) = \frac{3}{2}t^2 \Rightarrow y(6) = \frac{3}{2}(6)^2 = 54\text{m}$$

Hence the y co-ordinate is 54 m, when the x co-ordinate is 30 m.

(ii) Velocity at any instant of time

$$\therefore \vec{v}(t) = \frac{d}{dt}(x\hat{i} + y\hat{j})$$

$$\therefore \vec{v}(t) = \frac{d}{dt}[(2t + \frac{1}{2}t^2)\hat{i} + \frac{3}{2}t^2\hat{j}]$$

$$\therefore \vec{v}(t) = (2 + t)\hat{i} + 3t\hat{j}$$

$$\therefore \vec{v}(6) = 8\hat{i} + 18\hat{j}$$

$$v_x = 8\text{m s}^{-1} \text{ and } v_y = 18\text{m s}^{-1}$$

$$\therefore v = \sqrt{(8)^2 + (18)^2}$$

$$= \sqrt{64 + 324}$$

$$= 19.698\text{m s}^{-1}$$

4.15 Uniform Circular Motion

The motion of a particle moving on a circular path with constant speed is known as uniform circular motion. As shown in Fig. 4.30 a particle is moving on a circular path with radius r and its speed v is constant.

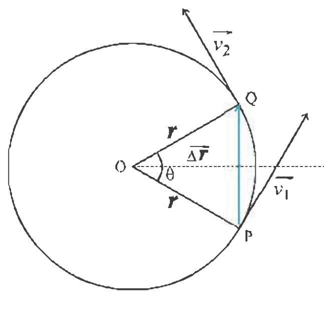


Figure 4.30

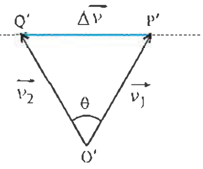


Figure 4.31

Velocity of the particle at a point moving on curved path is in the direction of the tangent drawn to the curved path at that point. Hence it is clear that for the particle moving on the circular path, with constant speed, the direction of velocity changes continuously but its magnitude remains constant.

Since the direction of velocity changes the motion of the particle is an accelerated motion. Thus the uniform circular motion of a particle is an illustration of accelerated motion. (Here the direction of the acceleration vector changes, hence this is also an illustration of the motion of a particle with variable acceleration of constant magnitude.) We have seen in the article 4.12 that in the case in where only the direction of velocity changes, the direction of acceleration is perpendicular to the direction of velocity. Now the velocity is in the direction of the tangent and the direction perpendicular to the tangent is the direction of radius (towards the centre). This acceleration is in the direction along radius towards the centre. This type of acceleration is called **radial acceleration a_r** or **centripetal acceleration a_c**.

To derive formula for radial acceleration, let the velocities of a particle performing uniform circular motion, be $\vec{v}_1$ and $\vec{v}_2$ at points P and Q respectively as shown in Fig 4.30 and the time it takes to go from P to Q be Δt. Thus the change in velocity in time interval Δt is $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ which is as shown in Fig. 4.31.

From the geometry of the figure it is clear that ΔOPQ and ΔO'P'Q' are similar triangles. Hence

$$\frac{P'Q'}{O'P'} = \frac{PQ}{OP} \quad \therefore \frac{\Delta v}{v_1} = \frac{\Delta r}{r}$$

$$\text{but } \left|\vec{v}_1\right| = \left|\vec{v}_2\right| = v$$

$$\therefore\ \Delta v = \frac{v}{r} \cdot \Delta r$$

The magnitude of average acceleration during time interval Δt is

$$\langle a \rangle = \frac{\Delta v}{\Delta t} = \frac{v}{r}\frac{\Delta r}{\Delta t}$$

Taking $\Delta t \rightarrow 0$ in this ratio we get the magnitude of instantaneous acceleration at time t.

$$\text{Acceleration } a_c = \lim_{\Delta t \rightarrow 0} \frac{v}{r}\frac{\Delta r}{\Delta t}$$

$$= \frac{v}{r}\left(\lim_{\Delta t \rightarrow 0} \frac{\Delta r}{\Delta t}\right)$$

$$= \frac{v}{r}\frac{dr}{dt}$$

but $\frac{dr}{dt} = v$ = instantaneous speed at time t

$$\text{Acceleration } a_c = \frac{v^2}{r} \qquad (4.15.1)$$

From Fig. 4.31 it is clear that direction of $\Delta\vec{v}$ is towards the centre. Hence the direction of acceleration (a_c) is towards the centre. Due to this fact this acceleration is called radial or centripetal acceleration. The force corresponding to this acceleration is obviously called centripetal force.

From the above discussion it is clear that in order to make a particle to move on a curved path it should be supplied necessary centripetal force.

The magnitude of centripetal acceleration is constant but its direction keeps on changing continuously so the vector representing the centripetal acceleration is not constant.

Illustration 22 : Nirav ties a small stone at the end of 1 meter long thread. He rotates the stone in the horizontal plane (Here neglect gravitational force). If the stone completes 100 rotations in 314 seconds then (i) what will be its linear speed ? (ii) What will be the magnitude of its centripetal acceleration ? Can we consider the vector representing its acceleration as a constant vector ?

Solution : Here the radius r of the circular path of stone is 1 meter.

(i) Stone completes 100 rotations in 314 seconds.

$\therefore$ time taken to complete one rotation i.e. periodic time T = $\frac{314}{100}$ = 3.14

Linear speed of the stone $v = \dfrac{\text{distance}}{\text{time}}$

$$= \frac{2\pi r}{T} = \frac{2 \times 3.14 \times 1}{3.14} = 2\text{m s}^{-1}$$

$\therefore\ v = 2\text{m s}^{-1}$

(ii) Magnitude of centripetal acceleration

$a_c = \dfrac{v^2}{r} = \dfrac{(2)^2}{1} = 4\text{m s}^{-2}$. Because the direction of this acceleration changes continuously, the vector representing acceleration can not be considered constant.

4.16 Projectile Motion

When an object is thrown in gravitational field of earth it moves with constant horizontal velocity and constant vertical acceleration. Such two dimentional motion is called a projectile motion and the object is called a projectile. If resistance offered by air is neglected, motion of a football kicked by a player and motion of a cricket ball thrown, in air, by a cricketer can be considered to be the projectile motion and the ball is called a projectile.

The projectile motion can be treated as the resultant motion of two independent component motions taking place simultaneously in mutually perpendicular directions. One component is along a horizontal direction without any acceleration and the other along the vertical direction with constant acceleration due to gravitational force. Galileo was the first to state this independency of the horizontal and vertical components of projectile motion.

In our discussion we will neglect air resistance.

Suppose the projectile is projected with velocity $\vec{v}_0$ and that makes an angle θ_0 with the X–axis (horizontal direction) as shown in Fig. 4.32.

The acceleration acting on the projectile is due to gravity which is directed vertically downward.

$$\therefore\ \vec{a} = -g\hat{j}$$

$$\text{or } a_x = 0,\ a_y = -g \qquad (4.16.1)$$

The components of initial velocity $\vec{v}_0$ are

$$v_{0x} = v_0 \cos\ \theta_0 \qquad (4.16.2a)$$

$$v_{0y} = v_0 \sin\ \theta_0 \qquad (4.16.2b)$$

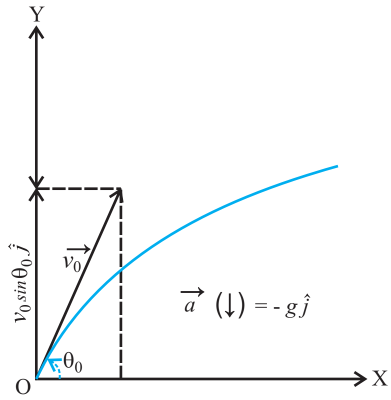


Figure 4.32

If we take the initial position to be the origin of the co–ordinate system, the co–ordinates of the point of projection would be $x_0 = 0$, $y_0 = 0$

Now using equations (4.14.6) and (4.14.7)

$$x = v_{0x}t = (v_0 cos\ \theta_0)t \qquad (4.16.3)$$

$$\text{and } y = (v_0 sin\ \theta_0)t - \frac{1}{2}gt^2 \qquad (4.16.4)$$

The component of velocity at any time t can be obtained from equation (4.14.2) and (4.14.3) as

$$v_x = v_{0x} = v_0 cos\ \theta_0 \qquad (4.16.5)$$

$$v_y = v_0 sin\ \theta_0 - gt \qquad (4.16.6)$$

Using equation (4.16.3) and (4.16.4) co-ordinates of the position of the projectile at any time in temrs of two parameters v_0 and θ_0 can be obtained. During the entire motion of the projectile, the x component of its velocity remains constant, while y component of velocity changes, like an object in free fall in the vertical direction.

Equation of trajectory of a projectile :

The equation giving relation between x and y co-ordinates of a projectile is known as equation of trajectory of a projectile.

To obtain the equation of trajectory of a projectile inserting value of t from equations (4.16.3) in equation (4.16.4). we get,

$$y = v_0\ sin\ \theta_0 \left(\frac{x}{(v_0 cos\ \theta_0)}\right) - \frac{1}{2}g\left(\frac{x^2}{(v_0{}^2 cos\ {}^2\theta_0)}\right)$$

$$y = (tan\theta_0)\ x - \frac{g}{2(v_0 cos\ \theta_0)^2} \cdot x^2 \qquad (4.16.7)$$

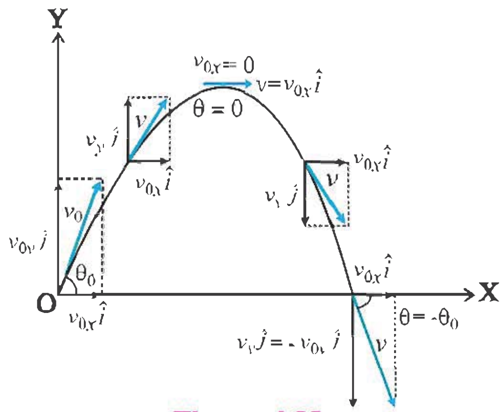


Figure 4.33

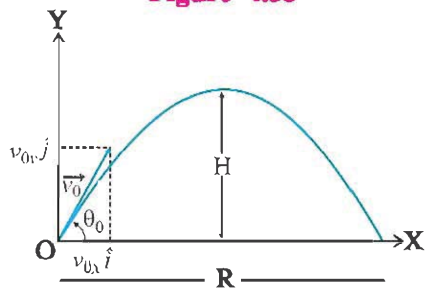


Figure 4.34

In this equation v_0, θ_0 and g are constant it is of the form $y = ax - bx^2$ in which a and b are constants. This is the equation of parabola. Hence we can say that the path of a projectile is a parabola. (See Fig. 4.33 and 4.34)

Time taken to achieve maximum height :

Suppose, the time taken by the projectile to reach maximum height H is t_m. (see Fig. 4.34) When projectile attains the maximum height, the y component of its velocity (v_y) becomes zero (See Fig. 4.33). Hence from equations (4.16.6)

$$v_y = v_0 \sin\theta_0 - gt_m = 0$$

$$\therefore t_m = \frac{v_0 \sin\theta_0}{g} \quad (4.16.8)$$

Maximum height (H) :

The maximum height (H) reached by the projectile can be calculated by substituting

$t = t_m = \dfrac{v_0 \sin\theta_0}{g}$ in equation (4.16.4).

Thus we get,

$$y = H = (v_0 \sin\theta_0)\left(\frac{v_0 \sin\theta_0}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\theta_0}{g}\right)^2$$

$$\therefore H = \frac{v_0^2 \sin^2\theta_0}{2g} \quad (4.16.9)$$

Time of flight (t_F) :

On substituting $y = 0$ and $t = t_F$ in equation (4.16.4).

$$0 = (v_0 \sin\theta_0)t_F - \frac{1}{2}gt_F^2$$

$$t_F = \frac{2v_0 \sin\theta_0}{g} = 2t_m \quad (4.16.10)$$

Range of a projectile (R) :

The horizontal distance covered by a projectile from its initial position ($x = y = 0$) to the final position (where it passes $y = 0$ during its fall) is called the range (R) of the projectile.

If is easy to under state that the range in the distance travelled by the projectic during its time of flight

To find the range (R) substitute $x = R$ and $t = t_F$ in equation (4.16.3)

$$R = (v_0 \cos\theta_0)(t_F)$$

$$= (v_0 \cos\theta_0)\left(\frac{2v_0 \sin\theta_0}{g}\right)$$

$$\therefore R = \frac{v_0^2 \sin 2\theta}{g}$$

$$\therefore R_{max} = \frac{v_0^2}{g} \quad (4.16.11)$$

It is clear from the above equations, $R = R_{max}$ is the maximum range for $\theta_0 = 45°$ for given v_0.

It is important to note that the magnitude of the range depends upon the projection velocity (v_0) and the projection angle (θ_0) while the maximum range R_{max} depend only on the projection velocity (v_0).

Find t_m and t_F for an object thrown in the vertical direction $\theta_0 = \dfrac{\pi}{2}$.

Illustration 23 : A football lying on the ground is kicked with velocity 28m s^{-1} in the direction making 30° with horizontal direction. Find (i) maximum height attained (ii) the time to return on the ground and (iii) the distance at which (from initial position) it will return on the earth. (take acceleration due to gravity $g = 9.8$m s^{-2})

Solution : (i) Maximum height (H) attained by the football

Here $v_0 = 28$m s^{-1}, $\theta_0 = 30°$ and $g = 9.8$m s^{-2}

$$\therefore H = \frac{v_0^2 \ sin^2 \ \theta_0}{2g} = \frac{(\ 28\ )^2 \ (sin30)^2}{2 \times 9.8}$$

$$= \frac{(\ 28\ )^2 \ (0.5)^2}{2 \times 9.8} = 10.0\text{m}$$

(ii) Time taken to return on the ground is the time of flight t_F

$$\therefore t_F = \frac{2v_0 \ sin \ \theta_0}{2g} = \frac{2 \times 28 \times sin30}{9.8}$$

$$= \frac{28}{9.8} = 2.9 \text{ s}$$

(iii) The distance at which the football returns on the ground from the place at which it was kicked is the range R.

$$\therefore R = \frac{v^2{}_0 \ sin \ 2\theta_0}{g}$$

$$= \frac{28 \times 28 \times sin60^o}{g}$$

$= 69$ m

Illustration 24 : Galileo in his book "Dialogues on the Two new sciences", stated that "for elevations which exceed or fall short of 45° by equal amounts, the ranges are equal". Prove this statement.

Solution : Suppose the ranges of two projectiles projected at angles $45^o - \theta$ and $45^o + \theta$ (having the same difference with 45°) are R_1 and R_2. Now onward θ is in degrees.

Using the formula : $R = \dfrac{v^2{}_0 \, sin \, 2\theta^\circ}{g}$, we get

$$R_1 = \frac{v_0^2 \ sin \ 2(45 - \theta^\circ)}{g}$$

$$= \frac{v_0^2 \ sin \ (90 - 2\theta^\circ)}{g}$$

$$= \frac{v_0^2 \ cos \ 2\theta^\circ}{g} \text{ and}$$

$$R_2 = \frac{v_0^2 \ sin \ 2(45 + \theta^\circ)}{g}$$

$$= \frac{v_0^2 \ sin \ (90 + 2\theta^\circ)}{g}$$

$$= \frac{v^2{}_0 \cos 2\theta^\circ}{g}$$

Thus we can see that $R_1 = R_2$

If two projectiles are throun with same speed with complementary angles of projection ($\theta_1 = \theta_2 = 90^o$) their ranges will be equal.

Illustration 25 : A water pipe lying on the ground has a hole in it. From this hole water stream shoots upwards at an angle 45° to the horizontal. The speed of water stream is 10m s^{-1}. At what height once this stream hit the wall which is 5m away from the hole ?

Solution : $\theta_0 = 45^o$, $v_0 = 10\text{m s}^{-1}$, $x = 5$ m

Using the formula,

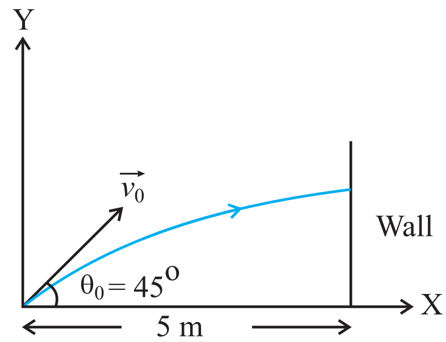


Figure 4.35

$$y = x(tan \ \theta_0) - \frac{g}{2(\ v_0 \ cos \ \theta_0\ )^2} \cdot x^2$$

$$y = 5(tan \ 45^o) - \frac{9.8 \times 25}{2 \times (10 \times cos \ 45^o)^2}$$

$$= 5 - \frac{9.8 \times 25}{2 \times 100 \times \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= 5 - \frac{9.8}{4} = 5 - 2.45$$

$= 2.55$ m

Thus water stream will hit the wall at the height of 2.55 m on the wall.

SUMMARY

1. In this chapter we have obtained information regarding vector and scalar quantities in detail. We have learned to represent vectors graphically. We have distinquished between position vector and displacement vector and seen how the displacement vector can be obtained.
2. As vectors do not obey the ordinary laws of algebra, we learnet vector algebra. Zero vector and unit vectors were defined and it was shown that how a vector can be represented using unit vector. How vectors can be resolved in a plain was explained. In the case of product of the vectors, scalar and vector products we are defined. We understood the meaning of instantaneous velocity and derived formula for acceleration. After understanding relative motion we have obtained expression for relative velocity.
3. Equations for motion in a plane were derived.
4. We discussed uniform circular motion in detail and derived expression for centripetal acceleration and shown that its direction is towards the centre along the radius.
5. We also learnt projectile motion and derived the equation for its trajectory is expressions for time required to achieve maximum height, maximum range and time of flight were derived. We have also shown that for any given velocity to obtain the maximum range a projectice should be projected an angle of 45°.

Multiplication of Vector

↓

with scalar

↓

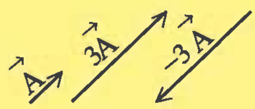
with number (simple multiplication)

vector $\vec{A}$ is multiplied by number m, new vector m$\vec{A}$ is obtained.

- If m is positive, then new vector is in the direction of $\vec{A}$
- If m is negative, then new vector is in the opposite direction of $\vec{A}$.

Ex. : $3 \times \vec{A} = 3\vec{A}$

$-3 \times \vec{A} = -3\vec{A}$



with physical quantity (simple multiplication)
(1) mass × velocity = momentum
(scalar) × (vector) = (vector)
(2) mass × acceleration = force Thus, when a vector quantity is multiplied by a scalar quantity, new vector quantity is obtained. Which is in the direction of original vector quantity.

with vector

↓

result is scalar
(vector)·(vector) = (scalar)
scalar product
or
(dot product)

result is vector
(vector) × (vector) = (vector)
vector product
or
(cross product)

EXERCISES

Choose the correct option from the given options :

1. Which quantity is a scalar from the following physical quantities.

(A) Acceleration (B) velocity

(C) linear momentum (D) Temperature

2. If $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{B} = 4\hat{i} + 6\hat{j} - 2\hat{k}$, what will be the angle between $\vec{A}$ and $\vec{B}$.

(A) π (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) 0^o

3. An object is moving on a circular path with velocity $\vec{v}$, at a given instant. When it completes half rotation, what will be the change in its velocity ?

(A) $\vec{v}$ (B) $-2\vec{v}$ (C) zero (D) $\sqrt{2}\,\vec{v}$

4. A vector representing a physical quantity is $\vec{C} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, the angle between X–axis and $\vec{C}$ is

(A) $cos^{-1}\frac{3}{\sqrt{29}}$ (B) $cos^{-1}\frac{4}{\sqrt{29}}$

(C) $cos^{-1}\frac{5}{\sqrt{29}}$ (D) $cos^{-1}\frac{2}{\sqrt{29}}$

5. Co-ordinates of *a* particle moving in *a* plane at any time *t* are given by equations $x = \alpha t^2$ and $y = \beta t^2$. Magnitude of the velocity of this particle is

(A) $2t\sqrt{\alpha^2 - \beta^2}$ (B) $2t\sqrt{\alpha^2 + \beta^2}$

(C) $2t(\alpha + \beta)$ (D) $\sqrt{\alpha^2 + \beta^2}$

6. In a projectile motion if the maximum height H is half the range (R) ($H = \frac{1}{2}R$) then angle of projections θ_0 is

(A) $tan^{-1}(1)$ (B) $tan^{-1}(2)$

(C) $tan^{-1}(3)$ (D) $tan^{-1}(4)$

7. An object is projected with velocity $\vec{v}$. If the range (R) of this object is double the maximum height H, then its range is

(A) $\frac{v^2}{g}$ (B) $\frac{3}{5}\frac{v^2}{g}$ (C) $\frac{4}{5}\frac{v^2}{g}$ (D) $\frac{1}{2}\frac{v^2}{g}$

[**Note :** Use result of above objective (6)]

8. $(\vec{A} \cdot \vec{B})^2 + |\vec{A} \times \vec{B}|^2 =$ ....

(A) AB (B) A^2B^2 (C) $\sqrt{AB}$ (C) zero

9. Rain falls in the downward direction with velocity 4 km/h. A man is walking on a straight road with velocity 3 km/h. The apprent velocity of rain relative to this man is

(A) 3 km h^{-1} (B) 4 km h^{-1} (C) 5 km h^{-1} (D) 7 km h^{-1}

10. For which angle of projection the range and its maximum height will be equal ?

(A) $\theta_0 = 45^o$ (B) $\theta_0 = tan^{-1}(4)$

(C) $\theta_0 = tan^{-1}\left(\frac{1}{4}\right)$ (D) $\theta_0 = 30^o$

11. A motorcar is moving northwards with velocity 30m s^{-1}. If it turns towards West with the same speed, then change in its velocity is

(A) 60m s^{-1} North–West (B) $30\sqrt{2}$ m s^{-1} North–West

(C) $30\sqrt{2}$ m s^{-1} South–West (D) 60m s^{-1} South–West

12. If the resultant vector of $\vec{A}$ and $\vec{B}$ makes an angle α with $\vec{A}$ and β with B. Then

(A) $\alpha < \beta$ always (B) If A < B, $\alpha < \beta$

(C) If A > B, $\alpha < \beta$ (D) If A = B, $\alpha < \beta$

13. The linear speed of the tip of second arm of a clock is v. The magnitude of change in its velocity in 15 second is

(A) zero (B) $\frac{v}{\sqrt{2}}$ (C) $\sqrt{2}\,v$ (D) $2v$

14. The velocity of a boat with respected to ground is $3\hat{i} + 4\hat{j}$ and the velocity of water with respected to ground is $-3\hat{i} - 4\hat{j}$. Hence the velocity of boat w.r.t. water is ....

(A) $8\hat{i}$ (B) $-6\hat{i} - 8\hat{j}$ (C) $6\hat{i} + 8\hat{j}$ (D) $6\hat{i}$

15. When the angle of projection is 25°, the range of the projectile is R. Now if the angle of projection is .... its range will remain same. (i.e. R)

(A) 40° (B) 45° (C) 65° (D) 60°

16. If the magnitude of the vector products of two vectors $|\vec{A} \times \vec{B}|$ is $\sqrt{3}$ times the magnitude of their scalar product $\vec{A} \cdot \vec{B}$ then the angle between them is ....

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

17. The acceleration of a projectile, at its maximum height is

(A) zero (B) g (C) maximum (D) minimum

18. An object is projected at angle of 45°, with the horizontal, with kinetic energy K. Its kinetic energy at maximum height is .... $[K = \frac{1}{2}mv^2]$

(A) 0 (B) $\frac{K}{2}$ (C) $\frac{K}{\sqrt{2}}$ (D) K

19. The velocity of a boat in a river of width 1.0 km, is 5 km h^{-1}. The boat crosses the river in 15 minutes, moving over the shortest path. Hence, the velocity of the flow of river is .... km h^{-1}.

(A) 1 (B) 3 (C) 4 (D) 5

20. Bullets are fired with the same initial velocity v in different directions on a plane surface. These bullets would fall on the maximum area of .... on this surface.

(A) $\frac{\pi v^2}{g}$ (B) $\frac{\pi v^2}{g^2}$ (C) $\frac{\pi^2 v^2}{g^2}$ (D) $\frac{\pi v^4}{g^2}$

21. For a projectile motion $y(t) = 8t - 5t^2$ and $x(t) = 6t$, where x and y are in metre and t is in second. The initial velocity of this projectile is ....

(A) 6 m/s (B) 8 m/s (C) 10 m/s (D) 14 m/s

22. If $|\vec{A} + \vec{B}| = |\vec{A}| = |\vec{B}|$, then the angle between $\vec{A}$ and $\vec{B}$ is ....

(A) 90° (B) 120° (C) 0° (D) 60°

23. If $\vec{A} + \vec{B} = \vec{C}$ and $A = \sqrt{3}$, $B = \sqrt{3}$ and $C = 3$, then angle between $\vec{A}$ and $\vec{B}$ is ....

(A) 0° (B) 30° (C) 60° (D) 90°

24. The unit vector is perpendicular to the two vectors $3\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} - 2\hat{j} + 4\hat{k}$ is ....

(A) $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$ (B) $\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

(C) $(\hat{i} - \hat{j} - \hat{k})$ (D) $\sqrt{3}(\hat{i} - \hat{j} - \hat{k})$

25. If the vectors $\vec{P} = a\hat{i} + a\hat{j} + 3\hat{k}$ and $\vec{Q} = a\hat{i} - 2\hat{j} - \hat{k}$ are mutually perpendicular, then the positive value of 'a' is ....

(A) 3 (B) 4 (C) 9 (D) 13

ANSWERS

1. (D)	2. (D)	3. (B)	4. (D)	5. (B)	6. (B)
7. (C)	8. (B)	9. (C)	10. (B)	11. (C)	12. (C)
13. (C)	14. (C)	15. (C)	16. (C)	17. (B)	18. (B)
19. (B)	20. (D)	21. (C)	22. (B)	23. (A)	24. (A) 25. (A)

Very Short Questions

1. What is the basic difference between vector and scalar ?
2. State names of two vectors quantities and two scalar quantities ?
3. What is needed to be stated to state when the position of an object is to be mentioned ?
4. Which vectors are called equal vectors ?
5. Define parallel vectors.
6. Define antiparallel vectors.
7. Which vectors are called non-parallel vectors.
8. State the two ways in which a vector is described.
9. How is the scalar product of two vectors defined ?
10. How is the vector product of two vectors defined ?
11. If the angle between two vectors is zero, the magnitude of their vector product is ....

12. If the angle between two vectors is 90°, their scalar product will be ....
13. If the angle between two vectors is zero, their scalar product will be ....
14. The direction of velocity of an object at any point on the path of its motion will along the ....
15. Velocity is a vector quantity. In how many ways can this vector be changed ?
16. Component of acceleration parallel to velocity (a_{11}) changes the .... velocity and perpendicular component ($a_{\perp}$) changes the .... velocity.
17. Acceleration in case of uniform circular motion along the tangent to the circular path is ....
18. What is called projectile motion ?
19. At the maximum height of the trajectory of a projectile, its velocity is ....
20. At the maximum height of the trajectory of a projectile, its acceleration is ....
21. To obtain the maximum range the object should be projected at an angle of .... with the horizontal.

Short Questions :

1. Describe the geometrical (graphical) method to represent vector quantities.
2. Distinguish between position vectors and displacement vectors.
3. Explain the multiplication of a vector by a real number.
4. Explain the subtraction of two vectors.
5. State the properties of addition of vectors.
6. Define unit vector and explain it in detail.
7. How are the two perpendicular components of a vector are obtained ?
8. State the properties of the scalar product of vectors.
9. In the case of the vector product of two vectors, explain the right handed screw rule for the direction of the resultant vector.
10. State the properties of vector products.
11. Obtain the formula for time t_m taken to achieve the maximum height of a projectile.
12. Derive formula for the maximum height H achieved by a projectile.
13. Obtain formula for the range of a projectile and using it obtain the formula for the maximum range.
14. Obtain the formula for total time of flight t_F of a projectile.

Answer the following questions in detail :

1. Drawing the necessary figure, explain the method of triangle for the addition of two vectors.
2. Drawing the necessary figure explain addition of two vectors using law of parallelogram. Obtain the magnitude and direction of the resultant vector using the components of the vectors.
3. Explain the resolution of vectors in a plane.
4. Describe the algebraic method for addition and subtraction of vectors.
5. Drawing the necessary figure to explain instantaneous velocity and derive the formula $\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$
6. Drawing the necessary figure explain acceleration and derive formula
   $$\vec{a} = \frac{d^2r}{dt^2} = \ddot{\vec{r}}$$
7. Using Derive an appropriate diagram explain relative velocity.
8. Obtain the equations for the motion in a plane.
   $$\vec{v} = \vec{v}_0 + \vec{a}\,t$$

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}\, t$$

$$v^2 - v_0^2 = 2\vec{a}\cdot(\vec{r} - \vec{r}_0)$$

9. Using an appropriate diagram derive formula for acceleration $a_c = \frac{v^2}{r}$ in the case of uniform circular motion and show that its direction is towards the centre along the radius.

Solve the following problems :

1. Two forces of equal magnitude act on a particle. If the angle between them is θ, find the magnitude of the resultant force. [**Ans. :** $2F\, cos\left(\frac{\theta}{2}\right)$]

2. Find the unit vector of vector $\vec{A} - \vec{B}$. Where vector $\vec{A} = 2\hat{i} - \hat{j} + 2\hat{k}$ unit and $\vec{B} = -\hat{i} - 2\hat{j} + 2\hat{k}$ unit [**Ans. :** $\frac{3\hat{i} + \hat{j}}{\sqrt{10}}$ unit]

3. If vectors $\vec{A} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{B} = 4\hat{i} + 6\hat{j} - 2\hat{k}$, then show that they are parallel vectors.

4. A passenger arriving in a new town has to go to the hotel located 10 km away on straight road from station. Due divergence taxi driver takes him along a circuitous path 23 km long and reaches the hotel in 28 minutes.

(a) What is the average speed of the taxi ?

(b) Find the magnitude of the average velocity. Are these two equal ?

[**Ans. :** (a) 49.3km h^{-1} (b) 21.26km h^{-1} These two are not equal]

5. A particle starts its motion at time $t = 0$ from the origin with velocity $10\hat{j}$ m s^{-1} and moves in the X–Y plane with constant acceleration $8\hat{i} + 2\hat{j}$.

(a) At what time is its x co-ordinate become 16m ? And at this time what will be its y co-ordinate ?

(b) What will be the speed of this particle at this time ?

[**Ans. :** (a) at 2s, y co-ordinate 24 m (b) Speed 21.26m s^{-1}]

6. An aricraft is flying at the height of 3600 m from the ground. If the angle subtended at the ground observation point by aircraft positions 10 seconds apart is 30°, what is the speed of the aircraft ? [**Ans. :** 60π m s^{-1}]

7. A bullet fired from a gun at an angle 30° with horizontal direction hits the ground 3 km away on the ground. By adjusting only the angle of projection is it possible to hit the target 5 km away ? Show with calculation. (neglect air–resistance)

8. Prove that the angle of projection $\theta_0 = tan^{-1}\left(\frac{4H}{R}\right)$ for a projectile, projected from the origin, where H = maximum height and R = range of projectile.

9. Three non–zero vector $\vec{A}$, $\vec{B}$ and $\vec{C}$ satisfy the vector equation $\vec{A} + \vec{B} = \vec{C}$ and their magnitudes are related by the scalar equation A + B = C. How would $\vec{A}$ be oriented w.r.t. $\vec{B}$? Account for your answer.

10. If the direction of vector $\vec{A}$ is reversed, find $\Delta\vec{A}$, $|\Delta\vec{A}|$ and $\Delta|\vec{A}|$.

11. By keeping the direction of a vector the same if its magnitude is doubled, would the magnitude of its every component be doubled ?

12. The tail of a vector is on the origin of X Y co-ordinate axes. The vector is in + X direction. If the vector rotates anti-clockwise, find its X and Y components for the rotation of (i) 90° (ii) 180° (iii) 270° and (iv) 360°.

13. Can the magnitude of the relative velocity of either of two objects be more than the magnitude of the velocities of these objects ? Give one example.

14. Determine the magnitude and direction of $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$.

[**Ans. :** magnitude of both = $\sqrt{2}$; 45° and 315° with X–axis]

15. $\vec{A}$ is in positive Y direction and its magnitude is 100 unit. $\vec{B}$ is in the direction making an angle of 60° (in upward direction) with positive X–axis and its magnitude is 200 unit. $\vec{C}$ is in positive X direction and its magnitude is 150 unit. Out of these vectors which one has the maximum value for its (i) x component (ii) y component ? [**Ans. :** (i) $\vec{C}$ (ii) $\vec{B}$]

16. The magnitude of x component of the position vector of a particle is 3 m and it is in negative X direction. The magnitude of y component of this vector is 4 m and it is in negative Y direction. Find the megnitude of this vector and its direction with respect to negative X–axis. [**Ans. :** 5 m, $tan^{-1}\frac{4}{3}$]

17. Which out of the following quantities are independent of the selection of axes ?

(A) $\vec{A} + \vec{B}$ (B) $\vec{A} - \vec{B}$ (C) $A_x + B_y$

[**Ans. :** (A) and (B)]

18. Find the angle between $\vec{A}$ and $\vec{B}$ if $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$

[**Ans. :** 90°]

19. Calculate the displacement of a particle, with position vector $\vec{r} = 3t^2\hat{i} + 4t^2\hat{j} + 7\hat{k}$ metre in 10 s. [**Ans. :** $300\hat{i} + 400\hat{j}$ (m)]

20. Obtain the component of vector $\vec{A} = 2\hat{i} + 3\hat{j}$ in the direction of vector $\hat{i} + \hat{j}$

[**Ans. :** $\frac{5}{\sqrt{2}}$]

21. Can the magnitude of a component of a vector be zero if the magnitude of the vector is not zero ? Can the magnitude of a vector be zero if one of its components is non–zero ?

22. Three non-zero vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ satisfy the equation $\vec{A} + \vec{B} = \vec{C}$ and their magnitude satisfy the equation $A^2 + B^2 = C^2$. How would $\vec{A}$ be oriented with respect to $\vec{B}$? Account for your answer.

23. Two obejcts are projected with the same velocity at different angles with the horizontal and if the range is same for both of them prove that $t_1 t_2 = \frac{2R}{g}$ where t_1 and t_2 are their time of flights.

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