Pseudo-Differential Operators
Theory and Applications
5

M. W. Wong

Discrete Fourier Analysis

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Vol. 5

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2010 Mathematics Subject Classification: 42-01, 47-01, 42A45, 47G30, 65T50, 65T60

ISBN 978-3-0348-0115-7 e-ISBN 978-3-0348-0116-4
DOI 10.1007/978-3-0348-0116-4

Library of Congress Control Number: 2011929860

Cover design: deblik

Printed on acid-free paper

Springer Basel AG is part of Springer Science+Business Media

www.birkhauser-science.com

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Preface

Fourier analysis is a prototype of beautiful mathematics with many-faceted applications not only in mathematics, but also in science and engineering. Since the work on heat flow of Jean Baptiste Joseph Fourier (March 21, 1768–May 16, 1830) in the treatise entitled “Théorie Analytique de la Chaleur”, Fourier series and Fourier transforms have gone from triumph to triumph, permeating mathematics such as partial differential equations, harmonic analysis, representation theory, number theory and geometry. Their societal impact can best be seen from spectroscopy to the effect that atoms, molecules and hence matters can be identified by means of the frequency spectrum of the light that they emit. Equipped with the fast Fourier transform in computations and fuelled by recent technological innovations in digital signals and images, Fourier analysis has stood out as one of the very top achievements of mankind comparable with the Calculus of Sir Isaac Newton. This sentiment is shared by David Mumford [28] among others.

This is a mathematical book on Fourier analysis, which best exemplifies interdisciplinary studies. The aim is to present the basic notions and techniques of Fourier transforms, wavelets, filter banks, signal analysis and pseudo-differential operators in discrete settings, thus making this fascinating area of mathematics accessible to as wide a readership as possible in mathematical sciences. It is my conviction that the beauty and the usefulness of the subject can be conveyed most effectively in the Definition-Theorem-Proof format interlaced with remarks, discussions and motivations from signal analysis.

The book consists of two parts. The first thirteen chapters contain topics related to the finite Fourier transform that can be understood completely by undergraduate students with basic knowledge of linear algebra and calculus. The last ten chapters are built on Hilbert spaces and Fourier series. A self-contained account on Hilbert spaces covering a broad spectrum of topics from basic definitions to the spectral theorem for self-adjoint and compact operators to Schatten–von Neumann classes is presented. The pointwise convergence and the L^2-theory of Fourier series are the contents of a chapter in the book. It is standard wisdom that the context for a proper study of Fourier series is the Lebesgue theory of measures and integrals. Notwithstanding the use of the language from measure theory in the second part of the book, much of the contents are accessible to students familiar with a solid undergraduate course in real analysis. An average graduate student in mathematics should be able to benefit from the entire book and hence equipped to do research in this subject.

Any book on Fourier analysis can be hoped to impart only a small part of the subject. This book is no exception and the emphasis is on the operator-theoretical aspects of the subject. The last two chapters of the book constitute an account of the published research works of mine and my former Ph.D. student, Dr. Shahla Molahajloo, on discrete pseudo-differential operators.

Chapter 1

The Finite Fourier Transform

A good starting point is the finite Fourier transform that underpins the contents of the first thirteen chapters of the book.

Let $\mathbb{C}$ be the set of all complex numbers. For a positive integer $N \geq 2$, we let $\mathbb{C}^N$ be the set defined by

$$\mathbb{C}^N = \left\{ \begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix} : z(n) \in \mathbb{C},\, n = 0, 1, \ldots, N-1 \right\}.$$

Then $\mathbb{C}^N$ is an N-dimensional complex vector space with respect to the usual addition and scalar multiplication of vectors. In fact, it is an inner product space in which the inner product $(\ ,\ )$ and norm $\|\ \|$ are defined by

$$(z, w) = \sum_{n=0}^{N-1} z(n)\overline{w(n)}$$

and

$$\|z\|^2 = (z, z) = \sum_{n=0}^{N-1} |z(n)|^2$$

for all $z = \begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix}$ and $w = \begin{pmatrix} w(0) \\ w(1) \\ \vdots \\ w(N-1) \end{pmatrix}$ in $\mathbb{C}^N$. Of particular importance in the first thirteen chapters is the space $\mathbb{Z}_N$ defined by

$$\mathbb{Z}_N = \{0, 1, \ldots, N-1\}.$$

M.W. Wong, *Discrete Fourier Analysis*, Pseudo-Differential Operators 5,
DOI 10.1007/978-3-0348-0116-4_1, © Springer Basel AG 2011

Let $z : \mathbb{Z}_N \to \mathbb{C}$ be a function. Then the function $z : \mathbb{Z}_N \to \mathbb{C}$ is completely specified by $\begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix}$. Thus, we can write

$$z = \begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix}.$$

In other words, we can think of the function $z : \mathbb{Z}_N \to \mathbb{C}$ as a finite sequence. If we let $L^2(\mathbb{Z}_N)$ be the set of all finite sequences, then we get

$$L^2(\mathbb{Z}_N) = \mathbb{C}^N.$$

Thus, $\mathbb{C}^N$ can be considered as the set of all finite sequences, or more precisely, functions on $\mathbb{Z}_N$. These finite sequences, i.e., functions on $\mathbb{Z}_N$, are in fact the mathematical analogs of digital signals in electrical engineering.

Definition 1.1. Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1} \in L^2(\mathbb{Z}_N)$ be defined by

$$\epsilon_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad m = 0, 1, \ldots, N-1,$$

where ϵ_m has 1 in the m^{th} position and zeros elsewhere.

Proposition 1.2. $\{\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1}\}$ *is an orthonormal basis for* $L^2(\mathbb{Z}_N)$.

The proof of Proposition 1.2 is left as an exercise.

The orthonormal basis $\{\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1}\}$ is the standard basis for $L^2(\mathbb{Z}_N)$. For another orthonormal basis for $L^2(\mathbb{Z}_N)$, we look at the signals in the following definition.

Definition 1.3. Let $e_0, e_1, \ldots, e_{N-1} \in L^2(\mathbb{Z}_N)$ be defined by

$$e_m = \begin{pmatrix} e_m(0) \\ e_m(1) \\ \vdots \\ e_m(N-1) \end{pmatrix}, \quad m = 0, 1, \ldots, N-1,$$

where

$$e_m(n) = \frac{1}{\sqrt{N}} e^{2\pi imn/N}, \quad n = 0, 1, \ldots, N-1.$$

Proposition 1.4. *$\{e_0, e_1, \ldots, e_{N-1}\}$ is an orthonormal basis for $L^2(\mathbb{Z}_N)$.*

Proof. For $j, k = 0, 1, \ldots, N-1$, we get

$$\begin{aligned}
(e_j, e_k) &= \sum_{n=0}^{N-1} e_j(n)\overline{e_k(n)} \\
&= \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} e^{2\pi ijn/N} \overline{\frac{1}{\sqrt{N}} e^{2\pi ikn/N}} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi ijn/N} e^{-2\pi ikn/N} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(j-k)n/N} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \left(e^{2\pi i(j-k)/N}\right)^n .
\end{aligned} \tag{1.1}$$

If $j = k$, then (1.1) gives

$$\|e_j\|^2 = (e_j, e_j) = \frac{1}{N} \sum_{n=0}^{N-1} 1 = 1, \quad j = 0, 1, \ldots, N-1.$$

If $j \neq k$, then $-N < j - k < N$ and hence $e^{2\pi i(j-k)/N} \neq 1$. Therefore for $j, k = 0, 1, \ldots, N-1$, (1.1) gives

$$(e_j, e_k) = \frac{1}{N} \frac{1 - \left(e^{2\pi i(j-k)/N}\right)^N}{1 - e^{2\pi i(j-k)/N}}. \tag{1.2}$$

But, for $j, k = 0, 1, \ldots, N-1$,

$$\left(e^{2\pi i(j-k)/N}\right)^N = e^{2\pi i(j-k)} = 1. \tag{1.3}$$

By (1.2) and (1,3), $(e_j, e_k) = 0$ if $j \neq k$. Hence $\{e_0, e_1, \ldots, e_{N-1}\}$ is orthonormal and hence linearly independent. Since $L^2(\mathbb{Z}_N)$ is N-dimensional, it follows that $\{e_0, e_1, \ldots, e_{N-1}\}$ is a basis for $L^2(\mathbb{Z}_N)$. This completes the proof. □

We call the basis $\{e_0, e_1, \ldots, e_{N-1}\}$ in Proposition 1.4 the orthonormal Fourier basis for $L^2(\mathbb{Z}_N)$. As an immediate consequence of Proposition 1.4, we have the following expression of a signal z in terms of the signals in the orthonormal Fourier basis.

Proposition 1.5. *Let z and w be signals in $L^2(\mathbb{Z}_N)$. Then*

$$z = \sum_{m=0}^{N-1} (z, e_m) e_m, \tag{1.4}$$

$$(z, w) = \sum_{m=0}^{N-1} (z, e_m) \overline{(w, e_m)} \tag{1.5}$$

and

$$\|z\|^2 = \sum_{m=0}^{N-1} |(z, e_m)|^2. \tag{1.6}$$

Proof. To prove (1.4), let $z \in L^2(\mathbb{Z}_N)$. Since $\{e_0, e_1, \ldots, e_{N-1}\}$ is a basis for $L^2(\mathbb{Z}_N)$, it follows that

$$z = \sum_{m=0}^{N-1} \alpha_m e_m, \tag{1.7}$$

where $\alpha_m \in \mathbb{C}$, $m = 0, 1, \ldots, N-1$. Thus, for $j = 0, 1, \ldots, N-1$, using the orthonormality of the basis $\{e_0, e_1, \ldots, e_{N-1}\}$, we obtain

$$(z, e_j) = \sum_{m=0}^{N-1} \alpha_m (e_m, e_j) = \alpha_j. \tag{1.8}$$

Hence by (1.7) and (1.8), the proof of (1.4) is complete. Now, let z and w be in $L^2(\mathbb{Z}_N)$. Then, using (1.4) and the orthonormality of $\{e_0, e_1, \ldots, e_{N-1}\}$, we get

$$\begin{aligned}
(z, w) &= \left(\sum_{m=0}^{N-1} (z, e_m) e_m, \sum_{l=0}^{N-1} (w, e_l) e_l \right) \\
&= \sum_{m=0}^{N-1} (z, e_m) \left(e_m, \sum_{l=0}^{N-1} (w, e_l) e_l \right) \\
&= \sum_{m=0}^{N-1} (z, e_m) \sum_{l=0}^{N-1} \overline{(w, e_l)} (e_m, e_l) \\
&= \sum_{m=0}^{N-1} (z, e_m) \overline{(w, e_m)}.
\end{aligned}$$

Thus, (1.5) is proved. Finally, if we put $w = z$ in (1.5), then we get

$$\|z\|^2 = (z, z) = \sum_{m=0}^{N-1} (z, e_m) \overline{(z, e_m)} = \sum_{m=0}^{N-1} |(z, e_m)|^2,$$

and the proof is complete. □

Let $z \in L^2(\mathbb{Z}_N)$. Then for $m = 0, 1, \ldots, N-1$,

$$(z, e_m) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} z(n) e^{-2\pi i m n/N}. \tag{1.9}$$

In view of the importance of the inner products (z, e_m), $m = 0, 1, \ldots, N-1$, revealed by Proposition 1.5, we need to study these inner products more carefully, and we introduce the following definition.

Definition 1.6. Let $z \in L^2(\mathbb{Z}_N)$. Then we let $\hat{z} \in L^2(\mathbb{Z}_N)$ be defined by

$$\hat{z} = \begin{pmatrix} \hat{z}(0) \\ \hat{z}(1) \\ \vdots \\ \hat{z}(N-1) \end{pmatrix},$$

where

$$\hat{z}(m) = \sum_{n=0}^{N-1} z(n) e^{-2\pi i m n/N}, \quad m = 0, 1, \ldots, N-1.$$

We call $\hat{z}$ the finite Fourier transform of z, which is sometimes denoted by $\mathcal{F}_{\mathbb{Z}_N} z$. It is important to note that by (1.9), we have

$$(\mathcal{F}_{\mathbb{Z}_N} z)(m) = \hat{z}(m) = \sqrt{N}(z, e_m), \quad m = 0, 1, \ldots, N-1. \tag{1.10}$$

We note that we have thrown away the factor $\frac{1}{\sqrt{N}}$ from the right-hand side of (1.9) in defining the finite Fourier transform. The advantage of doing this lies in the fact that in numerical computations, it is better to avoid computing $\sqrt{N}$.

Using the finite Fourier transform and (1.10), we can reformulate Proposition 1.5 in the following form.

Theorem 1.7. *Let z and w be signals in $L^2(\mathbb{Z}_N)$. Then*

(i) (The Fourier Inversion Formula)

$$z(n) = \frac{1}{N} \sum_{m=0}^{N-1} \hat{z}(m) e^{2\pi i m n/N}, \quad n = 0, 1, \ldots, N-1, \tag{1.11}$$

(ii) (Parseval's Identity)

$$(z, w) = \frac{1}{N} \sum_{m=0}^{N-1} \hat{z}(m) \overline{\hat{w}(m)} = \frac{1}{N} (\hat{z}, \hat{w}), \tag{1.12}$$

(iii) (Plancherel's Formula)

$$\|z\|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |\hat{z}(m)|^2 = \frac{1}{N}\|\hat{z}\|^2. \tag{1.13}$$

Proof. By (1.4) and (1.10), we get

$$z(n) = \sum_{m=0}^{N-1} \frac{1}{\sqrt{N}}\hat{z}(m)\frac{1}{\sqrt{N}}e^{2\pi imn/N} = \frac{1}{N}\sum_{m=0}^{N-1} \hat{z}(m)e^{2\pi imn/N}, \quad n = 0, 1, \ldots, N-1.$$

Thus, (1.11) follows. To prove (1.12), we note that by (1.5) and (1.10),

$$(z, w) = \sum_{m=0}^{N-1} \frac{1}{\sqrt{N}}\hat{z}(m)\frac{1}{\sqrt{N}}\overline{\hat{w}(m)} = \frac{1}{N}\sum_{m=0}^{N-1} \hat{z}(m)\overline{\hat{w}(m)} = \frac{1}{N}(\hat{z}, \hat{w}).$$

Finally, for (1.13), we use (1.6) and (1.10) to get

$$\|z\|^2 = \sum_{m=0}^{N-1} \left|\frac{1}{\sqrt{N}}\hat{z}(m)\right|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |\hat{z}(m)|^2 = \frac{1}{N}\|\hat{z}\|^2. \qquad \square$$

In order to understand the Fourier inversion formula in Theorem 1.7 better, we let $F_0, F_1, \ldots, F_{N-1} \in L^2(\mathbb{Z}_N)$ be defined by

$$F_m = \begin{pmatrix} F_m(0) \\ F_m(1) \\ \vdots \\ F_m(N-1) \end{pmatrix}, \quad m = 0, 1, \ldots, N-1,$$

where

$$F_m(n) = \frac{1}{N}e^{2\pi imn/N}, \quad n = 0, 1, \ldots, N-1. \tag{1.14}$$

Obviously, $\{F_0, F_1, \ldots, F_{N-1}\}$ is orthogonal, but not orthonormal in $L^2(\mathbb{Z}_N)$. Being an orthogonal set of N elements in the N-dimensional vector space $L^2(\mathbb{Z}_N)$, $\{F_0, F_1, \ldots, F_{N-1}\}$ is a basis for $L^2(\mathbb{Z}_N)$ and we call it the Fourier basis for $L^2(\mathbb{Z}_N)$. Using the Fourier basis for $L^2(\mathbb{Z}_N)$ defined by (1.14), the Fourier inversion formula in Theorem 1.7 becomes

$$z = \sum_{m=0}^{N-1} \hat{z}(m)F_m. \tag{1.15}$$

Thus, the components of the signal z with respect to the Fourier basis are the components of the finite Fourier transform $\hat{z}$.

That the finite Fourier transform $\mathcal{F}_{\mathbb{Z}_N} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ can be represented as a matrix can be seen as follows. Let $z \in L^2(\mathbb{Z}_N)$. Then for $m = 0, 1, \ldots, N-1$,

$$\begin{aligned}(\mathcal{F}_{\mathbb{Z}_N} z)(m) = \hat{z}(m) &= \sum_{n=0}^{N-1} z(n) e^{-2\pi i m n/N} \\ &= \sum_{n=0}^{N-1} z(n) \left(e^{-2\pi i/N}\right)^{mn} \\ &= \sum_{n=0}^{N-1} z(n) \omega_N^{mn}, \end{aligned} \tag{1.16}$$

where

$$\omega_N = e^{-2\pi i/N}.$$

If we let Ω_N be the matrix defined by

$$\Omega_N = \left(\omega_N^{mn}\right)_{0 \le m,n \le N-1},$$

then, by (1.16) and (1.17), we get

$$\Omega_N = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \omega_N^3 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \omega_N^6 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \omega_N^{3(N-1)} & \cdots & \omega_N^{(N-1)(N-1)} \end{pmatrix}, \tag{1.17}$$

and

$$\hat{z} = \Omega_N z. \tag{1.18}$$

Thus, the finite Fourier transform is the matrix Ω_N given by (1.17) and the formula (1.18) is often used in the computation of the finite Fourier transform $\hat{z}$ of the signal z. We call Ω_N the Fourier matrix of order $N \times N$ or simply the Fourier matrix.

Example 1.8. Let $N = 2$. Then using (1.17), we get

$$\Omega_2 = \begin{pmatrix} 1 & 1 \\ 1 & \omega_2 \end{pmatrix}.$$

But

$$\omega_2 = e^{-2\pi i/2} = e^{-\pi i} = -1.$$

So,

$$\Omega_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Example 1.9. Let $N = 4$. Then using (1.17) again, we get

$$\Omega_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ 1 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{pmatrix}.$$

But

$$\omega_4 = e^{-2\pi i/4} = e^{-\pi i/2} = -i.$$

So,

$$\Omega_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}.$$

Example 1.10. Let z be the signal in $L^2(\mathbb{Z}_4)$ given by

$$z = \begin{pmatrix} 1 \\ 0 \\ 1 \\ i \end{pmatrix}.$$

Then $\hat{z}$ can be computed using the definition of the finite Fourier transform. On the other hand, using the formula (1.18) and the formula for Ω_4 in Example 1.9, we get

$$\hat{z} = \Omega_4 z = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 2+i \\ -1 \\ 2-i \\ 1 \end{pmatrix}.$$

Remark 1.11. In order to compute the finite Fourier transform $\hat{z}$ of a signal z in $L^2(\mathbb{Z}_N)$, we multiply z by the Fourier matrix Ω_N to get $\hat{z}$. Hence it is nice if there are a lot of zeros in the entries of Ω_N, i.e., if Ω_N is a sparse matrix. But a look at the formula (1.17) for Ω_N immediately tells us that Ω_N has no nonzero entries. Thus, in order to compute the finite Fourier transform $\hat{z}$ of a signal z in $L^2(\mathbb{Z}_N)$ using the Fourier matrix Ω_N, N^2 complex multiplications are apparently necessary. In signal analysis, N is usually very big. A television signal, for instance, requires 10^8 pixel values per second, and hence one second of the sampled signal necessitates the use of a vector of length $N = 10^8$. This makes the computation of the finite Fourier transform using the Fourier matrix Ω_N a seemingly enormous task in signal processing. We shall pick up this interesting issue on computations later in the course.

Definition 1.12. Let w be a signal in $L^2(\mathbb{Z}_N)$. Then we define the signal $\check{w}$ in $L^2(\mathbb{Z}_N)$ by

$$\check{w} = \begin{pmatrix} \check{w}(0) \\ \check{w}(1) \\ \vdots \\ \check{w}(N-1) \end{pmatrix},$$

where

$$\check{w}(n) = \frac{1}{N} \sum_{m=0}^{N-1} w(m) e^{2\pi imn/N}, \quad n = 0, 1, \ldots, N-1.$$

We call $\check{w}$ the inverse finite Fourier transform of w and we sometimes denote it by $\mathcal{F}_{\mathbb{Z}_N}^{-1} w$. The full justification for the terminology will come later. Meanwhile, the following proposition provides a partial justification.

Proposition 1.13. *Let z be a signal in $L^2(\mathbb{Z}_N)$. Then $\check{\hat{z}} = z$.*

Proof. For $n = 0, 1, \ldots, N-1$, we can use the Fourier inversion formula in Theorem 1.7 to get

$$z(n) = \frac{1}{N} \sum_{m=0}^{N-1} \hat{z}(m) e^{2\pi imn/N},$$

and hence by Definition 1.12,

$$z(n) = \check{\hat{z}}(n).$$

This proves that $\check{\hat{z}} = z$. □

We have seen that the finite Fourier transform $\mathcal{F}_{\mathbb{Z}_N} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is the same as the Fourier matrix Ω_N. Let us now compute the matrix of the inverse finite Fourier transform $\mathcal{F}_{\mathbb{Z}_N}^{-1} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$. To this end, let $w \in L^2(\mathbb{Z}_N)$. Then for $m = 0, 1, \ldots, N-1$,

$$\begin{aligned}
(\mathcal{F}_{\mathbb{Z}_N}^{-1} w)(m) = \check{w}(m) &= \frac{1}{N} \sum_{n=0}^{N-1} w(n) e^{2\pi imn/N} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} w(n) \left(e^{2\pi i/N}\right)^{mn} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} w(n) \overline{\left(e^{-2\pi i/N}\right)^{mn}} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} w(n) \overline{\omega_N{}^{mn}} \\
&= \frac{1}{N} \left(\overline{\Omega_N} w\right)(m),
\end{aligned}$$

where $\overline{\Omega_N}$ is the matrix obtained from Ω_N by taking the complex conjugate of every entry in Ω_N. Thus,

$$\mathcal{F}_{\mathbb{Z}_N}^{-1} w = \check{w} = \frac{1}{N}\overline{\Omega_N} w, \quad w \in L^2(\mathbb{Z}_N).$$

So, the matrix of the inverse finite Fourier transform $\mathcal{F}_{\mathbb{Z}_N}^{-1} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is equal to $\frac{1}{N}\overline{\Omega_N}$.

The following proposition explains why we call $\mathcal{F}_{\mathbb{Z}_N}^{-1} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ the inverse finite Fourier transform.

Proposition 1.14. *$\mathcal{F}_{\mathbb{Z}_N}^{-1} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is the inverse of $\mathcal{F}_{\mathbb{Z}_N} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$.*

Proof. If we think of $\mathcal{F}_{\mathbb{Z}_N}$ and $\mathcal{F}_{\mathbb{Z}_N}^{-1}$ as matrices, then, by Proposition 1.13,

$$\mathcal{F}_{\mathbb{Z}_N}^{-1}\mathcal{F}_{\mathbb{Z}_N} = I, \tag{1.19}$$

where I is the identity matrix of order $N \times N$. Thus,

$$\det(\mathcal{F}_{\mathbb{Z}_N}^{-1}\mathcal{F}_{\mathbb{Z}_N}) = \det I = 1,$$

where $\det(\cdots)$ is the determinant of $(\cdots)$. Therefore

$$\det\mathcal{F}_{\mathbb{Z}_N}^{-1}\det\mathcal{F}_{\mathbb{Z}_N} = 1$$

and consequently $\det\mathcal{F}_{\mathbb{Z}_N} \neq 0$. So, the matrix $\mathcal{F}_{\mathbb{Z}_N}$ is invertible, i.e., the inverse of $\mathcal{F}_{\mathbb{Z}_N}$ exists and let us denote it by A. Now, we multiply both sides of (1.19) on the right by A and we get

$$\mathcal{F}_{\mathbb{Z}_N}^{-1}\mathcal{F}_{\mathbb{Z}_N} A = IA.$$

Hence

$$\mathcal{F}_{\mathbb{Z}_N}^{-1} = A$$

and the proof is complete. □

Corollary 1.15. $\Omega_N^{-1} = \frac{1}{N}\overline{\Omega_N}$.

Example 1.16. Using Example 1.8 and Corollary 1.15, we get

$$\Omega_2^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Example 1.17. Using Example 1.9 and Corollary 1.15, we get

$$\Omega_4^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

Example 1.18. Let w be the signal in $L^2(\mathbb{Z}_4)$ given by

$$w = \begin{pmatrix} 2+i \\ -1 \\ 2-i \\ 1 \end{pmatrix}.$$

Find the signal z in $L^2(\mathbb{Z}_4)$ such that $\hat{z} = w$.

Solution. We use the inverse of the Fourier matrix Ω_4 in Example 1.17 to get

$$z = \Omega_4^{-1} w = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} 2+i \\ 1 \\ 2-i \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ i \end{pmatrix}. \qquad \square$$

We are now in a position to give an interpretation of the finite Fourier transform in the context of signal analysis. The most fundamental formula is the Fourier inversion formula in Theorem 1.7. To recall, let $z \in L^2(\mathbb{Z}_N)$. Then (1.15) gives

$$z = \sum_{m=0}^{N-1} \hat{z}(m) F_m,$$

where $\{F_0, F_1, \ldots, F_{N-1}\}$ is the Fourier basis for $L^2(\mathbb{Z}_N)$. For $m = 0, 1, \ldots, N-1$,

$$F_m = \begin{pmatrix} F_m(0) \\ F_m(1) \\ \vdots \\ F_m(N-1) \end{pmatrix},$$

where

$$F_m(n) = \frac{1}{N} e^{2\pi i m n/N}, \quad n = 0, 1, \ldots, N-1.$$

For the sake of exposition, we assume that N is big and even. To simplify matters, we also drop the factor $\frac{1}{N}$ from $F_m(n)$ and look at only the real part of $e^{2\pi i m n/N}$, which is the same as $\cos(2\pi m n/N)$ for $m, n = 0, 1, \ldots, N-1$. The crucial idea is to look at $\cos(2\pi m n/N)$ as a function of n on $\mathbb{Z}_N$ for each value of m from 0 to $N-1$. For $m = 0$, we get the value 1 for each n in $\mathbb{Z}_N$. For $m = 1$, we have the function $\cos(2\pi n/N)$. To graph this function, we first draw the graph of the function f on $[0, N]$ given by

$$f(x) = \cos(2\pi x/N), \quad x \in [0, N],$$

and then mark the points on the curve corresponding to $n = 0, 1, \ldots, N-1$. The resulting picture is a set of N evenly spaced sampled points on the graph of one cycle of the cosine function. For $m = 2$, a similar argument will give the graph of $\cos(2\pi 2n/N)$ as a set of N evenly spaced sampled points on the graph of two cycles of the cosine function. For $m = 0, 1, \ldots, N/2$, the same argument applies and the graph of $\cos(2\pi mn/N)$ is a set of N evenly spaced sampled points on m cycles of the cosine function. To see what happens when $m = N/2, \ldots, N$, let us first look at the case when $m = N$. The function then is $\cos(2\pi n)$ and we are back to the value 1 for each n in $\mathbb{Z}_N$. In other words, we are back to the function when $m = 0$. For $m = N-1$, the function is $\cos(2\pi(N-1)n/N)$, which is exactly $\cos(2\pi n/N)$. So, we are back to the case when $m = 1$. These qualitative observations lead us to conclude that each F_m in the Fourier basis is a wave with pure frequency m when $m = 0, 1, \ldots, N/2$; and is a wave with pure frequency $N - m$ when $m = (N/2)+1, \ldots, N-1$. As m increases from 0 to $N/2$, the pure frequency given by F_m increases; and as m increases from $N/2$ to N, the pure frequency given by F_m decreases. The pure frequency of F_m is high when m is near the middle and is low when m is near the endpoints of $\mathbb{Z}_N$. With this interpretation of each F_m as a wave with pure frequency that depends on m, we can now give the meaning of the Fourier inversion formula in signal analysis. According to the Fourier inversion formula in Theorem 1.7, every signal z in $L^2(\mathbb{Z}_N)$ can be decomposed using the Fourier basis as

$$z = \sum_{m=0}^{N-1} \hat{z}(m) F_m.$$

Hence for $m = 0, 1, \ldots, N-1$, $\hat{z}(m)$ measures the "amount" of the wave F_m that is needed in composing the signal z. If $|\hat{z}(m)|$ is big (small) for values of m near $N/2$, then the signal z has strong (weak) high-frequency components. If $|\hat{z}(m)|$ is big (small) near 0 and near $N-1$, then the signal z has strong (weak) low-frequency components.

We end this chapter with a study of the interactions of the finite Fourier transform with translation and complex conjugation on $L^2(\mathbb{Z}_N)$. Let us first extend the definition of a signal in $L^2(\mathbb{Z}_N)$, initially defined only on $\mathbb{Z}_N$, to the set $\mathbb{Z}$ of all integers. We do this in such a way that the resulting function on $\mathbb{Z}$, again denoted by z, is periodic with period N. In other words, we demand that

$$z(n+N) = z(n), \quad n \in \mathbb{Z}.$$

From now on, we identify the function $z \in L^2(\mathbb{Z}_N)$ with its periodic extension z to all of $\mathbb{Z}$.

We give in the following proposition a basic property of periodic functions on $\mathbb{Z}$ with period N. It tells us that the sum on any interval of length N of a periodic function on $\mathbb{Z}$ with period N is the same as the sum of the function on the fundamental domain $\mathbb{Z}_N$. More precisely, we have

Proposition 1.19. *Let z be a periodic function on $\mathbb{Z}$ with period N. Then for every integer k,*

$$\sum_{n=k}^{k+N-1} z(n) = \sum_{n=0}^{N-1} z(n).$$

Proof. Let us first prove the formula for nonnegative integers k. For $k = 0$, the formula is trivially true. Let us assume that the formula is true for the positive integer k. Then

$$\sum_{n=k+1}^{k+N} z(n) = \sum_{n=k}^{k+N-1} z(n) - z(k) + z(k+N) = \sum_{n=k}^{k+N-1} z(n)$$

because z is periodic with period N. Hence the formula is also true for the positive integer $k+1$. To prove the formula for negative integers k, we let $k = -\kappa$, $\kappa = 1, 2, \ldots$. Then we need to prove that

$$\sum_{n=-\kappa}^{-\kappa+N-1} z(n) = \sum_{n=0}^{N-1} z(n), \quad \kappa = 1, 2, \ldots. \tag{1.20}$$

Let $\kappa = 1$. Then

$$\sum_{n=-1}^{N-2} z(n) = \sum_{n=0}^{N-1} z(n) + z(-1) - z(N-1) = \sum_{n=0}^{N-1} z(n)$$

because z is periodic with period N. Thus, (1.20) is valid for $\kappa = 1$. Now, suppose that (1.20) is true for the positive integer κ. Then

$$\sum_{n=-\kappa-1}^{-\kappa+N-2} z(n) = \sum_{n=-\kappa}^{-\kappa+N-1} z(n) + z(-\kappa-1) - z(-\kappa+N-1) = \sum_{n=-\kappa}^{-\kappa+N-1} z(n)$$

because z is periodic with period N. Therefore (1.20) is also true for the positive integer $\kappa + 1$. □

Remark 1.20. If, in the definition of the finite Fourier transform $\hat{z}$ of a signal $z \in L^2(\mathbb{Z}_N)$, we put

$$\hat{z}(m) = \sum_{n=0}^{N-1} z(n) e^{-2\pi i m n/N}, \quad m \in \mathbb{Z},$$

then $\hat{z}$ is periodic with period N. If, in the definition of the inverse finite Fourier transform $\check{w}$ of a signal $w \in L^2(\mathbb{Z}_N)$, we put

$$\check{w}(n) = \frac{1}{N} \sum_{m=0}^{N-1} w(m) e^{2\pi i m n/N}, \quad n \in \mathbb{Z},$$

then $\check{w}$ is periodic with period N. The proofs of these two facts are left as exercises.

Definition 1.21. Let z be a signal in $L^2(\mathbb{Z}_N)$ and let $k \in \mathbb{Z}$. Then we define the function $R_k z$ on $\mathbb{Z}$ by

$$(R_k z)(n) = z(n-k), \quad n \in \mathbb{Z}.$$

We call $R_k z$ the translation of z by k.

Example 1.22. Let z be the signal in $L^2(\mathbb{Z}_6)$ given by

$$z = \begin{pmatrix} 1 \\ 2+i \\ 8i \\ 3-i \\ 4 \\ 6 \end{pmatrix}.$$

Let $k = 2$. Then, by Definition 1.21 and the fact that z is periodic with period 6, we get

$$R_2 z = \begin{pmatrix} z(0-2) \\ z(1-2) \\ z(2-2) \\ z(3-2) \\ z(4-2) \\ z(5-2) \end{pmatrix} = \begin{pmatrix} z(-2) \\ z(-1) \\ z(0) \\ z(1) \\ z(2) \\ z(3) \end{pmatrix} = \begin{pmatrix} z(4) \\ z(5) \\ z(0) \\ z(1) \\ z(2) \\ z(3) \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 1 \\ 2+i \\ 8i \\ 3-i \end{pmatrix}.$$

From Example 1.22, we see that the effect of the translation by 2 is to push the entries except the last two in a signal down by two positions. The last two entries are rotated into the first two positions.

Proposition 1.23. *Let z be a signal in $L^2(\mathbb{Z}_N)$ and let $k \in \mathbb{Z}$. Then for all m in $\mathbb{Z}$,*

$$\widehat{R_k z}(m) = e^{-2\pi i m k/N} \hat{z}(m).$$

Proof. Let $m \in \mathbb{Z}$. Then we use the definition of the finite Fourier transform in Remark 1.20 and the definition of the translation by k to get

$$\widehat{R_k z}(m) = \sum_{n=0}^{N-1} (R_k z)(n) e^{-2\pi i m n/N} = \sum_{n=0}^{N-1} z(n-k) e^{-2\pi i m n/N}.$$

If we change the summation variable from n to l by the formula $l = n - k$, then

$$\widehat{R_k z}(m) = \sum_{l=-k}^{N-1-k} z(l) e^{-2\pi i m(l+k)/N} = e^{-2\pi i m k/N} \sum_{l=-k}^{N-1-k} z(l) e^{-2\pi i m l/N}. \quad (1.21)$$

Thus, applying Proposition 1.19 to the last term in (1.21), we get

$$\widehat{R_k z}(m) = e^{-2\pi i m k/N} \hat{z}(m).$$

□

Obviously, for $m \in \mathbb{Z}$, $|\widehat{R_k z}(m)| = |\hat{z}(m)|$. Thus, a translation cannot alter the amplitudes of the complex coefficients of the waves F_m that make up the signal. The phase of the finite Fourier transform $\hat{z}$, though, is changed in accordance with the formula in Proposition 1.23. Thus, the effect of the translation by k of a signal is not detected by the norm of the finite Fourier transform. In other words, information on the composition of z in terms of the waves F_m at a particular instant is not provided by the norm $\|\hat{z}\|$ of z. It is to be found in the phase of $\hat{z}$.

We can now come to complex conjugation on $L^2(\mathbb{Z}_N)$.

Definition 1.24. Let z be a signal in $L^2(\mathbb{Z}_N)$. Then we define the signal $\overline{z}$ in $L^2(\mathbb{Z}_N)$ by

$$\overline{z} = \begin{pmatrix} \overline{z(0)} \\ \overline{z(1)} \\ \vdots \\ \overline{z(N-1)} \end{pmatrix}.$$

We call $\overline{z}$ the complex conjugate of z.

Proposition 1.25. *Let z be a signal in $L^2(\mathbb{Z}_N)$. Then*

$$\hat{\overline{z}}(m) = \overline{\hat{z}(-m)}, \quad m \in \mathbb{Z}.$$

Proof. Using the definition of the finite Fourier transform and the definition of complex conjugation, we get

$$\hat{\overline{z}}(m) = \sum_{n=0}^{N-1} \overline{z}(n) e^{-2\pi imn/N} = \overline{\sum_{n=0}^{N-1} z(n) e^{2\pi imn/N}} = \overline{\hat{z}(-m)}$$

for all m in $\mathbb{Z}$. □

Definition 1.26. A signal z in $L^2(\mathbb{Z}_N)$ is said to be real if $\overline{z} = z$.

We can give a corollary of Proposition 1.25.

Corollary 1.27. *A signal z in $L^2(\mathbb{Z}_N)$ is real if and only if*

$$\hat{z}(m) = \overline{\hat{z}(-m)}, \quad m \in \mathbb{Z}. \tag{1.22}$$

Proof. Suppose that z is real. Then $\overline{z} = z$. Thus, by Proposition 1.25,

$$\hat{z}(m) = \hat{\overline{z}}(m) = \overline{\hat{z}(-m)}, \quad m \in \mathbb{Z}.$$

Conversely, suppose that (1.22) holds. Then, by Proposition 1.25 again,

$$\hat{z}(m) = \hat{\overline{z}}(m), \quad m \in \mathbb{Z}.$$

Thus, $\hat{z} = \hat{\overline{z}}$. So, by the Fourier inversion formula in Theorem 1.7, we get $z = \hat{z}$, i.e., z is a real signal. □

Exercises

1. For $m = 0, 1, \ldots, N-1$, let $\epsilon_m \in L^2(\mathbb{Z}_N)$ be defined by

$$\epsilon_m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

i.e., ϵ_m has 1 in the m^{th} position and zeros elsewhere. Prove that

$$\{\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1}\}$$

is an orthonormal basis for $L^2(\mathbb{Z}_N)$.

2. Let $z \in L^2(\mathbb{Z}_{216})$ be the signal given by

$$z(n) = 3\sin(2\pi 5n/216) - 2\cos(2\pi 6n/216), \quad n = 0, 1, \ldots, 215.$$

Compute $\mathcal{F}_{\mathbb{Z}_{216}} z$.

3. Compute Ω_3 and use it to compute $\mathcal{F}_{\mathbb{Z}_3} z$, where

$$z = \begin{pmatrix} 1 \\ i \\ 2 \end{pmatrix}.$$

4. Prove that for all positive integers $N \geq 2$,

$$\Omega_N^{-1} = \frac{1}{N}\Omega_N^*,$$

where Ω_N^* is the adjoint of Ω_N, i.e., the matrix obtained from Ω_N by taking the transpose of the complex conjugate of Ω_N.

5. Let $z \in L^2(\mathbb{Z}_N)$. Find a formula for $\hat{\hat{z}}$.

6. Let $z \in L^2(\mathbb{Z}_N)$. Prove that $\hat{z}$ and $\check{z}$ are periodic with period N.

Chapter 2

Translation-Invariant Linear Operators

We give in this chapter the most basic class of linear operators from $L^2(\mathbb{Z}_N)$ into $L^2(\mathbb{Z}_N)$ in signal analysis.

Definition 2.1. Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a linear operator, i.e.,

$$A(z + w) = Az + Aw$$

and

$$A(\alpha z) = \alpha A z$$

for all z and w in $L^2(\mathbb{Z}_N)$ and all α in $\mathbb{C}$. Then we say that A is translation-invariant if

$$AR_k = R_k A$$

for all k in $\mathbb{Z}$, where R_k is the translation by k on $L^2(\mathbb{Z}_N)$.

Remark 2.2. A translation-invariant linear operator A is the mathematical analog of a filter that transmits signals in electrical engineering. Its function is to transform an input signal z in $L^2(\mathbb{Z}_N)$ into an output signal Az in $L^2(\mathbb{Z}_N)$. The effect of a filter on two signals together is the sum of the effects of the filter on each signal separately. Also, if a signal is multiplied by a complex number, then the output signal should be multiplied by the same complex number. This explains why linearity is desirable. As for the condition that A should commute with translations, we note that if we delay or advance an input signal by a certain amount, then the output signal should be delayed or advanced by the same amount. In other words, if z is a signal in $L^2(\mathbb{Z}_N)$ and $k \in \mathbb{Z}$, then

$$A(R_k z) = R_k(Az).$$

Therefore the condition that $AR_k = R_k A$ is a natural one to impose on a filter.

The main result that we want to prove in this chapter is contained in the following theorem.

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DOI 10.1007/978-3-0348-0116-4_2, © Springer Basel AG 2011

Theorem 2.3. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a translation-invariant linear operator. Then for $m = 0, 1, \ldots, N-1$, F_m is an eigenfunction of A.*

Proof. Let $m \in \mathbb{Z}_N$. Then, using the fact that $\{F_0, F_1, \ldots, F_{N-1}\}$ is a basis for $L^2(\mathbb{Z}_N)$, we can find complex numbers $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}$ such that

$$(AF_m)(n) = \sum_{k=0}^{N-1} \alpha_k F_k(n), \quad n = 0, 1, \ldots, N-1. \tag{2.1}$$

Thus,

$$\begin{aligned}(R_1 AF_m)(n) &= \sum_{k=0}^{N-1} \alpha_k F_k(n-1) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \alpha_k e^{2\pi i k(n-1)/N} \\ &= \sum_{k=0}^{N-1} \alpha_k e^{-2\pi i k/N} F_k(n), \quad n = 0, 1, \ldots, N-1. \end{aligned} \tag{2.2}$$

Now, for $n = 0, 1, \ldots, N-1$,

$$(R_1 F_m)(n) = F_m(n-1) = \frac{1}{N} e^{2\pi i m(n-1)/N} = e^{-2\pi i m/N} F_m(n),$$

and hence, using the linearity of A, we get

$$(AR_1 F_m)(n) = e^{-2\pi i m/N}(AF_m)(n) = \sum_{k=0}^{N-1} \alpha_k e^{-2\pi i m/N} F_k(n). \tag{2.3}$$

Thus, by (2.2), (2.3) and equating coordinates, we get

$$\alpha_k e^{-2\pi i k/N} = \alpha_k e^{-2\pi i m/N}, \quad k = 0, 1, \ldots, N-1.$$

Thus, $\alpha_k = 0$ whenever $k \neq m$, and (2.1) becomes

$$AF_m = \alpha_m F_m.$$

Therefore F_m is an eigenfunction of A. □

In order to understand the full thrust of Theorem 2.3, a recall of some basic linear algebra is in order. Let $B = \{z_0, z_1, \ldots, z_{N-1}\}$ be a basis for $L^2(\mathbb{Z}_N)$. Then for any signal z in $L^2(\mathbb{Z}_N)$,

$$z = \sum_{k=0}^{N-1} \alpha_k z_k,$$

where $\alpha_0, \alpha_1, \dots, \alpha_{N-1} \in \mathbb{C}$. We define $(z)_B$ by

$$(z)_B = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{pmatrix}$$

and call $(z)_B$ the coordinates of z with respect to B. We can construct the matrix of an arbitrary linear operator $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ with respect to the basis B as follows. Using the fact that $B = \{z_0, z_1, \dots, z_{N-1}\}$ is a basis for $L^2(\mathbb{Z}_N)$, we can find complex numbers a_{mn}, $m, n = 0, 1, \dots, N-1$, such that

$$\begin{aligned} Az_0 &= a_{00}z_0 + a_{10}z_1 + \cdots + a_{N-1,0}z_{N-1}, \\ Az_1 &= a_{01}z_0 + a_{11}z_1 + \cdots + a_{N-1,1}z_{N-1}, \\ &\cdots \\ Az_{N-1} &= a_{0,N-1}z_0 + a_{1,N-1}z_1 + \cdots + a_{N-1,N-1}z_{N-1}. \end{aligned}$$

Now, we let $(A)_B$ be the matrix defined by

$$(A)_B = (a_{mn})_{0 \le m,n \le N-1} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0,N-1} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1,N-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{N-1,0} & a_{N-1,1} & a_{N-1,2} & \cdots & a_{N-1,N-1} \end{pmatrix}.$$

Then

$$(A)_B(z)_B = \begin{pmatrix} \sum_{k=0}^{N-1} a_{0k}\alpha_k \\ \sum_{k=0}^{N-1} a_{1k}\alpha_k \\ \vdots \\ \sum_{k=0}^{N-1} a_{N-1,k}\alpha_k \end{pmatrix}. \tag{2.4}$$

But

$$\begin{aligned} Az &= A\left(\alpha_0 z_0 + \alpha_1 z_1 + \cdots + \alpha_{N-1}z_{N-1}\right) \\ &= \alpha_0 Az_0 + \alpha_1 Az_1 + \cdots + \alpha_{N-1}Az_{N-1} \\ &= \left(\sum_{k=0}^{N-1} a_{0k}\alpha_k\right) z_0 + \left(\sum_{k=0}^{N-1} a_{1k}\alpha_k\right) z_1 + \cdots + \left(\sum_{k=0}^{N-1} a_{N-1,k}\alpha_k\right) z_{N-1}. \end{aligned}$$

Thus,

$$(Az)_B = \begin{pmatrix} \sum_{k=0}^{N-1} a_{0k}\alpha_k \\ \sum_{k=0}^{N-1} a_{1k}\alpha_k \\ \vdots \\ \sum_{k=0}^{N-1} a_{N-1,k}\alpha_k \end{pmatrix}. \tag{2.5}$$

So, by (2.4) and (2.5),

$$(Az)_B = (A)_B(z)_B.$$

We call $(A)_B$ the matrix of the linear operator $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ with respect to the basis B for $L^2(\mathbb{Z}_N)$. If B is the standard basis $S = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1}\}$, then

$$Az = (Az)_S = (A)_S(z)_S = (A)_S z, \quad z \in L^2(\mathbb{Z}_N).$$

Thus, the matrix $(A)_S$ of A with respect to S is equal to A. It is then of enormous interest to find a basis B for $L^2(\mathbb{Z}_N)$ such that the matrix $(A)_B$ of A with respect to B is as simple as, say, a diagonal matrix.

The following theorem is an immediate consequence of Theorem 2.3.

Theorem 2.4. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a translation-invariant linear operator and let $F = \{F_0, F_1, \ldots, F_{N-1}\}$ be the Fourier basis for $L^2(\mathbb{Z}_N)$. Then the matrix $(A)_F$ of A with respect to F is diagonal. In fact,*

$$(A)_F = \begin{pmatrix} \lambda_0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{N-1} \end{pmatrix},$$

where λ_m is the eigenvalue of A corresponding to the eigenfunction F_m, $m = 0, 1, \ldots, N-1$.

Proof. We get

$$AF_0 = \lambda_0 F_0 + 0F_1 + 0F_2 + \cdots + 0F_{N-1},$$
$$AF_1 = 0F_0 + \lambda_1 F_1 + 0F_2 + \cdots + 0F_{N-1},$$
$$AF_2 = 0F_0 + 0F_1 + \lambda_2 F_2 + \cdots + 0F_{N-1},$$
$$\cdots$$
$$AF_{N-1} = 0F_0 + 0F_1 + 0F_2 + \cdots + \lambda_{N-1} F_{N-1}.$$

Thus, the matrix $(A)_F$ of A with respect to the Fourier basis F is given by

$$(A)_F = \begin{pmatrix} \lambda_0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{N-1} \end{pmatrix}. \qquad \square$$

In order to study translation-invariant linear operators in detail, we introduce the notions of circulant matrices, convolution operators and Fourier multipliers, which are the topics in the following three chapters.

Exercises

1. Let $A : L^2(\mathbb{Z}_4) \to L^2(\mathbb{Z}_4)$ be the linear operator defined by

$$(Az)(n) = 3\,(R_1 z)(n) + z(n), \quad n = 0, 1, 2, 3.$$

   Prove that A is translation-invariant.

2. For the translation-invariant operator A in Exercise 1, find the matrix $(A)_S$ of A with respect to the standard basis S of $L^2(\mathbb{Z}_4)$.

3. For the translation-invariant operator A in Exercise 1, find the matrix $(A)_F$ of A with respect to the Fourier basis F of $L^2(\mathbb{Z}_4)$.

4. Do Exercises 1–3 again for the linear operator $A : L^2(\mathbb{Z}_4) \to L^2(\mathbb{Z}_4)$ defined by

$$(Az)(n) = (R_3 z)(n) + iz(n) - i(R_{-2} z)(n), \quad n = 0, 1, 2, 3.$$

5. Prove that a linear operator $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is translation-invariant if and only if A commutes with R_1.

Chapter 3

Circulant Matrices

We have seen that the matrix $(A)_F$ of a translation-invariant linear operator $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ with respect to the Fourier basis F for $L^2(\mathbb{Z}_N)$ is a diagonal matrix. Can we say something about the structure of the matrix $(A)_S$ with respect to the simplest basis for $L^2(\mathbb{Z}_N)$, namely, the standard basis S?

To answer this question, let $(a_{mn})_{0\leq m,n\leq N-1}$ be an $N \times N$ matrix. Then we define a_{mn} for all m and n in $\mathbb{Z}$ by periodic extension to all of $\mathbb{Z}$ in each of the variables m and n. In other words, we demand that

$$a_{m+N,n} = a_{mn}$$

and

$$a_{m,n+N} = a_{mn}$$

for all m and n in $\mathbb{Z}$. An $N \times N$ matrix is assumed to be so periodized from now on.

Definition 3.1. Let $C = (a_{mn})_{0\leq m,n\leq N-1}$ be an $N \times N$ matrix periodized as above. Then we say that C is circulant if

$$a_{m+1,n+1} = a_{mn}, \quad m, n \in \mathbb{Z}.$$

Let $C = (a_{mn})_{0\leq m,n\leq N-1}$ be a circulant matrix of order $N \times N$. Then the $(n+1)^{st}$ column of C is equal to

$$\begin{pmatrix} a_{0,n+1} \\ a_{1,n+1} \\ \vdots \\ a_{m+1,n+1} \\ \vdots \\ a_{N-1,n+1} \end{pmatrix} = \begin{pmatrix} a_{-1,n} \\ a_{0n} \\ \vdots \\ a_{mn} \\ \vdots \\ a_{N-2,n} \end{pmatrix} = \begin{pmatrix} a_{N-1,n} \\ a_{0n} \\ \vdots \\ a_{mn} \\ \vdots \\ a_{N-2,n} \end{pmatrix} = R_1 \begin{pmatrix} a_{0n} \\ a_{1n} \\ \vdots \\ a_{m+1,n} \\ \vdots \\ a_{N-1,n} \end{pmatrix}.$$

So, the $(n+1)^{st}$ column of C is obtained from the n^{th} column by the translation by 1. Similarly, the $(m+1)^{st}$ row of C is obtained from the m^{th} row by the translation

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DOI 10.1007/978-3-0348-0116-4_3, © Springer Basel AG 2011

by 1. Thus, it is easy to recognize a circulant matrix whenever we come across one.

Example 3.2. Let

$$C = \begin{pmatrix} \alpha & \delta & \gamma & \beta \\ \beta & \alpha & \delta & \gamma \\ \gamma & \beta & \alpha & \delta \\ \delta & \gamma & \beta & \alpha \end{pmatrix}.$$

Then C is a circulant matrix of order 4×4.

Proposition 3.3. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a translation-invariant linear operator. Then the matrix $(A)_S$ of A with respect to the standard basis S is circulant.*

Proof. Let $n \in \mathbb{Z}$. Then, by the division algorithm,

$$n + 1 = qN + r,$$

where r is some integer in $\mathbb{Z}_N$. Thus, letting

$$\epsilon_{n+1} = \epsilon_r,$$

we obtain

$$(A)_S \epsilon_{n+1} = (A)_S \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where the 1 is in the r^{th} position. Hence for all m and n in $\mathbb{Z}$,

$$(A)_S \epsilon_{n+1} = \begin{pmatrix} a_{0r} \\ a_{1r} \\ \vdots \\ a_{mr} \\ \vdots \\ a_{N-1,r} \end{pmatrix} = \begin{pmatrix} a_{0,n+1} \\ a_{1,n+1} \\ \vdots \\ a_{m,n+1} \\ \vdots \\ a_{N-1,n+1} \end{pmatrix},$$

and, using the translation-invariance of A, we get

$$\begin{aligned} a_{m+1,n+1} &= a_{m+1,r} = ((A)_S \epsilon_{n+1})(m+1) = (A\epsilon_{n+1})(m+1) \\ &= (AR_1\epsilon_n)(m+1) = (R_1 A\epsilon_n)(m+1) = (A\epsilon_n)(m) \\ &= ((A)_S \epsilon_n)(m) = a_{mn}. \end{aligned}$$

This proves that $(A)_S$ is circulant. □

Exercises

1. Find the circulant matrix corresponding to the linear operator $A : L^2(\mathbb{Z}_4) \to L^2(\mathbb{Z}_4)$ given by

$$(Az)(n) = z(n+2) - 2z(n+1) + z(n), \quad n = 0, 1, 2, 3.$$

2. Prove that the product of two circulant matrices is circulant.
3. Prove that the adjoint of a circulant matrix is circulant.
4. Let A be a circulant matrix. Prove that the linear operator $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is translation-invariant.
5. Prove that a circulant matrix A is normal, i.e., A commutes with its adjoint A^*.

Chapter 4

Convolution Operators

Translation-invariant linear operators from $L^2(\mathbb{Z}_N)$ into $L^2(\mathbb{Z}_N)$ can be given another representation that gives new insight into signal analysis. We first give a definition.

Definition 4.1. Let z and w be signals in $L^2(\mathbb{Z}_N)$. Then we define the signal $z * w$ in $L^2(\mathbb{Z}_N)$ by

$$(z * w)(m) = \sum_{n=0}^{N-1} z(m-n)w(n), \quad m \in \mathbb{Z}.$$

We call $z * w$ the convolution of z and w.

Let b be a signal in $L^2(\mathbb{Z}_N)$. Then we define the mapping $C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ by

$$C_b z = b * z, \quad z \in L^2(\mathbb{Z}_N).$$

Proposition 4.2. *$C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is a linear operator.*

Proof. Let z and w be signals in $L^2(\mathbb{Z}_N)$ and let $\alpha \in \mathbb{C}$. Then we get

$$\begin{aligned}(C_b(z+w))(m) &= \sum_{n=0}^{N-1} b(m-n)(z+w)(n) \\ &= \sum_{n=0}^{N-1} b(m-n)z(n) + \sum_{n=0}^{N-1} b(m-n)w(n) \\ &= (b*z)(m) + (b*w)(m) \\ &= (C_b z)(m) + (C_b w)(m)\end{aligned}$$

and

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DOI 10.1007/978-3-0348-0116-4_4, © Springer Basel AG 2011

$$\begin{aligned}(C_b(\alpha z))(m) &= \sum_{n=0}^{N-1} b(m-n)(\alpha z)(n)\\ &= \alpha \sum_{n=0}^{N-1} b(m-n)z(n)\\ &= \alpha(b*z)(m) = \alpha(C_b z)(m)\end{aligned}$$

for all m in $\mathbb{Z}$. Therefore

$$C_b(z+w) = C_b z + C_b w$$

and

$$C_b(\alpha z) = \alpha(C_b z).$$

This completes the proof. □

We call C_b the convolution operator associated to the kernel b.

Proposition 4.3. *Let $C = (a_{mn})_{0\le m,n\le N-1}$ be an $N \times N$ circulant matrix. Then for every z in $L^2(\mathbb{Z}_N)$,*

$$Cz = C_b z,$$

where b is the first column of C, i.e.,

$$b = \begin{pmatrix} a_{00} \\ a_{10} \\ \vdots \\ a_{N-1,0} \end{pmatrix}.$$

Proof. Since C is circulant, it follows that

$$a_{m-n,0} = a_{m-n+1,1} = \cdots = a_{mn}, \quad m,n \in \mathbb{Z}. \tag{4.1}$$

So, for $m \in \mathbb{Z}$, we get, by (4.1) and the definition of the convolution operator,

$$\begin{aligned}(Cz)(m) &= \sum_{n=0}^{N-1} a_{mn} z(n) = \sum_{n=0}^{N-1} a_{m-n,0} z(n)\\ &= \sum_{n=0}^{N-1} b(m-n)z(n) = (b*z)(m) = (C_b z)(m)\end{aligned}$$

for all m in $\mathbb{Z}$. □

Remark 4.4. Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a translation-invariant linear operator. Then, by Proposition 3.3, $A = (A)_S$ is a circulant matrix. Hence, by Proposition 4.3, A is also a convolution operator with kernel b given by the first column of the matrix A. The following proposition tells us that a convolution operator is translation-invariant.

Proposition 4.5. *Let $C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a convolution operator with kernel b, where $b \in L^2(\mathbb{Z}_N)$. Then C_b is translation-invariant.*

Proof. Let $k \in \mathbb{Z}$. Then for all $z \in L^2(\mathbb{Z}_N)$,

$$\begin{aligned}
(C_b R_k z)(m) &= \sum_{n=0}^{N-1} b(m-n)(R_k z)(n) \\
&= \sum_{n=0}^{N-1} b(m-n) z(n-k) \\
&= \sum_{l=-k}^{N-1-k} b(m-k-l) z(l), \quad m \in \mathbb{Z}. \qquad (4.2)
\end{aligned}$$

Since $b(m-k-l)z(l)$ is a periodic function of l on $\mathbb{Z}$ with period N, it follows from Proposition 1.19 that

$$\begin{aligned}
&\sum_{l=-k}^{N-1-k} b(m-k-l) z(l) \\
&= \sum_{l=0}^{N-1} b(m-k-l) z(l) \\
&= (b * z)(m-k) = (R_k C_b z)(m), \quad m \in \mathbb{Z}. \qquad (4.3)
\end{aligned}$$

Thus, by (4.2) and (4.3), $C_b R_k = R_k C_b$, and this proves that C_b is translation-invariant. □

Remark 4.6. Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a linear operator. Then, by Propositions 3.3, 4.3 and 4.5,

A is translation-invariant

⇔ the matrix $(A)_S$ of A with respect to the standard basis S for $L^2(\mathbb{Z}_N)$ is circulant

⇔ A is a convolution operator.

It is about time again for us to look at an application of translation-invariant linear operators in signal analysis. Let us begin with the important signal δ in $L^2(\mathbb{Z}_N)$ defined by

$$\delta = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The signal δ is called the Dirac delta or the unit impulse, and enjoys the important property that

$$z * \delta = z, \quad z \in L^2(\mathbb{Z}_N).$$

Indeed, for all $m \in \mathbb{Z}$,

$$(z * \delta)(m) = \sum_{n=0}^{N-1} z(m-n)\delta(n) = z(m).$$

We have seen that a filter in signal analysis is a translation-invariant linear operator $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ in Fourier analysis. By Remark 4.4, A is a convolution operator C_b with some kernel b in $L^2(\mathbb{Z}_N)$. To determine the kernel b, we note that

$$A\delta = C_b\delta = b * \delta = b.$$

Thus, the kernel b is the effect or response of the filter $A = C_b$ on the unit impulse δ. Hence the kernel b is just the impulse response of the filter $A = C_b$ in electrical engineering.

We end this chapter with a result, which tells us that the finite Fourier transform converts convolutions into multiplications.

Proposition 4.7. *Let z and w be signals in $L^2(\mathbb{Z}_N)$. Then*

$$\widehat{z * w}(m) = \hat{z}(m)\hat{w}(m), \quad m \in \mathbb{Z}.$$

Proof. Using the definition of the finite Fourier transform and the definition of convolution, we get

$$\begin{aligned}
\widehat{z * w}(m) &= \sum_{n=0}^{N-1} (z * w)(n) e^{-2\pi i m n/N} \\
&= \sum_{n=0}^{N-1} \left(\sum_{k=0}^{N-1} z(n-k)w(k) \right) e^{-2\pi i m(n-k)/N} e^{-2\pi i m k/N} \\
&= \sum_{k=0}^{N-1} \left(\sum_{n=0}^{N-1} z(n-k) e^{-2\pi i m(n-k)/N} \right) w(k) e^{-2\pi i m k/N} \\
&= \sum_{k=0}^{N-1} w(k) e^{-2\pi i m k/N} \left(\sum_{n=0}^{N-1} z(n-k) e^{-2\pi i m(n-k)/N} \right) \\
&= \sum_{k=0}^{N-1} w(k) e^{-2\pi i m k/N} \left(\sum_{l=-k}^{N-1-k} z(l) e^{-2\pi i m l/N} \right) \qquad (4.4)
\end{aligned}$$

for all m in $\mathbb{Z}$. Since $z(l)e^{-2\pi i m l/N}$ is a periodic function of l on $\mathbb{Z}$ with period N, it follows from Proposition 1.19 that

$$\sum_{l=-k}^{N-1-k} z(l) e^{-2\pi i m l/N} = \sum_{l=0}^{N-1} z(l) e^{-2\pi i m l/N} = \hat{z}(m), \quad m \in \mathbb{Z}. \qquad (4.5)$$

Thus, by (4.4) and (4.5),

$$\widehat{z * w}(m) = \hat{w}(m)\hat{z}(m), \quad m \in \mathbb{Z},$$

and the proof is complete. □

Exercises

1. Compute $\mathcal{F}_{\mathbb{Z}_N}\delta$.
2. Let C_{b_1} and C_{b_2} be convolution operators from $L^2(\mathbb{Z}_N)$ into $L^2(\mathbb{Z}_N)$ associated to kernels given, respectively, by b_1 and b_2 in $L^2(\mathbb{Z}_N)$. Prove that

$$C_{b_1}C_{b_2} = C_{b_1 * b_2}.$$

3. Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a linear operator. Then its adjoint $A^* : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is defined to be the linear operator such that

$$(Az, w) = (z, A^*w), \quad z, w \in L^2(Z_N).$$

Find the kernel of the adjoint of a convolution operator $C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$.

Chapter 5

Fourier Multipliers

We now come to the last ingredient in the analysis of translation-invariant linear operators. We begin with the multiplication of two signals in $L^2(\mathbb{Z}_N)$.

Definition 5.1. Let z and w be signals in $L^2(\mathbb{Z}_N)$. Then we define the signal zw in $L^2(\mathbb{Z}_N)$ by

$$zw = \begin{pmatrix} z(0)w(0) \\ z(1)w(1) \\ \vdots \\ z(N-1)w(N-1) \end{pmatrix},$$

i.e.,

$$(zw)(n) = z(n)w(n), \quad n \in \mathbb{Z}.$$

Let $\sigma \in L^2(\mathbb{Z}_N)$. Then we define the mapping $T_\sigma : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ by

$$T_\sigma z = (\sigma \hat{z})^\vee, \quad z \in L^2(\mathbb{Z}_N).$$

The proof of the following proposition is easy and is left as an exercise.

Proposition 5.2. *Let $\sigma \in L^2(\mathbb{Z}_N)$. Then $T_\sigma : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is a linear operator.*

We call T_σ the Fourier multiplier or the pseudo-differential operator associated to the symbol σ. It is more often and more instructive to call it a Fourier multiplier at this stage.

To see the role of a Fourier multiplier in signal analysis, let $\sigma \in L^2(\mathbb{Z}_N)$. Then for all signals z in $L^2(\mathbb{Z}_N)$, we get

$$(T_\sigma z)^\wedge(m) = \sigma(m)\hat{z}(m), \quad m \in \mathbb{Z}.$$

So, using the Fourier inversion formula in Theorem 1.7, we get

$$T_\sigma z = \sum_{m=0}^{N-1} (T_\sigma z)^\wedge(m) F_m = \sum_{m=0}^{N-1} \sigma(m)\hat{z}(m) F_m$$

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DOI 10.1007/978-3-0348-0116-4_5, © Springer Basel AG 2011

for all z in $L^2(\mathbb{Z}_N)$. Therefore the effect of the Fourier multiplier T_σ on a signal z is to pitch the amount of each of the waves $F_0, F_1, \ldots, F_{N-1}$ by the symbol σ in the composition of z in order to produce the output signal $T_\sigma z$. The heart of the matter is that a Fourier multiplier can be thought of as a frequency-pitching device in signal analysis.

Proposition 5.3. *Let $C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a convolution operator, where $b \in L^2(\mathbb{Z}_N)$. Then*

$$C_b = T_\sigma,$$

where $\sigma = \hat{b}$. Conversely, let $T_\sigma : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a Fourier multiplier, where $\sigma \in L^2(\mathbb{Z}_N)$. Then

$$T_\sigma = C_b,$$

where $b = \check{\sigma}$.

Proof. Let $z \in L^2(\mathbb{Z}_N)$. Then, using the Fourier inversion formula in Theorem 1.7 and Proposition 4.7,

$$C_b z = b * z = (\widehat{b * z})^\vee = (\hat{b}\hat{z})^\vee = (\sigma\hat{z})^\vee = T_\sigma z.$$

Conversely, using the Fourier inversion formula and Proposition 4.7 again, we get

$$T_\sigma z = (\sigma\hat{z})^\vee = (\hat{b}\hat{z})^\vee = (\widehat{b * z})^\vee = b * z = C_b z. \qquad \square$$

Proposition 5.4. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a linear operator. Then A is a Fourier multiplier if and only if the matrix $(A)_F$ of A with respect to the Fourier basis F is diagonal. Moreover, if A is the Fourier multiplier T_σ associated to the symbol σ in $L^2(\mathbb{Z}_N)$, then*

$$(A)_F = (T_\sigma)_F = \begin{pmatrix} \sigma(0) & 0 & 0 & \cdots & 0 \\ 0 & \sigma(1) & 0 & \cdots & 0 \\ 0 & 0 & \sigma(2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma(N-1) \end{pmatrix}.$$

Proof. Suppose that $A = T_\sigma$. Let $m \in \mathbb{Z}_N$. Then, using the Fourier inversion formula in Theorem 1.7 and the definition of the Fourier multiplier,

$$T_\sigma F_m = \sum_{n=0}^{N-1} (T_\sigma F_m)^\wedge(n) F_n = \sum_{n=0}^{N-1} \sigma(n)\widehat{F_m}(n) F_n. \tag{5.1}$$

Using the Fourier inversion formula again,

$$F_m = \sum_{n=0}^{N-1} \widehat{F_m}(n) F_n \Rightarrow \widehat{F_m}(n) = \begin{cases} 1, \, n = m, \\ 0, \, n \neq m. \end{cases} \tag{5.2}$$

So, by (5.1) and (5.2),

$$T_\sigma F_0 = \sigma(0)F_0 + 0F_1 + 0F_2 + \cdots + 0F_{N-1},$$

$$T_\sigma F_1 = 0F_0 + \sigma(1)F_1 + 0F_2 + \cdots + 0F_{N-1},$$

$$T_\sigma F_2 = 0F_0 + 0F_1 + \sigma(2)F_2 + \cdots + 0F_{N-1},$$

$$\cdots$$

$$T_\sigma F_{N-1} = 0F_0 + 0F_1 + 0F_2 + \cdots + \sigma(N-1)F_{N-1},$$

and hence

$$(T_\sigma)_F = \begin{pmatrix} \sigma(0) & 0 & 0 & \cdots & 0 \\ 0 & \sigma(1) & 0 & \cdots & 0 \\ 0 & 0 & \sigma(2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma(N-1) \end{pmatrix}.$$

Conversely, suppose that

$$(A)_F = D = \begin{pmatrix} d_0 & 0 & 0 & \cdots & 0 \\ 0 & d_1 & 0 & \cdots & 0 \\ 0 & 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & d_{N-1} \end{pmatrix},$$

where $d_0, d_1, \ldots, d_{N-1} \in \mathbb{C}$. Let $\sigma \in L^2(\mathbb{Z}_N)$ be defined by

$$\sigma = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{N-1} \end{pmatrix}.$$

Then for all $z \in L^2(\mathbb{Z}_N)$, we get, by the Fourier inversion formula in Theorem 1.7,

$$\begin{aligned}(Az)_F &= (A)_F(z)_F = D(z)_F \\ &= D\begin{pmatrix} \hat{z}(0) \\ \hat{z}(1) \\ \vdots \\ \hat{z}(N-1) \end{pmatrix} = \begin{pmatrix} d_0\hat{z}(0) \\ d_1\hat{z}(1) \\ \vdots \\ d_{N-1}\hat{z}(N-1) \end{pmatrix} \\ &= \sigma\hat{z} = (T_\sigma z)^\wedge = (T_\sigma z)_F.\end{aligned}$$

Hence

$$Az = T_\sigma z, \quad z \in L^2(\mathbb{Z}_N),$$

and this proves that $A = T_\sigma$. □

Remark 5.5. Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a linear operator. Then, by Propositions 5.3 and 5.4,

A is a convolution operator

$\Leftrightarrow$ A is a Fourier multiplier

$\Leftrightarrow$ the matrix $(A)_F$ of A with respect to the Fourier basis F for $L^2(\mathbb{Z}_N)$ is diagonal.

Remark 5.6. Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a linear operator. Then, by Remarks 4.6 and 5.5,

A is a translation-invariant linear operator

$\Leftrightarrow$ the matrix $(A)_S$ of A with respect to the standard basis S for $L^2(\mathbb{Z}_N)$ is circulant

$\Leftrightarrow$ A is a convolution operator

$\Leftrightarrow$ A is a Fourier multiplier

$\Leftrightarrow$ the matrix $(A)_F$ of A with respect to the Fourier basis F for $L^2(\mathbb{Z}_N)$ is diagonal.

Exercises

1. Prove that $T_\sigma : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is a linear operator.
2. Let σ and τ be signals in $L^2(\mathbb{Z}_N)$. Prove that

$$T_\sigma T_\tau = T_{\sigma\tau}$$

and

$$T_\sigma^* = T_{\overline{\sigma}}.$$

Chapter 6

Eigenvalues and Eigenfunctions

The results obtained in Chapters 2–5 can be used in the computation of eigenvalues of filters, which are given by translation-invariant linear operators. To recall, let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a filter, i.e., a translation-invariant linear operator. Then the matrix $(A)_S$ of A with respect to the standard basis S for $L^2(\mathbb{Z}_N)$ is circulant. The filter A is in fact a convolution operator C_b with impulse response b, where b is simply the first column of the matrix A. The filter A is also a Fourier multiplier T_σ with symbol σ and $\sigma = \hat{b}$. The matrix $(A)_F$ of the filter A with respect to the Fourier basis F for $L^2(\mathbb{Z}_N)$ is diagonal, and is given by

$$(A)_F = \begin{pmatrix} \lambda_0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{N-1} \end{pmatrix},$$

where λ_m is the eigenvalue of A corresponding to the eigenfunction F_m, $m = 0, 1, \ldots, N-1$.

We can now give an explicit formula for the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$. We want the formula to be so tractable that it can be used in computation. Such a formula comes readily from Propositions 5.3 and 5.4, and the discussion given in the first paragraph of this chapter.

Theorem 6.1. *Let $A : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be a translation-invariant linear operator. Then the eigenvalues of A are given by*

$$\sigma(0), \sigma(1), \ldots, \sigma(N-1),$$

where $\sigma = \hat{b}$ and b is the first column of the matrix A. Moreover, for $m = 0, 1, \ldots, N-1$, the eigenfunction of A corresponding to the eigenvalue $\sigma(m)$ is the wave F_m.

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DOI 10.1007/978-3-0348-0116-4_6,

Example 6.2. Let $\Delta : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be the linear operator defined by

$$(\Delta z)(n) = z(n+1) - 2z(n) + z(n-1), \quad n \in \mathbb{Z},$$

for all z in $L^2(\mathbb{Z}_N)$. Find all the eigenvalues of Δ.

Solution. That Δ is a linear operator is easy to check and is hence omitted. To check that Δ is translation-invariant, let $k \in \mathbb{Z}$. Then for all $z \in L^2(\mathbb{Z}_N)$,

$$\begin{aligned}(\Delta R_k z)(n) &= (R_k z)(n+1) - 2(R_k z)(n) + (R_k z)(n-1) \\ &= z(n-k+1) - 2z(n-k) + z(n-k-1), \quad n \in \mathbb{Z}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(R_k \Delta z)(n) &= (\Delta z)(n-k) \\ &= z(n-k+1) - 2z(n-k) + z(n-k-1), \quad n \in \mathbb{Z}.\end{aligned}$$

Therefore $\Delta R_k = R_k \Delta$, and hence Δ is translation-invariant. Let us now find the first column of the matrix $\Delta = (\Delta)_S$ of Δ with respect to the standard basis S for $L^2(\mathbb{Z}_N)$. To this end, we note that for all $z \in L^2(\mathbb{Z}_N)$,

$$(\Delta z)(0) = z(1) - 2z(0) + z(-1) = z(1) - 2z(0) + z(N-1)$$

and hence

$$\Delta = (\Delta)_S = \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix}.$$

So, the first column b of $\Delta = (\Delta)_S$ is given by

$$b = \begin{pmatrix} -2 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then the eigenvalues of Δ are $\sigma(0), \sigma(1), \ldots, \sigma(N-1)$, where σ is the symbol of Δ given by

$$
\begin{aligned}
\sigma(m) = \hat{b}(m) &= \sum_{n=0}^{N-1} b(n)e^{-2\pi imn/N} \\
&= -2 + e^{-2\pi im/N} + e^{-2\pi im(N-1)/N} \\
&= -2 + e^{-2\pi im/N} + e^{2\pi im/N} \\
&= -2 + 2\cos(2\pi m/N) \\
&= -4\sin^2\left(\frac{\pi m}{N}\right), \quad m \in \mathbb{Z}.
\end{aligned}
$$

□

Remark 6.3. Δ is the Laplacian on $\mathbb{Z}_N$.

Exercises

1. Find the eigenvalues of the linear operator $D : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ given by

$$(Dz)(n) = iz(n+1) - iz(n), \quad n = 0, 1, \ldots, N-1.$$

   (D is the Dirac operator on $\mathbb{Z}_N$.)

2. Let $\sigma \in L^2(\mathbb{Z}_N)$. Find the eigenvalues of the Fourier multiplier $T_\sigma : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$.

3. Let $b \in L^2(\mathbb{Z}_N)$. Find the eigenvalues of the convolution operator $C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$.

Chapter 7

The Fast Fourier Transform

The Fourier inversion formula in Theorem 1.7 states that for every signal z in $L^2(\mathbb{Z}_N)$, the coordinates $(z)_F$ and $(z)_S$ of z with respect to the Fourier basis F and the standard basis S for $L^2(\mathbb{Z}_N)$ respectively are related by

$$(z)_F = \hat{z} = \Omega_N z = \Omega_N (z)_S,$$

where Ω_N is the Fourier matrix of order $N \times N$. So, the change of basis from the standard basis S for $L^2(\mathbb{Z}_N)$ to the Fourier basis F for $L^2(\mathbb{Z}_N)$ is the same as multiplying an $N \times 1$ column vector by the $N \times N$ matrix Ω_N, and it has been pointed out in Remark 1.11 that this entails N^2 multiplications of complex numbers. In view of the fact that in signal analysis, N is usually very big and therefore the task of carrying out N^2 complex multiplications is formidable even for high-speed computers. As a matter of fact, the number of additions should have been taken into account. Due to the fact that multiplication is much slower than addition on a computer, an idea of the speed of computer time required for the computation of the finite Fourier transform can be gained by just counting the number of complex multiplications.

Do we really need N^2 complex multiplications? Apparently, the answer is yes and we cannot do better than this because there is not even one single zero entry in the matrix Ω_N. The key that helps us in reducing the number of multiplications by a great deal lies in the structure of the matrix Ω_N. The entries in Ω_N are just powers of the single complex number ω_N given by

$$\omega_N = e^{-2\pi i/N}.$$

This structure enables us to decompose Ω_N into factors with many zeros. This factorization, first envisaged by Gauss in 1805 and developed by Cooley and Tukey in 1965, is the basic idea behind the fast Fourier transform, of which the acronym is justifiably FFT.

We illustrate the basic ideas of the fast Fourier transform in this chapter. To make life simple, we only look at the case when N is a power of 2. The trick then

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DOI 10.1007/978-3-0348-0116-4_7, © Springer Basel AG 2011

is to work with the matrix $\Omega_{N/2}$ instead of Ω_N. To wit, suppose that $N = 4$. Then

$$\Omega_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & \omega_4^4 & \omega_4^6 \\ 1 & \omega_4^3 & \omega_4^6 & \omega_4^9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix},$$

and we can write

$$\begin{aligned} \Omega_4 &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & i \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} I_2 & D_2 \\ I_2 & -D_2 \end{pmatrix} \begin{pmatrix} \Omega_2 & 0 \\ 0 & \Omega_2 \end{pmatrix} P_4, \end{aligned} \qquad (7.1)$$

where I_2 is the identity matrix of order 2×2, D_2 is the diagonal matrix of which the diagonal entries are 1 and ω_4 and P_4 is the permutation matrix that puts $z(0)$ and $z(2)$ before $z(1)$ and $z(3)$ in the signal z in $L^2(\mathbb{Z}_4)$. The formula (7.1) is the decomposition of Ω_4 into factors with many zeros. It is also the reduction of Ω_4 to Ω_2. The first matrix on the right-hand side of (7.1) aligns the two half-size outputs to produce the desired $\Omega_4 z$. A generalization of (7.1) is the following formula of Cooley and Tukey to the effect that

$$\Omega_N = \begin{pmatrix} I_{N/2} & D_{N/2} \\ I_{N/2} & D_{N/2} \end{pmatrix} \begin{pmatrix} \Omega_{N/2} & 0 \\ 0 & \Omega_{N/2} \end{pmatrix} P_N, \qquad (7.2)$$

where $I_{N/2}$ is the identity matrix of order $\frac{N}{2} \times \frac{N}{2}$, $D_{N/2}$ is the diagonal matrix of which the diagonal entries are $1, \omega_N, \omega_N^2, \ldots, \omega_N^{(N-2)/2}$, and P_N is the permutation matrix that puts the evens before the odds. The Cooley and Tukey formula (7.2) is the FFT or the first step of the FFT.

What is then the next step? It is of course the reduction of the matrix $\Omega_{N/2}$ to the matrix $\Omega_{N/4}$ by means of the Cooley–Tukey formula (7.2) where the N is now replaced by $N/2$. Then we keep going from $N/4$ to $N/8$ and so on.

To see how much is saved in using the FFT, let us recall that without the FFT, direct matrix multiplication requires N^2 complex multiplications. What do we gain with the FFT? To answer this question, let $N = 2^l$, where l is a positive integer. Then we have the following theorem.

Theorem 7.1. *The number of complex multiplications using the* FFT *is at most* $\frac{1}{2}Nl = \frac{1}{2}N\log_2 N$.

Proof. Let $l = 1$. Then $N = 2$. Let

$$z = \begin{pmatrix} z(0) \\ z(1) \end{pmatrix}.$$

Then

$$\hat{z} = \Omega_2 z = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} z(0) \\ z(1) \end{pmatrix} = \begin{pmatrix} z(0) + z(1) \\ z(0) - z(1) \end{pmatrix}$$

and no multiplication is required. Suppose that the theorem is true for l. Then the number of complex multiplications required at level l is at most $\frac{1}{2}2^l l$. Using the Cooley-Tukey formula (7.2), we need 2^l multiplications using the diagonal D's to put together the half-size products from the level l. Thus, we need at most $2^l + 2^l l = \frac{1}{2}2^{l+1}(l+1)$. By the principle of mathematical induction, the proof is complete. □

Remark 7.2. In order to appreciate the power of the FFT, let $l = 10$. Then

$$N = 2^{10} = 1024.$$

Direct matrix multiplication without the FFT requires $(1024)^2$ complex multiplications. We need to perform more than a million complex multiplications. Using the FFT, we need at most $\frac{1}{2} \times 1024 \times 10 = 5120$ complex multiplications. The save in computer time is enormous.

Remark 7.3. Convolutions can also be computed rapidly with the FFT. To see how, let us recall that, by Proposition 4.7 and the Fourier inversion formula, we get

$$z * w = (\widehat{z * w})^\vee = (\hat{z}\hat{w})^\vee, \quad z, w \in L^2(\mathbb{Z}_N).$$

If N is a power of 2, then, by Theorem 7.1, we need at most $\frac{1}{2}N\log_2 N$ complex multiplications to compute $\hat{z}$, at most $\frac{1}{2}N\log_2 N$ complex multiplications to compute $\hat{w}$, at most N complex multiplications to compute $\hat{z}\hat{w}$ and at most $\frac{1}{2}N\log_2 N$ complex multiplications to compute the inverse finite Fourier transform of $\hat{z}\hat{w}$. So, using the FFT, we need at most $N + \frac{3N}{2}\log_2 N$ complex multiplications to compute $z * w$. This observation will be useful to us in the study of wavelets.

Exercises

1. Let $\sigma \in L^2(\mathbb{Z}_N)$, where N is a power of 2. Use the FFT to find an upper bound on the number of complex multiplications required to compute $T_\sigma z$, where $z \in L^2(\mathbb{Z}_N)$.

2. Let b_1 and b_2 be signals in $L^2(\mathbb{Z})$, where N is a power of 2. Use the FFT to find an upper bound on the number of complex multiplications required to compute the product $C_{b_1} C_{b_2}$.

Chapter 8

Time-Frequency Analysis

Let z be a signal in $L^2(\mathbb{Z}_N)$. Then we say that z is time-localized near n_0 if all components $z(n)$ are 0 or relatively small except for a few values of n near n_0. An orthonormal basis B for $L^2(\mathbb{Z}_N)$ is said to be time-localized if every signal in B is time-localized.

Let $B = \{z_0, z_1, \ldots, z_{N-1}\}$ be a time-localized orthonormal basis for $L^2(\mathbb{Z}_N)$. Then for every signal z in $L^2(\mathbb{Z}_N)$,

$$z = \sum_{n=0}^{N-1} \alpha_n z_n, \tag{8.1}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{N-1} \in \mathbb{C}$. If we are interested in the part of the signal z near a point n_0, then we can concentrate on the terms for which the basis signals are localized near n_0 and ignore the rest. An advantage then is that the full sum (8.1) is replaced by a much smaller one. We have performed signal compression in so doing.

Let us for a moment think of n as the space variable instead of the time variable. Suppose that a coefficient in the sum (8.1) is big. Then, using a space-localized orthonormal basis, we can locate and concentrate on this big coefficient. This is the idea underlying medical imaging.

Let z be a signal in $L^2(\mathbb{Z}_N)$. Then we say that z is frequency-localized near m_0 if all components $\hat{z}(m)$ are 0 or relatively small except for a few values of m near m_0. An orthonormal basis B for $L^2(\mathbb{Z}_N)$ is said to be frequency-localized if every signal in B is frequency-localized.

Example 8.1. The standard basis $S = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_{N-1}\}$ for $L^2(\mathbb{Z}_N)$ is time-localized, but not frequency-localized.

Proof. Indeed, let $k \in \mathbb{Z}$. Then

$$\epsilon_k(n) = \begin{cases} 1, & n = k, \\ 0, & n \neq k, \end{cases}$$

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DOI 10.1007/978-3-0348-0116-4_8, © Springer Basel AG 2011

and hence ϵ_k in $L^2(\mathbb{Z}_N)$ is localized at k. On the other hand,

$$\widehat{\epsilon_k}(m) = \sum_{n=0}^{N-1} \epsilon_k(n) e^{-2\pi imn/N} = e^{-2\pi imk/N}, \quad m \in \mathbb{Z}.$$

Since $|\widehat{\epsilon_k}(m)| = 1$, $m \in \mathbb{Z}$, it follows that ϵ_k is not frequency-localized. □

Example 8.2. The Fourier basis $F = \{F_0, F_1, \ldots, F_{N-1}\}$ for $L^2(\mathbb{Z}_N)$ is frequency-localized, but not time-localized.

Proof. The argument used in the proof of Example 8.1 shows that F is not time-localized. To see that F is frequency-localized, let $k \in \mathbb{Z}$. Then, using the Fourier inversion formula, we get

$$F_k = \sum_{m=0}^{N-1} \widehat{F_k}(m) F_m.$$

Hence

$$\widehat{F_k}(m) = \begin{cases} 1, \, m = k, \\ 0, \, m \neq k, \end{cases}$$

and this proves that F_k is frequency-localized at k. □

Why are frequency-localized orthonormal bases good? First of all, we can expect that such a basis, like the Fourier basis, is amenable to fast computation. To see another benefit that can be obtained from a frequency-localized orthonormal basis, let us suppose that we need to remove the high-frequency components of a signal without affecting adversely the quality of the resulting signal. Then we need to know which frequencies to remove. This information is provided by a frequency-localized orthonormal basis. To render this idea concrete and transparent, let us return to the Fourier basis F, which is the prototype of a frequency-localized orthonormal basis. By the Fourier inversion formula in Theorem 1.7, every signal z in $L^2(\mathbb{Z}_N)$ is of the form

$$z = \sum_{m=0}^{N-1} \hat{z}(m) F_m.$$

We see from our discussions in Chapter 1 that this is a high-frequency signal if $|\hat{z}(m)|$ is big for frequencies m near $N/2$. Thus, we see clearly that a frequency-localized orthonormal basis such as the Fourier basis F locates the high-frequency components of a signal if any.

An important objective is to construct orthonormal bases B for $L^2(\mathbb{Z}_N)$ such that B is time-localized, frequency-localized and there is a fast algorithm for the computation of $(z)_B$ for all z in $L^2(\mathbb{Z}_N)$. To this end, we need some preparations.

Definition 8.3. Let z be a signal in $L^2(\mathbb{Z}_N)$. Then we define the involution z^* of z by

$$z^*(n) = \overline{z(-n)}, \quad n \in \mathbb{Z}.$$

Proposition 8.4. *Let z be a signal in $L^2(\mathbb{Z}_N)$. Then*

$$\widehat{z^*}(m) = \overline{\hat{z}(m)}, \quad m \in \mathbb{Z}.$$

Proof. Let $m \in \mathbb{Z}$. Then, using the definition of the finite Fourier transform and the involution, we get

$$\begin{aligned}
\widehat{z^*}(m) &= \sum_{n=0}^{N-1} z^*(n) e^{-2\pi i m n/N} \\
&= \sum_{n=0}^{N-1} \overline{z(-n)} e^{-2\pi i m n/N} \\
&= \sum_{l=-(N-1)}^{0} \overline{z(l)} e^{2\pi i m l/N} \\
&= \overline{\sum_{l=-(N-1)}^{0} z(l) e^{-2\pi i m l/N}}. \qquad (8.2)
\end{aligned}$$

Since $z(l)e^{-2\pi i m l/N}$ is a periodic function of l on $\mathbb{Z}$ with period N, we can use Proposition 1.19 and (8.2) to get

$$\widehat{z^*}(m) = \overline{\sum_{l=0}^{N-1} z(l) e^{-2\pi i m l/N}} = \overline{\hat{z}(m)}. \qquad \square$$

Proposition 8.5. *Let z and w be signals in $L^2(\mathbb{Z}_N)$. Then*

$$(z * w^*)(m) = (z, R_m w) \qquad (8.3)$$

and

$$(z * w)(m) = (z, R_m w^*) \qquad (8.4)$$

for all m in $\mathbb{Z}$.

To prove Proposition 8.5, we need a lemma.

Lemma 8.6. *Let z and w be signals in $L^2(\mathbb{Z}_N)$. Then*

$$z * w = w * z.$$

Proof. Using the definition of the convolution of z and w, we get

$$\begin{aligned}
(z * w)(m) &= \sum_{n=0}^{N-1} z(m-n) w(n) \\
&= \sum_{l=m}^{m-(N-1)} w(m-l) z(l), \quad m \in \mathbb{Z}. \qquad (8.5)
\end{aligned}$$

Since $w(m-l)z(l)$ is a periodic function of l on $\mathbb{Z}$ with period N, we can use Proposition 1.19 and (8.5) to get

$$(z * w)(m) = \sum_{l=0}^{N-1} w(m-l)z(l) = (w * z)(m), \quad m \in \mathbb{Z}. \qquad \square$$

To prove (8.3), we note that, by Lemma 8.6 and the definition of the involution,

$$\begin{aligned}(z, R_m w) &= \sum_{n=0}^{N-1} z(n)\overline{(R_m w)(n)} \\ &= \sum_{n=0}^{N-1} z(n)\overline{w(n-m)} \\ &= \sum_{n=0}^{N-1} z(n)w^*(m-n) \\ &= (w^* * z)(m) = (z * w^*)(m), \quad m \in \mathbb{Z}.\end{aligned}$$

To prove (8.4), we first note that

$$(w^*)^*(n) = \overline{w^*(-n)} = \overline{\overline{w(n)}} = w(n), \quad n \in \mathbb{Z},$$

and hence, by (8.3),

$$(z, R_m w^*) = (z * (w^*)^*)(m) = (z * w)(m), \quad m \in \mathbb{Z},$$

and (8.4) is proved.

Let us now suppose that there exists a signal w in $L^2(\mathbb{Z}_N)$ such that

$$B = \{R_0 w, R_1 w, \ldots, R_{N-1} w\}$$

is an orthonormal basis for $L^2(\mathbb{Z}_N)$. We note that B is an orthonormal basis for $L^2(\mathbb{Z}_N)$ generated by successive translations of the single signal w. Then for every z in $L^2(\mathbb{Z}_N)$,

$$z = \sum_{n=0}^{N-1} \alpha_n R_n w,$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{N-1} \in \mathbb{C}$. But for $k \in \mathbb{Z}$,

$$(z, R_k w) = \sum_{n=0}^{N-1} \alpha_n (R_n w, R_k w).$$

Since $\{R_0 w, R_1 w, \ldots, R_{N-1} w\}$ is orthonormal, we see that

$$(z, R_k w) = \alpha_k, \quad k \in \mathbb{Z},$$

and hence

$$z = \sum_{n=0}^{N-1} (z, R_n w) R_n w.$$

So, by (8.3),

$$z = \sum_{n=0}^{N-1} (z * w^*)(n) R_n w$$

and consequently

$$(z)_B = \begin{pmatrix} (z * w^*)(0) \\ (z * w^*)(1) \\ \vdots \\ (z * w^*)(N-1) \end{pmatrix} = z * w^*.$$

So, the computation of $(z)_B$ for all z in $L^2(\mathbb{Z}_N)$, or the change of basis from the standard basis S for $L^2(\mathbb{Z}_N)$ to the basis B, can be performed by a fast algorithm, i.e., the FFT. See Remark 7.3 in this connection.

By choosing the signal w in $L^2(\mathbb{Z}_N)$ to be such that w is time-localized near some point n_0, the orthonormal basis $B = \{R_0 w, R_1 w, \ldots, R_{N-1} w\}$ is of course a time-localized basis. We have just seen that the change of basis from the standard basis S for $L^2(\mathbb{Z}_N)$ to the orthonormal basis B can be computed by a fast algorithm. So, if B is also frequency-localized, then the objective of this chapter is achieved. Is the orthonormal basis B frequency-localized? Unfortunately, the answer is no and this can be seen from Proposition 8.8. First we need a lemma.

Lemma 8.7. *Let z and w be signals in $L^2(\mathbb{Z}_N)$. Then*

$$(R_j z, R_k w) = (z, R_{k-j} w)$$

for all j and k in $\mathbb{Z}_N$ with $j \le k$.

Proof. Let $j, k \in \mathbb{Z}$. Then

$$\begin{aligned} (R_j z, R_k w) &= \sum_{n=0}^{N-1} (R_j z)(n) \overline{(R_k w)(n)} \\ &= \sum_{n=0}^{N-1} z(n-j) \overline{w(n-k)} \\ &= \sum_{l=-j}^{N-1-j} z(l) \overline{w(l+j-k)}. \end{aligned} \tag{8.6}$$

Since $z(l)\overline{w(l+j-k)}$ is a periodic function of l on $\mathbb{Z}$ with period N, it follows from Proposition 1.19 and (8.6) that

$$\begin{aligned}(R_j z, R_k w) &= \sum_{l=0}^{N-1} z(l)\overline{w(l+j-k)} \\ &= \sum_{l=0}^{N-1} z(l)\overline{(R_{k-j}w)(l)} \\ &= (z, R_{k-j}w).\end{aligned}$$

□

Proposition 8.8. *Let w be a signal in $L^2(\mathbb{Z}_N)$. Then $\{R_0 w, R_1 w, \ldots, R_{N-1}w\}$ is an orthonormal basis for $L^2(\mathbb{Z}_N)$ if and only if*

$$|\hat{w}(m)| = 1, \quad m \in \mathbb{Z}.$$

Proof. Let us begin with the formula for the finite Fourier transform of the unit impulse δ and this is the answer to Question 1 in Chapter 4. Indeed,

$$\hat{\delta}(m) = \sum_{n=0}^{N-1} \delta(n) e^{-2\pi i m n/N} = 1, \quad m \in \mathbb{Z}.$$

Suppose that $\{R_0 w, R_1 w, \ldots, R_{N-1}w\}$ is an orthonormal basis for $L^2(\mathbb{Z}_N)$. Then

$$(w, R_k w) = \begin{cases} 1, \ k = 0, \\ 0, \ k \neq 0, \end{cases} \tag{8.7}$$

for all k in $\mathbb{Z}_N$. By (8.3), we get

$$(w, R_k w) = (w * w^*)(k), \quad k \in \mathbb{Z},$$

and hence, by (8.7),

$$w * w^* = \delta.$$

So, by Propositions 4.7 and 8.4, we get

$$1 = \hat{\delta}(m) = (w * w^*)^\wedge(m) = \hat{w}(m)\widehat{w^*}(m) = \hat{w}(m)\overline{\hat{w}(m)} = |\hat{w}(m)|^2$$

for all m in $\mathbb{Z}$. Conversely, suppose that $|\hat{w}(m)|^2 = 1$, $m \in \mathbb{Z}$. Then, by Propositions 4.7 and 8.4,

$$(w * w^*)^\wedge(m) = \hat{w}(m)\widehat{w^*}(m) = \hat{w}(m)\overline{\hat{w}(m)} = |\hat{w}(m)|^2 = 1$$

for all m in $\mathbb{Z}$. Thus,

$$w * w^* = \delta$$

and by (8.3),

$$(w, R_k w) = (w * w^*)(k) = \begin{cases} 1, \ k = 0, \\ 0, \ k \neq 0. \end{cases}$$

Hence, by Lemma 8.7 and (8.7),

$$(R_j w, R_k w) = \begin{cases} 1, \ j = k, \\ 0, \ j \neq k. \end{cases}$$

So, $\{R_0 w, R_1 w, \ldots, R_{N-1} w\}$ is an orthonormal basis for $L^2(\mathbb{Z}_N)$. □

Exercises

1. Let $k \in \mathbb{Z}$. Then we define $M_k : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ by

$$(M_k z)(n) = e^{-2\pi i n k/N} z(n), \quad n \in \mathbb{Z}_N.$$

   Prove that

$$\widehat{M_k z}(m) = (R_{-k} \hat{z})(m), \quad m \in \mathbb{Z}_N.$$

2. Give a characterization of all signals w in $L^2(\mathbb{Z}_N)$ such that

$$\{M_0 w, M_1 w, \ldots, M_{N-1} w\}$$

   is an orthonormal basis for $L^2(\mathbb{Z}_N)$.

3. Explain how to obtain a frequency-localized orthonormal basis for $L^2(\mathbb{Z}_N)$.

Chapter 9

Time-Frequency Localized Bases

We have seen in the previous chapter that successive translations of a single signal in $L^2(\mathbb{Z}_N)$ can give us a time-localized orthonormal basis B for $L^2(\mathbb{Z}_N)$ such that the change of basis from the standard basis S for $L^2(\mathbb{Z}_N)$ to the basis B can be implemented by the FFT. The only thing that is missing from the basis B, though, is frequency-localization. Can we still do something with translations of signals in the construction of orthonormal bases, which are time-localized, frequency-localized and amenable to computation by a fast algorithm? If we cannot do it with one signal, can we do it with two signals? The answer is amazingly yes. We explain how this can be done in this chapter.

Throughout this chapter, we assume that N is an even integer, say, $N = 2M$, where M is a positive integer.

Definition 9.1. Suppose that there exist signals φ and ψ in $L^2(\mathbb{Z}_N)$ for which the set B given by

$$B = \{R_0\varphi, R_2\varphi, \ldots, R_{2M-2}\varphi\} \cup \{R_0\psi, R_2\psi, \ldots, R_{2M-2}\psi\}$$

is an orthonormal basis for $L^2(\mathbb{Z}_N)$. Then we call B a time-frequency localized basis for $L^2(\mathbb{Z}_N)$. The signals φ and ψ are called the mother wavelet and the father wavelet for the time-frequency localized basis B respectively.

Remark 9.2. The basis $\{R_0\varphi, R_2\varphi, \ldots, R_{2M-2}\varphi\} \cup \{R_0\psi, R_2\psi, \ldots, R_{2M-2}\psi\}$ is more conveniently written as $\{R_{2k}\varphi\}_{k=0}^{M-1} \cup \{R_{2k}\psi\}_{k=0}^{M-1}$.

We need some preliminary results for the construction of wavelet bases for $L^2(\mathbb{Z}_N)$.

Definition 9.3. Let $N = 2M$, where M is a positive integer, and let z be a signal in $L^2(\mathbb{Z}_N)$. Then we define the signal z^+ in $L^2(\mathbb{Z}_N)$ by

$$z^+(n) = (-1)^n z(n), \quad n \in \mathbb{Z}.$$

Proposition 9.4. *Let z be a signal in $L^2(\mathbb{Z}_N)$, where $N = 2M$. Then*

$$\widehat{z^+}(m) = \hat{z}(m + M), \quad m \in \mathbb{Z}.$$

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DOI 10.1007/978-3-0348-0116-4_9, © Springer Basel AG 2011

Proof. Using the definition of z^+ and the definition of the finite Fourier transform, we get

$$\begin{aligned}\widehat{z^+}(m) &= \sum_{n=0}^{N-1} z^+(n)e^{-2\pi imn/N} \\ &= \sum_{n=0}^{N-1} (-1)^n z(n)e^{-2\pi imn/N} \\ &= \sum_{n=0}^{N-1} e^{-2\pi inM/N} z(n)e^{-2\pi imn/N} \\ &= \sum_{n=0}^{N-1} z(n)e^{-2\pi in(m+M)/N} = \hat{z}(m+M)\end{aligned}$$

for all m in $\mathbb{Z}$. □

The following remark on the signals z and z^+ in $L^2(\mathbb{Z}_N)$ is useful to us.

Remark 9.5. Let z be any signal in $L^2(\mathbb{Z}_N)$. Then for all n in $\mathbb{Z}$,

$$(z + z^+)(n) = 2z(n)$$

if n is even, and is 0 if n is odd.

The following technical lemma is the key to the construction of wavelet bases.

Lemma 9.6. *Let $N = 2M$, where M is a positive integer. Let w be a signal in $L^2(\mathbb{Z}_N)$. Then $\{R_0 w, R_2 w, \ldots, R_{2M-2} w\}$ is an orthonormal set with M distinct signals in $L^2(\mathbb{Z}_N)$ if and only if*

$$|\hat{w}(m)|^2 + |\hat{w}(m+M)|^2 = 2, \quad m = 0, 1, \ldots, M-1. \tag{9.1}$$

Proof. Suppose that $\{R_0 w, R_2 w, \ldots, R_{2M-2} w\}$ is an orthonormal set with M distinct signals. Then, by (8.3),

$$\begin{aligned}(w * w^*)(2k) &= (w, R_{2k} w) \\ &= \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, \ldots, M-1. \end{cases}\end{aligned} \tag{9.2}$$

By (9.2) and Remark 9.5,

$$\begin{aligned}&((w * w^*) + (w * w^*)^+)(2k) \\ &= 2(w * w^*)(2k) = \begin{cases} 2, & k = 0, \\ 0, & k = 1, 2, \ldots, M-1. \end{cases}\end{aligned} \tag{9.3}$$

Therefore, by (9.3),

$$(w * w^*) + (w * w^*)^+ = 2\delta, \tag{9.4}$$

where δ is the unit impulse in $L^2(\mathbb{Z}_N)$. Taking the finite Fourier transform on both sides of (9.4), we get, by Propositions 4.7 and 9.4, and the fact that $\hat{\delta}(m) = 1$ for all m in $\mathbb{Z}$,

$$\hat{w}(m)\widehat{w^*}(m) + (w * w^*)^\wedge(m + M) = 2, \quad m \in \mathbb{Z}. \tag{9.5}$$

Applying Propositions 4.7 and 8.4 to (9.5), we get

$$\hat{w}(m)\overline{\hat{w}(m)} + \hat{w}(m + M)\overline{\hat{w}(m + M)} = 2, \quad m \in \mathbb{Z},$$

and (9.1) is proved. Conversely, suppose that (9.1) is valid. Then

$$|\hat{w}(m)|^2 + |\hat{w}(m + M)|^2 = 2, \quad m \in \mathbb{Z}. \tag{9.6}$$

Indeed, let $m \in \mathbb{Z}$. Then, by the division algorithm,

$$m = qM + r,$$

where r is some integer in $\{0, 1, \ldots, M - 1\}$. Thus,

$$|\hat{w}(m)|^2 + |\hat{w}(m + M)|^2 = |\hat{w}(r + qM)|^2 + |\hat{w}(r + (q + 1)M)|^2. \tag{9.7}$$

If q is even, then, by (9.1), (9.7) and the fact that $\hat{w}$ is periodic with period $2M$, we get

$$|\hat{w}(m)|^2 + |\hat{w}(m + M)|^2 = |\hat{w}(r)|^2 + |\hat{w}(r + M)|^2 = 2.$$

If q is odd, then the analog of (9.6) is

$$|\hat{w}(m)|^2 + |\hat{w}(m + M)|^2 = |\hat{w}(r + (q - 1)M + M)|^2 + |\hat{w}(r + (q - 1)M)|^2. \tag{9.8}$$

Thus, by (9.1), (9.8) and the fact that $\hat{w}$ is periodic with period $2M$, we get

$$|\hat{w}(m)|^2 + |\hat{w}(m + M)|^2 = |\hat{w}(r + M)|^2 + |\hat{w}(r)|^2 = 2,$$

and (9.6) is established. Now, for all m in $\mathbb{Z}$, we use Propositions 4.7, 8.4 and 9.4 as in the first part of the proof and (9.5) to get

$$(w * w^*)^\wedge(m) + ((w * w^*)^+)^\wedge(m) = 2$$

and hence

$$(w * w^*) + (w * w^*)^+ = 2\delta. \tag{9.9}$$

Thus, by (8.3) and (9.9), we get, for $k = 0, 1, \ldots, M - 1$,

$$\begin{aligned} (w, R_{2k}w) &= (w * w^*)(2k) \\ &= \frac{1}{2}\left((w * w^*) + (w * w^*)^+\right)(2k) \\ &= \begin{cases} 1, \ k = 0, \\ 0, \ k = 1, 2, \ldots, M - 1. \end{cases} \end{aligned} \tag{9.10}$$

Now, let $j, k \in \{0, 1, \ldots, M-1\}$ be such that $j \neq k$. Suppose that $j < k$. Then, by Lemma 8.7 and (9.10),

$$(R_{2j}w, R_{2k}w) = (w, R_{2k-2j}w) = 0.$$

Also, for $k = 0, 1, \ldots, M-1$, we get, by Lemma 8.7 and (9.10) again,

$$\|R_{2k}w\|^2 = (R_{2k}w, R_{2k}w) = (w, R_0 w) = 1.$$

This proves that $\{R_0 w, R_2 w, \ldots, R_{2M-2}w\}$ is an orthonormal set in $L^2(\mathbb{Z}_N)$. □

Definition 9.7. Let $N = 2M$, where M is a positive integer. Let φ and ψ be signals in $L^2(\mathbb{Z}_N)$. Then for all m in $\mathbb{Z}$, we define the 2×2 matrix $A_{\varphi,\psi}(m)$ by

$$A_{\varphi,\psi}(m) = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{\varphi}(m) & \hat{\psi}(m) \\ \hat{\varphi}(m+M) & \hat{\psi}(m+M) \end{pmatrix}.$$

We call $A_{\varphi,\psi}(m)$ the system matrix of the signals φ and ψ at the integer m.

We can now give the result, which we can use to construct wavelet bases.

Theorem 9.8. *Let $N = 2M$, where M is a positive integer. Let φ and ψ be signals in $L^2(\mathbb{Z}_N)$. Then the set $B = \{R_{2k}\varphi\}_{k=0}^{M-1} \cup \{R_{2k}\psi\}_{k=0}^{M-1}$ is a time-frequency localized basis for $L^2(\mathbb{Z}_N)$ if and only if $A_{\varphi,\psi}(m)$ is a unitary matrix for $m = 0, 1, \ldots, M-1$. Equivalently, B is a time-frequency localized basis for $L^2(\mathbb{Z}_N)$ if and only if*

$$|\hat{\varphi}(m)|^2 + |\hat{\varphi}(m+M)|^2 = 2, \tag{9.11}$$

$$|\hat{\psi}(m)|^2 + |\hat{\psi}(m+M)|^2 = 2, \tag{9.12}$$

and

$$\hat{\varphi}(m)\overline{\hat{\psi}(m)} + \hat{\varphi}(m+M)\overline{\hat{\psi}(m+M)} = 0 \tag{9.13}$$

for $m = 0, 1, \ldots, M-1$.

Proof. Suppose that B is an orthonormal basis for $L^2(\mathbb{Z}_N)$. Then $\{R_{2k}\varphi\}_{k=0}^{M-1}$ and $\{R_{2k}\psi\}_{k=0}^{M-1}$ are orthonormal sets with M distinct signals in $L^2(\mathbb{Z}_N)$. So, by Lemma 9.6, we get

$$|\hat{\varphi}(m)|^2 + |\hat{\varphi}(m+M)|^2 = 2$$

and

$$|\hat{\psi}(m)|^2 + |\hat{\psi}(m+M)|^2 = 2$$

for $m = 0, 1, \ldots, M-1$. Hence (9.11) and (9.12) are proved. To prove (9.13), we note that orthonormality gives

$$(\varphi, R_{2k}\psi) = 0, \quad k = 0, 1, \ldots, M-1. \tag{9.14}$$

By (8.3) and (9.14), we get

$$(\varphi * \psi^*)(2k) = (\varphi, R_{2k}\psi) = 0, \quad k = 0, 1, \ldots, M-1. \tag{9.15}$$

By (9.15) and Remark 9.5,

$$(\varphi * \psi^*) + (\varphi * \psi^*)^+ = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{9.16}$$

So, taking the finite Fourier transform on both sides of (9.16), using Propositions 4.7, 8.4 and 9.4, we obtain

$$\hat{\varphi}(m)\overline{\hat{\psi}(m)} + \hat{\varphi}(m+M)\overline{\hat{\psi}(m+M)} = 0$$

for $m = 0, 1, \ldots, M-1$. Conversely, suppose that (9.11)–(9.13) are valid. Then, by (9.11), (9.12) and Lemma 9.6, $\{R_{2k}\varphi\}_{k=0}^{M-1}$ and $\{R_{2k}\psi\}_{k=0}^{M-1}$ are both orthonormal sets with M distinct signals in $L^2(\mathbb{Z}_N)$. Now, by (9.13) and the periodicity argument used in the proof of Lemma 9.6, we get

$$\hat{\varphi}(m)\overline{\hat{\psi}(m)} + \hat{\varphi}(m+M)\overline{\hat{\psi}(m+M)} = 0, \quad m \in \mathbb{Z}.$$

(See Exercise 4 in this regard.) Thus, by Propositions 4.7, 8.4 and 9.4, we get

$$((\varphi * \psi^*) + (\varphi * \psi^*)^+)^\wedge(m) = 0, \quad m \in \mathbb{Z}.$$

Then

$$(\varphi * \psi^*) + (\varphi * \psi^*)^+ = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and hence

$$(\varphi, R_{2k}\psi) = (\varphi * \psi^*)(2k) = 0 \tag{9.17}$$

for $k = 0, 1, \ldots, M-1$. Now, let $j, k \in \{0, 1, \ldots, M-1\}$ be such that $j \leq k$. Then, by Lemma 8.7 and (9.17),

$$(R_{2j}\varphi, R_{2k}\psi) = (\varphi, R_{2k-2j}\psi) = 0.$$

Thus, $B = \{R_{2k}\varphi\}_{k=0}^{M-1} \cup \{R_{2k}\psi\}_{k=0}^{M-1}$ is an orthonormal set with N distinct signals in $L^2(\mathbb{Z}_N)$, and is hence an orthonormal basis for $L^2(\mathbb{Z}_N)$. This completes the proof. □

Remark 9.9. Let us look at the relations between the mother wavelet and the father wavelet given by (9.11)–(9.13). By (9.11), we can have, say,

$$|\hat{\varphi}(m_0)|^2 = 2$$

and

$$|\hat{\varphi}(m_0 + M)|^2 = 0$$

where m_0 is some integer in $\mathbb{Z}_M$. For such an integer m_0, we get, by (9.13),

$$|\hat{\psi}(m_0 + M)|^2 = 2$$

and hence, by (9.12),

$$|\hat{\psi}(m_0)|^2 = 0.$$

This means that for such an integer m_0, the amount of the wave F_{m_0} in the father wavelet ψ is 0, while the amount of the same wave F_{m_0} in the mother wavelet φ is full. So, we can construct the mother wavelet φ and the father wavelet ψ in such a way that φ contains only low-frequency waves and ψ contains only high-frequency waves. Thus, in the culture of signal analysis, a mother wavelet is a low pass filter and a father wavelet is a high pass filter.

Exercises

1. Let $\varphi \in L^2(\mathbb{Z}_4)$ be given by

$$\hat{\varphi} = \begin{pmatrix} \sqrt{2} \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

   Find a father wavelet ψ in $L^2(\mathbb{Z}_4)$ such that $\{R_0\varphi, R_2\varphi, R_0\psi, R_2\psi\}$ is an orthonormal basis for $L^2(\mathbb{Z}_4)$.

2. Let φ and ψ be signals in $L^2(\mathbb{Z}_N)$, $N = 2M$, where M is a positive integer, be given by

$$\varphi = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

   and

$$\psi = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

   Prove that $\{R_{2k}\varphi\}_{k=0}^{M-1} \cup \{R_{2k}\psi\}_{k=0}^{M-1}$ is an orthonormal basis for $L^2(\mathbb{Z}_N)$. (We call this orthonormal basis a Haar basis for $L^2(\mathbb{Z}_N)$.)

3. Let $N = 2M$, where M is a positive integer. Let φ and ψ be signals in $L^2(\mathbb{Z}_N)$. Prove that the set $B = \{R_{2k}\varphi\}_{k=0}^{M-1} \cup \{R_{2k}\psi\}_{k=0}^{M-1}$ is a time-frequency localized basis for $L^2(\mathbb{Z}_N)$ if and only if

$$|\hat{\varphi}(m)|^2 + |\hat{\psi}(m)|^2 = 2,$$

$$|\hat{\varphi}(m+M)|^2 + |\hat{\psi}(m+M)|^2 = 2$$

and

$$\hat{\varphi}(m)\overline{\hat{\varphi}(m+M)} + \hat{\psi}(m)\overline{\hat{\psi}(m+M)} = 0$$

for $m = 0, 1, \ldots, M-1$.

4. Let $N = 2M$, where M is a positive integer. Let φ and ψ be signals in $L^2(\mathbb{Z}_N)$ such that (9.11)–(9.13) are satisfied for $m = 0, 1, \ldots, M-1$. Prove that (9.11)–(9.13) are also valid for all m in $\mathbb{Z}$.

Chapter 10

Wavelet Transforms and Filter Banks

Let $B = \{R_{2k}\varphi\}_{k=0}^{M-1} \cup \{R_{2k}\psi\}_{k=0}^{M-1}$ be a time-frequency localized basis for $L^2(\mathbb{Z}_N)$, where φ is the mother wavelet and ψ is the father wavelet. For every signal z in $L^2(\mathbb{Z}_N)$, we get, by the fact that B is an orthonormal basis for $L^2(\mathbb{Z}_N)$ and (8.3),

$$z = \sum_{k=0}^{M-1} (z, R_{2k}\varphi)R_{2k}\varphi + \sum_{k=0}^{M-1} (z, R_{2k}\psi)R_{2k}\psi. \tag{10.1}$$

So, by (8.3),

$$(z)_B = \begin{pmatrix} (z, R_0\varphi) \\ (z, R_2\varphi) \\ \vdots \\ (z, R_{2M-2}\varphi) \\ (z, R_0\psi) \\ (z, R_2\psi) \\ \vdots \\ (z, R_{2M-2}\psi) \end{pmatrix} = \begin{pmatrix} (z * \varphi^*)(0) \\ (z * \varphi^*)(2) \\ \vdots \\ (z * \varphi^*)(2M-2) \\ (z * \psi^*)(0) \\ (z * \psi^*)(2) \\ \vdots \\ (z * \psi^*)(2M-2) \end{pmatrix}, \quad z \in L^2(\mathbb{Z}_N). \tag{10.2}$$

Let $V_{\varphi,\psi}$ be the $N \times N$ matrix defined by

$$V_{\varphi,\psi} = \left(R_0\varphi|\cdots|R_{2M-2}\varphi|R_0\psi|\cdots|R_{2M-2}\psi\right).$$

Then, by (10.1) and (10.2), we get

$$z = (z)_S = V_{\varphi,\psi}(z)_B$$

or

$$(z)_B = V_{\varphi,\psi}^{-1} z, \quad z \in L^2(\mathbb{Z}_N).$$

Definition 10.1. Let $W_{\varphi,\psi} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be defined by

$$W_{\varphi,\psi} = V_{\varphi,\psi}^{-1}.$$

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DOI 10.1007/978-3-0348-0116-4_10, © Springer Basel AG 2011

Then we call $W_{\varphi,\psi}$ the wavelet transform associated to the mother wavelet φ and the father wavelet ψ.

Remark 10.2. The wavelet transform $W_{\varphi,\psi}$ is the change of basis matrix from the standard basis S for $L^2(\mathbb{Z}_N)$ to the time-frequency localized basis B generated by φ and ψ.

Remark 10.3. Since $V_{\varphi,\psi}$ is a unitary matrix, it follows that

$$W_{\varphi,\psi} = V_{\varphi,\psi}^{-1} = V_{\varphi,\psi}^*,$$

where $V_{\varphi,\psi}^*$ is the adjoint of $V_{\varphi,\psi}$, i.e., the transpose of the conjugate of $V_{\varphi,\psi}$. So, an explicit formula for the wavelet transform $W_{\varphi,\psi}$ is available.

Using the explicit formula for the wavelet transform $W_{\varphi,\psi}$, we can compute the coordinates $(z)_B$ of every signal z in $L^2(\mathbb{Z}_N)$ by means of the formula

$$(z)_B = W_{\varphi,\psi} z.$$

As has been pointed out, this computation may entail up to N^2 complex multiplications on a computer, and hence is not a feasible formula for the computation of $(z)_B$ for z in $L^2(\mathbb{Z}_N)$. As a matter of fact, it is the formula (10.2) that people use to compute $(z)_B$ for every z in $L^2(\mathbb{Z}_N)$. A look at (10.2) reveals that for every z in $L^2(\mathbb{Z}_N)$, the components in $(z)_B$ are given by convolutions of z with the involutions of the mother and father wavelets, and hence $(z)_B$ can be computed rapidly by the FFT. In order to exploit the structure of (10.2), we need a definition.

Definition 10.4. Let $N = 2M$, where M is a positive integer. Then we define the linear operator $D : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_M)$ by

$$(Dz)(n) = z(2n), \quad n = 0, 1, \ldots, M-1,$$

for all z in $L^2(\mathbb{Z}_N)$.

What the linear operator D does to a signal z in $L^2(\mathbb{Z}_N)$ can best be seen by an example.

Example 10.5. Let z be the signal in $L^2(\mathbb{Z}_8)$ defined by

$$z = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \\ 6 \\ 5 \\ 8 \\ 7 \end{pmatrix}.$$

Then Dz is the signal in $L^2(\mathbb{Z}_4)$ given by

$$Dz = \begin{pmatrix} 2 \\ 3 \\ 6 \\ 8 \end{pmatrix}.$$

Note that D discards the entries of a signal evaluated at the odd integers. We call D the downsampling or the decimation operator. The downsampling operator is also denoted frequently in the engineering literature by $\downarrow 2$. Using the downsampling operator, the computation of $(z)_B$ for every z in $L^2(\mathbb{Z}_N)$ is given schematically by the following process.

$$z \mapsto \left\{ \begin{array}{c} z * \varphi^* \mapsto D(z * \varphi^*) \\ z * \psi^* \mapsto D(z * \psi^*) \end{array} \right\} \mapsto \left(\begin{array}{c} D(z * \varphi^*) \\ D(z * \psi^*) \end{array} \right) = (z)_B. \tag{10.3}$$

The prcocess described by (10.3) is an example of a filter bank in multi-rate signal analysis or subband coding, which is a prosperous field in electrical engineering.

The filter bank (10.3) is what engineers use to compute the wavelet transform $W_{\varphi,\psi}z$ of a signal z in $L^2(\mathbb{Z}_N)$. Mathematically, we have the following theorem.

Theorem 10.6. *Let φ and ψ be, respectively, the mother wavelet and the father wavelet of a time-frequency localized basis for $L^2(\mathbb{Z}_N)$. Then*

$$W_{\varphi,\psi}z = \left(\begin{array}{c} D(z * \varphi^*) \\ D(z * \psi^*) \end{array} \right), \quad z \in L^2(\mathbb{Z}_N).$$

We end this chapter with a filter bank to compute the inverse wavelet transform $W_{\varphi,\psi}^{-1}z$ of a signal z in $L^2(\mathbb{Z}_N)$. First, we need a definition.

Definition 10.7. Let $N = 2M$, where M is a positive integer. Then we define the linear operator $U : L^2(\mathbb{Z}_M) \to L^2(\mathbb{Z}_N)$ by $(Uz)(n) = z\left(\frac{n}{2}\right)$ if n is even, and is 0 if n is odd.

The linear operator U doubles the size of every signal z by inserting an entry 0 after every entry in z. We call U the upsampling operator and it is sometimes denoted by $\uparrow 2$.

Example 10.8. Let us look at the signal

$$w = \begin{pmatrix} 2 \\ 3 \\ 6 \\ 8 \end{pmatrix},$$

which is obtained by downsampling the signal

$$z = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \\ 6 \\ 5 \\ 8 \\ 7 \end{pmatrix}$$

in Example 10.5. Now, upsampling w gives us

$$UDz = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 6 \\ 0 \\ 8 \\ 0 \end{pmatrix}.$$

It is important to note that $UDz \neq z$. In fact,

$$UDz = \frac{1}{2}(z + z^+).$$

We can now give a filter bank consisting of two phases. The first phase is the analysis of a signal z in $L^2(\mathbb{Z}_N)$ using the wavelet transform $W_{\varphi,\psi}$ and the second phase is the reconstruction of the signal from the wavelet transform using the inverse wavelet transform $W_{\varphi,\psi}^{-1}$. The entire computation is captured by the following process.

Theorem 10.9. *Let $N = 2M$, where M is a positive integer, and let z be a signal in $L^2(\mathbb{Z}_N)$. Then*

$$z \mapsto \left\{ \begin{array}{c} z * \varphi^* \mapsto D(z * \varphi^*) \mapsto UD(z * \varphi^*) \mapsto \varphi * UD(z * \varphi^*) \\ z * \psi^* \mapsto D(z * \psi^*) \mapsto UD(z * \psi^*) \mapsto \psi * UD(z * \psi^*) \end{array} \right\} + \mapsto z,$$

where the final step in the filter bank is given by

$$z = (\varphi * UD(z * \varphi^*)) + (\psi * UD(z * \psi^*)).$$

Proof. Let

$$w = \begin{pmatrix} w(0) \\ w(1) \\ \vdots \\ w(N-1) \end{pmatrix}.$$

Then, using the definition of the downsampling operator,

$$Dw = \begin{pmatrix} w(0) \\ w(2) \\ \vdots \\ w(2M-2) \end{pmatrix}.$$

Now, using the definition of the upsampling operator and Remark 9.5,

$$UDw = \begin{pmatrix} w(0) \\ 0 \\ w(2) \\ 0 \\ \vdots \\ w(2M-2) \\ 0 \end{pmatrix} = \frac{1}{2}(w + w^+). \tag{10.4}$$

Thus, by (10.4),

$$(UD(z * \varphi^*)) = \frac{1}{2}((z * \varphi^*) + (z * \varphi^*)^+). \tag{10.5}$$

If we take the finite Fourier transform on both sides of (10.5), then, by Propositions 4.7, 8.4 and 9.4, we get

$$\begin{aligned} &(\varphi * UD(z * \varphi^*))^\wedge(m) \\ &= \hat{\varphi}(m)(UD(z * \varphi^*))^\wedge(m) \\ &= \hat{\varphi}(m)\frac{1}{2}(\hat{z}(m)\overline{\hat{\varphi}(m)} + \hat{z}(m+M)\overline{\hat{\varphi}(m+M)}) \end{aligned} \tag{10.6}$$

for all m in $\mathbb{Z}$. Similarly,

$$\begin{aligned} &(\psi * UD(z * \psi^*))^\wedge(m) \\ &= \hat{\psi}(m)\frac{1}{2}(\hat{z}(m)\overline{\hat{\psi}(m)} + \hat{z}(m+M)\overline{\hat{\psi}(m+M)}) \end{aligned} \tag{10.7}$$

for all m in $\mathbb{Z}$. Adding (10.6) and (10.7), we get

$$\begin{aligned} &\{(\varphi * UD(z * \varphi^*)) + (\psi * UD(z * \psi^*))\}^\wedge(m) \\ &= \frac{1}{2}\hat{z}(m)\{|\hat{\varphi}(m)|^2 + |\hat{\psi}(m)|^2\} \\ &\quad + \frac{1}{2}\hat{z}(m+M)\{\hat{\varphi}(m)\overline{\hat{\varphi}(m+M)} + \hat{\psi}(m)\overline{\hat{\psi}(m+M)}\} \end{aligned} \tag{10.8}$$

for all m in $\mathbb{Z}$. Since φ is the mother wavelet and ψ is the father wavelet of the time-frequency localized basis B, it follows that the system matrix $A_{\varphi,\psi}$ is unitary. So,

$$|\hat{\varphi}(m)|^2 + |\hat{\psi}(m)|^2 = 2 \tag{10.9}$$

and

$$\hat{\varphi}(m)\overline{\hat{\varphi}(m+M)} + \hat{\psi}(m)\overline{\hat{\psi}(m+M)} = 0 \tag{10.10}$$

for all m in $\mathbb{Z}$. Hence, by (10.8), (10.9) and (10.10),

$$\{(\varphi * UD(z * \varphi^*)) + (\psi * UD(z * \psi^*))\}^\wedge(m) = \hat{z}(m)$$

for all m in $\mathbb{Z}$. Therefore, using the inverse finite Fourier transform, we conclude that

$$(\varphi * UD(z * \varphi^*)) + (\psi * UD(z * \psi^*)) = z$$

and the proof is complete. □

From Theorem 10.9, we see that the inverse wavelet transform $W_{\varphi,\psi}^{-1}$ can in fact be computed using the filter bank

$$\left\{ \begin{array}{l} UD(z * \varphi^*) \mapsto \varphi * UD(z * \varphi^*) \\ UD(z * \psi^*) \mapsto \psi * UD(z * \psi^*) \end{array} \right\} + \mapsto z$$

for every z in $L^2(\mathbb{Z}_N)$.

Exercises

1. Prove the downsampling operator D and the upsampling operator U on $L^2(\mathbb{Z}_N)$, where $N = 2M$ and M is a positive integer, are adjoint to each other in the sense that
$$(Dz, w) = (z, Uw)$$
for all z in $L^2(\mathbb{Z}_N)$ and all w in $L^2(\mathbb{Z}_M)$.

2. Let φ and ψ be, respectively, the mother wavelet and the father wavelet of the Haar basis for $L^2(\mathbb{Z}_N)$ in Exercise 2 of the preceding chapter. Compute the wavelet transform $W_{\varphi,\psi}z$ and the inverse wavelet transform $W_{\varphi,\psi}^{-1}z$ of z for all z in $L^2(\mathbb{Z}_N)$.

Chapter 11

Haar Wavelets

Let z be a signal in $L^2(\mathbb{Z}_N)$, where we now assume that $N = 2^l$ for some positive integer l. We let $N = 2M$, where M is a positive integer. As usual, we write

$$z = \begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix}.$$

Let a be defined by

$$a = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{M-1} \end{pmatrix},$$

where

$$a_n = \frac{z(2n) + z(2n+1)}{\sqrt{2}}, \quad n = 0, 1, \ldots, M-1,$$

and let d be defined by

$$d = \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{M-1} \end{pmatrix},$$

where

$$d_n = \frac{z(2n) - z(2n+1)}{\sqrt{2}}, \quad n = 0, 1, \ldots, M-1.$$

We call a the trend and d the fluctuation of the signal z.

Definition 11.1. Let $W : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ be the linear operator defined by

$$Wz = \begin{pmatrix} a \\ d \end{pmatrix}, \quad z \in L^2(\mathbb{Z}_N),$$

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DOI 10.1007/978-3-0348-0116-4_11, © Springer Basel AG 2011

where a is the trend and d is the fluctuation of the signal z.

We call W the Haar transform. More precisely, it should be called the first-level Haar transform. It is easy to see that the Haar transform W has the inverse W^{-1} given by

$$W^{-1}\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{M-1} \\ d_0 \\ d_1 \\ \vdots \\ d_{M-1} \end{pmatrix} = \begin{pmatrix} (a_0+d_0)/\sqrt{2} \\ (a_0-d_0)/\sqrt{2} \\ (a_1+d_1)/\sqrt{2} \\ (a_1-d_1)/\sqrt{2} \\ (a_2+d_2)/\sqrt{2} \\ \vdots \\ (a_{M-1}+d_{M-1})/\sqrt{2} \\ (a_{M-1}-d_{M-1})/\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{M-1} \\ d_0 \\ d_1 \\ \vdots \\ d_{M-1} \end{pmatrix} \in L^2(\mathbb{Z}_N).$$

In order to understand the Haar transform, which transforms the input signal z into its trend a and fluctuation d, let us study the small fluctuation property and the similar trend property of a signal in $L^2(\mathbb{Z}_N)$.

Proposition 11.2 (Small Fluctuation Property). *The fluctuation signal d is very small in the sense that*

$$d \doteqdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

To see why the small fluctuation property is valid, let us recall that the components of z are samples of a continuous analog signal g with a very short, but the same, time interval between the consecutive samples. In other words,

$$z(n) = g(t_n), \quad n \in \mathbb{Z}_N,$$

and

$$t_{n+1} - t_n = h, \quad n \in \mathbb{Z}_N,$$

where h is some very small positive number, which people call the step size. So, using the continuity of the function g, we get

$$d_n = \frac{z(2n) - z(2n+1)}{\sqrt{2}} = \frac{g(t_{2n}) - g(t_{2n+1})}{\sqrt{2}} \doteqdot 0$$

for $n = 0, 1, \ldots, M-1$.

A similar continuity argument can be used to explain the trend property of a signal.

Proposition 11.3 (Similar Trend Property). *The trend signal behaves like the original signal.*

Indeed, using the continuity of the function g, we get

$$a_n = \frac{z(2n) + z(2n+1)}{\sqrt{2}} = \frac{g(t_{2n}) + g(t_{2n+1})}{\sqrt{2}} \doteqdot \sqrt{2}g(t_{2n+1})$$

for $n = 0, 1, \ldots, M-1$. Thus, the trend a is the signal obtained by sampling the values of g at the equally spaced instants $t_1, t_3, \ldots, t_{N-1}$.

Remark 11.4. Recall that the Haar transform splits a given signal z in $L^2(\mathbb{Z}_N)$ into its trend a and fluctuation d. The fluctuation d is small in the sense of the small fluctuation property and the trend a is like the original z in the sense of the similar trend property. Thus, we can transmit the trend a instead of the original z without affecting much the originality of the signal z. The advantage accrued is that only half the number of bits of the original z need to be transmitted. This process of transmitting the trend instead of the original is known as signal compression.

The signal compression described in Remark 11.4 or the application of the Haar transform can certainly be iterated on the trends of the original signal. Thus, we have

$$L^2(\mathbb{Z}_N) \ni z \mapsto \begin{pmatrix} a^{(1)} \\ d^{(1)} \end{pmatrix} \mapsto \begin{pmatrix} a^{(2)} \\ d^{(2)} \\ d^{(1)} \end{pmatrix} \mapsto \begin{pmatrix} a^{(3)} \\ d^{(3)} \\ d^{(2)} \\ d^{(1)} \end{pmatrix} \mapsto \cdots .$$

To compute the second-level trend $a^{(2)}$ and the second-level fluctuation $d^{(2)}$ of a signal z in $L^2(\mathbb{Z}_N)$, we note that

$$a^{(2)} = \begin{pmatrix} \frac{a_0^{(1)}+a_1^{(1)}}{\sqrt{2}} \\ \frac{a_2^{(1)}+a_3^{(1)}}{\sqrt{2}} \\ \vdots \\ \frac{a_{M-2}^{(1)}+a_{M-1}^{(1)}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{z(0)+z(1)+z(2)+z(3)}{2} \\ \frac{z(4)+z(5)+z(6)+z(7)}{2} \\ \vdots \\ \frac{z(N-4)+z(N-3)+z(N-2)+z(N-1)}{2} \end{pmatrix}$$

and

$$d^{(2)} = \begin{pmatrix} \frac{a_0^{(1)}-a_1^{(1)}}{\sqrt{2}} \\ \frac{a_2^{(1)}-a_3^{(1)}}{\sqrt{2}} \\ \vdots \\ \frac{a_{M-2}^{(1)}-a_{M-1}^{(1)}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{z(0)+z(1)-z(2)-z(3)}{2} \\ \frac{z(4)+z(5)-z(6)-z(7)}{2} \\ \vdots \\ \frac{z(N-4)+z(N-3)-z(N-2)-z(N-1)}{2} \end{pmatrix}.$$

In order to get more transparent formulas for the trends $a^{(1)}, a^{(2)}$ and the fluctuations $d^{(1)}, d^{(2)}$ of a signal, we introduce N signals

$$V_0^{(1)}, V_1^{(1)}, \ldots, V_{M-1}^{(1)}, W_0^{(1)}, W_1^{(1)}, \ldots, W_{M-1}^{(1)},$$

at the first level and $M = \frac{N}{2}$ signals

$$V_0^{(2)}, V_1^{(2)}, \ldots, V_{\frac{M}{2}-1}^{(2)}, W_0^{(2)}, W_1^{(2)}, \ldots, W_{\frac{M}{2}-1}^{(2)},$$

at the second level.

Definition 11.5. Let

$$V_0^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad V_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, V_{M-1}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

We call $V_0^{(1)}, V_1^{(1)}, \ldots, V_{M-1}^{(1)}$ the first-level Haar scaling signals and we note that

$$\{V_0^{(1)}, V_1^{(1)}, \ldots, V_{M-1}^{(1)}\} = \{R_{2k} V_0^{(1)}\}_{k=0}^{M-1}.$$

Definition 11.6. Let

$$W_0^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad W_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, W_{M-1}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

We call $W_0^{(1)}, W_1^{(1)}, \ldots, W_{M-1}^{(1)}$ the first-level Haar wavelets and we note that

$$\{W_0^{(1)}, W_1^{(1)}, \ldots, W_{M-1}^{(1)}\} = \{R_{2k} W_0^{(1)}\}_{k=0}^{M-1}.$$

In fact, by Theorem 9.8, $\{R_{2k} V_0^{(1)}\}_{k=0}^{M-1} \cup \{R_{2k} W_0^{(1)}\}_{k=0}^{M-1}$ can be shown to be a time-frequency localized basis for $L^2(\mathbb{Z}_N)$, where $V_0^{(1)}$ is the father wavelet and $W_0^{(1)}$ is the mother wavelet. This is left as Exercise 1. A similar structure exists for the Haar scaling signals and the Haar wavelets at the second level, which we introduce in the following two definitions.

Definition 11.7. Let

$$V_0^{(2)} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ V_1^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ \dots, \ V_{\frac{M}{2}-1}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

We call $V_0^{(2)}, V_1^{(2)}, \dots, V_{\frac{M}{2}-1}^{(2)}$ the second-level Haar scaling signals.

Definition 11.8. Let

$$W_0^{(2)} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ W_1^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ \dots, \ W_{\frac{M}{2}-1}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}.$$

We call $W_0^{(2)}, W_1^{(2)}, \dots, W_{\frac{M}{2}-1}^{(2)}$ the second-level Haar wavelets.

Remark 11.9. The first-level and the second-level Haar scaling signals and Haar wavelets are generated from the standard basis. To see how, we note that for $\alpha_1 = \alpha_2 = \frac{1}{\sqrt{2}}$,

$$V_0^{(1)} = \alpha_1 \epsilon_0 + \alpha_2 \epsilon_1,$$

$$V_1^{(1)} = \alpha_1 \epsilon_2 + \alpha_2 \epsilon_3,$$

$$\cdots$$

and hence

$$V_m^{(1)} = \alpha_1 \epsilon_{2m} + \alpha_2 \epsilon_{2m+1}, \quad m = 0, 1, \dots, M-1.$$

Also,

$$V_0^{(2)} = \alpha_1 V_0^{(1)} + \alpha_2 V_1^{(1)},$$

$$V_1^{(2)} = \alpha_1 V_2^{(1)} + \alpha_2 V_3^{(1)},$$

$$\cdots$$

and hence

$$V_m^{(2)} = \alpha_1 V_{2m}^{(1)} + \alpha_2 V_{2m+1}^{(1)}$$

for $m = 0, 1, \ldots, \frac{M}{2} - 1$. Similarly, if we let $\beta_1 = \frac{1}{\sqrt{2}}$ and $\beta_2 = -\frac{1}{\sqrt{2}}$, then

$$W_m^{(1)} = \beta_1 \epsilon_{2m} + \beta_2 \epsilon_{2m+1}$$

for $m = 0, 1, \ldots, M - 1$, and

$$W_m^{(2)} = \beta_1 V_{2m}^{(1)} + \beta_2 V_{2m+1}^{(1)}$$

for $m = 0, 1, \ldots, \frac{M}{2} - 1$. We call α_1, α_2 the scaling numbers and β_1, β_2 the wavelet numbers respectively.

The first-level Haar scaling signals and the first-level Haar wavelets can be used to represent, respectively, the first-level trend $a^{(1)}$ and the first-level fluctuation $d^{(1)}$ of a signal z.

Proposition 11.10. *Let z be a signal in $L^2(\mathbb{Z}_N)$. Then*

$$a^{(1)} = \begin{pmatrix} (z, V_0^{(1)}) \\ (z, V_1^{(1)}) \\ \vdots \\ (z, V_{\frac{N}{2}-1}^{(1)}) \end{pmatrix}, \quad d^{(1)} = \begin{pmatrix} (z, W_0^{(1)}) \\ (z, W_1^{(1)}) \\ \vdots \\ (z, W_{\frac{N}{2}-1}^{(1)}) \end{pmatrix}.$$

Applying translations, (8.3) and the downsampling operator to Proposition 11.10, we leave it as Exercise 4 to prove that for every signal z in $L^2(\mathbb{Z}_N)$,

$$a^{(1)} = D(z * (V_0^{(1)})^*)$$

and

$$d^{(1)} = D(z * (W_0^{(1)})^*).$$

However, we prefer to look at the components of $a^{(1)}$ and $d^{(1)}$ as inner products in this chapter.

The second-level Haar scaling signals and the second-level Haar wavelets can be used to represent, respectively, the second-level trend $a^{(2)}$ and the second-level fluctuation $d^{(2)}$ of a signal z.

Proposition 11.11. *Let z be a signal in $L^2(\mathbb{Z}_N)$. Then*

$$a^{(2)} = \begin{pmatrix} (z, V_0^{(2)}) \\ (z, V_1^{(2)}) \\ \vdots \\ (z, V_{\frac{M}{2}-1}^{(2)}) \end{pmatrix}, \; d^{(2)} = \begin{pmatrix} (z, W_0^{(2)}) \\ (z, W_1^{(2)}) \\ \vdots \\ (z, W_{\frac{M}{2}-1}^{(2)}) \end{pmatrix}.$$

The proofs of Propositions 11.10 and 11.11 follow immediately from the definitions for trends, fluctuations, Haar scaling signals and Haar wavelets given at the first and second levels.

We can now give a discussion of the very important multiresolution analysis, which we abbreviate as MRA. Let us recall that the formula for the inverse Haar transform gives for every signal z in $L^2(\mathbb{Z}_N)$,

$$\begin{aligned} z &= \begin{pmatrix} \frac{a_0+d_0}{\sqrt{2}} \\ \frac{a_0-d_0}{\sqrt{2}} \\ \vdots \\ \frac{a_{M-1}+d_{M-1}}{\sqrt{2}} \\ \frac{a_{M-1}-d_{M-1}}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{a_0}{\sqrt{2}} \\ \frac{a_0}{\sqrt{2}} \\ \vdots \\ \frac{a_{M-1}}{\sqrt{2}} \\ \frac{a_{M-1}}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \frac{d_0}{\sqrt{2}} \\ -\frac{d_0}{\sqrt{2}} \\ \vdots \\ \frac{d_{M-1}}{\sqrt{2}} \\ -\frac{d_{M-1}}{\sqrt{2}} \end{pmatrix} \\ &= \sum_{n=0}^{M-1} a_n V_n^{(1)} + \sum_{n=0}^{M-1} d_n W_n^{(1)} \\ &= \sum_{n=0}^{M-1} (z, V_n^{(1)}) V_n^{(1)} + \sum_{n=0}^{M-1} (z, W_n^{(1)}) W_n^{(1)}. \end{aligned}$$

If for every z in $L^2(\mathbb{Z}_N)$, we define $A^{(1)}$ and $D^{(1)}$ by

$$A^{(1)} = \sum_{n=0}^{M-1} (z, V_n^{(1)}) V_n^{(1)}$$

and

$$D^{(1)} = \sum_{n=1}^{M} (z, W_n^{(1)}) W_n^{(1)},$$

then

$$z = A^{(1)} + D^{(1)}.$$

$A^{(1)}$ is called the first-level average signal and $D^{(1)}$ is called the first-level detail signal of the given signal z. Iterations give

$$\begin{aligned} z &= A^{(1)} + D^{(1)} = A^{(2)} + D^{(2)} + D^{(1)} \\ &= A^{(3)} + D^{(3)} + D^{(2)} + D^{(1)} \\ &= \cdots = A^{(l)} + \sum_{j=1}^{l} D^{(j)}, \quad z \in L^2(\mathbb{Z}_N), \end{aligned}$$

where $A^{(j)}$ and $D^{(j)}$ are, respectively, the j^{th}-level average signal and the j^{th}-level detail signal of z. To get the formula for $A^{(2)}$ and the formula for $D^{(2)}$, we note that if we apply the inverse of the first-level Haar transform to $A^{(1)}$, then we get

$$\begin{aligned} A^{(1)} &= \begin{pmatrix} \frac{a_0^{(1)}}{\sqrt{2}} \\ \frac{a_0^{(1)}}{\sqrt{2}} \\ \frac{a_1^{(1)}}{\sqrt{2}} \\ \frac{a_1^{(1)}}{\sqrt{2}} \\ \vdots \\ \frac{a_{M-1}^{(1)}}{\sqrt{2}} \\ \frac{a_{M-1}^{(1)}}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{a_0^{(2)}+d_0^{(2)}}{2} \\ \frac{a_0^{(2)}+d_0^{(2)}}{2} \\ \frac{a_0^{(2)}-d_0^{(2)}}{2} \\ \frac{a_0^{(2)}-d_0^{(2)}}{2} \\ \vdots \\ \frac{a_{\frac{M}{2}-1}^{(2)}-d_{\frac{M}{2}-1}^{(2)}}{2} \\ \frac{a_{\frac{M}{2}-1}^{(2)}-d_{\frac{M}{2}-1}^{(2)}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{a_0^{(2)}}{2} \\ \frac{a_0^{(2)}}{2} \\ \frac{a_0^{(2)}}{2} \\ \frac{a_0^{(2)}}{2} \\ \vdots \\ \frac{a_{\frac{M}{2}-1}^{(2)}}{2} \\ \frac{a_{\frac{M}{2}-1}^{(2)}}{2} \end{pmatrix} + \begin{pmatrix} \frac{d_0^{(2)}}{2} \\ \frac{d_0^{(2)}}{2} \\ -\frac{d_0^{(2)}}{2} \\ -\frac{d_0^{(2)}}{2} \\ \vdots \\ -\frac{d_{\frac{M}{2}-1}^{(2)}}{2} \\ -\frac{d_{\frac{M}{2}-1}^{(2)}}{2} \end{pmatrix} \\ &= \sum_{n=0}^{\frac{M}{2}-1} a_n^{(2)} V_n^{(2)} + \sum_{n=0}^{\frac{M}{2}-1} d_n^{(2)} W_n^{(2)} \\ &= \sum_{n=0}^{\frac{M}{2}-1} (z, V_n^{(2)}) V_n^{(2)} + \sum_{n=0}^{\frac{M}{2}-1} (z, W_n^{(2)}) W_n^{(2)} \end{aligned}$$

for all z in $L^2(\mathbb{Z}_N)$. Thus,

$$A^{(2)} = \sum_{n=0}^{\frac{M}{2}-1} (z, V_n^{(2)}) V_n^{(2)}$$

and

$$D^{(2)} = \sum_{n=0}^{\frac{M}{2}-1} (z, W_n^{(2)}) W_n^{(2)}.$$

Remark 11.12. The MRA gives us

$$\begin{aligned} z &= A^{(1)} + D^{(1)} = A^{(2)} + D^{(2)} + D^{(1)} = \cdots \\ &= A^{(l-1)} + \sum_{j=1}^{l-1} D^{(j)} = A^{(l)} + \sum_{j=1}^{l} D^{(j)}, z \in L^2(\mathbb{Z}_N). \end{aligned}$$

Thus,

$$\{W_0^{(1)}, \ldots, W_{M-1}^{(1)}\}, \{W_0^{(2)}, \ldots, W_{\frac{M}{2}-1}^{(2)}\}, \ldots, \{W_0^{(l-1)}, W_1^{(l-1)}\}, \{W_0^{(l)}\}, \{V_0^{(l)}\}$$

is an orthonormal basis for $L^2(\mathbb{Z}_N)$. See Exercise 6. We call it the multiresolution basis for $L^2(\mathbb{Z}_N)$. For $j = 1, 2, \ldots, l$, the average signal $A^{(j)}$ and the detail signal $D^{(j)}$ at level j are, respectively, given by

$$A^{(j)} = \sum_{n=0}^{\frac{N}{2^j}-1} (z, V_n^{(j)}) V_n^{(j)}$$

and

$$D^{(j)} = \sum_{n=0}^{\frac{N}{2^j}-1} (z, W_n^{(j)}) W_n^{(j)}.$$

At level j, $j = 1, 2, \ldots, l$, the signal z is the sum of the average signal at level j and the detail signals from the first level to the j^{th} level. For the sake of illustration, let us examine the first two levels in some detail. For $j = 1$, the signal z is given by

$$z = A^{(1)} + D^{(1)}.$$

The first-level average signal $A^{(1)}$ is a linear combination of the Haar scaling signals $V_0^{(1)}, V_1^{(1)}, \ldots, V_{M-1}^{(1)}$. Each of these Haar scaling signals is a short-lived signal moving across the time axis in steps of two time units and lives for only two time units. The Haar scaling signals measure short-lived trends in the signal z. The first-level detail signal $D^{(1)}$ is a linear combimation of the Haar wavelets $W_0^{(1)}, W_1^{(1)}, \ldots, W_{M-1}^{(1)}$. Each of these Haar wavelets is also short-lived, moves across the time axis in steps of two time units and lives for two time units. These

Haar wavelets can detect short-lived fluctuations in the signal z. For $j = 2$, the signal z is given by

$$z = A^{(2)} + D^{(2)} + D^{(1)}.$$

The second-level average signal $A^{(2)}$ is a linear combination of the second-level Haar scaling signals $V_0^{(2)}, V_1^{(2)}, \ldots, V_{\frac{M}{2}-1}^{(2)}$. Each of these moves across the time axis in steps of four time units and lives for four time units. Similarly, the second-level detail signal $D^{(2)}$ is a linear combination of the second-level Haar wavelets $W_0^{(2)}, W_1^{(2)}, \ldots, W_{\frac{M}{2}-1}^{(2)}$. Each of them moves across the time axis in steps of four time units and lives for four time units. The second-level Haar scaling signals measure short-lived trends and the second-level Haar wavelets detect short-lived fluctuations in the signal z. The scale on which these transients can be measured and detected at the second level is twice as long as the scale at the first level.

Another way to see that the detail signals of a signal can be ignored is in terms of the energy of a signal. The energy $E(z)$ of a signal z in $L^2(\mathbb{Z}_N)$ is simply given by

$$E(z) = \|z\|^2.$$

We need the following lemma in order to explain the concentration of the energy of a signal.

Lemma 11.13. *Let z and w be orthogonal signals in $L^2(\mathbb{Z}_N)$. Then*

$$\|z + w\|^2 = \|z\|^2 + \|w\|^2.$$

Proof. Using the orthogonality of z and w, we get

$$\|z+w\|^2 = (z+w, z+w) = (z,z) + (z,w) + (w,z) + (w,w) = \|z\|^2 + \|w\|^2. \qquad \square$$

Proposition 11.14. *Let z be a signal in $L^2(\mathbb{Z}_N)$, $N = 2^l$, where l is a positive integer. Then*

$$E(z) \doteqdot E(A^{(j)}), \quad j = 1, 2, \ldots, l.$$

The proposition tells us that the energy of a signal is concentrated in the average signals. To see why, let us use the small fluctuation property and Lemma 11.13 to get

$$\begin{aligned} E(D^{(k)}) &= E\left(\sum_{n=0}^{\frac{N}{2^k}-1} (z, W_n^{(k)}) W_n^{(k)}\right) \\ &= \left\| \sum_{n=0}^{\frac{N}{2^k}-1} (z, W_n^{(k)}) W_n^{(k)} \right\|^2 \\ &= \sum_{n=0}^{\frac{N}{2^k}-1} \left| (z, W_n^{(k)}) \right|^2 \end{aligned}$$

$$= \sum_{n=0}^{\frac{N}{2^k}-1} \{d_n^{(k)}\}^2 \doteqdot 0$$

for $k = 1, 2, \ldots, l$. Thus, by Lemma 11.13 again,

$$\begin{aligned} E(z) &= E\left(A^{(j)} + \sum_{k=0}^{j} D^{(k)}\right) \\ &= \left\| A^{(j)} + \sum_{k=0}^{j} D^{(k)} \right\|^2 \\ &= \|A^{(j)}\|^2 + \sum_{k=0}^{j} \|D^{(k)}\|^2 \\ &\doteqdot \|A^{(j)}\|^2 = E\left(A^{(j)}\right) \end{aligned}$$

for $j = 1, 2, \ldots, l$.

The concentration of energy of a signal in the average signals can best be seen by means of the following example.

Example 11.15. Let z be the signal in $L^2(\mathbb{Z}_8)$ defined by

$$z = \begin{pmatrix} 4 \\ 6 \\ 10 \\ 12 \\ 8 \\ 6 \\ 5 \\ 5 \end{pmatrix}.$$

Then the first-level trend $a^{(1)}$ and the first-level fluctuation $d^{(1)}$ are given by

$$a^{(1)} = \begin{pmatrix} 5\sqrt{2} \\ 11\sqrt{2} \\ 7\sqrt{2} \\ 5\sqrt{2} \end{pmatrix}, \; d^{(1)} = \begin{pmatrix} -\sqrt{2} \\ -\sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}.$$

The second-level trend $a^{(2)}$ and the second-level fluctuation $d^{(2)}$ are given by

$$a^{(2)} = \begin{pmatrix} 16 \\ 12 \end{pmatrix}, \; d^{(2)} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}.$$

Now,

$$A^{(1)} = \sum_{j=0}^{3} a_j^{(1)} V_j^{(1)}$$

and hence

$$E(A^{(1)}) = \|A^{(1)}\|^2 = \|a^{(1)}\|^2 = 440.$$

Since

$$E(z) = \|z\|^2 = 446,$$

it follows that 98.7% of the energy of the signal z is in the first-level average. Now, note that

$$A^{(2)} = \sum_{j=0}^{1} a_j^{(2)} V_j^{(2)}$$

and hence

$$E(A^{(2)}) = \|A^{(2)}\|^2 = \|a^{(2)}\|^2 = 400,$$

which is 89.7% of the energy of z.

Exercises

1. Prove that $\{R_{2k}V_0^{(1)}\}_{k=0}^{M-1} \cup \{R_{2k}W_0^{(1)}\}_{k=0}^{M-1}$ is a time-frequency localized basis for $L^2(\mathbb{Z}_N)$.
2. Prove Proposition 11.10.
3. Prove Proposition 11.11.
4. Prove that
$$a^{(1)} = D(z * (V_0^{(1)})^*)$$
and
$$d^{(1)} = D(z * (W_0^{(1)})^*).$$
5. Use the Haar wavelets to write down explicitly the full MRA
$$A^{(l)} + \sum_{j=1}^{l} D^{(j)}$$
of the signal z in $L^2(\mathbb{Z}_8)$ given by
$$z = \begin{pmatrix} 4 \\ 6 \\ 10 \\ 12 \\ 8 \\ 6 \\ 5 \\ 5 \end{pmatrix}.$$
6. Prove that
$$\{W_0^1, \ldots, W_{M-1}^{(1)}\}, \{W_0^{(2)}, \ldots W_{\frac{M}{2}-1}^{(2)}\}, \ldots, \{W_0^{l-1}, W_1^{l-1}\}, \{W_0^{(l)}\}, \{V_0^{(l)}\}$$
is an orthonormal basis for $L^2(\mathbb{Z}_N)$.

Chapter 12

Daubechies Wavelets

We first give a more detailed discussion of the small fluctuation property for the first-level Haar wavelets. Once this is understood, we can ask whether or not the same property can be upheld for other wavelets. It is in this context that we introduce the Daubechies wavelets in this chapter.

Let z be a signal in $L^2(\mathbb{Z}_N)$. We again suppose that $N = 2M$, where M is a positive integer. Let us write

$$z = \begin{pmatrix} z(0) \\ z(1) \\ \vdots \\ z(N-1) \end{pmatrix}.$$

Suppose that z is obtained by sampling an analog signal g and we assume that g has a continuous second derivative. So,

$$z(n) = g(t_n), \quad n = 0, 1, \ldots, N-1.$$

We also suppose that the sample values are obtained in such a way that the step size h at time t_n given by

$$h = t_{n+1} - t_n$$

is the same for $n = 0, 1, \ldots, N-1$. Let us now compute the fluctuation $(z, W_0^{(1)})$, where $W_0^{(1)}$ is the first Haar wavelet at the first level, i.e.,

$$W_0^{(1)} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

M.W. Wong, *Discrete Fourier Analysis*, Pseudo-Differential Operators 5,
DOI 10.1007/978-3-0348-0116-4_12, © Springer Basel AG 2011

Then, using Taylor's theorem,

$$\begin{aligned}(z, W_0^{(1)}) &= z(0)\beta_1 + z(1)\beta_2 \\ &= z(0)\beta_1 + g(t_1)\beta_2 \\ &= z(0)\beta_1 + g(t_0 + h)\beta_2 \\ &= z(0)\beta_1 + \{g(t_0) + O(h)\}\beta_2 \\ &= z(0)\beta_1 + \{z(0) + O(h)\}\beta_2 \\ &= z(0)(\beta_1 + \beta_2) + O(h)\end{aligned}$$

as $h \to 0$. It is physically intuitive that if the step size goes to zero, then the fluctuation of the signal also goes to zero. Thus,

$$(z, W_0^{(1)}) = O(h)$$

as $h \to 0$, or equivalently, $\beta_1 + \beta_2 = 0$. This is indeed the case for the Haar wavelet numbers

$$\beta_1 = \frac{1}{\sqrt{2}}, \ \beta_2 = -\frac{1}{\sqrt{2}}.$$

Let us now go one step further and look for new wavelets by determining wavelet numbers β_1, β_2, β_3 and β_4. What does the small fluctuation property impose on these four numbers? To answer this question, let us note that the first wavelet at the first level is given by

$$W_0^{(1)} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So, by Taylor's theorem, the fluctuation $(z, W_0^{(1)})$ of the signal z is given by

$$\begin{aligned}(z, W_0^{(1)}) &= z(0)\beta_1 + z(1)\beta_2 + z(2)\beta_3 + z(3)\beta_4 \\ &= z(0)\beta_1 + g(t_1)\beta_2 + g(t_2)\beta_3 + g(t_3)\beta_4 \\ &= z(0)\beta_1 + g(t_0 + h)\beta_2 + g(t_0 + 2h)\beta_3 + g(t_0 + 3h)\beta_4 \\ &= z(0)\beta_1 + \{g(t_0) + g'(t_0)h + O(h^2)\}\beta_2 \\ &\quad + \{g(t_0) + g'(t_0)2h + O(h^2)\}\beta_3 \\ &\quad + \{g(t_0) + g'(t_0)3h + O(h^2)\}\beta_4 \\ &= z(0)(\beta_1 + \beta_2 + \beta_3 + \beta_4) \\ &\quad + g'(t_0)h(\beta_2 + 2\beta_3 + 3\beta_4) + O(h^2) \qquad (12.1)\end{aligned}$$

as $h \to 0$. Again, we invoke the physical intuition that if the step size goes to 0, then the fluctuation of a signal goes to 0. In addition to this intuition, we expect from (12.1) that the more refined wavelet $W_0^{(1)}$ should be able to detect the $O(h^2)$-fluctuation in this case. Thus, it is reasonable to expect that

$$(z, W_0^{(1)}) = O(h^2)$$

as $h \to 0$. So, by (12.1), we get

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0 \tag{12.2}$$

and

$$\beta_2 + 2\beta_3 + 3\beta_4 = 0. \tag{12.3}$$

The equations (12.2) and (12.3) on β_1, β_2, β_3 and β_4 are two of the most important ingredients in the construction of the Daubechies wavelets to be studied next.

We begin with introducing another orthonormal basis for $L^2(\mathbb{Z}_N)$ consisting of the first-level scaling signals $V_0^{(1)}, V_1^{(1)}, \ldots, V_{\frac{N}{2}-1}^{(1)}$ and the first-level wavelets $W_0^{(1)}, W_1^{(1)}, \ldots, W_{\frac{N}{2}-1}^{(1)}$ due to Daubechies. Motivated by the small fluctuation property, we suppose that

$$W_0^{(1)} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \; W_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \; \ldots, \; W_{\frac{N}{2}-1}^{(1)} = \begin{pmatrix} \beta_3 \\ \beta_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \beta_1 \\ \beta_2 \end{pmatrix},$$

where the wavelet numbers β_1, β_2, β_3 and β_4 are configured in such a way that in $W_{\frac{N}{2}-1}^{(1)}$, β_3 and β_4 are rotated to the top. Furthermore, the wavelet numbers β_1, β_2, β_3 and β_4 satisfy the system of nonlinear equations given by

$$\beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2 = 1, \tag{12.4}$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0, \tag{12.5}$$

$$\beta_2 + 2\beta_3 + 3\beta_4 = 0, \tag{12.6}$$

$$\beta_1\beta_3 + \beta_2\beta_4 = 0. \tag{12.7}$$

It should be noted that (12.4) and (12.7) follow from the orthonormality of the wavelets $\{W_0^{(1)}, W_1^{(1)}, \ldots, W_{\frac{N}{2}-1}^{(1)}\}$, and (12.5) and (12.6) are just, respectively, (12.2) and (12.6) in the last chapter. Solving the nonlinear system for β_1, β_2, β_3 and β_4 gives us

$$\beta_1 = \frac{1-\sqrt{3}}{4\sqrt{2}},\ \beta_2 = \frac{\sqrt{3}-3}{4\sqrt{2}},\ \beta_3 = \frac{3+\sqrt{3}}{4\sqrt{2}},\ \beta_4 = \frac{-1-\sqrt{3}}{4\sqrt{2}},$$

or

$$\beta_1 = -\frac{1-\sqrt{3}}{4\sqrt{2}},\ \beta_2 = -\frac{\sqrt{3}-3}{4\sqrt{2}},\ \beta_3 = -\frac{3+\sqrt{3}}{4\sqrt{2}},\ \beta_4 = -\frac{-1-\sqrt{3}}{4\sqrt{2}}.$$

We take the first set of values for β_1, β_2, β_3 and β_4 as the wavelet numbers.

In order to determine the first-level scaling signals, we use the following fact.

Lemma 12.1. *Let $\psi \in L^2(\mathbb{Z}_N)$, $N = 2M$, be such that $\{R_{2k}\psi\}_{k=0}^{M-1}$ is an orthonormal set with M distinct signals in $L^2(\mathbb{Z}_N)$. If we define $\varphi \in L^2(\mathbb{Z}_N)$ by*

$$\varphi(n) = (-1)^{n-1}\overline{\psi(1-n)}, \quad n \in \mathbb{Z},$$

then $\{R_{2k}\varphi\}_{k=0}^{M-1} \cup \{R_{2k}\psi\}_{k=0}^{M-1}$ is a time-frequency localized basis for $L^2(\mathbb{Z}_N)$.

Proof. For all m in $\mathbb{Z}$, we can use the definition of the finite Fourier transform and periodicity to get

$$\begin{aligned}
\hat{\varphi}(m) &= \sum_{n=0}^{N-1} \varphi(n) e^{-2\pi i m n/N} \\
&= \sum_{n=0}^{N-1} (-1)^{n-1}\overline{\psi(1-n)} e^{-2\pi i m n/N} \\
&= \sum_{k=1}^{2-N} (-1)^{-k}\overline{\psi(k)} e^{-2\pi i (1-k)m/N} \\
&= e^{-2\pi i m/N} \sum_{k=0}^{N-1} (e^{\pi i})^{-k}\overline{\psi(k)} e^{2\pi i k m/N} \\
&= e^{-2\pi i m/N} \overline{\sum_{k=0}^{N-1} \psi(k) e^{-2\pi i (m+M)k/N}} \\
&= e^{-2\pi i m/N} \overline{\hat{\psi}(m+M)}.
\end{aligned} \tag{12.8}$$

So, by (12.8) and the periodicity of $\hat{\psi}$, we get

$$\begin{aligned}
&\hat{\varphi}(m+M) \\
&= e^{-2\pi i (m+M)/N} \overline{\hat{\psi}(m+2M)}
\end{aligned}$$

$$= e^{-2\pi iM/N} e^{-2\pi im/N} \overline{\hat{\psi}(m)}$$
$$= -e^{-2\pi im/N} \overline{\hat{\psi}(m)} \tag{12.9}$$

for all m in $\mathbb{Z}$. Hence, by (12.8), (12.9) and Lemma 9.6, we get

$$|\hat{\varphi}(m)|^2 + |\hat{\varphi}(m+M)|^2 = |\hat{\psi}(m)|^2 + |\hat{\psi}(m+M)|^2 = 2 \tag{12.10}$$

for $m = 0, 1, \ldots, M-1$. Moreover,

$$\begin{aligned} &\hat{\varphi}(m)\overline{\hat{\psi}(m)} + \hat{\varphi}(m+M)\overline{\hat{\psi}(m+M)} \\ &= e^{-2\pi im/N}\overline{\hat{\psi}(m+M)\hat{\psi}(m)} - e^{-2\pi im/N}\overline{\hat{\psi}(m)\hat{\psi}(m+M)} \\ &= 0 \end{aligned} \tag{12.11}$$

for $m = 0, 1, \ldots, M-1$. Thus, by (12.10), (12.11) and Theorem 9.8, the proof is complete. □

We can now use Lemma 12.1 to determine the first-level scaling signals. Indeed, we define $V_0^{(1)}$ by

$$V_0^{(1)}(n) = (-1)^{n-1}\overline{W^{(1)}_{\frac{N}{2}-1}(1-n)}, \quad n \in \mathbb{Z}, \tag{12.12}$$

and then define $V_k^{(1)}$, $k =, 1, 2, \ldots, M-1$, by

$$V_k^{(1)} = R_{2k} V_0^{(1)}.$$

We note that, by (12.12),

$$V_0^{(1)} = \begin{pmatrix} -\beta_4 \\ \beta_3 \\ -\beta_2 \\ \beta_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

and hence the scaling numbers α_1, α_2, α_3 and α_4 are given by

$$\alpha_1 = -\beta_4,\ \alpha_2 = \beta_3,\ \alpha_3 = -\beta_2,\ \alpha_4 = \beta_1.$$

We call $V_0^{(1)}, V_1^{(1)}, \ldots, V_{M-1}^{(1)}$ the first-level Daubechies scaling signals, and $W_0^{(1)}, W_1^{(1)}, \ldots, W_{M-1}^{(1)}$ the first-level Daubechies wavelets.

As in the case of Haar wavelets, the Daubechies scaling signals and the Daubechies wavelets at the first level and the second level are generated from the standard basis for $L^2(\mathbb{Z}_N)$. Indeed, we get

$$V_m^{(1)} = \alpha_1\epsilon_{2m} + \alpha_2\epsilon_{2m+1} + \alpha_3\epsilon_{2m+2} + \alpha_4\epsilon_{2m+3}, \quad m = 0, 1, \ldots, \frac{N}{2} - 1,$$

$$V_m^{(2)} = \alpha_1 V_{2m}^{(1)} + \alpha_2 V_{2m+1}^{(1)} + \alpha_3 V_{2m+2}^{(1)} + \alpha_4 V_{2m+3}^{(1)}, \quad m = 0, 1, \ldots, \frac{N}{4} - 1,$$

$$W_m^{(1)} = \beta_1\epsilon_{2m} + \beta_2\epsilon_{2m+1} + \beta_3\epsilon_{2m+2} + \beta_4\epsilon_{2m+3}, \quad m = 0, 1, \ldots, \frac{N}{2} - 1,$$

and

$$W_m^{(2)} = \beta_1 V_{2m}^{(1)} + \beta_2 V_{2m+1}^{(1)} + \beta_3 V_{2m+2}^{(1)} + \beta_4 V_{2m+3}^{(1)}, \quad m = 0, 1, \ldots, \frac{N}{4} - 1,$$

with the understanding that

$$V_{m+\frac{N}{2}}^{(1)} = V_m^{(1)}, \quad m \in \mathbb{Z}.$$

Let $z \in L^2(\mathbb{Z}_N)$, $N = 2^l$, where l is a positive integer. Then for $j = 1, 2, \ldots, l$, we can define the j^{th}-level trend $a^{(j)}$ and the j^{th}-level fluctuation $d^{(j)}$ by

$$a^{(j)} = \begin{pmatrix} (z, V_0^{(j)}) \\ (z, V_1^{(j)}) \\ \vdots \\ (z, V_{\frac{N}{2^j}-1}^{(j)}) \end{pmatrix}$$

and

$$d^{(j)} = \begin{pmatrix} (z, W_0^{(j)}) \\ (z, W_1^{(j)}) \\ \vdots \\ (z, W_{\frac{N}{2^j}-1}^{(j)}) \end{pmatrix}.$$

The MRA of a signal z in $L^2(\mathbb{Z}_N)$ is then given by

$$\begin{aligned} z &= A^{(1)} + D^{(1)} = A^{(2)} + D^{(2)} + D^{(1)} \\ &= A^{(3)} + D^{(3)} + D^{(2)} + D^{(1)} \\ &= \cdots = A^{(l)} + \sum_{j=1}^{l} D^{(j)}, \end{aligned}$$

where $A^{(j)}$ and $D^{(j)}$ are the j^{th}-level average signal and the detail signal of z given, respectively, by

$$A^{(j)} = \sum_{n=0}^{\frac{N}{2^j}-1} (z, V_n^{(j)}) V_n^{(j)}$$

and

$$D^{(j)} = \sum_{n=0}^{\frac{N}{2^j}-1} (z, W_n^{(j)}) W_n^{(j)}.$$

By now, we have enough evidence and hence confidence to conclude that the mathematics for the Daubechies wavelets is the same as the mathematics for the Haar wavelets. What is then the point of studying Daubechies wavelets when the Haar wavelets are already there to give the mathematics of wavelets? The answer is that, in general, the Daubechies wavelets are much more refined tools in signal analysis than the Haar wavelets. However, signals abound in such great complexity and variety that we cannot expect a single method or a few techniques can serve them all. A time-frequency localized basis that is good for one kind of signals may not work as well for another kind. Thus, it is desirable to have as many good wavelet bases as possible.

Example 12.2. For the signal z in Example 11.15 given by

$$z = \begin{pmatrix} 4 \\ 6 \\ 10 \\ 12 \\ 8 \\ 6 \\ 5 \\ 5 \end{pmatrix},$$

if we use the Haar wavelets, then 98.7% of the energy is in the first-level average. Let us see how it works out when we use the Daubechies wavelets. We get

$$a^{(1)} = \begin{pmatrix} \frac{16-3\sqrt{3}}{\sqrt{2}} \\ \frac{19+2\sqrt{3}}{\sqrt{2}} \\ \frac{11.5+\sqrt{3}}{\sqrt{2}} \\ \frac{9.5}{\sqrt{2}} \end{pmatrix}.$$

Then, as in Example 11.15,

$$\|A^{(1)}\|^2 = \|a^{(1)}\|^2 = 443.35,$$

which is 99.4% of the energy of the original signal z. Thus, we can improve the concentration of energy when the Daubechies wavelets are used instead of the Haar wavelets.

We end this chapter with some remarks. The Haar wavelets have two nonzero wavelet numbers given by

$$\beta_1 = \frac{1}{\sqrt{2}}$$

and

$$\beta_2 = -\frac{1}{\sqrt{2}}.$$

We may call these Haar wavelets Haar(2) wavelets. The $N/2$ first-level Daubechies wavelets in this chapter, also referred to as the Daub(4) wavelets, have four nonozero wavelet numbers β_1, β_2, β_3 and β_4, which can be solved from a nonlinear system of four equations. The Daubechies wavelet $W^{(1)}_{\frac{N}{2}-1}$ has a wrap-around in which β_3 and β_4 are rotated to the top. Using the methodology of the previous chapter and this chapter, wavelets having six nonzero wavelet numbers with wrap-around in $W^{(1)}_0$ and in $W^{(1)}_{\frac{N}{2}}$ have also been devised and they are dubbed Coiflets or Coif(6) wavelets in deference to Coifman. In addition to the Haar(2), Daub(4) and Coif(6) wavelets, there are many other kinds of wavelets. Details can be found in the book [45] by Walker.

Exercises

1. Solve Equations (12.4)–(12.7) for the wavelet numbers β_1, β_2, β_3 and β_4.
2. Repeat Exercise 5 of the preceding chapter using the Daubechies wavelets.
3. For the signal z in $L^2(\mathbb{Z}_8)$ given in Exercise 5 of the preceding chapter, use the Daubechies wavelets to find the % of the energy of the signal saved in the second-level average $A^{(2)}$.

Chapter 13

The Trace

We give in this chapter a class of linear operators A from $L^2(\mathbb{Z}_N)$ into $L^2(\mathbb{Z}_N)$ for which the trace $\mathrm{tr}(A)$ of A can be computed.

Let σ and φ be signals in $L^2(\mathbb{Z}_N)$. Then we define the linear operator $T_{\sigma,\varphi}:L^2(\mathbb{Z}_N)\to L^2(\mathbb{Z}_N)$ by

$$T_{\sigma,\varphi}z=\sum_{k=0}^{N-1}\sigma(k)(z,\pi_k\varphi)\pi_k\varphi,\quad z\in L^2(\mathbb{Z}_N),$$

where

$$\pi_k\varphi=\sqrt{N}(R_k\varphi)^\vee,\quad k=0,1,\ldots,N-1.$$

The following fact tells us that if we choose the signal φ to be the unit impulse δ, then the linear operator $T_{\sigma,\varphi}$ is a scalar multiple of the Fourier multiplier associated to the symbol σ.

Proposition 13.1. *Let σ be any signal in $L^2(\mathbb{Z}_N)$. Then*

$$T_{\sigma,\delta}=T_\sigma.$$

Proof. For $k=0,1,\ldots,N-1$, we get, by the definition of the inverse finite Fourier transform and Proposition 1.23,

$$(R_k\delta)^\vee(n)=\frac{1}{N}(R_k\delta)^\wedge(-n)=\frac{1}{N}e^{2\pi ink/N}\hat{\delta}(-n),\quad n\in\mathbb{Z}.\tag{13.1}$$

Using (13.1) and the fact that $\hat{\delta}(m)=1$ for all m in $\mathbb{Z}$, we get for $k=0,1,\ldots,N-1$,

$$(R_k\delta)^\vee(n)=\frac{1}{N}e^{2\pi ink/N}=F_k(n),\quad n\in\mathbb{Z}.$$

Therefore

$$\pi_k\delta=\sqrt{N}F_k,\quad k=0,1,\ldots,N-1.\tag{13.2}$$

M.W. Wong, *Discrete Fourier Analysis*, Pseudo-Differential Operators 5,
DOI 10.1007/978-3-0348-0116-4_13, © Springer Basel AG 2011

Since

$$(z, F_k) = \frac{1}{N}\hat{z}(k), \quad k = 0, 1, \ldots, N-1,$$

it follows from (13.2) and the definition of $T_{\sigma,\delta}$ that

$$\begin{aligned} T_{\sigma,\delta} z &= N \sum_{k=0}^{N-1} \sigma(k)(z, F_k) F_k \\ &= \sum_{k=0}^{N-1} \sigma(k)\hat{z}(k) F_k, \quad z \in L^2(\mathbb{Z}_N). \end{aligned} \tag{13.3}$$

Using the definition of the Fourier multiplier and the Fourier inversion formula, we can see that

$$T_\sigma z = \sum_{m=0}^{N-1} \sigma(m)\hat{z}(m) F_m, \quad z \in L^2(\mathbb{Z}_N). \tag{13.4}$$

So, by (13.3) and (13.4),

$$T_{\sigma,\delta} = T_\sigma.$$

□

Let us choose a signal φ in $L^2(\mathbb{Z}_N)$ such that

$$|\hat{\varphi}(m)| = 1, \quad m \in \mathbb{Z}.$$

Then, by Proposition 8.8, we know that $\{R_0\varphi, R_1\varphi, \ldots, R_{N-1}\varphi\}$ is an orthonormal basis for $L^2(\mathbb{Z}_N)$. Hence, using Parseval's identity and the Fourier inversion formula, we get for $j, k = 0, 1, \ldots, N-1$,

$$\begin{aligned} (\pi_j\varphi, \pi_k\varphi) &= N((R_j\varphi)^\vee, (R_k\varphi)^\vee) \\ &= (R_j\varphi, R_k\varphi) = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases} \end{aligned}$$

So, $\{\pi_0\varphi, \pi_1\varphi, \ldots, \pi_{N-1}\varphi\}$ is an orthonormal basis for $L^2(\mathbb{Z}_N)$. With this observation, complete information on the eigenvalues and eigenfunctions for the linear operator $T_{\sigma,\varphi}$ is contained in the following proposition, which is an extension of Theorem 6.1.

Proposition 13.2. *Let σ be a signal in $L^2(\mathbb{Z}_N)$ and let φ be a signal in $L^2(\mathbb{Z}_N)$ such that*

$$|\hat{\varphi}(m)| = 1, \quad m \in \mathbb{Z}. \tag{13.5}$$

Then the eigenvalues of the linear operator $T_{\sigma,\varphi}$ are given by

$$\sigma(0), \sigma(1), \ldots, \sigma(N-1).$$

Moreover, for $m = 0, 1, \ldots, N-1$, the eigenfunction of $T_{\sigma,\varphi}$ corresponding to the eigenvalue $\sigma(m)$ is the signal $\pi_m\varphi$.

Proof. For $m = 0, 1, \ldots, N-1$, we get, by the definition of $T_{\sigma,\varphi}$ and the orthonormality of $\{\pi_0\varphi, \pi_1\varphi, \ldots, \pi_{N-1}\varphi\}$,

$$T_{\sigma,\varphi}(\pi_m\varphi) = \sum_{k=0}^{N-1} \sigma(k)(\pi_m\varphi, \pi_k\varphi)\pi_k\varphi = \sigma(m)\pi_m\varphi,$$

and this completes the proof. □

In the more often case when the signal φ does not fulfil the condition (13.5), the computation of the eigenvalues and eigenfunctions of $T_{\sigma,\varphi}$ is more difficult. However, some information on the eigenvalues, i.e., the trace $\mathrm{tr}(T_{\sigma,\varphi})$, can still be obtained in terms of the symbol σ. To this end, a brief and self-contained recapitulation of the trace of a square matrix may be in order.

The trace $\mathrm{tr}(A)$ of an $N \times N$ matrix A with complex entries is the sum of the eigenvalues, repeated according to multiplicity, of A. The best scenario is that we are able to compute all the eigenvalues of A explicitly. This is very often very difficult. In the case when we do not know the eigenvalues explicitly, the trace will give us some information about the eigenvalues.

Theorem 13.3. *Let A be an $N \times N$ matrix with complex entries. Then*

$$\mathrm{tr}(A) = \sum_{j=0}^{N-1} a_{jj},$$

where $a_{00}, a_{11}, \ldots, a_{N-1,N-1}$ are the diagonal entries of A.

Proof. The characteristic polynomial $p(\lambda)$ of the matrix A is given by

$$p(\lambda) = \det(A - \lambda I),$$

where $\det\{\cdots\}$ is the determinant of $\{\cdots\}$ and I is the identity matrix of order $N \times N$. By the fundamental theorem of algebra, we can write

$$p(\lambda) = (\lambda_0 - \lambda)(\lambda_1 - \lambda)\cdots(\lambda_{N-1} - \lambda),$$

where $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$ are the eigenvalues of A, which are counted according to multiplicity. The coefficient of λ^{N-1} is then easily seen to be $(-1)^{N-1}\mathrm{tr}(A)$. On the other hand,

$$p(\lambda) = \det\begin{pmatrix} a_{00} - \lambda & a_{01} & \cdots & a_{0,N-1} \\ a_{10} & a_{11} - \lambda & \cdots & a_{1,N-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N-1,0} & a_{N-1,1} & \cdots & a_{N-1,N-1} - \lambda \end{pmatrix}.$$

It follows from the definition of the determinant that the determinant is a sum of terms in which each term is a product of entries contributed by each row and each column exactly once. Thus, the terms of degree $N-1$ in $p(\lambda)$ can only come from $(a_{00}-\lambda)(a_{11}-\lambda)\cdots(a_{N-1,N-1}-\lambda)$. So, the coefficient of λ^{N-1} is equal to $(-1)^{N-1}\sum_{j=0}^{N-1} a_{jj}$. This completes the proof. □

The following theorem on the computation of the trace is often useful.

Theorem 13.4. *Let A be an $N\times N$ matrix with complex entries. Then*

$$\operatorname{tr}(A)=\sum_{j=0}^{N-1}(A\varphi_j,\varphi_j),$$

where $\{\varphi_0,\varphi_1,\ldots,\varphi_{N-1}\}$ is any orthonormal basis for $L^2(\mathbb{Z}_N)$.

Proof. Let $B=\{\varphi_0,\varphi_1,\ldots,\varphi_{N-1}\}$ be an orthonormal basis for $L^2(\mathbb{Z}_N)$. Then

$$A\varphi_0=\alpha_{00}\varphi_0+\alpha_{10}\varphi_1+\cdots+\alpha_{N-1,0}\varphi_{N-1},$$

$$A\varphi_1=\alpha_{01}\varphi_0+\alpha_{11}\varphi_1+\cdots+\alpha_{N-1,1}\varphi_{N-1},$$

$$\cdots$$

$$A\varphi_{N-1}=\alpha_{0,N-1}\varphi_0+\alpha_{1,N-1}\varphi_1+\cdots+\alpha_{N-1,N-1}\varphi_{N-1}.$$

In other words, for $0\le i\le N-1$,

$$A\varphi_i=\alpha_{0i}\varphi_0+\alpha_{1i}\varphi_1+\cdots+\alpha_{N-1,i}\varphi_{N-1}. \tag{13.6}$$

Using the orthonormality of the basis B, we get

$$\alpha_{ij}=(A\varphi_j,\varphi_i),\quad i,j\in\mathbb{Z}_N.$$

So, the matrix $(A)_B$ of A with respect to the basis B is given by

$$(A)_B=(\alpha_{ij})_{0\le i,j\le N-1}=((A\varphi_j,\varphi_i))_{0\le i,j\le N-1}. \tag{13.7}$$

Let $S=\{\epsilon_0,\epsilon_1,\ldots,\epsilon_{N-1}\}$ be the standard basis for $L^2(\mathbb{Z}_N)$. The matrix $(A)_S$ of A with respect to S for $L^2(\mathbb{Z}_N)$ is the same as the matrix $A=(a_{ij})_{0\le i,j\le N-1}$. Let us now compute the change of basis matrix from S to B. To do this, we note that

$$\epsilon_0=c_{00}\varphi_0+c_{10}\varphi_1+\cdots+c_{N-1,0}\varphi_{N-1},$$

$$\epsilon_1=c_{01}\varphi_0+c_{11}\varphi_1+\cdots+c_{N-1,1}\varphi_{N-1},$$

$$\cdots$$

$$\epsilon_{N-1}=c_{0,N-1}\varphi_0+c_{1,N-1}\varphi_1+\cdots+c_{N-1,N-1}\varphi_{N-1}.$$

More succinctly, we can write for $0\le i\le N-1$,

$$\epsilon_i=c_{0i}\varphi_0+c_{1i}\varphi_1+\cdots+c_{i,N-1}\varphi_{N-1}. \tag{13.8}$$

Using the orthonormality of the basis B, we get

$$c_{ij} = (\epsilon_j, \varphi_i), \quad i, j \in \mathbb{Z}.$$

Let

$$C = (c_{ij})_{0 \le i,j \le N-1} = ((\epsilon_j, \varphi_i))_{0 \le i,j \le N-1}.$$

Since

$$\begin{aligned}
\sum_{j=0}^{N-1} c_{lj}\overline{c_{kj}} &= \sum_{j=0}^{N-1} (\epsilon_j, \varphi_l)(\varphi_k, \epsilon_j) \\
&= \sum_{j=0}^{N-1} (\varphi_k, (\varphi_l, \epsilon_j)\epsilon_j) \\
&= \left(\varphi_k, \sum_{j=0}^{N-1} (\varphi_l, \epsilon_j)\epsilon_j\right) \\
&= (\varphi_k, \varphi_l) = \begin{cases} 1, \ k = l, \\ 0, \ k \ne l, \end{cases}
\end{aligned}$$

it follows that C is a unitary matrix. Now, we note that the coordinates $(z)_B$ with respect to the basis B are given by

$$(z)_B = Cz, \quad z \in L^2(\mathbb{Z}_N),$$

and

$$A = C^{-1}(A)_B C. \tag{13.9}$$

To see that (13.9) is true, we use (13.6) and (13.8) to obtain for $0 \le i \le N-1$,

$$\begin{aligned}
A\epsilon_1 &= \sum_{l=0}^{N-1} c_{li} A\varphi_l = \sum_{l=0}^{N-1} c_{li} \sum_{k=0}^{N-1} \alpha_{kl}\varphi_k \\
&= \sum_{k=0}^{N-1} \left(\sum_{l=0}^{N-1} \alpha_{kl} c_{li}\right) \varphi_k = \sum_{k=0}^{N-1} ((A)_B C)_{ki}\varphi_k,
\end{aligned} \tag{13.10}$$

where $((A)_B C)_{ki}$ is the entry in the k^{th} row and i^{th} column of the matrix $(A)_B C$. On the other hand, using (13.8) again,

$$\begin{aligned}
A\epsilon_i &= \sum_{l=0}^{N-1} a_{li} A\epsilon_l = \sum_{l=0}^{N-1} a_{li} \sum_{k=0}^{N-1} c_{kl}\varphi_k \\
&= \sum_{k=0}^{N-1} \left(\sum_{l=0}^{N-1} c_{kl} a_{li}\right) \varphi_k = \sum_{k=0}^{N-1} (CA)_{ki}\varphi_k,
\end{aligned} \tag{13.11}$$

where $(C^{-1}A)_{ki}$ is the entry in the k^{th} row and i^{th} column of the matrix CA. Thus, by (13.10) and (13.11),

$$(A)_B C = CA$$

and hence

$$A = C^{-1}(A)_B A,$$

as asserted. So, by (13.7) and (13.9),

$$\begin{aligned}(a_{ij})_{0\le i,j\le N-1} &= C^{-1}\left(\sum_{l=0}^{N-1}\alpha_{il}c_{lj}\right)_{0\le i,j\le N-1} \\ &= \left(\sum_{k=0}^{N-1}\overline{c_{ki}}\sum_{l=0}^{N-1}\alpha_{kl}c_{lj}\right)_{0\le i,j\le N-1}. \qquad (13.12)\end{aligned}$$

Thus, by (13.12) ,

$$a_{jj} = \sum_{k,l=0}^{N-1}\alpha_{kl}c_{lj}\overline{c_{kj}}, \quad j = 0, 1, \ldots, N-1,$$

and hence, using the unitarity of C,

$$\sum_{j=0}^{N-1} a_{jj} = \sum_{k,l=0}^{N-1}\alpha_{kl}\sum_{j=0}^{N-1} c_{lj}\overline{c_{kj}} = \sum_{k=0}^{N-1}\alpha_{kk} = \sum_{k=0}^{N-1}(A\varphi_k, \varphi_k).$$

Thus, by Theorem 13.3 and the definition of the trace, the proof is complete. □

We can now compute the trace of $T_{\sigma,\varphi}$.

Theorem 13.5. *Let σ and φ be signals in $L^2(\mathbb{Z}_N)$. Then the trace $\operatorname{tr}(T_{\sigma,\varphi})$ of the linear operator $T_{\sigma,\varphi}$ associated to the signals σ and φ is given by*

$$\operatorname{tr}(T_{\sigma,\varphi}) = \|\varphi\|^2\sum_{k=0}^{N-1}\sigma(k).$$

Proof. Let $\{z_0, z_1, \ldots, z_{N-1}\}$ be any orthonormal basis for $L^2(\mathbb{Z}_N)$. Then

$$\begin{aligned}\operatorname{tr}(T_{\sigma,\varphi}) &= \sum_{j=0}^{N-1}(T_{\sigma,\varphi}z_j, z_j) \\ &= \sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\sigma(k)|(z_j, \pi_k\varphi)|^2 \\ &= \sum_{k=0}^{N-1}\sigma(k)\sum_{j=0}^{N-1}|(z_j, \pi_k\varphi)|^2. \qquad (13.13)\end{aligned}$$

Since $\{z_0, z_1, \ldots, z_{N-1}\}$ is an orthonormal basis for $L^2(\mathbb{Z}_N)$, it follows that for $k = 0, 1, \ldots, N-1$, we get

$$\pi_k\varphi = \sum_{j=0}^{N-1} (\pi_k\varphi, z_j)z_j,$$

and hence, by Lemma 11.13,

$$\|\pi_k\varphi\|^2 = \left\|\sum_{j=0}^{N-1} (\pi_k\varphi, z_j)z_j\right\|^2 = \sum_{j=0}^{N-1} |(\pi_k\varphi, z_j)|^2. \tag{13.14}$$

So, by (13.13) and (13.14), we get

$$\operatorname{tr}(T_{\sigma,\varphi}) = \sum_{k=0}^{N-1} \sigma(k)\|\pi_k\varphi\|^2. \tag{13.15}$$

By Plancherel's theorem, we get for $k = 0, 1, \ldots, N-1$,

$$\|\pi_k\varphi\|^2 = N\|(R_k\varphi)^\vee\|^2 = \|R_k\varphi\|^2 = \|\varphi\|^2. \tag{13.16}$$

Thus, by (13.15) and (13.16),

$$\operatorname{tr}(T_{\sigma,\varphi}) = \|\varphi\|^2 \sum_{k=0}^{N-1} \sigma(k). \qquad \square$$

Exercises

1. Let $b \in L^2(\mathbb{Z}_N)$. Prove that the convolution operator $C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is a linear operator of the form $T_{\sigma,\varphi} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$, where σ and φ are signals in $L^2(\mathbb{Z}_N)$.
2. Let $b \in L^2(\mathbb{Z}_N)$. Compute the trace $\operatorname{tr}(C_b)$ of the convolution operator $C_b : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$.
3. For all signals σ and φ in $L^2(\mathbb{Z}_N)$, the linear operator $A_{\sigma,\varphi} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ is defined by

$$A_{\sigma,\varphi}z = \sum_{k=0}^{N-1} \sigma(k)(z, \rho_k\varphi)\rho_k\varphi, \quad z \in L^2(\mathbb{Z}_N),$$

where

$$\rho_k\varphi = (M_{-k}\varphi)^\vee, \quad k = 0, 1, \ldots, N-1,$$

and M_{-k} is defined in Exercise 1 of Chapter 8. Suppose that

$$|\varphi(m)| = \frac{1}{N}, \quad m \in \mathbb{Z}. \tag{13.17}$$

Find all the eigenvalues and the corresponding eigenfunctions of $A_{\sigma,\varphi}$.

4. Let σ and φ be signals in $L^2(\mathbb{Z}_N)$ Compute the trace $\mathrm{tr}(A_{\sigma,\varphi})$ of the linear operator $A_{\sigma,\varphi} : L^2(\mathbb{Z}_N) \to L^2(\mathbb{Z}_N)$ defined as in the preceding exercise, but without the condition (13.17).

5. Prove that for all $N \times N$ matrices A and B,

$$\mathrm{tr}\left([A, B]\right) = 0,$$

where $[A, B]$ is the commutator of A and B defined by

$$[A, B] = AB - BA.$$

6. Let A and B be $N \times N$ matrices such that

$$B = C^{-1}AC,$$

where C is an $N \times N$ invertible matrix. Prove that

$$\mathrm{tr}(A) = \mathrm{tr}(B).$$

Chapter 14

Hilbert Spaces

Our starting point is an infinite-dimensional complex vector space X. An inner product $(\ ,\ )$ in X is a mapping from $X \times X$ into $\mathbb{C}$ such that

- $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$,
- $(x, \alpha y + \beta z) = \overline{\alpha}(x, y) + \overline{\beta}(x, z)$,
- $(x, x) \geq 0$,
- $(x, x) = 0 \Leftrightarrow x = 0$

for all x, y and z in X and all complex numbers α and β. Given an inner product $(\ ,\ )$ in X, the induced norm $\|\ \|$ in X is given by

$$\|x\|^2 = (x, x), \quad x \in X.$$

Let us make a very useful remark that from the definition of an inner product we can prove that

$$(x, y) = \overline{(y, x)}, \quad x, y \in X.$$

We leave this as an exercise. See Exercise 1.

The most fundamental fact about an inner product is the Schwarz inequality to the effect that for all x and y in X,

$$|(x, y)| \leq \|x\|\,\|y\|$$

and equality occurs if and only if x and y are linearly dependent. We leave the proof of the Schwarz inequality as an exercise. See Exercise 2. Let us also recall that $\|\ \|$ is a norm in X in the sense that for all x and y in X and all complex numbers α,

$$\|x\| \geq 0, \tag{14.1}$$

$$\|x\| = 0 \Leftrightarrow x = 0, \tag{14.2}$$

$$\|\alpha x\| = |\alpha|\,\|x\|, \tag{14.3}$$

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DOI 10.1007/978-3-0348-0116-4_14, © Springer Basel AG 2011

$$\|x + y\| \le \|x\| + \|y\|. \tag{14.4}$$

The easy proof is left as an exercise. See Exercise 3.

Let X be an infinite-dimensional complex vector space in which the inner product and the norm are denoted by, respectively, $(\ ,\ )$ and $\|\ \|$.

Let $\{x_j\}_{j=1}^\infty$ be a sequence in X. Then we say that $\{x_j\}_{j=1}^\infty$ converges to x in X if

$$\|x_j - x\| \to 0$$

as $j \to \infty$. It is easy to to prove that if $\{x_j\}_{j=1}^\infty$ converges to x in X, then

$$(x_j, y) \to (x, y)$$

for all y in X. The simple proof is left as an exercise. See Exercise 5.

A sequence $\{x_j\}_{j=1}^\infty$ in X is said to be a Cauchy sequence if

$$\|x_j - x_k\| \to 0$$

as $j, k \to \infty$. If every Cauchy sequence in X converges, then we call X a Hilbert space.

Remark 14.1. A complex vector space X is said to be a normed vector space if there exists a function $\|\ \| : X \to [0, \infty)$ satisfying (14.1)–(14.4). Such a function is a norm in X. A normed vector space in which every Cauchy sequence in X converges in X is said to be complete. A complete normed vector space is called a Banach space. While Banach spaces do not belong to the main focus in this book, they do come up frequently in our discussions.

A sequence $\{x_j\}_{j=1}^\infty$ in a Hilbert space X is said to be orthogonal if

$$(x_j, x_k) = 0$$

for all positive integers j and k with $j \neq k$. An orthogonal sequence $\{x_j\}_{j=1}^\infty$ in a Hilbert space X is said to be an orthonormal sequence if

$$\|x_j\| = 1, \quad j = 1, 2, \ldots.$$

An orthonormal sequence $\{x_j\}_{j=1}^\infty$ in a Hilbert space X is said to be complete if every element x in X with the property that

$$(x, x_j) = 0, \quad j = 1, 2, \ldots,$$

is the zero element in X. We call a complete orthonormal sequence in a Hilbert space X an orthonormal basis for X.

We assume that every Hilbert space X encountered in this book has an orthonormal basis $\{\varphi_j\}_{j=1}^\infty$ in the sense described in the preceding paragraph.

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in a Hilbert space X. For every positive integer n, we let s_n be the partial sum defined by

$$s_n = \sum_{j=1}^{n} x_j.$$

If the sequence $\{s_n\}_{n=1}^{\infty}$ converges to s in X, then we say that the series $\sum_{j=1}^{\infty} x_j$ converges to s in X and we write

$$\sum_{j=1}^{\infty} x_j = s.$$

The series $\sum_{j=1}^{\infty} x_j$ is said to be absolutely convergent if

$$\sum_{j=1}^{\infty} \|x_j\| < \infty.$$

Proposition 14.2. *Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in a Hilbert space X such that the series $\sum_{j=1}^{\infty} x_j$ is absolutely convergent. Then $\sum_{j=1}^{\infty} x_j$ converges in X.*

The proof of Proposition 14.2 is very easy and is left as an exercise. See Exercise 6.

We now give two useful lemmas. The first lemma is an inequality, which is known as the Bessel inequality.

Lemma 14.3. *Let $\{\varphi_j\}_{j=1}^{\infty}$ be an orthonormal sequence for a Hilbert space X. Then for all elements x in X,*

$$\sum_{j=1}^{\infty} |(x, \varphi_j)|^2 \leq \|x\|^2.$$

Proof. For all N in $\mathbb{N}$, let s_N be the partial sum given by

$$s_N = \sum_{j=1}^{N} (x, \varphi_j)\varphi_j.$$

Then

$$\begin{aligned} \|x - s_N\|^2 &= (x, x) - (x, s_N) - (s_N, x) + (s_N, s_N) \\ &= \|x\|^2 - 2\,\mathrm{Re}(x, s_N) + \|s_N\|^2. \end{aligned}$$

But

$$(x, s_N) = \sum_{j=1}^{N} \overline{(x, \varphi_j)}(x, \varphi_j) = \sum_{j=1}^{N} |(x, \varphi_j)|^2$$

and

$$\|s_N\|^2 = \left(\sum_{j=1}^{N}(x,\varphi_j)\varphi_j, \sum_{k=1}^{N}(x,\varphi_k)\varphi_k\right)$$
$$= \sum_{j=1}^{N}\sum_{k=1}^{N}(x,\varphi_j)\overline{(x,\varphi_k)}(\varphi_j,\varphi_k) = \sum_{j=1}^{N}|(x,\varphi_j)|^2.$$

So,

$$\|x - s_N\|^2 = \|x\|^2 - \sum_{j=1}^{N}|(x,\varphi_j)|^2,$$

or equivalently,

$$\sum_{j=1}^{N}|(x,\varphi_j)|^2 = \|x\|^2 - \|x - s_N\|^2.$$

Therefore

$$\sum_{j=1}^{N}|(x,\varphi_j)|^2 \le \|x\|^2.$$

Letting $N \to \infty$, we get

$$\sum_{j=1}^{\infty}|(x,\varphi_j)|^2 \le \|x\|^2. \qquad \square$$

The second lemma is a criterion for a series to converge in X.

Lemma 14.4. *Let $\{\varphi_j\}_{j=1}^\infty$ be an orthonormal sequence in a Hilbert space X. Let $\{\alpha_j\}_{j=1}^\infty$ be a sequence of complex numbers. Then $\sum_{j=1}^\infty \alpha_j\varphi_j$ converges in X if and only if*

$$\sum_{j=1}^{\infty}|\alpha_j|^2 < \infty.$$

Proof. Suppose that $\sum_{j=1}^\infty \alpha_j\varphi_j$ converges to x in X. Then for all positive integers M and N with $M \ge N$,

$$\left(\sum_{j=1}^{M}\alpha_j\varphi_j, \varphi_N\right) = \sum_{j=1}^{M}\alpha_j(\varphi_j,\varphi_N) = \alpha_N.$$

Letting $M \to \infty$, we get

$$(x,\varphi_N) = \left(\sum_{j=1}^{\infty}\alpha_j\varphi_j, \varphi_N\right) = \alpha_N.$$

By Bessel's inequality in Lemma 14.3, we get

$$\sum_{N=1}^{\infty} |\alpha_N|^2 = \sum_{N=1}^{\infty} |(x, \varphi_N)|^2 \leq \|x\|^2 < \infty.$$

Conversely, suppose that

$$\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty.$$

For all positive integers M, let

$$s_M = \sum_{j=1}^{M} \alpha_j \varphi_j.$$

Then for all positive integers M and N with $M \geq N$,

$$\|s_M - s_N\|^2 = \sum_{j=N}^{M} \|\alpha_j \varphi_j\|^2 = \sum_{j=N}^{M} |\alpha_j|^2 \to 0$$

as $N \to \infty$. So, $\{s_M\}_{M=1}^{\infty}$ is a Cauchy sequence in X. Since X is a Hilbert space, it follows that $s_M \to x$ for some x in X as $M \to \infty$, and the proof is complete. □

The following simple facts on series in Hilbert spaces are easy to be established and are also left as exercises. See Exercises 7 and 8. The first proposition is an infinite-dimensional analog of Pythagoras' theorem.

Proposition 14.5 (Pythagoras' Theorem). *Let $\{x_j\}_{j=1}^{\infty}$ be an orthonormal sequence in a Hilbert space X and let $\{\alpha_j\}_{j=1}^{\infty}$ be a sequence of complex numbers such that $\sum_{j=1}^{\infty} \alpha_j x_j$ converges in X. Then*

$$\left\| \sum_{j=1}^{\infty} \alpha_j x_j \right\|^2 = \sum_{j=1}^{\infty} |\alpha_j|^2.$$

The second proposition is a formula for the inner product of two series in a Hilbert space.

Proposition 14.6. *Let $\{x_j\}_{j=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ be sequences in a Hilbert space X such that $\sum_{j=1}^{\infty} x_j$ and $\sum_{j=1}^{\infty} y_j$ both converge in X. Then*

$$\left(\sum_{j=1}^{\infty} x_j, \sum_{j=1}^{\infty} y_j \right) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (x_j, y_k).$$

Remark 14.7. In an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for a Hilbert space, we have used the set $\mathbb{N}$ of all positive integers for the index set of the basis. In fact, any infinite and countable index set J can be used instead of $\mathbb{N}$ because J can be put in a one-to-one correspondence with $\mathbb{N}$.

The following theorem contains all that we need to know about orthonormal bases for Hilbert spaces.

Theorem 14.8. *Let $\{\varphi_j\}_{j=1}^{\infty}$ be an orthonormal basis for a Hilbert space X. Then for all x and y in X, we have the following conclusions.*

(i) (The Fourier Inversion Formula)

$$x = \sum_{j=1}^{\infty} (x, \varphi_j)\varphi_j.$$

(ii) (Parseval's Identity)

$$(x, y) = \sum_{j=1}^{\infty} (x, \varphi_j)(\varphi_j, y).$$

(iii) (Plancherel's Theorem)

$$\|x\|^2 = \sum_{j=1}^{\infty} |(x, \varphi_j)|^2.$$

Proof. By Bessel's inequality in Lemma 14.3, the series $\sum_{j=1}^{\infty}(x, \varphi_j)\varphi_j$ converges in X. Let y be the element in X such that

$$y = x - \sum_{j=1}^{\infty} (x, \varphi_j)\varphi_j.$$

Then for $k = 1, 2, \ldots,$

$$(y, \varphi_k) = (x, \varphi_k) - \sum_{j=1}^{\infty} (x, \varphi_j)(\varphi_j, \varphi_k) = (x, \varphi_k) - (x, \varphi_k) = 0.$$

Since $\{\varphi_j\}_{j=1}^{\infty}$ is an orthonormal basis for X and hence complete, it follows that $y = 0$. Therefore

$$x = \sum_{j=1}^{\infty} (x, \varphi_j)\varphi_i.$$

This proves the Fourier inversion formula. As for Parseval's identity, we write

$$x = \sum_{j=1}^{\infty} (x, \varphi_j)\varphi_j$$

and

$$y = \sum_{k=1}^{\infty} (y, \varphi_k)\varphi_k.$$

Then by Proposition 14.6,

$$\begin{aligned}(x, y) &= \left(\sum_{j=1}^{\infty} (x, \varphi_j)\varphi_j, \sum_{k=1}^{\infty} (y, \varphi_k)\varphi_k \right) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (x, \varphi_j)\overline{(y, \varphi_k)}(\varphi_j, \varphi_k) \\ &= \sum_{j=1}^{\infty} (x, \varphi_j)(\varphi_j, y).\end{aligned}$$

Then Plancherel's formula follows immediately from Parseval's identity by letting $y = x$. □

We end this chapter with a more in-depth study of the inner products in Hilbert spaces culminating in the Riesz representation theorem.

Let X be an infinite-dimensional Hilbert space in which the inner product and norm are denoted by, respectively, $(\,,\,)$ and $\|\;\|$. A linear functional on X is a linear transformation from X into $\mathbb{C}$, where $\mathbb{C}$ is to be understood as a one-dimensional complex vector space. A linear functional T on X is said to be a bounded linear functional on X if there exists a positive constant C such that

$$|T(x)| \leq C\|x\|, \quad x \in X.$$

It can be shown that for every fixed element y in X, the linear transformation $T : X \to \mathbb{C}$ defined by

$$T(x) = (x, y), \quad x \in X,$$

is a bounded linear functional on X. See Exercise 9. That all bounded linear functionals on X come from inner products in this way is the content of the Riesz representation theorem. Before embarking on a proof of this fact, we first have a glimpse into the geometry of Hilbert spaces.

Theorem 14.9. *Let M be a closed subspace of X in the sense that the limits of all sequences in M lie in M. Let $x \in X \setminus M$ and let d be the distance between x and M defined by*

$$d = \inf_{z \in M} \|x - z\|.$$

Then there exists an element z in M such that

$$\|x - z\| = d.$$

Proof. There exists a sequence $\{z_j\}_{j=1}^{\infty}$ in M such that

$$\|x - z_j\| \to d$$

as $j \to \infty$. By the law of the parallelogram in Exercise 4, we get for $j, k = 1, 2, \ldots,$

$$\begin{aligned}&2\|x - z_j\|^2 + 2\|x - z_k\|^2 \\ &= \|(x - z_j) + (x - z_k)\|^2 + \|(x - z_j) - (x - z_k)\|^2,\end{aligned}$$

which can be rewritten as

$$4\left\|x - \frac{z_j + z_k}{2}\right\|^2 + \|z_j - z_k\|^2 = 2\|x - z_j\|^2 + 2\|x - z_k\|^2. \tag{14.5}$$

Since

$$d \leq \left\|x - \frac{z_j + z_k}{2}\right\|, \quad j, k = 1, 2, \ldots,$$

it follows from (14.5) that for $j, k = 1, 2, \ldots,$

$$\|z_j - z_k\|^2 \leq 2\|x - z_j\|^2 + 2\|x - z_k\|^2 - 4d^2 \to 0$$

as $j, k \to \infty$. Therefore $\{z_j\}_{j=1}^{\infty}$ is a Cauchy sequence in X. Since X is complete, there exists an element z in X such that

$$z_j \to z$$

in X as $j \to \infty$. But $\{z_j\}_{j=1}^{\infty}$ is a sequence in M and M is closed. So, $z \in M$ and

$$d = \lim_{j\to\infty} \|x - z_j\| = \|x - z\|. \qquad \square$$

Let M be a closed subspace of a Hilbert space X. Then the orthogonal complement $M^{\perp}$ of M is defined by

$$M^{\perp} = \{x \in X : (x, y) = 0,\ y \in M\}.$$

Theorem 14.10. *Let M be a closed subspace of a Hilbert space X. Then for every x in X, we can find unique elements v and w such that $v \in M$, $w \in M^{\perp}$ and*

$$x = v + w.$$

Proof. If $x \in M$, then we can let $v = x$ and $w = 0$. If $x \notin M$, then we can apply Theorem 14.9 to obtain an element z in M such that

$$\|x - z\| = d,$$

where

$$d = \inf_{z\in M} \|x - z\|.$$

Then we can write

$$x = z + (x - z)$$

and note that $x - z \in M^{\perp}$. Indeed, let $w = x - z$. Then $w \notin M$. Hence for all nonzero elements u in M and all complex numbers α, we get

$$d^2 \leq \|w - \alpha u\|^2 = \|w\|^2 - 2\,\mathrm{Re}(\alpha(u, w)) + |\alpha|^2\|u\|^2. \tag{14.6}$$

If we write

$$(u, w) = |(u, w)|e^{i\arg(u,w)}$$

and choose α such that

$$\alpha = te^{-i\arg(u,w)},$$

where t is any positive number, then by (14.6),

$$\begin{aligned} d^2 &\leq \|w\|^2 - 2t|(u, w)| + t^2\|u\|^2 \\ &= \|u\|^2 \left(t^2 - 2t\frac{|(u, w)|}{\|u\|^2} + \frac{|(u, w)|^2}{\|u\|^4} \right) + d^2 - \frac{|(u, w)|^2}{\|u\|^2} \\ &= \|u\|^2 \left(t - \frac{|(u, w)|}{\|u\|^2} \right)^2 + d^2 - \frac{|(u, w)|^2}{\|u\|^2}. \end{aligned} \tag{14.7}$$

If we let $t = \frac{|(u,w)|}{\|u\|^2}$ in (14.7), then

$$|(u, w)|^2 \leq 0.$$

Thus, $(u, w) = 0$, which is the same as saying that

$$x - z \in M^{\perp}.$$

Finally, for uniqueness, let us assume that

$$x = v_1 + w_1$$

and

$$x = v_2 + w_2,$$

where $v_1, v_2 \in M$ and $w_1, w_2 \in M^{\perp}$. Then

$$v_1 - v_2 = w_2 - w_1.$$

Thus,

$$v_1 - v_2 \in M \cap M^{\perp},$$

which means that

$$\|v_1 - v_2\|^2 = (v_1 - v_2, v_1 - v_2) = 0.$$

So, $v_1 - v_2 = 0$, i.e., $v_1 = v_2$. Similarly, $w_1 = w_2$, and the proof is complete. $\square$

As a corollary of Theorem 14.10, we give the following result.

Corollary 14.11. *Let M be a proper and closed subspace of a Hilbert space X. Then there exists a nonzero element y in X such that*

$$(y, z) = 0, \quad z \in M.$$

Proof. Let $x \in X \setminus M$. Then by Theorem 14.10,

$$x = v + w,$$

where $v \in M$ and $w \in M^{\perp}$. Let $y = w$. Then the corollary is proved if we can show that $w \neq 0$. But

$$w = 0 \Rightarrow x = v \in M,$$

which is a contradiction. □

Here is the Riesz representation theorem as promised.

Theorem 14.12 (The Riesz Representation Theorem). *Let T be a bounded linear functional on a Hilbert space X. Then there exists a unique element y in X such that*

$$T(x) = (x, y), \quad x \in X.$$

Proof. If $T(x) = 0$ for all x in X, then we take $y = 0$ and the theorem is proved. So, suppose that T is not identically zero. Let M be the subspace of X defined by

$$M = \{x \in X : T(x) = 0\}.$$

Then M is the null space of T and, by Exercise 10, must be closed. By Corollary 14.11, there exists a nonzero element w in X such that $w \in M^{\perp}$. Then $w \notin M$. Hence $T(w) \neq 0$ and for all x in X,

$$T(T(w)x - T(x)w) = T(w)T(x) - T(x)T(w) = 0.$$

So,

$$T(w)x - T(x)w \in M.$$

Since $w \in M^{\perp}$, it follows that

$$(T(w)x - T(x)w, w) = 0,$$

which can be expressed as

$$T(x)\|w\|^2 = T(w)(x, w).$$

Thus,

$$T(x) = \left(x, \frac{\overline{T(w)}}{\|w\|^2} w\right), \quad x \in X,$$

and the theorem is proved by taking $y = \frac{\overline{T(w)}}{\|w\|^2} w$. To prove uniqueness, let y_1 and y_2 be elements in X such that

$$T(x) = (x, y_1)$$

and

$$T(x) = (x, y_2)$$

for all x in X. Then

$$(x, y_1 - y_2) = 0, \quad x \in X.$$

If we let $x = y_1 - y_2$, then

$$\|y_1 - y_2\|^2 = (y_1 - y_2, y_1 - y_2) = 0$$

and hence $y_1 = y_2$. □

Exercises

1. Use the definition of an inner product to prove that for all x and y in a complex vector space X with inner product $(\ ,\ )$,

$$(x, y) = \overline{(y, x)}.$$

2. Let X be a complex vector space equipped with an inner product $(\ ,\ )$ and the induced norm $\|\ \|$. Prove that for all x and y in X,

$$|(x, y)| \leq \|x\|\,\|y\|$$

and equality occurs if and only if x and y are linearly dependent.

3. Prove that the norm $\|\ \|$ induced by an inner product $(\ ,\ )$ in a complex vector space X has the properties (14.1)–(14.4).

4. Let X be a complex vector space in which the norm $\|\ \|$ is induced by the inner product $(\ ,\ )$. Prove that for all x and y in X,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Explain why this is sometimes known as the law of the parallelogram.

5. Let X be an infinite-dimensional complex vector space with inner product $(\ ,\ )$. Prove that if $\{x_j\}_{j=1}^{\infty}$ converges to x in X, then for all y in X,

$$(x_j, y) \to (x, y)$$

as $j \to \infty$. Is the converse true?

6. Prove Proposition 14.2.

7. Prove Proposition 14.5.

8. Prove Proposition 14.6.

9. Let X be a Hilbert space in which the inner product and norm are denoted by, respectively, $(\ ,\ )$ and $\|\ \|$. Let $y \in X$. Then prove that the mapping $T : X \to \mathbb{C}$ defined by

$$T(x) = (x, y), \quad x \in X,$$

is a bounded linear functional.

10. Prove that the null space of a bounded linear functional T on X must be a closed subspace of X.

11. Prove that for all elements x in a Hilbert space X with inner product $(\ ,\ )$ and norm $\|\ \|$,

$$\|x\| = \sup_{\|y\|=1} |(x, y)|.$$

Chapter 15

Bounded Linear Operators

A linear operator A on a Hilbert space X is said to be a bounded linear operator on X if there exists a positive constant C such that

$$\|Ax\| \le C\|x\|, \quad x \in X.$$

For a bounded linear operator A on a Hilbert space X, we define the norm $\|A\|_*$ of A by

$$\|A\|_* = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|. \tag{15.1}$$

It follows from (15.1) that if A is a bounded linear operator on a Hilbert space X, then

$$\|Ax\| \le \|A\|_*\|x\|, \quad x \in X.$$

A linear operator A on a Hilbert space X is said to be continuous at a point x in X if for every sequence $\{x_j\}_{j=1}^{\infty}$ converging to x in X,

$$Ax_j \to Ax$$

in X as $j \to \infty$.

It is a basic fact in functional analysis that a linear operator A on a Hilbert space X is a bounded linear operator on X if and only if A is continuous at a point in X. See Exercise 1.

Let $B(X)$ be the set of all bounded linear operators on a Hilbert space X. Then it is easy to prove that $B(X)$ is a complex vector space with respect to the usual addition of two bounded linear operators and the usual scalar multiplication of a bounded linear operator by a complex number. In fact, $B(X)$ is a normed vector space with respect to the norm $\| \|_*$ given by (15.1). See Exercise 2.

In fact, we have the following theorem.

Theorem 15.1. *$B(X)$ is a Banach space with respect to the norm $\| \|_*$.*

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DOI 10.1007/978-3-0348-0116-4_15, © Springer Basel AG 2011

Proof. Let $\{A_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $B(X)$, i.e.,

$$\|A_j - A_k\|_* \to 0$$

as $j, k \to \infty$. Then all we need to prove is that there exists an element A in $B(X)$ such that

$$\|A_j - A\|_* \to 0$$

as $j \to \infty$. We first note that there exists a positive constant M such that

$$\|A_j x\| \le M\|x\|, \quad x \in X,\, j = 1, 2, \ldots. \tag{15.2}$$

Indeed, there exists a positive integer N such that

$$\|A_j - A_k\|_* < 1, \quad j, k \ge N.$$

So, using the triangle inequality for the norm $\| \; \|_*$, we see that

$$\|A_j\|_* < 1 + \|A_N\|_*, \quad j \ge N.$$

If we let

$$M = \max\{\|A_1\|_*, \ldots, \|A_{N-1}\|_*, 1 + \|A_N\|_*\},$$

then

$$\|A_j\|_* \le M, \quad j = 1, 2, \ldots,$$

establishing (15.2). Now, for all x in X,

$$\|A_j x - A_k x\| = \|(A_j - A_k)x\| \le \|A_j - A_k\|_* \|x\| \to 0$$

as $j, k \to \infty$. So, $\{A_j x\}_{j=1}^{\infty}$ is a Cauchy sequence in X. Since X is a Hilbert space, $\{A_j x\}_{j=1}^{\infty}$ converges to an element in X, which we denote by Ax. That $A : X \to X$ so defined is a linear operator follows from the linearity of taking limits. (This is Exercise 3.) Furthermore, we see from (15.2) that

$$\|Ax\| = \lim_{j\to\infty} \|A_j x\| \le M\|x\|, \quad x \in X.$$

Therefore $A \in B(X)$. It remains to prove that

$$\|A_j - A\|_* \to 0$$

as $j \to \infty$. But for all positive numbers ε, there exists a positive integer N such that

$$\|A_j - A_k\|_* < \varepsilon$$

whenever $j, k \ge N$. So, for all x in X,

$$\|A_j x - A_k x\| \le \|A_j - A_k\|_* \|x\| \le \varepsilon\|x\|$$

whenever $j, k \geq N$. Letting $k \to \infty$, we get for all x in X,

$$\|A_j x - Ax\| \leq \varepsilon \|x\|$$

whenever $j \geq N$. Thus,

$$\|A_j - A\|_* \leq \varepsilon$$

whenever $j \geq N$ and the proof is complete. □

We now give a result on the boundedness of a family of bounded linear operators on a Hilbert space. The result is known as the principle of uniform boundedness or the Banach–Steinhaus theorem and is dependent on the fine structure of Banach spaces, i.e., complete normed vector spaces. To explicate the structure, let X be a Banach space in which the norm is denoted by $\| \, \|$. We denote by $B(x_0, r)$ the open ball with center x_0 and radius r in X given by

$$B(x_0, r) = \{x \in X : \|x - x_0\| < r\}.$$

A subset W of X is said to be nowhere dense in X if the closure $\overline{W}$ of W contains no open balls. Then we have the following result known as Baire's category theorem.

Theorem 15.2. *A Banach space cannot be expressed as a countable union of nowhere dense sets.*

Proof. Let X be a Banach space with norm $\| \, \|$. Suppose by way of contradiction that

$$X = \bigcup_{k=1}^{\infty} W_k,$$

where W_k is nowhere dense for $k = 1, 2, \ldots$. Since W_1 is nowhere dense, there exists a point x_1 in X such that $x_1 \notin \overline{W_1}$. Thus, there exists a number r_1 in $(0, 1)$ such that

$$\overline{B(x_1, r_1)} \cap W_1 = \emptyset.$$

Since W_2 is nowhere dense, the open ball $B(x_1, r_1)$ is not a subset of $\overline{W_2}$. So, there exists a point x_2 in $B(x_1, r_1) \setminus \overline{W_2}$. Hence there exists a number r_2 in $\left(0, \frac{1}{2}\right)$ such that

$$\overline{B(x_2, r_2)} \cap W_2 = \emptyset$$

and

$$B(x_2, r_2) \subset B(x_1, r_1).$$

Thus, we can find a sequence $\{B(x_k, r_k)\}_{k=1}^{\infty}$ of open balls in X such that

$$r_k \in \left(0, \frac{1}{k}\right),$$

$$\overline{B(x_k, r_k)} \cap W_k = \emptyset$$

and

$$B(x_k, r_k) \subset B(x_{k-1}, r_{k-1}).$$

Now, for $j > k$, $x_j \in B(x_k, r_k)$ and hence

$$\|x_j - x_k\| < r_k < \frac{1}{k}. \tag{15.3}$$

So, $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence in X. Since X is complete, it follows that there exists a point x_0 in X such that

$$x_k \to x_0$$

in X as $k \to \infty$. Letting $j \to \infty$ in (15.3), we get

$$\|x_k - x_0\| \le r_k, \quad k = 1, 2, \ldots.$$

Therefore

$$x_0 \in \overline{B(x_k, r_k)}, \quad k = 1, 2, \ldots.$$

Then

$$x_0 \notin W_k, \quad k = 1, 2, \ldots.$$

So,

$$x_0 \notin \bigcup_{k=1}^\infty W_k = X$$

and this is a contradiction. □

We can now give the Banach–Steinhaus theorem for a family of bounded linear operators on a Hilbert space. The full-fledged version is presented as an exercise in this chapter.

Theorem 15.3. *For every Hilbert space X, let $W \subset B(X)$ be such that for all x in X,*

$$\sup_{A \in W} \|Ax\| < \infty.$$

Then

$$\sup_{A \in W} \|A\|_* < \infty.$$

Proof. For all positive integers n, let S_n be the subset of X defined by

$$S_n = \{x \in X : \|Ax\| \le n, A \in W\}.$$

Then S_n is closed in X. Indeed, let $\{x_k\}_{k=1}^\infty$ be a sequence in S_n such that $x_k \to x$ in X. Then for all A in W,

$$\|Ax_k\| \le n, \quad k = 1, 2, \ldots,$$

and

$$Ax_k \to Ax$$

in X as $k \to \infty$. So, for all A in W,

$$\|Ax\| = \lim_{k\to\infty} \|Ax_k\| \le n.$$

Therefore $x \in S_n$. Let $x \in X$. Since

$$\sup_{A\in W} \|Ax\| < \infty,$$

it follows that $x \in S_n$ for some n. Hence

$$X = \bigcup_{n=1}^{\infty} S_n.$$

By Theorem 15.2, there exists a positive integer N such that S_N is not nowhere dense. So, we can find an open ball $B(x_0, r)$ in X such that

$$B(x_0, r) \subset S_N.$$

Let x be any nonzero element in X and let z be the element in X defined by

$$z = x_0 + \frac{r}{2\|x\|}x.$$

Then

$$\|z - x_0\| = \frac{r}{2} < r$$

and hence

$$z \in B(x_0, r) \subset S_N.$$

Thus, for all A in W,

$$\|Az\| \le N$$

and hence

$$\|Ax\| = \left\| \frac{2\|x\|}{r}(z - x_0) \right\| \le \frac{4N}{r}\|x\|.$$

This proves that

$$\sup_{A\in W} \|A\|_* \le \frac{4N}{r}. \qquad \square$$

Let A be a bounded linear operator on a Hilbert space X. A complex number λ is said to be an eigenvalue of A if there exists a nonzero element x in X such that

$$Ax = \lambda x. \tag{15.4}$$

Let λ be an eigenvalue of A. Then a nonzero element x in X for which (15.4) holds is called an eigenvector of A corresponding to λ.

Remark 15.4. The set of all eigenvalues of a bounded linear operator A on a Hilbert space X is in general a proper subset of the spectrum $\Sigma(A)$ of A. In fact,

$$\Sigma(A) = \mathbb{C} \setminus \rho(A),$$

where $\rho(A)$ is the resolvent set of A defined by

$$\rho(A) = \{\lambda \in \mathbb{C} : A - \lambda I : X \to X \text{ is bijective}\},$$

and I is the identity operator on X. It is then easy to see that an eigenvalue λ of A is in $\Sigma(A)$ because $A - \lambda I : X \to X$ is not injective.

A crowning achievement in linear algebra is that an $n \times n$ self-adjoint matrix A can be transformed into a diagonal matrix in which the diagonal entries are the eigenvalues of A. Moreover, there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A. We need an analogous diagonalization of self-adjoint and compact operators on Hilbert spaces.

Exercises

1. Prove that a linear operator A on a Hilbert space X is a bounded linear operator on X if and only if A is continuous at a point in X.
2. Prove that $\| \,\|_*$ given by (15.1) is a norm in $B(X)$.
3. Prove that in the proof of Theorem 15.1, $A : X \to X$ is a linear operator.
4. Let X and Y be Banach spaces with norms denoted, respectively, by $\| \,\|_X$ and $\| \,\|_Y$, and let $A : X \to Y$ be a bounded linear operator, i.e., a linear operator such that there exists a positive constant C such that

   $$\|Ax\|_Y \le C\|x\|_X, \quad x \in X.$$

   Then the set $B(X, Y)$ of all bounded linear operators from X into Y is a Banach space with respect to the norm $\| \,\|_{B(X,Y)}$ given by

   $$\|A\|_{B(X,Y)} = \sup_{x \ne 0} \frac{\|Ax\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Ax\|_Y$$

   for all A in $B(X, Y)$. Prove the Banach–Steinhaus theorem to the effect that if W is a subset of $B(X, Y)$ such that for all x in X,

   $$\sup_{A \in W} \|Ax\|_Y < \infty,$$

   then

   $$\sup_{A \in W} \|A\|_{B(X,Y)} < \infty.$$

Chapter 16

Self-Adjoint Operators

Of particular importance in operator theory are self-adjoint operators. To this end, we first recall the notion of the adjoint of a bounded linear operator A on a Hilbert space X. A linear operator B on X is said to be an adjoint of A if

$$(Ax, y) = (x, By), \quad x, y \in X.$$

It is easy to see that a bounded linear operator A on X has at most one adjoint. See Exercise 1.

Theorem 16.1. *Every bounded linear operator A on a Hilbert space X has an adjoint, which is also a bounded linear operator on X.*

Proof. We first observe that for each fixed element y in X, the linear functional $T_y : X \to \mathbb{C}$ defined by

$$T_y(x) = (Ax, y), \quad x \in X,$$

is a bounded linear functional on X. Indeed, using the Schwarz inequality and the assumption that A is a bounded linear operator on X, there exists a positive constant C such that

$$|T_y(x)| = |(Ax, y)| \leq \|Ax\| \, \|y\| \leq C \, \|y\| \, \|x\|.$$

So, by the Riesz representation theorem, there exists a unique element z in X such that

$$T_y(x) = (Ax, y) = (x, z), \quad x \in X. \tag{16.1}$$

We now define the mapping $A^* : X \to X$ by

$$A^* y = z$$

for all y in X, where z is the unique element in X that depends on y and is guaranteed for each y in X by the Riesz representation theorem. To see that

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DOI 10.1007/978-3-0348-0116-4_16, © Springer Basel AG 2011

$A^* : X \to X$ is a linear operator, let α_1 and α_2 be complex numbers, and let y_1 and y_2 be elements in X. Then by (16.1), we get for all x in X,

$$\begin{aligned}
&(x, A^*(\alpha_1 y_1 + \alpha_2 y_2)) \\
&= (Ax, \alpha_1 y_1 + \alpha_2 y_2) = \overline{\alpha_1}(Ax, y_1) + \overline{\alpha_2}(Ax, y_2) \\
&= \overline{\alpha_1}(x, A^* y_1) + \overline{\alpha_2}(x, A^* y_2) = (x, \alpha_1 A^* y_1) + (x, \alpha_2 A^* y_2) \\
&= (x, \alpha_1 A^* y_1 + \alpha_2 A^* y_2).
\end{aligned}$$

Thus,

$$A^*(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 A^* y_1 + \alpha_2 A^* y_2.$$

That A^* is bounded follows because for all y in X, we can use the Schwarz inequality and the assumption that A is bounded to obtain a positive constant C such that

$$\|A^* y\|^2 = (A^* y, A^* y) = |(AA^* y, y)| \leq C\|A^* y\| \, \|y\|.$$

Thus,

$$\|A^* y\| \leq C\|y\|, \quad y \in X,$$

and the proof is complete. □

A bounded linear operator A from a Hilbert space X into X is said to be self-adjoint if $A = A^*$.

Self-adjoint operators enjoy special properties not shared by bounded linear operators in general. The following interesting result on the norm of a self-adjoint operator should be compared with the formula (15.1).

Theorem 16.2. *Let A be a self-adjoint operator on a Hilbert space X. Then*

$$\|A\|_* = \sup_{\|x\|=1} |(Ax, x)|.$$

Proof. Let $x \in X$ be such that $\|x\| = 1$. Then by the Schwarz inequality and the fact that A is a bounded linear operator,

$$|(Ax, x)| \leq \|Ax\| \, \|x\| \leq \|A\|_* \|x\|^2 = \|A\|_*.$$

Thus,

$$\sup_{\|x\|=1} |(Ax, x)| \leq \|A\|_*.$$

To prove the converse, we let

$$M = \sup_{\|x\|=1} |(Ax, x)|.$$

Then for all x in X,

$$|(Ax, x)| \leq M\|x\|^2. \tag{16.2}$$

It is left as Exercise 3 to prove that for all x and y in X,

$$\begin{aligned} 4(Ax,y) = \{&(A(x+y),x+y)-(A(x-y),x-y)\} \\ &+ i\{(A(x+iy),x+iy)-(A(x-iy),x-iy)\}. \end{aligned} \tag{16.3}$$

Then for all z in X with $z \neq 0$ and $Az \neq 0$, let

$$u = \frac{Az}{\alpha},$$

where $\alpha^2 = \frac{\|Az\|}{\|z\|}$. Then by (16.2), (16.3) and Exercise 4, we get

$$\begin{aligned} \|Az\|^2 &= (A(\alpha z),u) \\ &= \frac{1}{4}\{(A(\alpha z+u),\alpha z+u)-(A(\alpha z-u),\alpha z-u)\} \\ &\leq \frac{1}{4}M(\|\alpha z+u\|^2+\|\alpha z-u\|^2) \\ &= \frac{1}{2}M(\|\alpha z\|^2+\|u\|^2) \\ &= \frac{1}{2}M\left(\alpha^2\|z\|^2+\frac{1}{\alpha^2}\|Az\|^2\right) \\ &= M\|Az\|\,\|z\|. \end{aligned}$$

Thus, for all z in X with $z \neq 0$ and $Az \neq 0$,

$$\|Az\| \leq M\|z\|. \tag{16.4}$$

It is clear that (16.4) is trivially true for $Az = 0$ or $z = 0$. Thus,

$$\|Az\| \leq M\|z\|, \quad z \in X,$$

and therefore

$$\|A\|_* \leq M = \sup_{\|x\|=1} |(Ax,x)|. \qquad \square$$

Theorem 16.3. *Let A be a self-adjoint operator on a Hilbert space X. Then all eigenvalues of A are real. Moreover, if x and y are eigenvectors of A corresponding to, respectively, eigenvalues λ and μ, where $\lambda \neq \mu$, then*

$$(x,y) = 0.$$

Proof. Let λ be an eigenvalue of A and let φ be a corresponding eigenvector with $\|\varphi\| = 1$. Then

$$\lambda = \lambda(\varphi,\varphi) = (A\varphi,\varphi) = (\varphi,A\varphi) = \overline{\lambda}.$$

So, λ is a real number. Now, using the fact that λ and μ are real and the self-adjointness of A,

$$\lambda(x,y) = (Ax,y) = (x,Ay) = \mu(x,y).$$

So,

$$(\lambda - \mu)(x, y) = 0.$$

Since $\lambda \neq \mu$, it follows that

$$(x, y) = 0. \qquad \square$$

The second part of Theorem 16.3 tells us that eigenvectors corresponding to distinct eigenvalues of a self-adjoint operator are orthogonal.

Exercises

1. Prove that a linear operator on a Hilbert space has at most one adjoint.
2. Let A be a bounded linear operator on a Hilbert space X. Let M be a closed subspace of X such that M is invariant with respect to A, i.e.,

   $$x \in M \Rightarrow Ax \in M.$$

   Prove that the orthogonal complement $M^\perp$ of M is invariant with respect to A^*.
3. Prove (16.3) for all bounded linear operators A on a Hilbert space X.
4. Prove that a bounded linear operator A is self-adjoint on a Hilbert space X if and only if

   $$(Ax, x) \in \mathbb{R}$$

   for all x in X.
5. Let A be a bounded linear operator on a Hilbert space X. Prove that

   $$(A^*)^* = A.$$

6. Prove that for all bounded linear operators A on a Hilbert space X,

   $$\|A^*\|_* = \|A\|_*.$$

Chapter 17

Compact Operators

A sequence $\{x_j\}_{j=1}^{\infty}$ in a Hilbert space X is said to be bounded if there exists a positive constant C such that

$$\|x_j\| \leq C, \quad j = 1, 2, \dots.$$

A bounded linear operator A on a Hilbert space X is said to be compact if for every bounded sequence $\{x_j\}_{j=1}^{\infty}$ in X, the sequence $\{Ax_j\}_{j=1}^{\infty}$ has a convergent subsequence in X.

A bounded linear operator A on a Hilbert space X is said to be an operator of finite rank if the range $R(A)$ of A given by

$$R(A) = \{Ax : x \in X\}$$

is finite-dimensional. It can be proved that an operator of finite rank must be compact. See Exercise 1.

Theorem 17.1. *Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of compact operators on a Hilbert space X such that*

$$\|A_j - A\|_* \to 0$$

as $j \to \infty$, where A is a bounded linear operator on X. Then A is a compact operator on X.

Proof. Let $\{x_j\}_{j=1}^{\infty}$ be a bounded sequence in X. Then there exists a positive constant C such that

$$\|x_j\| \leq C, \quad j = 1, 2, \dots.$$

Since A_1 is compact, there exists a subsequence $\{x_{1,j}\}_{j=1}^{\infty}$ of $\{x_j\}_{j=1}^{\infty}$ such that $\{A_1 x_{1,j}\}_{j=1}^{\infty}$ converges in X. Since A_2 is compact, there is a subsequence $\{x_{2,j}\}_{j=1}^{\infty}$ of $\{x_{1,j}\}_{j=1}^{\infty}$ such that $\{A_2 x_{2,j}\}_{j=1}^{\infty}$ converges in X. Thus, repeating this argument, there exists a subsequence $\{x_{n,j}\}_{j=1}^{\infty}$ of $\{x_{n-1,j}\}_{j=1}^{\infty}$ such that $\{A_n x_{n,j}\}_{j=1}^{\infty}$ converges in X. For $j = 1, 2, \dots$, let $z_j = x_{j,j}$. Then for $k = 1, 2, \dots$, $\{A_k z_j\}_{j=1}^{\infty}$

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DOI 10.1007/978-3-0348-0116-4_17, © Springer Basel AG 2011

converges in X. So, for $j, k, l = 1, 2, \ldots,$

$$\begin{aligned}\|Az_j - Az_l\| &\le \|Az_j - A_k z_j\| + \|A_k z_j - A_k z_l\| + \|A_k z_l - Az_l\| \\ &\le 2C\|A - A_k\|_* + \|A_k z_j - A_k z_l\|. \end{aligned} \tag{17.1}$$

Now, for every positive number ε, there exists a positive integer K such that

$$2C\|A - A_K\|_* < \frac{\varepsilon}{2}. \tag{17.2}$$

Since $\{A_K z_j\}_{j=1}^{\infty}$ converges in X, it follows that there exists a positive integer N such that

$$j, l \ge N \Rightarrow \|A_K z_j - A_K z_l\| < \frac{\varepsilon}{2}. \tag{17.3}$$

So, by (17.1)–(17.3),

$$j, l \ge N \Rightarrow \|Az_j - Az_l\| < \varepsilon.$$

Thus, $\{Az_j\}_{j=1}^{\infty}$ is a Cauchy sequence in X and hence convergent in X. □

A sequence $\{x_j\}_{j=1}^{\infty}$ in a Hilbert space X is said to converge weakly to x in X if

$$(x_j, y) \to (x, y)$$

for all y in X as $j \to \infty$. By Exercise 5 in Chapter 14, convergence in X implies weak convergence in X, but the converse is not true. It is also an exercise in this chapter to show that a weakly convergent sequence in X has to be bounded.

Theorem 17.2. *Let A be a compact operator on a Hilbert space X. Then A maps weakly convergent sequences into convergent sequences.*

Proof. Let $\{x_j\}_{j=1}^{\infty}$ be a sequence in X such that $x_j \to x$ weakly in X as $j \to \infty$. Then for all y in X,

$$(Ax_j - Ax, y) = (x_j - x, A^* y) \to 0$$

as $j \to \infty$. Therefore $Ax_j \to Ax$ weakly in X as $j \to \infty$. Suppose that $\{Ax_j\}_{j=1}^{\infty}$ does not converge to Ax in X. Then there exists a positive number ε such that

$$\|Ax_{j_k} - Ax\| \ge \varepsilon,$$

where $\{x_{j_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_j\}_{j=1}^{\infty}$. It is clear that $\{x_{j_k}\}_{k=1}^{\infty}$ is a weakly convergent sequence in X and hence it is bounded. Since A is compact, we can find a subsequence of $\{x_{j_k}\}_{k=1}^{\infty}$, again denoted by $\{x_{j_k}\}_{k=1}^{\infty}$, such that

$$Ax_{j_k} \to y$$

for some y in X as $k \to \infty$. Thus, $\{Ax_{j_k}\}_{k=1}^{\infty}$ converges weakly to y in X as $k \to \infty$. Thus, $Ax = y$, and hence

$$Ax_{j_k} \to Ax$$

in X as $k \to \infty$. □

Theorem 17.3. *Let A be a self-adjoint and compact operator on a Hilbert space X. Then $\|A\|_*$ or $-\|A\|_*$ is an eigenvalue of A.*

Proof. By Theorem 16.2,

$$\|A\|_* = \sup_{\|x\|=1} |(Ax, x)|.$$

So, we can find a sequence $\{x_j\}_{j=1}^{\infty}$ of elements in X such that

$$\|x_j\| = 1, \quad j = 1, 2, \ldots,$$

and

$$|(Ax_j, x_j)| \to \|A\|_*$$

as $j \to \infty$. There are three cases to be considered. The first case is when there exists a positive integer J such that

$$j \geq J \Rightarrow (Ax_j, x_j) > 0.$$

The second case is when > 0 is replaced by < 0. The third case is when the first case and the second case do not stand. For the first case, let $\lambda = \|A\|_*$. Then

$$\begin{aligned} \|Ax_j - \lambda x_j\|^2 &= \|Ax_j\|^2 - 2\lambda(Ax_j, x_j) + \lambda^2\|x_j\|^2 \\ &\leq 2\lambda^2 - 2\lambda(Ax_j, x_j) \to 0 \end{aligned} \tag{17.4}$$

as $j \to \infty$. Since A is a compact operator, there exists a subsequence of $\{x_j\}_{j=1}^{\infty}$, again denoted by $\{x_j\}_{j=1}^{\infty}$, such that

$$Ax_j \to y \tag{17.5}$$

for some y in X as $j \to \infty$. By (17.4) and (17.5), we see that

$$\lambda x_j \to y \tag{17.6}$$

as $j \to \infty$. Using the boundedness and hence continuity of A,

$$\lambda A x_j \to Ay$$

as $j \to \infty$. But by (17.5) again,

$$\lambda A x_j \to \lambda y.$$

Thus,

$$Ay = \lambda y.$$

So, $\lambda = \|A\|_*$ is an eigenvalue of A if we can show that $y \neq 0$. But by (17.6),

$$\|y\| = \lim_{j\to\infty} \|\lambda x_j\| = \|A\|_* \neq 0.$$

This completes the proof for the first case. The proof for the second case is the same if we let $\lambda = -\|A\|_*$. The third case is the same as the first case or the second case if we pass to a subsequence of $\{x_j\}_{j=1}^{\infty}$. □

Exercises

1. Prove that an operator of finite rank on a Hilbert space is compact.
2. Let $K(X)$ be the set of all compact operators on a Hilbert space X. Prove that $K(X)$ is a two-sided ideal in $B(X)$, i.e., for all $K \in K(X)$ and all $A \in B(X)$,
$$KA \in K(X)$$
and
$$AK \in K(X).$$
3. Prove that the limit of a weakly convergent sequence in X is unique.
4. Let $\{x_j\}_{j=1}^\infty$ be a weakly convergent sequence in a Hilbert space X. Prove that $\{x_j\}_{j=1}^\infty$ is bounded.
5. Let X be a Hilbert space with inner product $(\ ,\ )$ and norm $\|\ \|$. Let $\{x_j\}_{j=1}^\infty$ be a sequence in X such that $x_j \to x$ in X weakly and
$$\|x_j\| \to \|x\|$$
as $j \to \infty$. Prove that $x_j \to x$ in X as $j \to \infty$.

Chapter 18

The Spectral Theorem

We are now in a good position to state and prove the spectral theorem for self-adjoint and compact operators on Hilbert spaces.

Theorem 18.1 (The Spectral Theorem). *Let A be a self-adjoint and compact operator on a Hilbert space X. Then there exists an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for X consisting of eigenvectors of A. Moreover, for all x in X,*

$$Ax = \sum_{j=1}^{\infty} \lambda_j (x, \varphi_j)\varphi_j,$$

where λ_j is the eigenvalue of A corresponding to the eigenvector φ_j.

Proof. By Theorem 17.3, $\|A\|_*$ or $-\|A\|_*$ is an eigenvalue of A. Let $\lambda_1 = \pm\|A\|_*$ and let φ_1 be a corresponding eigenvector with $\|\varphi_1\| = 1$. Let $\Phi_1 = \text{Span}\{\varphi_1\}$ and let $X_2 = \Phi_1^{\perp}$. Φ_1 is obviously invariant with respect to A. Then by Exercise 2 in Chapter 16, X_2 is invariant with respect to A^* and hence A. Let A_2 be the restriction of A to X_2. Then $A_2 : X_2 \to X_2$ is obviously compact. It is also self-adjoint because for all x and y in X_2,

$$(A_2x, y) = (Ax, y) = (x, Ay) = (x, A_2y) = (A_2^*x, y).$$

By Theorem 17.3 again, $\|A_2\|_*$ or $-\|A_2\|_*$ is an eigenvalue of A_2. Let $\lambda_2 = \pm\|A_2\|_*$ and let φ_2 be a corresponding eigenvector with $\|\varphi_2\| = 1$. Repeating this construction, let us suppose that we have eigenvectors

$$\varphi_1, \varphi_2, \ldots, \varphi_n$$

of A and corresponding eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

such that

$$|\lambda_j| = \|A_j\|_*, \quad j = 1, 2, \ldots, n,$$

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DOI 10.1007/978-3-0348-0116-4_18, © Springer Basel AG 2011

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|, \tag{18.1}$$

$A_1 = A$ and for $2 \leq j \leq n$, A_j is the restriction to the orthogonal complement

$$X_j = (\mathrm{Span}\{\varphi_1, \varphi_2, \ldots, \varphi_{j-1}\})^\perp.$$

Moreover, for $2 \leq j \leq n$, $A_j : X_j \to X_j$ is self-adjoint and compact. We repeat this process and stop if $A_n = 0$. If $A_n = 0$, then for all x in X,

$$x_n = x - \sum_{j=1}^{n-1} (x, \varphi_j)\varphi_j \in X_n,$$

and so is the same as

$$A_n x_n = Ax - \sum_{j=1}^{n-1} (x, \varphi_j) A\varphi_j.$$

Thus,

$$Ax = \sum_{j=1}^{n-1} \lambda_j (x, \varphi_j)\varphi_j, \quad x \in X,$$

as required. Now, suppose that

$$A_n \neq 0, \quad n = 1, 2, \ldots.$$

Let $x \in X$. Then the element x_n defined by

$$x_n = x - \sum_{j=1}^{n-1} (x, \varphi_j)\varphi_j$$

is in X_n. So, by Proposition 14.5, we get

$$\|x\|^2 = \|x_n\|^2 + \sum_{j=1}^{n-1} |(x, \varphi_j)|^2.$$

Therefore

$$\|x_n\| \leq \|x\|, \quad n = 1, 2, \ldots.$$

But

$$\|Ax_n\| = \|A_n x_n\| \leq \|A_n\|_* \|x_n\| \leq |\lambda_n| \, \|x\|, \quad n = 1, 2, \ldots.$$

So,

$$\left\| Ax - \sum_{j=1}^{n-1} \lambda_j (x, \varphi_j)\varphi_j \right\| \leq |\lambda_n| \, \|x\|.$$

Now, we note that $\lambda_n \to 0$ as $n \to \infty$. Indeed, using the compactness of A, there exists a subsequence $\{\varphi_{n_j}\}_{j=1}^\infty$ of $\{\varphi_n\}_{n=1}^\infty$ such that $\{A\varphi_{n_j}\}_{j=1}^\infty$ converges in X. Thus,

$$\|\lambda_{n_j}\varphi_{n_j} - \lambda_{n_k}\varphi_{n_k}\|^2 = \|A\varphi_{n_j} - A\varphi_{n_k}\|^2 \to 0$$

as $j, k \to \infty$. So,

$$|\lambda_{n_j}|^2 + |\lambda_{n_k}|^2 = \|\lambda_{n_j}\varphi_{n_j} - \lambda_{n_k}\|^2 \to 0$$

as $j, k \to \infty$. Hence by (18.1),

$$|\lambda_n| \to 0$$

as $n \to \infty$. So,

$$Ax = \sum_{j=1}^{\infty} \lambda_j (x, \varphi_j)\varphi_j, \quad x \in X. \tag{18.2}$$

Note that in (18.2),

$$\lambda_j \neq 0, \quad j = 1, 2, \ldots.$$

Let $\{\psi_j\}_{j=1}^K$ be an orthonormal basis for the null space $N(A)$ of A, where $K \leq \infty$. By Theorem 16.3, we see that

$$(\varphi_j, \psi_k) = 0, \quad j = 1, 2, \ldots, \; k = 1, 2, \ldots, K.$$

Now, let $x \in X$. Then by (18.2),

$$x - \sum_{j=1}^{J} (x, \varphi_j)\varphi_j \in N(A).$$

Therefore

$$x = \sum_{j=1}^{J} (x, \varphi_j)\varphi_j + \sum_{k=1}^{K} (x, \psi_k)\psi_k, \quad x \in X,$$

and the proof is complete. □

Exercises

1. Let A be a self-adjoint and compact operator on a Hilbert space X such that

$$Ax = \sum_{j=1}^{\infty} \lambda_j (x, \varphi_j)\varphi_j, \quad x \in X,$$

   where $\{\varphi_j\}_{j=1}^\infty$ is an orthonormal basis for X consisting of eigenvectors of A and λ_j is a corresponding eigenvalue of A corresponding to φ_j. Prove that if a complex number λ is an eigenvalue of A, then there exists a positive integer j such that $\lambda = \lambda_j$.

2. Let A be a positive operator on a Hilbert space X, i.e.,

$$(Ax, x) \geq 0, \quad x \in X.$$

Furthermore, suppose that A is compact and can be written as in Exercise 1, i.e.,

$$Ax = \sum_{j=1}^{\infty} \lambda_j (x, \varphi_j)\varphi_j, \quad x \in X.$$

Then we define the square root $A^{1/2}$ of A by

$$A^{1/2}x = \sum_{j=1}^{\infty} \lambda_j^{1/2} (x, \varphi_j)\varphi_j, \quad x \in X.$$

Prove that for $j = 1, 2, \ldots,$ $\lambda_j^{1/2}$ is an eigenvalue of $A^{1/2}$.

3. Let A and $A^{1/2}$ be as in Exercise 2. Prove that for every eigenvalue λ of $A^{1/2}$, there exists a positive integer j such that $\lambda = \lambda_j^{1/2}$.

Chapter 19

Schatten–von Neumann Classes

We consider special classes of compact operators in this chapter known as Schatten–von Neumann classes S_p, $1 \le p < \infty$. The most distinguished class is S_2, which is made up of Hilbert–Schmidt operators.

Let A be a compact operator on a Hilbert space X. Then by Exercise 2 in Chapter 17, A^*A is compact. It is easy to show that A^*A is self-adjoint. See Exercise 1. Then by Exercises 2 and 3 in Chapter 18, we can look at all eigenvalues of $(A^*A)^{1/2}$ and we enumerate them as

$$s_1, s_2, \ldots .$$

We call these positive numbers the singular values of A. For $1 \le p < \infty$, the compact operator A is said to be in the Schatten–von Neumann class S_p if

$$\sum_{j=1}^{\infty} s_j^p < \infty.$$

For all $A \in S_p$, $1 \le p < \infty$, we define $\|A\|_{S_p}$ by

$$\|A\|_{S_p} = \left(\sum_{j=1}^{\infty} s_j^p \right)^{1/p} .$$

To complete the picture, we define S_∞ to be simply $B(X)$. The classes S_1 and S_2 are, respectively, the trace class and the Hilbert–Schmidt class. Now, we study the Hilbert–Schmidt class in some detail. We begin with a lemma, which we leave as Exercise 3.

Lemma 19.1. *Let A be a bounded linear operator on a Hilbert space X. Then for all orthonormal bases $\{\varphi_j\}_{j=1}^{\infty}$ and $\{\psi_j\}_{j=1}^{\infty}$ for X,*

$$\sum_{j=1}^{\infty} \|A\varphi_j\|^2 = \sum_{j=1}^{\infty} \|A^*\psi_j\|^2,$$

where the sums may be ∞.

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DOI 10.1007/978-3-0348-0116-4_19, © Springer Basel AG 2011

Theorem 19.2. *Let A be a bounded linear operator on a Hilbert space X. Then $A \in S_2$ if and only if there exists an orthonormal basis $\{\varphi_j\}_{j=1}^\infty$ for X such that*

$$\sum_{j=1}^\infty \|A\varphi_j\|^2 < \infty.$$

Proof. Suppose that $A \in S_2$. Then let $\{\varphi_j\}_{j=1}^\infty$ be an orthonormal basis for X consisting of eignvectors of $(A^*A)^{1/2}$. For $j = 1, 2, \ldots,$ let s_j be the eigenvalue of $(A^*A)^{1/2}$ corresponding to the eigenvector φ_j. Then for $j = 1, 2, \ldots,$ s_j^2 is an eigenvalue of A^*A with φ_j as a corresponding eigenvector. Thus,

$$\sum_{j=1}^\infty \|A\varphi_j\|^2 = \sum_{j=1}^\infty (A^*A\varphi_j, \varphi_j) = \sum_{j=1}^\infty s_j^2 < \infty.$$

The proof of the converse is the same if we can prove that A is compact. For all positive integers N, we define the linear operator A_N on X by

$$A_N x = \sum_{j=1}^N (Ax, \varphi_j)\varphi_j, \quad x \in X.$$

Then the range of A_N is spanned by $\varphi_1, \varphi_2, \ldots, \varphi_N$ and is hence finite-dimensional. Thus, A_N is an operator of finite rank on X. Moreover, we obtain by means of Lemma 14.3 and the Schwarz inequality for all x in X,

$$\begin{aligned}
\|(A - A_N)x\|^2 &= \left\| \sum_{j=N+1}^\infty (Ax, \varphi_j)\varphi_j \right\|^2 \\
&\le \sum_{j=N+1}^\infty |(Ax, \varphi_j)|^2 \\
&= \sum_{j=N+1}^\infty |(x, A^*\varphi_j)|^2 \\
&\le \|x\|^2 \sum_{j=N+1}^\infty \|A^*\varphi_j\|^2.
\end{aligned}$$

Thus,

$$\|A - A_N\|_* \le \sum_{j=N+1}^\infty \|A^*\varphi_j\|^2.$$

By Lemma 19.1,

$$\sum_{j=1}^\infty \|A^*\varphi_j\|^2 < \infty$$

and hence

$$\|A - A_N\|_* \leq \sum_{j=N+1}^{\infty} \|A^*\varphi_j\|^2 \to 0$$

as $N \to \infty$. So, A is the limit in norm of a sequence of compact operators on X. By Theorem 17.1, A is compact. □

Exercises

1. Let A be a bounded linear operator on a Hilbert space. Prove that A^*A is self-adjoint.
2. Prove that if $1 \leq p \leq q \leq \infty$, then $S_p \subseteq S_q$.
3. Prove Lemma 19.1.

Chapter 20

Fourier Series

In this chapter we give a succinct and sufficiently self-contained treatment of Fourier series that we need for the rest of this book. A proper treatment of Fourier series entails the use of measure theory and the corresponding theory of integration, which we assume as prerequisites.

Let us begin with the space $L^1[-\pi, \pi]$ of all functions f on $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi} |f(\theta)|\, d\theta < \infty.$$

Let $f \in L^1[-\pi, \pi]$. Then we define the Fourier transform $\hat{f}$ on the set $\mathbb{Z}$ of all integers by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta, \quad n \in \mathbb{Z}. \tag{20.1}$$

We also call $\hat{f}(n)$ the Fourier transform or the Fourier coefficient of the function f at frequency n. The formal sum $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}$ is referred to as the Fourier series of f on $[-\pi, \pi]$.

Remark 20.1. We take to heart the convention that the interval $[-\pi, \pi]$ can be identified with the unit circle $\mathbb{S}^1$ centered at the origin. With this identification, functions on $[-\pi, \pi]$ can be identified as functions on $\mathbb{S}^1$ or as periodic functions with period 2π on $\mathbb{R}$. Fourier series can then be thought of as Fourier analysis on $\mathbb{S}^1$ and we have this picture in mind for the rest of the book.

The most obvious and fundamental problem at this point is the mathematical interpretation of the equation

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}, \quad \theta \in [-\pi, \pi]. \tag{20.2}$$

A natural interpretation of (20.2) is that the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}$ converges pointwise to $f(\theta)$ for all θ in $[-\pi, \pi]$. To be more precise, let $\{s_N\}_{N=0}^{\infty}$ be

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DOI 10.1007/978-3-0348-0116-4_20, © Springer Basel AG 2011

the sequence of partial sums of the Fourier series defined by

$$s_N(\theta) = \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta}, \quad \theta \in [-\pi, \pi].$$

Then

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} = f(\theta)$$

for all θ in $[-\pi, \pi]$ means that

$$\lim_{N\to\infty} s_N(\theta) = f(\theta)$$

for all θ in $[-\pi, \pi]$. Sufficient conditions on f ensuring the pointwise convergence as defined abound and we give in the first part of this chapter a small sample of such sufficient conditions. In order to do this, we need some preparation.

For $1 \le p < \infty$, let $L^p(\mathbb{S}^1)$ be the set of all measurable functions f on $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi} |f(\theta)|^p d\theta < \infty.$$

Then $L^p(\mathbb{S}^1)$ is a vector space endowed with the norm $\| \, \|_p$ given by

$$\|f\|_p = \left(\int_{-\pi}^{\pi} |f(\theta)|^p d\theta\right)^{1/p}, \quad f \in L^p(\mathbb{S}^1).$$

In fact, $L^p(\mathbb{S}^1)$ is a Banach space with respect to the norm $\| \, \|_p$. This simply means that every sequence $\{f_j\}_{j=1}^{\infty}$ with

$$\|f_j - f_k\|_p \to 0$$

as $j, k \to \infty$ has to converge to some f in $L^p(\mathbb{S}^1)$ as $j \to \infty$, i.e.,

$$\|f_j - f\|_p \to 0.$$

For $p = \infty$, $L^\infty(\mathbb{S}^1)$ is the space of all essentially bounded functions on $[-\pi, \pi]$. It turns out that $L^\infty(\mathbb{S}^1)$ is a Banach space with respect to the norm $\| \, \|_\infty$ given by the essential supremum of f on $[-\pi, \pi]$. In fact, for all f in $L^\infty(\mathbb{S}^1)$,

$$\|f\|_\infty = \inf\{M > 0 : m\{\theta \in [-\pi, \pi] : |f(\theta)| > M\} = 0\},$$

where $m\{\cdots\}$ is the Lebesgue measure of the set $\{\cdots\}$.

Of particular importance is the space $L^2(\mathbb{S}^1)$, which is a Hilbert space in which the inner product $(\, , \,)_2$ and norm $\| \, \|_2$ are given, respectively, by

$$(f, g)_2 = \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\, d\theta$$

and

$$\|f\|_2 = \left(\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta \right)^{1/2}$$

for all f and g in $L^2(\mathbb{S}^1)$.

For $n \in \mathbb{Z}$, we define the function e_n on $[-\pi, \pi]$ by

$$e_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}, \quad \theta \in [-\pi, \pi]. \tag{20.3}$$

Then we have the following simple but useful result.

Lemma 20.2. $\{e_n\}_{n=-\infty}^{\infty}$ *is an orthonormal set in* $L^2(\mathbb{S}^1)$.

Proof. For $m, n \in \mathbb{Z}$ with $m \neq n$, we have

$$\begin{aligned}(e_m, e_n)_2 &= \int_{-\pi}^{\pi} e_m(\theta)\overline{e_n(\theta)}\, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)\theta} d\theta \\ &= \frac{1}{2\pi} \frac{1}{m-n} e^{i(m-n)\theta}\Big|_{-\pi}^{\pi} = 0.\end{aligned}$$

If $m = n$, then

$$(e_m, e_n)_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta = 1. \qquad \square$$

We have the following useful corollary, which is known as the Riemann–Lebesgue lemma.

Corollary 20.3. *Let* $f \in L^1(\mathbb{S}^1)$. *Then* $\lim_{|n|\to\infty} \hat{f}(n) = 0$.

To prove the Riemann–Lebesgue lemma and other results in analysis, we invoke the space of all C^∞ functions φ on $(-\pi, \pi)$ such that the support $\operatorname{supp}(\varphi)$ of φ is contained in $(-\pi, \pi)$. Let us also recall that $\operatorname{supp}(\varphi)$ is the closure of the set $\{\theta \in (-\pi, \pi) : \varphi(\theta) \neq 0\}$.

Lemma 20.4. $C_0^\infty(\mathbb{S}^1)$ *is dense in* $L^p(\mathbb{S}^1)$, $1 \le p < \infty$.

For a proof of Lemma 20.4, see, for instance, [19, 49].

Proof of Corollary 20.3. Let $f \in L^2(\mathbb{S}^1)$. Then using the orthonormality of the sequence $\{e_n\}_{n=-\infty}^{\infty}$ and the Bessel inequality in Lemma 14.3, we get by (20.1) and (20.3)

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \le \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} |(f, e_n)_2|^2 \le \frac{1}{2\pi} \|f\|_2^2.$$

So,

$$\hat{f}(n) \to 0$$

as $|n| \to \infty$. Now, let $f \in L^1(\mathbb{S}^1)$ and let ε be a given positive number. Then by Lemma 20.4, there exists a function φ in $C_0^\infty(-\pi, \pi)$ such that

$$\|f - \varphi\|_1 < \pi\varepsilon.$$

Thus, by (20.1) and the triangle inequality, we get for all n in $\mathbb{Z}$,

$$|\hat{f}(n)| \le |\hat{\varphi}(n)| + \frac{1}{2\pi}\|f - \varphi\|_1 < |\hat{\varphi}(n)| + \frac{\varepsilon}{2}.$$

Since $\varphi \in L^2(\mathbb{S}^1)$, it follows that

$$\hat{\varphi}(n) \to 0$$

as $|n| \to \infty$. So, there exists a positive integer N such that

$$|n| \ge N \Rightarrow |\hat{\varphi}(n)| < \frac{\varepsilon}{2}.$$

Therefore

$$|n| \ge N \Rightarrow |\hat{f}(n)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and the proof is complete. □

The next lemma gives an integral representation for the partial sum of a Fourier series.

Lemma 20.5. *Let* $f \in L^1(\mathbb{S}^1)$. *Then for all* $N = 0, 1, 2, \ldots$,

$$s_N(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta - \phi) f(\phi)\, d\phi, \quad \phi \in [-\pi, \pi],$$

where

$$D_N(\theta) = \frac{\sin\left(N + \frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}, \quad \theta \in [-\pi, \pi].$$

Remark 20.6. The function D_N is usually dubbed the Dirichlet kernel of the Fourier series and Lemma 20.5 gives an expression for the partial sum of a Fourier series of a function as the convolution of the Dirichlet kernel and the function. See Exercise 3 for convolutions in general.

Proof of Lemma 20.5. For $N = 0, 1, 2, \ldots$, and $\theta \in [-\pi, \pi]$,

$$\begin{aligned} s_N(\theta) &= \sum_{n=-N}^{N} \hat{f}(n) e^{in\theta} \\ &= \frac{1}{2\pi}\sum_{n=-N}^{N}\left(\int_{-\pi}^{\pi} e^{-in\phi} f(\phi)\, d\phi\right) e^{in\theta} \\ &= \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{-\pi}^{\pi} e^{in(\theta-\phi)} f(\phi)\, d\phi. \end{aligned}$$

Let us now compute $\sum_{n=-N}^{N} e^{in\theta}$ explicitly. Indeed,

$$\sum_{n=-N}^{N} e^{in\theta} = 1 + \sum_{n=1}^{N}(e^{in\theta} + e^{-in\theta}) = 1 + 2\mathrm{Re}\sum_{n=1}^{N} e^{in\theta}.$$

But

$$\begin{aligned}\sum_{n=1}^{N} e^{in\theta} &= e^{i\theta}\sum_{n=0}^{N-1} e^{in\theta} = e^{i\theta}\frac{1-e^{iN\theta}}{1-e^{i\theta}} = \frac{e^{\frac{1}{2}i\theta} - e^{i\left(N+\frac{1}{2}\right)\theta}}{e^{-\frac{1}{2}i\theta} - e^{\frac{1}{2}i\theta}} \\ &= -\frac{\cos\frac{1}{2}\theta - \cos\left(N+\frac{1}{2}\right)\theta + i\left(\sin\frac{1}{2}\theta - \sin\left(N+\frac{1}{2}\right)\theta\right)}{2i\sin\frac{1}{2}\theta}.\end{aligned}$$

So,

$$\mathrm{Re}\sum_{n=1}^{N} e^{in\theta} = -\frac{1}{2}\left(1 - \frac{\sin\left(N+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}\right).$$

Thus,

$$\sum_{n=-N}^{N} e^{in\theta} = \frac{\sin\left(N+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta} \tag{20.4}$$

and we get

$$s_N(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta-\phi)f(\phi)\,d\phi,$$

as asserted. □

Corollary 20.7. *For $N = 0, 1, 2, \ldots,$ D_N is an even function such that*

$$\int_{-\pi}^{\pi} D_N(\theta)\,d\theta = 2\pi.$$

Proof. That D_N is an even function is obvious. By (20.4),

$$\int_{-\pi}^{\pi} D_N(\theta)\,d\theta = \int_{-\pi}^{\pi}\sum_{n=-N}^{N} e^{in\theta}d\theta = \sum_{n=-N}^{N}\int_{-\pi}^{\pi} e^{in\theta}d\theta.$$

Since

$$\int_{-\pi}^{\pi} e^{in\theta}d\theta = \begin{cases} 0, & n \neq 0, \\ 2\pi, & n = 0, \end{cases}$$

the proof is complete. □

The following theorem gives a criterion for the Fourier series of a function to converge to the function at a point.

Theorem 20.8. (Dini's Test) *Let $f \in L^1(\mathbb{S}^1)$. If $\theta \in [-\pi, \pi]$ is such that*

$$\int_{-\pi}^{\pi} \frac{|f(\theta+\phi) - f(\theta)|}{|\phi|}\, d\phi < \infty,$$

then

$$s_N(\theta) \to f(\theta)$$

as $N \to \infty$.

Proof. By Lemma 20.5 and Corollary 20.7,

$$\begin{aligned} s_N(\theta) - f(\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\phi) f(\theta-\phi)\, d\phi - f(\theta)\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\phi)\, d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\phi)(f(\theta-\phi) - f(\theta))\, d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\phi)(f(\theta+\phi) - f(\theta))\, d\phi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(\theta+\phi) - f(\theta)}{\sin\frac{1}{2}\phi} \sin\left(N + \frac{1}{2}\phi\right) d\phi. \end{aligned}$$

Without loss of generality, we assume that f is a real-valued function. Since

$$\frac{f(\theta+\phi) - f(\theta)}{\phi}\frac{\phi}{\sin\frac{1}{2}\phi} \in L^1(\mathbb{S}^1)$$

as a function of ϕ, it follows from the Riemann–Lebesgue lemma in Corollary 20.3 that

$$s_N(\theta) - f(\theta) = \frac{1}{2\pi}\mathrm{Im}\left(\int_{-\pi}^{\pi} \frac{f(\theta+\phi) - f(\theta)}{\phi}\frac{\phi}{\sin\frac{1}{2}\phi} e^{i\left(N+\frac{1}{2}\right)\phi} d\phi\right) \to 0$$

as $N \to \infty$. □

In order to obtain a familiar class of functions satisfying the Dini condition in Theorem 20.8, we say that a function f in $L^1(\mathbb{S}^1)$ is Lipschitz continuous at a point θ in $[-\pi, \pi]$ if there exist positive constants M and α such that

$$|f(\theta) - f(\phi)| \le M|\theta - \phi|^\alpha, \quad \phi \in [-\pi, \pi].$$

The number α is the order of Lipschitz continuity of the function f at the point θ. A function f in $L^1(\mathbb{S}^1)$ is said to be Lipschitz continuous of order α if it is Lipschitz continuous of order α at all points in $[-\pi, \pi]$.

Theorem 20.9. *Let $f \in L^1(\mathbb{S}^1)$ be such that f is Lipschitz continuous of order α at θ in $[-\pi, \pi]$, where α is a positive number. Then the Fourier series of f converges to f at the point θ.*

Proof. Let $\theta \in [-\pi, \pi]$. Then using the Lipschitz continuity of f,

$$\int_{-\pi}^{\pi} \frac{|f(\theta+\phi) - f(\theta)|}{|\phi|} d\phi \le \int_{-\pi}^{\pi} |\phi|^{\alpha-1} d\phi < \infty.$$

So, by Theorem 20.8, the theorem is proved. □

Theorem 20.10. *Let f be a continuous function on $[-\pi, \pi]$ such that*

$$f(-\pi) = f(\pi),$$

f' exists at all but possibly a finite number of points in $[-\pi, \pi]$ and

$$\int_{-\pi}^{\pi} |f'(\theta)|^2 d\theta < \infty.$$

Then the Fourier series of f converges to f absolutely and uniformly on $[-\pi, \pi]$.

We need the following lemma for a proof of Theorem 20.10.

Lemma 20.11. *Let f be as in Theorem* 20.10*. Then f is Lipschitz continuous of order $\frac{1}{2}$ on $[-\pi, \pi]$.*

Proof. For all θ and ϕ in $[-\pi, \pi]$ with $\theta \ge \phi$, we get by means of the Schwarz inequality,

$$\begin{aligned}
|f(\theta) - f(\phi)| &= \left| \int_{\phi}^{\theta} f'(\omega)\, d\omega \right| \\
&\le \left(\int_{\phi}^{\theta} d\omega \right)^{1/2} \left(\int_{\phi}^{\theta} |f'(\omega)|^2 d\omega \right)^{1/2} \\
&\le |\theta - \phi|^{1/2} \left(\int_{-\pi}^{\pi} |f'(\omega)|^2 d\omega \right)^{1/2} \\
&= \|f'\|_2 |\theta - \phi|^{1/2}
\end{aligned}$$

and the proof is complete. □

Proof of Theorem 20.10*.* Let $g = f'$. Then by integration by parts, we get for all n in $\mathbb{Z}$,

$$\begin{aligned}
\hat{g}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f'(\theta)\, d\theta \\
&= \frac{1}{2\pi}\, e^{-in\theta} f(\theta) \Big|_{-\pi}^{\pi} + \frac{in}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta \\
&= in\hat{f}(n). \qquad (20.5)
\end{aligned}$$

Now, for all positive integers M and N with $M < N$, we get by Schwarz inequality, (20.5) and the Bessel inequality in Lemma 14.3

$$\begin{aligned}
|s_N(\theta) - s_M(\theta)| &= \left| \sum_{n=-N}^{N} \hat{f}(n)e^{in\theta} - \sum_{n=-M}^{M} \hat{f}(n)e^{in\theta} \right| \\
&= \left| \sum_{M<|n|\le N} \hat{f}(n) \right| \le \sum_{M<|n|\le N} |\hat{f}(n)| \\
&\le \left(\sum_{M<|n|\le N} \frac{1}{n^2} \right)^{1/2} \left(\sum_{M<|n|\le N} n^2 |\hat{f}(n)|^2 \right)^{1/2} \\
&\le \left(\sum_{M<|n|\le N} \frac{1}{n^2} \right)^{1/2} \left(\sum_{n=-\infty}^{\infty} |\hat{g}(n)|^2 \right)^{1/2} \\
&\le \left(\sum_{M<|n|\le N} \frac{1}{n^2} \right)^{1/2} (2\pi)^{-1/2} \|f'\|_2.
\end{aligned}$$

Since $\sum_{n=-\infty}^{\infty} \frac{1}{n^2} < \infty$, it follows that for every positive number ε, there exists a positive integer K such that

$$N > M \ge K \Rightarrow \left(\sum_{M<|n|\le N} \frac{1}{n^2} \right)^{1/2} (2\pi)^{-1/2} \|f'\|_2 < \varepsilon.$$

Thus,

$$N > M \ge K \Rightarrow |s_N(\theta) - s_M(\theta)| < \varepsilon$$

for all θ in $[-\pi, \pi]$. So, there exists a continuous function h on $[-\pi, \pi]$ such that

$$s_N \to h$$

absolutely and uniformly on $[-\pi, \pi]$ as $N \to \infty$. By Lemma 20.11, f is Lipschitz continuous of order $\frac{1}{2}$ on $[-\pi, \pi]$. So, by Theorem 20.9,

$$s_N(\theta) \to f(\theta)$$

for all θ in $[-\pi, \pi]$. Thus,

$$f(\theta) = h(\theta), \quad \theta \in [-\pi, \pi],$$

and so

$$s_N \to f$$

absolutely and uniformly on $[-\pi, \pi]$ as $N \to \infty$. □

All these positive results on pointwise convergence notwithstanding, a surprising result of Andrey Kolmogorov [21, 22] tells us that there exists a function f in $L^1(\mathbb{S}^1)$ such that the Fourier series of f diverges at every point in $[-\pi, \pi]$. It has long been an outstanding open problem on whether or not there exists a continuous function in $L^1(\mathbb{S}^1)$ such that the Fourier series of f diverges on a set of positive measure. A stunning result in 1966 due to Lennart Carleson states that if $f \in L^2(\mathbb{S}^1)$, then the Fourier series of f converges to f almost everywhere on $[-\pi, \pi]$. A more precise result due to Richard Hunt gives the same conclusion on almost everywhere convergence on $[-\pi, \pi]$ if $f \in L^p(\mathbb{S}^1)$ for $1 < p \leq \infty$.

We now change the point of view from almost everywhere convergence to convergence in the mean. First, we need to sharpen Lemma 20.2.

Theorem 20.12. $\{e_n\}_{n=-\infty}^{\infty}$ *is an orthonormal basis for* $L^2(\mathbb{S}^1)$.

Proof. Let $f \in L^2(\mathbb{S}^1)$. Then for every positive number ε, we can use Lemma 20.4 to find a function φ in $C_0^\infty(-\pi, \pi)$ such that

$$\|f - \varphi\|_2 < \frac{\varepsilon}{3}.$$

Then by Theorem 20.10,

$$\sum_{n=-N}^{N} (\varphi, e_n)_2 e_n(\theta) = \sum_{n=-N}^{N} \hat{\varphi}(n) e^{in\theta} \to \varphi(\theta)$$

uniformly with respect to θ on $[-\pi, \pi]$ as $N \to \infty$. So,

$$\left\| \sum_{n=-N}^{N} (\varphi, e_n)_2 e_n - \varphi \right\|_2 \to 0$$

as $N \to \infty$. Now, by Pythagoras' theorem in Proposition 14.5 and the Bessel inequality in Lemma 14.3, we have

$$\begin{aligned}
&\left\| \sum_{n=-N}^{N} (f, e_n)_2 e_n - \sum_{n=-N}^{N} (\varphi, e_n)_2 e_n \right\|_2 \\
&= \left(\sum_{n=-N}^{N} |(f, e_n)_2 - (\varphi, e_n)_2|^2 \right)^{1/2} \\
&\leq \left(\sum_{n=-\infty}^{\infty} |(f, e_n)_2 - (\varphi, e_n)_2|^2 \right)^{1/2} \\
&\leq \|f - \varphi\|_2 < \frac{\varepsilon}{3}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|f - s_N\|_2 \\
&= \left\| f - \sum_{n=-N}^{N} (f, e_n)_2 e_n \right\|_2 \\
&\le \|f - \varphi\|_2 + \left\| \varphi - \sum_{n=-N}^{N} (\varphi, e_n)_2 e_n \right\|_2 \\
&\quad + \left\| \sum_{n=-N}^{N} (\varphi, e_n)_2 e_n - \sum_{n=-N}^{N} (f, e_n)_2 e_n \right\|_2 \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus,

$$s_N = \sum_{n=-N}^{N} (f, e_n)_2 e_n \to f$$

in $L^2(\mathbb{S}^1)$ as $N \to \infty$. It is then obvious to see that if $f \in L^2(\mathbb{S}^1)$ is such that

$$\hat{f}(n) = 0, \quad n \in \mathbb{Z},$$

then $f = 0$. Therefore $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{S}^1)$. □

The L^2-theory of Fourier series can now be easily obtained as corollaries of Theorem 20.12.

Theorem 20.13 (Plancherel's Formula). *Let f and g be functions in $L^2(\mathbb{S}^1)$. Then*

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} = \frac{1}{2\pi}(f, g)_2.$$

Proof. By (20.1) and (20.3), we get for all n in $\mathbb{Z}$,

$$\hat{f}(n) = \frac{1}{\sqrt{2\pi}}(f, e_n)_2$$

and

$$\hat{g}(n) = \frac{1}{\sqrt{2\pi}}(g, e_n)_2.$$

By Parseval's identity in Theorem 14.8, we get

$$(f, g)_2 = \sum_{n=-\infty}^{\infty} (f, e_n)_2 \overline{(g, e_n)_2} = 2\pi \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.$$

This completes the proof of the theorem. □

We can now give another interpretation of (20.2).

Theorem 20.14. *Let $f \in L^2(\mathbb{S}^1)$. Then* (20.2) *is valid in the sense that*

$$s_N \to f$$

in $L^2(\mathbb{S}^1)$ as $N \to \infty$.

Proof. By the Fourier inversion formula in Theorem 14.8,

$$f = \sum_{n=-\infty}^{\infty} (f, e_n)_2 e_n.$$

This is then the same as saying that

$$s_N = \sum_{n=-N}^{N} (f, e_n)_2 e_n \to f$$

in $L^2(\mathbb{S}^1)$ as $N \to \infty$. □

Remark 20.15. Theorem 20.14 can be reformulated conveniently, albeit less precisely, as

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta}, \quad \theta \in [-\pi, \pi],$$

and the convergence of the Fourier series is in $L^2(\mathbb{S}^1)$. This is the Fourier inversion formula for Fourier series.

Exercises

1. Prove that for $1 \leq p < q \leq \infty$,

$$L^q(\mathbb{S}^1) \subset L^p(\mathbb{S}^1)$$

   and the inclusion is proper.

2. Let f be the Dirichlet function on $[-\pi, \pi]$, i.e., for all θ in $[-\pi, \pi]$, $f(\theta) = 1$ if θ is irrational and $f(\theta) = 0$ if θ is rational. Compute $\hat{f}(n)$ for all n in $\mathbb{Z}$. What is the Fourier series of f?

3. Let f and g be functions in $L^1(\mathbb{S}^1)$. Let $f * g$ be the convolution of f and g defined by

$$(f * g)(\theta) = \int_{-\pi}^{\pi} f(\theta - \phi) g(\phi)\, d\phi, \quad \theta \in [-\pi, \pi].$$

   Prove that $f * g$ is a periodic function with period 2π on $\mathbb{R}$ and $f * g \in L^1(\mathbb{S}^1)$. Compute $\widehat{f * g}(n)$ for all n in $\mathbb{Z}$.

4. Let f be a periodic function with period 2π on $\mathbb{R}$ such that f and its first N derivatives are continuous on $\mathbb{R}$. Prove that

$$\hat{f}(n) = O(|n|^{-N})$$

as $|n| \to \infty$.

5. Prove that

$$\int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta \to \infty$$

as $N \to \infty$.

6. Let X be the Banach space of all continuous functions f on $[-\pi, \pi]$ such that

$$f(-\pi) = f(\pi)$$

and the norm $\| \, \|_X$ is given by

$$\|f\|_X = \sup_{\theta \in [-\pi,\pi]} |f(\theta)|$$

for all f in X. For $N = 0, 1, 2, \ldots$, prove that the mapping $A_N : X \to \mathbb{C}$ defined by

$$A_N f = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(\theta) f(\theta)\, d\theta, \quad f \in X,$$

is a bounded linear functional and

$$\|A_N\|_{B(X,\mathbb{C})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta.$$

7. Use the Banach–Steinhaus theorem in Exercise 4 of Chapter 15 to prove the existence of a continuous function f on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$ and the Fourier series of f diverges at 0.

Chapter 21

Fourier Multipliers on $\mathbb{S}^1$

The Plancherel theorem and the Fourier inversion formula for Fourier series are the basic ingredients for the study of filters on the unit circle $\mathbb{S}^1$ with center at the origin. As a motivation, let us recall that for all functions f in $L^2(\mathbb{S}^1)$,

$$f = \sum_{n=-\infty}^{\infty} (f, e_n)_2 e_n, \tag{21.1}$$

where the convergence is in $L^2(\mathbb{S}^1)$. Equation (21.1) can be recast as

$$I = \sum_{n=-\infty}^{\infty} (\cdot, e_n)_2 e_n, \tag{21.2}$$

where I is the identity operator on $L^2(\mathbb{S}^1)$. Equation (21.2) vividly reveals the fact that the identity operator I on $L^2(\mathbb{S}^1)$ can be decomposed into an infinite sum of one-dimensional projections, i.e., $(\cdot, e_n)_2 e_n$, $n \in \mathbb{Z}$. More interesting operators with useful applications can then be obtained by inserting into the Fourier inversion formula (21.1) or (21.2) a suitable function σ on $\mathbb{Z}$. To see how this is done, we first introduce the function spaces from which we choose the functions σ.

For $1 \le p < \infty$, we let $L^p(\mathbb{Z})$ be the set of all sequences $a = \{a_n\}_{n=-\infty}^{\infty}$ such that

$$\sum_{n=-\infty}^{\infty} |a_n|^p < \infty.$$

It can be shown that $L^p(\mathbb{Z})$ is a Banach space in which the norm $\| \ \|_{L^p(\mathbb{Z})}$ is given by

$$\|a\|_{L^p(\mathbb{Z})} = \left(\sum_{n=-\infty}^{\infty} |a_n|^p \right)^{1/p}, \quad a \in L^p(\mathbb{Z}).$$

We let $L^\infty(\mathbb{Z})$ be the set of all bounded sequences $a = \{a_n\}_{n=-\infty}^{\infty}$. It can also be shown that $L^\infty(\mathbb{Z})$ is a Banach space in which the norm $\| \ \|_{L^\infty(\mathbb{Z})}$ is given by

$$\|a\|_{L^\infty(\mathbb{Z})} = \sup_{n \in \mathbb{Z}} |a_n|, \quad a \in L^\infty(\mathbb{Z}).$$

M.W. Wong, *Discrete Fourier Analysis*, Pseudo-Differential Operators 5,
DOI 10.1007/978-3-0348-0116-4_21, © Springer Basel AG 2011

It is left as an exercise to prove that for $1 \le p \le q \le \infty$,

$$L^p(\mathbb{Z}) \subset L^q(\mathbb{Z}),$$

the inclusion is proper and

$$\|a\|_{L^q(\mathbb{Z})} \le \|a\|_{L^p(\mathbb{Z})}, \quad a \in L^p(\mathbb{Z}).$$

See Exercise 1.

We note that $L^2(\mathbb{Z})$ is a Hilbert space in which the inner product $(\ ,\ )_{L^2(\mathbb{Z})}$ is given by

$$(a, b)_{L^2(\mathbb{Z})} = \sum_{n=-\infty}^{\infty} a_n \overline{b_n}$$

for all $a = \{a_n\}_{n=-\infty}^{\infty}$ and $b = \{b_n\}_{n=-\infty}^{\infty}$ in $L^2(\mathbb{Z})$. We need the following result that follows from the Plancherel formula in Theorem 20.13.

Theorem 21.1 (Plancherel's Theorem). *The linear operator $\mathcal{F}_{\mathbb{S}^1} : L^2(\mathbb{S}^1) \to L^2(\mathbb{Z})$ defined by*

$$(\mathcal{F}_{\mathbb{S}^1} f)(n) = \hat{f}(n), \quad n \in \mathbb{Z},$$

is a bijection such that

$$(\mathcal{F}_{\mathbb{S}^1} f, \mathcal{F}_{\mathbb{S}^1} g)_{L^2(\mathbb{Z})} = \frac{1}{2\pi}(f, g)_2 \tag{21.3}$$

for all f and g in $L^2(\mathbb{S}^1)$.

The linear operator $\mathcal{F}_{\mathbb{S}^1}$ in Theorem 21.1 is the Fourier transform on $\mathbb{S}^1$

Proof. Linearity is obvious. (21.3) is a reformulation of the Plancherel formula in Theorem 20.13. Injectivity follows immediately from (21.3). To prove surjectivity, let $a = \{a_n\}_{n=-\infty}^{\infty} \in L^2(\mathbb{Z})$. Then for $N = 0, 1, 2, \ldots,$ we define the function s_N on $[-\pi, \pi]$ by

$$s_N(\theta) = \sum_{n=-N}^{N} a_n e^{in\theta} = \sum_{n=-N}^{N} a_n \sqrt{2\pi} e_n(\theta), \quad \theta \in [-\pi, \pi].$$

Then for all positive integers M and N with $N > M$,

$$\begin{aligned} \|s_N - s_M\|_2^2 &= \left\| \sum_{M<|n|\le N} a_n \sqrt{2\pi} e_n \right\|_2^2 \\ &= \left(\sum_{M<|n|\le N} a_n \sqrt{2\pi} e_n, \sum_{M<|n|\le N} a_n \sqrt{2\pi} e_n \right)_2 \\ &= 2\pi \sum_{M<|n|\le N} |a_n|^2 \to 0 \end{aligned}$$

as $M \to \infty$. Therefore $\{s_N\}_{N=0}^\infty$ is a Cauchy sequence in $L^2(\mathbb{S}^1)$. Since $L^2(\mathbb{S}^1)$ is complete,

$$s_N \to f$$

for some f in $L^2(\mathbb{S}^1)$ as $N \to \infty$. Now, for a fixed m in $\mathbb{Z}$,

$$\widehat{s_N}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta} s_N(\theta)\, d\theta = \frac{1}{2\pi}\sum_{n=-N}^{N} a_n \int_{-\pi}^{\pi} e^{i(n-m)\theta} d\theta = a_m \tag{21.4}$$

for sufficiently large N. Moreover, for all m in $\mathbb{Z}$, using the Schwarz inequality, we get

$$\begin{aligned} |\widehat{s_N}(m) - \hat{f}(m)| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |s_N(\theta) - f(\theta)| d\theta \\ &\le \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} |s_N(\theta) - f(\theta)|^2 d\theta\right)^{1/2} \left(\int_{-\pi}^{\pi} d\theta\right)^{1/2} \\ &= \frac{1}{\sqrt{2\pi}} \|s_N - f\|_2 \to 0 \end{aligned} \tag{21.5}$$

as $N \to \infty$. So, by (21.4) and (21.5), $\hat{f}(m) = a_m$ and hence

$$\mathcal{F}_{\mathbb{S}^1} f = a.$$

This completes the proof. □

From the proof of Theorem 21.1, we have the following corollary.

Corollary 21.2. *For all a in $L^2(\mathbb{Z})$, $\mathcal{F}_{\mathbb{S}^1}^{-1} a$ is the function in $L^2(\mathbb{S}^1)$ defined by*

$$(\mathcal{F}_{\mathbb{S}^1}^{-1} a)(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, \quad \theta \in [-\pi, \pi].$$

Let $\sigma \in L^\infty(\mathbb{Z})$. Then for all f in $L^2(\mathbb{S}^1)$, we define the function $T_\sigma f$ on $\mathbb{S}^1$ by

$$(T_\sigma f)(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} \sigma(n) \hat{f}(n), \quad \theta \in [-\pi, \pi]. \tag{21.6}$$

It is worth pointing out that in (21.6), the role of σ is to assign different weights to different frequencies in the frequency space $\mathbb{Z}$ so as to contribute to a new or filtered signal $T_\sigma f$. As such, either the function σ or the operator T_σ can be looked at as a filter . As the filter is only applied on the frequency domain, we call it a frequency-modulation filter or FM-filter. As the operator T_σ is completely specified by the function σ, we call σ the symbol of the operator T_σ.

Remark 21.3. The operator T_σ is customarily dubbed a Fourier multiplier on $\mathbb{S}^1$ in mathematics. It is easy to understand why it is Fourier. In order to understand why it is a multiplier, see Exercise 2.

We develop in this chapter the theory of Fourier multipliers on $\mathbb{S}^1$. The choice of $L^\infty(\mathbb{Z})$ for symbols is reasonable in view of the following result.

Theorem 21.4. *Let σ be a measurable function on $\mathbb{Z}$, i.e., a sequence. Then T_σ is a bounded linear operator from $L^2(\mathbb{S}^1)$ into $L^2(\mathbb{S}^1)$ if and only if $\sigma \in L^\infty(\mathbb{Z})$. Moreover, if $\sigma \in L^\infty(\mathbb{Z})$, then*

$$\|T_\sigma\|_* = \|\sigma\|_{L^\infty(\mathbb{Z})}.$$

Proof. Suppose that $\sigma \in L^\infty(\mathbb{Z})$. Then for all f in $L^2(\mathbb{S}^1)$,

$$\sigma\hat{f} \in L^2(\mathbb{Z}).$$

By (21.6) and Corollary 21.2,

$$T_\sigma f \in L^2(\mathbb{S}^1)$$

and hence by (21.3),

$$\|T_\sigma f\|_2 = \sqrt{2\pi}\|\sigma\hat{f}\|_{L^2(\mathbb{Z})} \le \sqrt{2\pi}\|\sigma\|_{L^\infty(\mathbb{Z})}\|\hat{f}\|_{L^2(\mathbb{Z})} = \|\sigma\|_{L^\infty(\mathbb{Z})}\|f\|_2.$$

Therefore $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a bounded linear operator with

$$\|T_\sigma\|_* \le \|\sigma\|_{L^\infty(\mathbb{Z})}. \tag{21.7}$$

To prove the converse, let $\sigma \notin L^\infty(\mathbb{Z})$. Suppose that $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a bounded linear operator. Then there exists a positive constant C such that

$$\|T_\sigma f\|_2 \le C\|f\|_2, \quad f \in L^2(\mathbb{S}^1). \tag{21.8}$$

For $N = 1, 2, \ldots$, there exists an integer n_N such that

$$|\sigma(n_N)| > N.$$

Without loss of generality, we can assume that

$$|n_1| < |n_2| < \cdots .$$

For $N = 1, 2, \ldots$, let f_{n_N} be the function on $\mathbb{S}^1$ defined by

$$f_{n_N} = e^{in_N\theta}, \quad \theta \in [-\pi, \pi].$$

Then for $N = 1, 2, \ldots$,

$$\widehat{f_{n_N}}(n) = \begin{cases} 1, & n = n_N, \\ 0, & n \ne n_N. \end{cases}$$

So, for $N = 1, 2, \ldots$,

$$(T_\sigma f_{n_N})(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta}\sigma(n)\widehat{f_{n_N}}(n) = e^{in_N\theta}\sigma(n_N),$$

which gives us

$$\|T_\sigma f_{n_N}\|_2 = \sqrt{2\pi}|\sigma(n_N)| > \sqrt{2\pi}N. \tag{21.9}$$

Thus, by (21.8) and (21.9), we get for $N = 1, 2, \ldots,$

$$\sqrt{2\pi}N < C\|f_{n_N}\|_2 = \sqrt{2\pi}C.$$

This contradiction proves that $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is not a bounded linear operator. Finally, we need to prove that

$$\|T_\sigma\|_* = \|\sigma\|_{L^\infty(\mathbb{Z})}.$$

Suppose by way of contradiction that

$$\|T_\sigma\|_* \neq \|\sigma\|_{L^\infty(\mathbb{Z})}.$$

Then by (21.7),

$$\|T_\sigma\|_* < \|\sigma\|_{L^\infty(\mathbb{Z})}.$$

Hence there exists an integer m such that

$$|\sigma(m)| > \|T_\sigma\|_*.$$

So, for all nonzero functions f in $L^2(\mathbb{S}^1)$,

$$\|T_\sigma f\|_2^2 \leq \|T_\sigma\|_*^2 \|f\|_2^2 < |\sigma(m)|^2 \|f\|_2^2,$$

and in view of (21.3), we get

$$\sum_{n=-\infty}^{\infty} |\sigma(n)|^2 |\hat{f}(n)|^2 < |\sigma(m)|^2 \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2. \tag{21.10}$$

Now, let $f \in L^2(\mathbb{S}^1)$ be the function such that

$$\hat{f}(n) = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

Then by (21.10),

$$|\sigma(m)|^2 < |\sigma(m)|^2,$$

and this contradiction proves the theorem. □

We can now look at the spectral theory of Fourier multipliers on $\mathbb{S}^1$.

Theorem 21.5. *Let $\sigma \in L^\infty(\mathbb{Z})$. Then for all n in $\mathbb{Z}$, $\sigma(n)$ is an eigenvalue of $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ and e_n is a corresponding eigenfunction. Moreover, the spectrum $\Sigma(T_\sigma)$ of $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is precisely given by*

$$\Sigma(T_\sigma) = \{\sigma(n) : n \in \mathbb{Z}\}^c,$$

where $\{\cdots\}^c$ denotes the closure in $\mathbb{C}$ of the set $\{\cdots\}$.

Proof. Let $m \in \mathbb{Z}$. Then

$$T_\sigma e_m = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \sigma(n)\widehat{e_m}(n)e_n,$$

where the convergence is in $L^2(\mathbb{S}^1)$. But

$$\widehat{e_m}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i(m-n)\theta} d\theta = \begin{cases} \frac{1}{\sqrt{2\pi}}, & n = m, \\ 0, & n \neq m. \end{cases}$$

Therefore

$$T_\sigma e_m = \sigma(m)e_m,$$

i.e., $\sigma(m)$ is an eigenvalue of $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ and e_m is a corresponding eigenfunction. Let $\lambda \notin \{\sigma(n) : n \in \mathbb{C}\}^c$. Then there exists a positive constant C such that

$$|\sigma(n) - \lambda| \geq C, \quad n \in \mathbb{Z}. \tag{21.11}$$

By Theorem 21.1, we get for all f in $L^2(\mathbb{S}^1)$,

$$\|(T_\sigma - \lambda I)f\|_2^2 = 2\pi \sum_{n=-\infty}^{\infty} |\sigma(n) - \lambda|^2 |\hat{f}(n)|^2 \geq 2\pi C \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = C\|f\|_2^2.$$

Thus, $T_\sigma - \lambda I : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is injective. To show surjectivity, let $g \in L^2(\mathbb{S}^1)$. Then by Theorem 21.1,

$$\{\hat{g}(n)\}_{n=-\infty}^{\infty} \in L^2(\mathbb{Z}).$$

By (21.11),

$$\frac{\hat{g}}{\sigma - \lambda} \in L^2(\mathbb{Z}).$$

By Theorem 21.1, there exists a function f in $L^2(\mathbb{S}^1)$ such that

$$\hat{f}(n) = \frac{\hat{g}(n)}{\sigma(n) - \lambda}, \quad n \in \mathbb{Z}.$$

Therefore

$$(\sigma(n) - \lambda)\hat{f}(n) = \hat{g}(n), \quad n \in \mathbb{Z}.$$

This gives

$$(T_\sigma - \lambda I)f = g$$

and the proof is complete. □

We give in the following theorem a characterization of compact Fourier multipliers on $\mathbb{S}^1$.

Theorem 21.6. *Let $\sigma \in L^\infty(\mathbb{Z})$. Then $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is compact if and only if*

$$\lim_{|n|\to\infty} \sigma(n) = 0.$$

Proof. We first prove the necessity. Indeed, for all f in $L^2(\mathbb{S}^1)$, using the Riemann–Lebesgue lemma in Corollary 20.3,

$$(e_n, f) = \frac{1}{\sqrt{2\pi}} \overline{\int_{-\pi}^{\pi} e^{-in\theta} f(\theta)\, d\theta} \to 0$$

as $|n| \to \infty$. Therefore

$$e_n \to 0$$

weakly in $L^2(\mathbb{S}^1)$ as $|n| \to \infty$. So, it follows from the compactness of $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ that

$$\|T_\sigma e_n\|_2 \to 0$$

as $|n| \to \infty$. By Theorem 21.5,

$$T_\sigma e_n = \sigma(n) e_n, \quad n \in \mathbb{Z}.$$

Therefore

$$|\sigma(n)| \to 0$$

as $|n| \to \infty$, and the necessity is established. To prove the sufficiency, we define for all positive integers N, the function σ_N on $\mathbb{Z}$ by

$$\sigma_N(n) = \begin{cases} \sigma(n), & |n| \le N, \\ 0, & |n| > N. \end{cases}$$

Then for $N = 1, 2, \ldots$, we get for all f in $L^2(\mathbb{S}^1)$,

$$T_{\sigma_N} f = \sum_{|n|\le N} \sigma(n)(f, e_n)_2 e_n,$$

which is an operator of finite rank and hence compact. Now, for all f in $L^2(\mathbb{S}^1)$,

$$\|T_{\sigma_N} f - T_\sigma f\|_2^2 = 2\pi \sum_{|n|>N} |\sigma(n)|^2 |\hat{f}(n)|^2.$$

For every positive number ε, we can find a positive integer N_0 such that

$$|n| \ge N_0 \Rightarrow |\sigma(n)| < \varepsilon.$$

So,

$$|n| \ge N_0 \Rightarrow \|T_{\sigma_N} f - T_\sigma f\|_2^2 \le 2\pi\varepsilon^2 \sum_{|n|>N} |\hat{f}(n)|^2 \le \varepsilon^2 \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \le \varepsilon^2 \|f\|_2^2$$

whenever $|n| \geq N_0$. Therefore

$$|n| \geq N_0 \Rightarrow \|T_{\sigma_N} - T_\sigma\|_* \leq \varepsilon.$$

In other words, T_σ is the limit in norm of the sequence $\{T_{\sigma_N}\}_{N=1}^\infty$ of compact operators and so must be compact by Theorem 17.1. □

Theorem 21.4 on the L^2-boundedness of Fourier multipliers on $\mathbb{S}^1$ is only the tip of an iceberg.

Theorem 21.7. *For $1 \leq p < \infty$, the Fourier multiplier $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is in the Schatten–von Neumann class S_p if and only if $\sigma \in L^p(\mathbb{Z})$. Moreover, if $\sigma \in L^p(\mathbb{Z})$, then*

$$\|T_\sigma\|_{S_p} = \|\sigma\|_{L^p(\mathbb{Z})}.$$

To prove Theorem 21.7, we use the following lemma.

Lemma 21.8. *Let $\sigma \in L^\infty(\mathbb{Z})$. If we let*

$$|T_\sigma| = (T_\sigma^* T_\sigma)^{1/2},$$

then

$$|T_\sigma| = T_{|\sigma|}.$$

Proof. By Exercise 2 in this chapter,

$$T_\sigma^* T_\sigma = T_{\overline{\sigma}\sigma} = T_{|\sigma|^2} = T_{|\sigma|}^2.$$

Therefore

$$|T_\sigma| = (T_\sigma^* T_\sigma)^{1/2} = T_{|\sigma|},$$

as claimed. □

Proof of Theorem 21.7. Since $\sigma \in L^p(\mathbb{Z})$, it follows that

$$\lim_{|n|\to\infty} \sigma(n) = 0.$$

Hence $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is compact. By Lemma 21.8 and Theorem 21.5, we get for all n in $\mathbb{Z}$,

$$|T_\sigma| e_n = T_{|\sigma|} e_n = |\sigma(n)| e_n.$$

So, the singular values of T_σ are given by $|\sigma(n)|$, $n \in \mathbb{Z}$. Therefore

$$T_\sigma \in S_p \Leftrightarrow \sum_{n=-\infty}^{\infty} |\sigma(n)|^p < \infty \Leftrightarrow \sigma \in L^p(\mathbb{Z}).$$

Moreover, if $\sigma \in L^p(\mathbb{Z})$, then

$$\|T_\sigma\|_{S_p} = \left(\sum_{n=-\infty}^{\infty} |\sigma(n)|^p \right)^{1/p} = \|\sigma\|_{L^p(\mathbb{Z})},$$

as required. □

Exercises

1. Prove that for $1 \leq p < q \leq \infty$,

$$L^p(\mathbb{Z}) \subset L^q(\mathbb{Z}),$$

   the inclusion is proper and

$$\|a\|_{L^q(\mathbb{Z})} \leq \|a\|_{L^p(\mathbb{Z})}, \quad a \in L^p(\mathbb{Z}).$$

2. Let σ and τ be symbols in $L^\infty(\mathbb{Z})$. Prove that

$$T_\sigma T_\tau = T_\tau T_\sigma$$

   and

$$T_\sigma^* = T_{\overline{\sigma}}.$$

Chapter 22

Pseudo-Differential Operators on $\mathbb{S}^1$

We present in this chapter time-varying FM-filters that extend the FM-filters in the preceding chapter. These time-varying FM-filters are pseudo-differential operators on the unit circle $\mathbb{S}^1$ with center at the origin.

For $1 \le p < \infty$, we denote by $L^p(\mathbb{S}^1 \times \mathbb{Z})$ the set of all measurable functions σ on $\mathbb{S}^1 \times \mathbb{Z}$ such that

$$\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |\sigma(\theta, n)|^p d\theta < \infty.$$

Then $L^p(\mathbb{S}^1 \times \mathbb{Z})$ is a Banach space in which the norm $\| \, \|_{L^p(\mathbb{S}^1 \times \mathbb{Z})}$ is given by

$$\|\sigma\|_{L^p(\mathbb{S}^1 \times \mathbb{Z})} = \left(\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |\sigma(\theta, n)|^p d\theta \right)^{1/p}, \quad \sigma \in L^p(\mathbb{S}^1 \times \mathbb{Z}).$$

$L^2(\mathbb{S}^1 \times \mathbb{Z})$ is a Hilbert space with inner product $(\, , \,)_{L^2(\mathbb{S}^1 \times \mathbb{Z})}$ and norm $\| \, \|_{L^2(\mathbb{S}^1 \times \mathbb{Z})}$ given, respectively, by

$$(\sigma, \tau)_{L^2(\mathbb{S}^1 \times \mathbb{Z})} = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \sigma(\theta, n) \overline{\tau(\theta, n)} \, d\theta$$

and

$$\|\sigma\|_{L^2(\mathbb{S}^1 \times \mathbb{Z})} = \left(\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |\sigma(\theta, n)|^2 d\theta \right)^{1/2}$$

for all σ and τ in $L^2(\mathbb{S}^1 \times \mathbb{Z})$.

Let σ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Then for all f in $L^2(\mathbb{S}^1)$, we define the function $T_\sigma f$ on $\mathbb{S}^1$ formally by

$$(T_\sigma f)(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} \sigma(\theta, n) \hat{f}(n), \quad \theta \in [-\pi, \pi].$$

We call T_σ the pseudo-differential operator corresponding to the symbol σ.

M.W. Wong, *Discrete Fourier Analysis*, Pseudo-Differential Operators 5,
DOI 10.1007/978-3-0348-0116-4_22, © Springer Basel AG 2011

Why is T_σ so defined differential? To understand this, let us look at the linear differential operator $P(\theta, D)$ on $\mathbb{S}^1$ defined by

$$P(\theta, D) = \sum_{j=0}^{m} a_j(\theta) D^j,$$

where $D = -i\frac{d}{d\theta}$, and $a_0, a_1, \ldots, a_m$ are measurable functions on $\mathbb{S}^1$. Let $f \in C^\infty(\mathbb{S}^1)$. Then by (20.5),

$$(\mathcal{F}_{\mathbb{S}^1} D^j f)(n) = (-i)^j (in)^j \hat{f}(n) = n^j \hat{f}(n), \quad n \in \mathbb{Z}.$$

So, by Corollary 21.2,

$$\begin{aligned}
(P(\theta, D)f)(\theta) &= \sum_{j=0}^{m} a_j(\theta)(D^j f)(\theta) \\
&= \sum_{j=0}^{m} a_j(\theta)(\mathcal{F}_{\mathbb{S}^1}^{-1} \mathcal{F}_{\mathbb{S}^1} D^j f)(\theta) \\
&= \sum_{j=0}^{m} a_j(\theta) \sum_{n=-\infty}^{\infty} n^j \hat{f}(n) e^{in\theta} \\
&= \sum_{n=-\infty}^{\infty} e^{in\theta} \left(\sum_{j=0}^{m} a_j(\theta) n^j \right) \hat{f}(n) \qquad (22.1)
\end{aligned}$$

for all θ in $[-\pi, \pi]$. The message of the integral representation (22.1) for $P(\theta, D)f$ is that a pseudo-differential operator T_σ corresponding to a polynomial

$$\sigma(\theta, n) = \sum_{j=0}^{m} a_j(\theta) n^j$$

is a linear differential operator. In the context of filters in signal analysis, a pseudo-differential operator corresponding to a symbol σ depending on both time θ in $\mathbb{S}^1$ and frequency n in $\mathbb{Z}$ is a time-varying FM-filter. In view of the fact that a pseudo-differential operator is a filter in which time and frequency are to be controlled simultaneously, it is a much more intricate object to work with precisely because of the Heisenberg uncertainty principle.

Theorem 22.1. *Let $\sigma \in L^2(\mathbb{S}^1 \times \mathbb{Z})$. Then $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a bounded linear operator and*

$$\|T_\sigma\|_* \le (2\pi)^{-1/2} \|\sigma\|_{L^2(\mathbb{S}^1 \times \mathbb{Z})}.$$

For a proof of Theorem 22.1, we use Minkowski's inequality in integral form to the effect that for $1 \le p < \infty$, if f is a measurable function on $X \times Y$, where

(X, μ) and (Y, ν) are measure spaces, then

$$\left(\int_X \left|\int_Y f(x,y)\, d\nu(y)\right|^p d\mu(x)\right)^{1/p} \le \int_Y \left(\int_X |f(x,y)|^p d\mu(x)\right)^{1/p} d\nu(y).$$

Instead of giving a rigorous proof, we give a heuristic argument for why it is true. To do this, we note that $\int_Y f(\cdot, y)\, d\nu(y)$ can be considered as a sum of functions indexed by y. So, the left-hand side is in fact the L^p norm of a sum of functions, which according to the triangle inequality is at most the sum of the L^p norms of the functions. But the sum of the L^p norms of the functions is just $\int_Y \left(\int_X |f(x,y)|^p d\mu(x)\right)^{1/p} d\nu(y)$.

Proof of Theorem 22.1. Let $f \in L^2(\mathbb{S}^1)$. Then by Minkowski's inequality in integral form,

$$\begin{aligned}
\|T_\sigma f\|_2 &= \left(\int_{-\pi}^{\pi} \left|\sum_{n=-\infty}^{\infty} e^{in\theta}\sigma(\theta,n)\hat{f}(n)\right|^2 d\theta\right)^{1/2} \\
&\le \sum_{n=-\infty}^{\infty} \left(\int_{-\pi}^{\pi} |\sigma(\theta,n)|^2 |\hat{f}(n)|^2 d\theta\right)^{1/2} \\
&= \sum_{n=-\infty}^{\infty} |\hat{f}(n)| \left(\int_{-\pi}^{\pi} |\sigma(\theta,n)|^2 d\theta\right)^{1/2}.
\end{aligned}$$

So, by the Schwarz inequality and the Parseval identity for Fourier series in Theorem 21.1,

$$\begin{aligned}
\|T_\sigma f\|_2 &\le \left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\right)^{1/2} \left(\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |\sigma(\theta,n)|^2 d\theta\right)^{1/2} \\
&= (2\pi)^{-1/2} \|\sigma\|_{L^2(\mathbb{S}^1\times\mathbb{Z})} \|f\|_2.
\end{aligned}$$

□

The following L^p formula is useful to us.

Theorem 22.2. *Let* $\sigma \in L^p(\mathbb{S}^1 \times \mathbb{Z})$, $1 \le p < \infty$. *Then*

$$\sum_{n=-\infty}^{\infty} \|T_\sigma e_n\|_p^p = (2\pi)^{-p/2} \|\sigma\|_{L^p(\mathbb{S}^1\times\mathbb{Z})}^p.$$

Proof. For $j \in \mathbb{Z}$, we get

$$(T_\sigma e_j)(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta}\sigma(\theta,n)\widehat{e_j}(n), \quad \theta \in [-\pi, \pi]. \tag{22.2}$$

But

$$\widehat{e_j}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} e^{i(j-n)\theta} d\theta = \begin{cases} \frac{1}{\sqrt{2\pi}}, & n = j, \\ 0, & n \neq j. \end{cases} \tag{22.3}$$

So, by (22.2) and (22.3),

$$(T_\sigma e_j)(\theta) = \sigma(\theta, j)\frac{1}{\sqrt{2\pi}} e^{ij\theta} = \sigma(\theta, j)e_j(\theta), \quad \theta \in [-\pi, \pi]. \tag{22.4}$$

Hence

$$\sum_{j=-\infty}^{\infty} \|T_\sigma e_j\|_p^p = \sum_{j=-\infty}^{\infty} \int_{-\pi}^{\pi} |\sigma(\theta, j)|^p (2\pi)^{-p/2} d\theta = (2\pi)^{-p/2}\|\sigma\|^p_{L^p(\mathbb{S}^1\times\mathbb{Z})},$$

as asserted. □

Using Theorem 19.2, Theorem 22.2 and the fact that $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{S}^1)$, we have the following result.

Theorem 22.3. *The pseudo-differential operator $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a Hilbert–Schmidt operator if and only if $\sigma \in L^2(\mathbb{S}^1 \times \mathbb{Z})$. Moreover, if $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a Hilbert–Schmidt operator, then*

$$\|T_\sigma\|_{S_2} = (2\pi)^{-1/2}\|\sigma\|_{L^2(\mathbb{S}^1\times\mathbb{Z})}.$$

The proof of Theorem 22.2 gives a necessary condition on a measurable function σ on $\mathbb{S}^1 \times \mathbb{Z}$ for $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ to be a bounded linear operator. To wit, let σ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$ and let $f \in L^2(\mathbb{S}^1)$. Then for $n \in \mathbb{Z}$, we can use (22.4) to obtain

$$(T_\sigma e_n)(\theta) = \sigma(\theta, n)e_n(\theta), \quad \theta \in [-\pi, \pi],$$

and hence

$$\|T_\sigma e_n\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sigma(\theta, n)|^2 d\theta.$$

So, if $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a bounded linear operator, we can get a positive constant C such that

$$\|T_\sigma e_n\|_2^2 \leq C\|e_n\|_2^2, \quad n \in \mathbb{Z},$$

which is then the same as

$$\int_{-\pi}^{\pi} |\sigma(\theta, n)|^2 d\theta \leq 2\pi C, \quad n \in \mathbb{Z}.$$

Therefore a necessary condition for $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ to be a bounded linear operator is

$$\sup_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} |\sigma(\theta, n)|^2 d\theta < \infty. \tag{22.5}$$

The following example shows that (22.5) is not sufficient for $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ to be a bounded linear operator.

Example 22.4. Let σ be the function on $\mathbb{S}^1 \times \mathbb{Z}$ defined by

$$\sigma(\theta, n) = e^{-in\theta}, \quad \theta \in [-\pi, \pi],\, n \in \mathbb{Z}.$$

Then

$$\sup_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} |\sigma(\theta, n)|^2 d\theta = 2\pi < \infty.$$

Let f be the function on $\mathbb{S}^1$ defined by

$$f(\theta) = \sum_{n=1}^{\infty} \frac{1}{n}\, e^{in\theta}, \quad \theta \in [-\pi, \pi].$$

Since the sequence $\{a_n\}_{n=-\infty}^{\infty}$ defined by

$$a_n = \begin{cases} \frac{1}{n}, & n \geq 1, \\ 0, & n < 1, \end{cases}$$

is in $L^2(\mathbb{Z})$, it follows from Theorem 21.1 that $f \in L^2(\mathbb{S}^1)$. But for all $\theta \in [-\pi, \pi]$,

$$(T_\sigma f)(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} \sigma(\theta, n) \hat{f}(n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

The following theorem tells us when a pseudo-differential operator $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a bounded linear operator.

Theorem 22.5. *Let σ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Suppose that we can find a positive constant C and a function w in $L^1(\mathbb{Z})$ such that*

$$|\hat{\sigma}(m, n)| \leq C|w(m)|, \quad m, n \in \mathbb{Z}, \tag{22.6}$$

where

$$\hat{\sigma}(m, n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-im\theta} \sigma(\theta, n)\, d\theta. \tag{22.7}$$

Then $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a bounded linear operator and

$$\|T_\sigma\|_* \leq C\|w\|_{L^1(\mathbb{Z})}.$$

In order to prove Theorem 22.5, we first establish Young's inequality for the convolution of two functions on $\mathbb{Z}$.

Theorem 22.6. *Let*

$$a = \{a_n\}_{n=-\infty}^{\infty} \in L^1(\mathbb{Z})$$

and

$$b = \{b_n\}_{n=-\infty}^{\infty} \in L^p(\mathbb{Z}),$$

where $1 \le p \le \infty$. *Then the function* $a * b$ *on* $\mathbb{Z}$ *defined by*

$$(a * b)_n = \sum_{k=-\infty}^{\infty} a_{n-k} b_k, \quad n \in \mathbb{Z},$$

is in $L^p(\mathbb{Z})$ *and*

$$\|a * b\|_{L^p(\mathbb{Z})} \le \|a\|_{L^1(\mathbb{Z})} \|b\|_{L^p(\mathbb{Z})}.$$

Remark 22.7. The sequence $a * b$ is known as the convolution of a and b. The inequality in Theorem 22.6 is known as Young's inequality.

Proof of Lemma 22.6. For $1 \le p < \infty$, we note that by Minkowski's inequality in integral form,

$$\begin{aligned}
\|a * b\|_p &= \left(\sum_{n=-\infty}^{\infty} |(a * b)_n|^p \right)^{1/p} = \left(\sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} a_{n-k} b_k \right|^p \right)^{1/p} \\
&= \left(\sum_{n=-\infty}^{\infty} \left| \sum_{k=-\infty}^{\infty} a_k b_{n-k} \right|^p \right)^{1/p} \le \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |a_k|^p |b_{n-k}|^p \right)^{1/p} \\
&= \sum_{k=-\infty}^{\infty} |a_k| \left(\sum_{n=-\infty}^{\infty} |b_{n-k}|^p \right)^{1/p} = \sum_{k=-\infty}^{\infty} |a_k| \left(\sum_{n=-\infty}^{\infty} |b_n|^p \right)^{1/p} \\
&= \|a\|_{L^1(\mathbb{Z})} \|b\|_{L^p(\mathbb{Z})}.
\end{aligned}$$

For $p = \infty$, we get for all n in $\mathbb{Z}$,

$$\begin{aligned}
|(a * b)_n| &= \left| \sum_{k=-\infty}^{\infty} a_{n-k} b_k \right| \le \sum_{k=-\infty}^{\infty} |a_{n-k}| \, |b_k| \\
&\le \|b\|_{L^\infty(\mathbb{Z})} \sum_{k=-\infty}^{\infty} |a_{n-k}| = \|b\|_{L^\infty(\mathbb{Z})} \sum_{k=-\infty}^{\infty} |a_k| \\
&= \|a\|_{L^1(\mathbb{Z})} \|b\|_{L^\infty(\mathbb{Z})}.
\end{aligned}$$

□

Proof of Theorem 22.5. Let $f \in C^\infty(\mathbb{S}^1)$. Then we have

$$\begin{aligned}
\|T_\sigma f\|_2^2 &= \int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} e^{in\theta} \sigma(\theta, n) \hat{f}(n) \right|^2 d\theta \\
&= \int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{i(m+n)\theta} \hat{\sigma}(m, n) \hat{f}(n) \right|^2 d\theta.
\end{aligned}$$

So,

$$\|T_\sigma f\|_2^2 = \int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{ik\theta} \hat{\sigma}(k-n,n)\hat{f}(n) \right|^2 d\theta$$

$$= \int_{-\pi}^{\pi} \left| \sum_{k=-\infty}^{\infty} e^{ik\theta} \left(\sum_{n=-\infty}^{\infty} \hat{\sigma}(k-n,n)\hat{f}(n) \right) \right|^2 d\theta. \tag{22.8}$$

Using (22.8) and the orthogonality of the functions $\{e_j\}_{j=-\infty}^{\infty}$ in $L^2(\mathbb{S}^1)$,

$$\|T_\sigma f\|_2^2 = 2\pi \sum_{k=-\infty}^{\infty} \left| \sum_{n=-\infty}^{\infty} \hat{\sigma}(k-n,n)\hat{f}(n) \right|^2$$

$$\leq 2\pi \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |\hat{\sigma}(k-n,n)||\hat{f}(n)| \right)^2. \tag{22.9}$$

Using (22.6), we get

$$\|T_\sigma f\|_2^2 \leq 2\pi C^2 \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |w(k-n)|\,|\hat{f}(n)| \right)^2$$

$$\leq 2\pi C^2 \sum_{k=-\infty}^{\infty} |(|w| * |\hat{f}|)(k)|^2,$$

where $|w| * |\hat{f}|$ is the convolution of $|w|$ and $|\hat{f}|$. Finally, using Young's inequality and the Parseval identity for Fourier series, we have

$$\|T_\sigma f\|_2^2 \leq 2\pi C^2 \|w\|_{L^1(\mathbb{Z})}^2 \|\hat{f}\|_{L^2(\mathbb{Z})}^2 = C^2 \|w\|_{L^1(\mathbb{Z})}^2 \|f\|_2^2. \tag{22.10}$$

By Lemma 20.4, $C^\infty(\mathbb{S}^1)$ is dense in $L^2(\mathbb{S}^1)$ and it follows that (22.10) holds for all functions f in $L^2(\mathbb{S}^1)$. □

Remark 22.8. In order to justify the interchange of the two sums in (22.8), we note that by Fubini's theorem, it is sufficient to prove that

$$\sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |\hat{\sigma}(k-n,n)||\hat{f}(n)| < \infty.$$

Since $f \in C^2(\mathbb{S}^1)$, it follows from Exercise 4 in Chapter 20 that

$$|\hat{f}(n)| \leq O(n^{-2})$$

as $|n| \to \infty$. Hence $\hat{f} \in L^1(\mathbb{Z})$. Using an argument in the proof of Theorem 22.5, we have

$$\sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |\hat{\sigma}(k-n,n)||\hat{f}(n)| \leq C\|w\|_{L^1(\mathbb{Z})} \|\hat{f}\|_{L^1(\mathbb{Z})} < \infty.$$

Remark 22.9. The sufficient conditions on the symbol σ to ensure the boundedness of $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ in [30, 31] require a certain number of derivatives of σ with respect to θ. By means of Theorem 22.5 and Bernstein's theorem, i.e., Theorem 3.1 in Chapter VI of the book [53], we see that all we need is that the symbol σ is Lipschitz continuous of order α, $\alpha > \frac{1}{2}$.

Let σ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$ such that $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is compact. From the proof of Theorem 21.6, we know that $e_n \to 0$ weakly in $L^2(\mathbb{S}^1)$ as $|n| \to \infty$. Then it follows from the compactness of $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ that $\|T_\sigma e_n\|_2 \to 0$ as $|n| \to \infty$. By (22.4), we see that

$$\int_{-\pi}^{\pi} |\sigma(\theta, n)|^2 d\theta \to 0 \tag{22.11}$$

as $|n| \to \infty$. That the condition (22.11) is not enough for $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ to be compact can be illustrated by the following example.

Example 22.10. Let σ be the function on $\mathbb{S}^1 \times \mathbb{Z}$ defined by

$$\sigma(\theta, n) = \begin{cases} \frac{1}{\ln n} e^{-in\theta}, & n > 1, \\ 0, & n \leq 1, \end{cases} \tag{22.12}$$

for all θ in $[-\pi, \pi]$. Then it can be seen easily that σ satisties (22.11). If we let f be the function on $\mathbb{S}^1$ defined by

$$f(\theta) = \sum_{n=1}^{\infty} \frac{1}{n} e^{in\theta}, \quad \theta \in [-\pi, \pi],$$

then as in Example 22.4, $f \in L^2(\mathbb{S}^1)$. But

$$(T_\sigma f)(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta} \sigma(\theta, n) \hat{f}(n) = \sum_{n=2}^{\infty} \frac{1}{n \ln n}, \quad \theta \in [-\pi, \pi].$$

So, $T_\sigma f$ is not even in $L^2(\mathbb{S}^1)$.

The following theorem gives the L^2-compactness of pseudo-differential operators on $L^2(\mathbb{S}^1)$.

Theorem 22.11. *Let σ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Suppose that we can find a function w in $L^1(\mathbb{Z})$ and a function C on $\mathbb{Z}$ such that*

$$\lim_{|n| \to \infty} C(n) = 0$$

and

$$|\hat{\sigma}(m, n)| \leq C(n) |w(m)|, \quad m, n \in \mathbb{Z}. \tag{22.13}$$

Then $T_\sigma : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is compact.

Proof. For all positive integers N, we define the function σ_N on $\mathbb{S}^1 \times \mathbb{Z}$ by

$$\sigma_N(\theta, n) = \begin{cases} \sigma(\theta, n), & |n| \le N, \\ 0, & |n| > N, \end{cases}$$

for all θ in $[-\pi, \pi]$ and n in $\mathbb{Z}$. Then for $N = 1, 2, \ldots,$

$$\sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |\sigma_N(\theta, n)|^2 d\theta = \sum_{n=-N}^{N} \int_{-\pi}^{\pi} |\sigma(\theta, n)|^2 d\theta < \infty$$

and hence by Theorem 22.3, $T_{\sigma_N} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a Hilbert–Schmidt operator. Let $\tau_N = \sigma - \sigma_N$. Then by the definition of σ_N, we have

$$\tau_N(\theta, n) = \begin{cases} 0, & |n| \le N, \\ \sigma(\theta, n), & |n| > N, \end{cases}$$

for all θ in $[-\pi, \pi]$. Let ε be a given positive number. Then there exists a positive integer N_0 such that

$$|C(n)| < \varepsilon$$

whenever $|n| > N_0$. So, for $N \ge N_0$,

$$\begin{aligned} \|(T_{\sigma_N} - T_\sigma)f\|_2^2 &= \int_{-\pi}^{\pi} |(T_{\tau_N} f)(\theta)|^2 d\theta \\ &= \int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} e^{in\theta} \tau_N(n, \theta) \hat{f}(n) \right|^2 d\theta. \end{aligned}$$

Now, we use the same argument in the derivation of (22.9) to get

$$\begin{aligned} \|(T_{\sigma_N} - T_\sigma)f\|_2^2 &\le 2\pi \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |\widehat{\tau_N}(k-n, n)| |\hat{f}(n)| \right)^2 \\ &= 2\pi \sum_{k=-\infty}^{\infty} \left(\sum_{|n|>N} |\hat{\sigma}(k-n, n)| |\hat{f}(n)| \right)^2 . \end{aligned}$$

Using (22.13), we get

$$\begin{aligned} \|(T_{\sigma_N} - T_\sigma)f\|_2^2 &\le 2\pi \sum_{k=-\infty}^{\infty} \left(\sum_{|n|>N} C(n) |w(k-n)| \, |\hat{f}(n)| \right)^2 \\ &\le 2\pi\varepsilon^2 \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |w(k-n)| \, |\hat{f}(n)| \right)^2 . \end{aligned}$$

Using the same argument in the derivation of (22.10), we have

$$\|(T_{\sigma_N} - T_\sigma)f\|_2^2 \le \varepsilon^2 B\|f\|_2^2,$$

where $B = \|w\|^2_{L^1(\mathbb{Z})}$. Thus, for $N \ge N_0$,

$$\|T_\sigma - T_{\sigma_N}\|_* \le \sqrt{B}\varepsilon.$$

In other words, T_σ is the limit in norm of the sequence $\{T_{\sigma_N}\}_{N=1}^\infty$ and so must be compact by Theorem 17.1. □

We end this chapter with a glimpse into the L^p-boundedness of pseudo-differential operators.

We first give sufficient conditions on a function σ on $\mathbb{Z}$ to ensure that the corresponding Fourier multiplier $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$, $1 < p < \infty$, is a bounded linear operator.

Lemma 22.12. *Let σ be a measurable function on $\mathbb{Z}$ and let k be the smallest integer greater than $\frac{1}{2}$. Suppose that there exists a positive constant C such that*

$$|(\partial^j \sigma)(n)| \le C\langle n\rangle^{-j}, \quad n \in \mathbb{Z},$$

for $0 \le j \le k$, where ∂^j is the difference operator given by

$$(\partial^j \sigma)(n) = \sum_{l=0}^{j} (-1)^{j-l}\binom{j}{l}\sigma(n+l), \quad n \in \mathbb{Z},$$

and

$$\langle n\rangle = (1 + |n|^2)^{1/2}, \quad n \in \mathbb{Z}.$$

Then for $1 < p < \infty$, $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is a bounded linear operator and there exists a positive constant B, depending on p only, such that

$$\|T_\sigma f\|_p \le BC\|f\|_p, \quad f \in L^p(\mathbb{S}^1).$$

Remark 22.13. The condition on the number k of "derivatives" in Lemma 22.12 can best be understood in the context of the multi-dimensional torus in which k should be the smallest integer greater than $d/2$, where d is the dimension of the torus.

The proof of Lemma 22.12 entails the use of the Littlewood–Paley theory in Fourier series in, *e.g.*, Chapter XV of the book [53] by Zygmund. See, in particular, Theorem 4.14 in Chapter XV of [53] in this connection. Extensions of Lemma 22.12 to the context of compact Lie groups are attributed to Weiss [46, 47] and the Littlewood–Paley theory for compact Lie groups can be found in Stein [36]. Analogs of Lemma 22.12 for $\mathbb{R}^n$ can be found in the works of Hörmander [18] and Stein [35].

The following theorem gives sufficient conditions for the L^p-boundedness of pseudo-differential operators on $\mathbb{S}^1$. The ideas for the result and its proof come from Theorem 10.7 in the book [49] by Wong.

Theorem 22.14. *Let σ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$ and let k be the smallest integer greater than $\frac{1}{2}$. Suppose that we can find a positive constant C and a function w in $L^1(\mathbb{Z})$ such that*

$$|(\partial_n^j \hat{\sigma})(m,n)| \le C|w(m)|\langle n\rangle^{-j}, \quad m,n \in \mathbb{Z}, \tag{22.14}$$

for $0 \le j \le k$, where $\hat{\sigma}(m,n)$ is as defined in (22.7) and ∂_n^j is the partial difference operator with respect to the variable n in $\mathbb{Z}$. Then for $1 < p < \infty$, $T_\sigma : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ is a bounded linear operator. Moreover, there exists a positive constant B depending only on p such that

$$\|T_\sigma\|_{B(L^p(\mathbb{S}^1))} \le BC\|w\|_{L^1(\mathbb{Z})},$$

where $\| \|_{B(L^p(\mathbb{S}^1))}$ is the norm in the Banach space of all bounded linear operators from $L^p(\mathbb{S}^1)$ into $L^p(\mathbb{S}^1)$.

Proof. Let $f \in L^p(\mathbb{S}^1)$. Then by Fubini's theorem,

$$\begin{aligned}(T_\sigma f)(\theta) &= \sum_{n=-\infty}^{\infty} e^{in\theta}\sigma(\theta,n)\hat{f}(n) \\ &= \sum_{n=-\infty}^{\infty} \hat{f}(n)\left\{\sum_{m=-\infty}^{\infty} \hat{\sigma}(m,n)e^{im\theta}\right\}e^{in\theta} \\ &= \sum_{m=-\infty}^{\infty} e^{im\theta}\left\{\sum_{n=-\infty}^{\infty} \hat{\sigma}(m,n)\hat{f}(n)e^{in\theta}\right\} \\ &= \sum_{m=-\infty}^{\infty} e^{im\theta}(T_{\sigma_m} f)(\theta)\end{aligned}$$

for all θ in $[-\pi,\pi]$, where

$$\sigma_m(n) = \hat{\sigma}(m,n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\sigma(\theta,n)\,d\theta, \quad m,n \in \mathbb{Z}.$$

By (22.14),

$$|(\partial^j \sigma_m)(n)| = |(\partial_n^j \hat{\sigma})(m,n)| \le C|w(m)|\langle n\rangle^{-j}, \quad m,n \in \mathbb{Z},$$

for $0 \le j \le k$. Therefore by Lemma 22.12, there exists a positive constant B depending only on p such that

$$\|T_{\sigma_m} f\|_p \le BC|w(m)|\,\|f\|_p, \quad m \in \mathbb{Z}. \tag{22.15}$$

Then by (22.15) and Minkowski's inequality in integral form, we get

$$\|T_\sigma f\|_p = \left\{\int_{-\pi}^{\pi} \left|\sum_{m=-\infty}^{\infty} e^{im\theta}(T_{\sigma_m} f)(\theta)\right|^p d\theta\right\}^{1/p}$$

$$\leq \sum_{m=-\infty}^{\infty} \left\{ \int_{-\pi}^{\pi} |(T_{\sigma_m} f)(\theta)|^p d\theta \right\}^{1/p}$$
$$= \sum_{m=-\infty}^{\infty} \|T_{\sigma_m} f\|_p$$
$$\leq BC\|w\|_{L^1(\mathbb{Z})}\|f\|_p,$$

and this completes the proof. □

Exercises

1. Let $a \in L^1(\mathbb{Z})$ and $b \in L^p(\mathbb{Z})$, $1 \leq p \leq \infty$. Prove that

$$a * b = b * a.$$

2. Prove that the function σ in (22.12) satisfies the condition (22.11).
3. The Rihaczek transform $R(f,g)$ of two functions f and g in $L^2(\mathbb{S}^1)$ is the function on $\mathbb{S}^1 \times \mathbb{Z}$ defined by

$$R(f,g)(\theta,n) = e^{in\theta}\hat{f}(n)\overline{g(\theta)}, \quad \theta \in [-\pi,\pi],\, n \in \mathbb{Z}.$$

Let $\sigma \in L^2(\mathbb{S}^1 \times \mathbb{Z})$. Prove that for all f and g in $L^2(\mathbb{S}^1)$,

$$(T_\sigma f, g)_2 = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \sigma(\theta,n) R(f,g)(\theta,n)\, d\theta.$$

4. Prove that for all functions f_1, g_1, f_2 and g_2 in $L^2(\mathbb{S}^1)$,

$$(R(f_1,f_2), R(g_1,g_2))_{L^2(\mathbb{S}^1\times\mathbb{Z})} = \frac{1}{2\pi}(f_1,g_1)_2\overline{(f_2,g_2)_2}.$$

(This is the Moyal identity for the Rihaczek transform.)

Chapter 23

Pseudo-Differential Operators on $\mathbb{Z}$

This chapter is a "dual" of the preceding chapter. Presenting this duality can contribute to a better understanding of pseudo-differential operators.

Let $a \in L^2(\mathbb{Z})$. Then the Fourier transform $\mathcal{F}_{\mathbb{Z}}a$ of a is the function on the unit circle $\mathbb{S}^1$ centered at the origin defined by

$$(\mathcal{F}_{\mathbb{Z}}a)(\theta) = \sum_{n=-\infty}^{\infty} a(n)e^{in\theta}, \quad \theta \in [-\pi, \pi].$$

Then by Theorem 21.1,

$$\mathcal{F}_{\mathbb{Z}} = \mathcal{F}_{\mathbb{S}^1}^{-1}.$$

Thus, $\mathcal{F}_{\mathbb{Z}}a \in L^2(\mathbb{S}^1)$ and the Plancherel formula for Fourier series gives

$$\sum_{n=-\infty}^{\infty} |a(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |(\mathcal{F}_{\mathbb{Z}}a)(\theta)|^2 d\theta.$$

The Fourier inversion formula for Fourier series gives

$$a(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} (\mathcal{F}_{\mathbb{Z}}a)(\theta)\, d\theta, \quad n \in \mathbb{Z}.$$

Let $\sigma : \mathbb{Z} \times \mathbb{S}^1 \to \mathbb{C}$ be a measurable function. Then for every sequence a in $L^2(\mathbb{Z})$, we define the sequence $T_\sigma a$ formally by

$$(T_\sigma a)(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \sigma(n, \theta) (\mathcal{F}_{\mathbb{Z}}a)(\theta)\, d\theta, \quad n \in \mathbb{Z}.$$

T_σ is called the pseudo-differential operator on $\mathbb{Z}$ corresponding to the symbol σ whenever the integral exists for all n in $\mathbb{Z}$. It is the natural analog on $\mathbb{Z}$ of the standard pseudo-differential operators on $\mathbb{R}^n$ explained in, *e.g.*, [49].

M.W. Wong, *Discrete Fourier Analysis*, Pseudo-Differential Operators 5, DOI 10.1007/978-3-0348-0116-4_23, © Springer Basel AG 2011

Theorem 23.1. *The pseudo-differential operator $T_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is a Hilbert–Schmidt operator if and only if $\sigma \in L^2(\mathbb{Z} \times \mathbb{S}^1)$. Moreover, if $T_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is a Hilbert–Schmidt operator, then*

$$\|T_\sigma\|_{S_2} = (2\pi)^{-1/2}\|\sigma\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)}.$$

Proof. The starting point is the standard orthonormal basis $\{\epsilon_k\}_{k\in\mathbb{Z}}$ for $L^2(\mathbb{Z})$ given by

$$\epsilon_k(n) = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$

For $k \in \mathbb{Z}$, we get

$$(\mathcal{F}_{\mathbb{Z}}\epsilon_k)(\theta) = \sum_{n=-\infty}^{\infty} \epsilon_k(n)e^{in\theta} = e^{ik\theta}, \quad \theta \in [-\pi, \pi],$$

and hence

$$\begin{aligned} (T_\sigma\epsilon_k)(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}\sigma(n,\theta)(\mathcal{F}_{\mathbb{Z}}\epsilon_k)(\theta)\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n-k)\theta}\sigma(n,\theta)\,d\theta \\ &= (\mathcal{F}_{\mathbb{S}^1}\sigma)(n, n-k) \end{aligned}$$

for all $n \in \mathbb{Z}$, where $(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,\cdot)$ is the Fourier transform of the function $\sigma(n,\cdot)$ on $\mathbb{S}^1$ given by

$$(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\sigma(n,\theta)\,d\theta, \quad m, n \in \mathbb{Z}.$$

So, using Fubini's theorem and the Plancherel formula for Fourier series, we get

$$\begin{aligned} \|T_\sigma\|_{S_2}^2 &= \sum_{k=-\infty}^{\infty} \|T_\sigma\epsilon_k\|_{L^2(\mathbb{Z})}^2 = \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} |(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,n-k)|^2 \\ &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,n-k)|^2 = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,k)|^2 \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} |\sigma(n,\theta)|^2 d\theta = \frac{1}{2\pi}\|\sigma\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)}^2 \end{aligned}$$

and this completes the proof. □

As for the L^p-boundedness and the L^p-compactness, $1 \le p < \infty$, of pseudo-differential operators on $\mathbb{Z}$, let us begin with a simple and elegant result on the L^2-boundedness of pseudo-differential operators on $\mathbb{Z}$.

Theorem 23.2. *Let σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$ such that we can find a positive constant C and a function $w \in L^2(\mathbb{Z})$ for which*

$$|\sigma(n, \theta)| \leq C|w(n)|$$

for all $n \in \mathbb{Z}$ and almost all θ in $[-\pi, \pi]$. Then $T_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is a bounded linear operator. Furthermore,

$$\|T_\sigma\|_* \leq C\|w\|_{L^2(\mathbb{Z})}.$$

Proof. Let $a \in L^2(\mathbb{Z})$. Then by the Schwarz inequality and the Plancherel formula,

$$\begin{aligned}
\|T_\sigma a\|^2_{L^2(\mathbb{Z})} &= \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \left| \int_{-\pi}^{\pi} e^{-in\theta} \sigma(n, \theta)(\mathcal{F}_\mathbb{Z} a)(\theta)\, d\theta \right|^2 \\
&\leq \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |\sigma(n, \theta)|^2 |(\mathcal{F}_\mathbb{Z} a)(\theta)|^2 d\theta \\
&\leq \frac{C^2}{2\pi} \sum_{n=-\infty}^{\infty} |w(n)|^2 \int_{-\pi}^{\pi} |(\mathcal{F}_\mathbb{Z} a)(\theta)|^2 d\theta \\
&= C^2 \|w\|^2_{L^2(\mathbb{Z})} \|a\|^2_{L^2(\mathbb{Z})}.
\end{aligned}$$

□

The next theorem gives a single sufficient condition on the symbols σ for the corresponding pseudo-differential operators $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ to be bounded for $1 \leq p < \infty$.

Theorem 23.3. *Let σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$ such that we can find a positive constant C and a function w in $L^1(\mathbb{Z})$ for which*

$$|(\mathcal{F}_{\mathbb{S}^1} \sigma)(n, m)| \leq C|w(m)|, \quad m, n \in \mathbb{Z}.$$

Then the pseudo-differential operator $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is a bounded linear operator for $1 \leq p < \infty$. Furthermore,

$$\|T_\sigma\|_{B(L^p(\mathbb{Z}))} \leq C\|w\|_{L^1(\mathbb{Z})},$$

where $\| \, \|_{B(L^p(\mathbb{Z})}$ is the norm in the Banach space of all bounded linear operators from $L^p(\mathbb{Z})$ into $L^p(\mathbb{Z})$.

Proof. Let $a \in L^1(\mathbb{Z})$. Then for all $n \in \mathbb{Z}$, we get

$$\begin{aligned}
(T_\sigma a)(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \sigma(n, \theta)(\mathcal{F}_\mathbb{Z} a)(\theta)\, d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \sigma(n, \theta) \left(\sum_{m=-\infty}^{\infty} a(m) e^{im\theta} \right) d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} a(m) \int_{-\pi}^{\pi} e^{-i(n-m)\theta} \sigma(n,\theta)\, d\theta \\
&= \sum_{m=-\infty}^{\infty} (\mathcal{F}_{\mathbb{S}^1}\sigma)(n, n-m) a(m) \\
&= ((\mathcal{F}_{\mathbb{S}^1}\sigma)(n,\cdot) * a)(n). \qquad (23.1)
\end{aligned}
$$

So,

$$
\begin{aligned}
\|T_\sigma a\|^p_{L^p(\mathbb{Z})} &= \sum_{n=-\infty}^{\infty} |((\mathcal{F}_{\mathbb{S}^1}\sigma)(n,\cdot) * a)(n)|^p \\
&\leq \sum_{n=-\infty}^{\infty} (|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,\cdot)| * |a|)(n))^p \\
&\leq C^p \sum_{n=-\infty}^{\infty} |(|w| * |a|)(n)|^p. \qquad (23.2)
\end{aligned}
$$

Thus, by Young's inequality in Theorem 22.6,

$$\|T_\sigma a\|^p_{L^p(\mathbb{Z})} \leq C^p \|w\|^p_{L^1(\mathbb{Z})} \|a\|^p_{L^p(\mathbb{Z})},$$

which is equivalent to

$$\|T_\sigma\|_{B(L^p(\mathbb{Z}))} \leq C\|w\|_{L^1(\mathbb{Z})}.$$

Since $L^1(\mathbb{Z})$ is dense in $L^p(\mathbb{Z})$ by Exercise 3, the proof is complete for $1 \leq p < \infty$. □

The very mild condition in Theorem 23.3 on the L^p-boundedness of pseudo-differential operators on $\mathbb{Z}$ is dramatically different from the condition for L^p-boundedness of pseudo-differential operators on $\mathbb{R}^n$ in which derivatives with respect to the configuration variables and the dual variables are essential. See Chapter 10 of [49] for boundedness of pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

The following theorem is a result on L^p-compactness.

Theorem 23.4. *Let σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$ such that we can find a positive function C on $\mathbb{Z}$ and a function w in $L^1(\mathbb{Z})$ for which*

$$|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m)| \leq C(n)|w(m)|, \quad m,n \in \mathbb{Z},$$

and

$$\lim_{|n|\to\infty} C(n) = 0.$$

Then the pseudo-differential operator $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is a compact operator for $1 \leq p < \infty$.

Proof. For every positive integer N, we define the symbol σ_N on $\mathbb{Z} \times \mathbb{S}^1$ by

$$\sigma_N(n,\theta) = \begin{cases} \sigma(n,\theta), & |n| \le N, \\ 0, & |n| > N. \end{cases}$$

Now, by (23.1), we get for all $a \in L^p(\mathbb{Z})$,

$$(T_{\sigma_N} a)(n) = \begin{cases} ((\mathcal{F}_{\mathbb{S}^1}\sigma)(n,\cdot) * a)(n), & |n| \le N, \\ 0, & |n| > N. \end{cases}$$

Therefore the range of $T_{\sigma_N} : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is finite-dimensional, i.e., $T_{\sigma_N} : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is a finite rank operator. Let ε be a positive number. Then there exists a positive integer N_0 such that

$$|C(n)| < \varepsilon$$

whenever $|n| > N_0$. So, as in the derivation of (23.2), we get for $N > N_0$,

$$\begin{aligned} \|(T_\sigma - T_{\sigma_N})a\|^p_{L^p(\mathbb{Z})} &= \sum_{n=-\infty}^{\infty} |((\mathcal{F}_{\mathbb{S}^1}(\sigma - \sigma_N))(n,\cdot) * a)(n)|^p \\ &= \sum_{|n|>N} |((\mathcal{F}_{\mathbb{S}^1}\sigma)(n,\cdot) * a)(n)|^p \\ &\le \sum_{|n|>N} |(|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,\cdot)| * |a|)(n)|^p \\ &\le \sum_{|n|>N} C(n)^p |(|w| * |a|)(n)|^p \\ &\le \varepsilon^p \sum_{|n|>N} |(|w| * |a|)(n)|^p. \end{aligned}$$

By Young's inequality, we get for $N > N_0$,

$$\|(T_\sigma - T_{\sigma_N})a\|^p_{L^p(\mathbb{Z})} \le \varepsilon^p \|w\|^p_{L^1(\mathbb{Z})} \|a\|^p_{L^p(\mathbb{Z})}.$$

Hence for $N > N_0$,

$$\|T_\sigma - T_{\sigma_N}\|_{B(L^p(\mathbb{Z}))} \le \varepsilon \|w\|_{L^1(\mathbb{Z})}.$$

So, $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is the limit in norm of a sequence of compact operators on $L^p(\mathbb{Z})$ and hence by Theorem 17.1 must be compact. □

We end this chapter with a result on the numerical analysis of pseudo-differential operators on $\mathbb{Z}$.

Theorem 23.5. *Let σ be a symbol satisfying the hypotheses of Theorem* 23.3. *Then for $1 \le p < \infty$, the matrix A_σ of the pseudo-differential operator $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is given by*

$$A_\sigma = (\sigma_{nk})_{n,k\in\mathbb{Z}},$$

where

$$\sigma_{nk} = (\mathcal{F}_{\mathbb{S}^1}\sigma)(n, n-k).$$

Furthermore, the matrix A_σ is almost diagonal in the sense that

$$|\sigma_{nk}| \leq C|w(n-k)|, \quad n, k \in \mathbb{Z}.$$

Remark 23.6. Since $w \in L^1(\mathbb{Z})$, it follows that, roughly speaking,

$$w(m) = O(|m|^{-(1+\alpha)})$$

as $|m| \to \infty$, where α is a positive number. So, the entry σ_{nk} in the n^{th} row and the k^{th} column of the matrix A_σ decays in such a way that

$$|\sigma_{nk}| = O(|n-k|^{-(1+\alpha)})$$

as $|n-k| \to \infty$. In other words, the off-diagonal entries in A_σ are small and the matrix A_σ can be seen as almost diagonal. This fact is very useful for the numerical analysis of pseudo-differential operators on $\mathbb{Z}$. See [32] for the numerical analysis of pseudo-differential operators and related topics.

Proof of Theorem 23.5. By (23.1), we get for all $n \in \mathbb{Z}$,

$$(T_\sigma a)(n) = (\mathcal{F}_{\mathbb{S}^1}\sigma)(n, \cdot) * a)(n) = \sum_{k=-\infty}^{\infty} (\mathcal{F}_{\mathbb{S}^1}\sigma)(n, n-k)a(k).$$

So, $T_\sigma a$ is the same as the product $A_\sigma a$ of the matrices A_σ and a. □

We give a numerical example to illustrate the almost diagonalization.

Example 23.7. Let

$$\sigma(n, \theta) = \left(n + \frac{1}{2}\right)^{-2} \sum_{k=-\infty}^{\infty} e^{ik\theta} \left(k + \frac{1}{2}\right)^{-2}, \quad n \in \mathbb{Z},\, \theta \in \mathbb{S}^1.$$

Then

$$\sigma_{nk} = \left(n + \frac{1}{2}\right)^{-2} \left(n - k + \frac{1}{2}\right)^{-2}, \quad k, n \in \mathbb{Z}.$$

Computing the 7×7 matrix $A_\sigma = (\sigma_{nk})_{-3 \leq k, n \leq 3}$ numerically, we get the following matrix in which the entries are generated by MATLAB.

$$\begin{pmatrix}
0.6400 & 0.6400 & 0.0711 & 0.0256 & 0.0131 & 0.0079 & 0.0053 \\
0.1975 & 1.7778 & 1.7778 & 0.1975 & 0.0711 & 0.0363 & 0.0219 \\
0.6400 & 1.7778 & 16.000 & 16.000 & 1.7778 & 0.6400 & 0.3265 \\
0.3265 & 0.6400 & 1.7778 & 16.000 & 16.000 & 1.7778 & 0.6400 \\
0.0219 & 0.0363 & 0.0711 & 0.1975 & 1.7778 & 1.7778 & 0.1975 \\
0.0053 & 0.0079 & 0.0131 & 0.0256 & 0.0711 & 0.6400 & 0.6400 \\
0.0019 & 0.0027 & 0.0040 & 0.0067 & 0.0131 & 0.0363 & 0.3265
\end{pmatrix}$$

Example 23.8. As another example with a closed formula for the symbol, we let

$$\sigma(n,\theta) = e^{-n^2}\theta^2/2, \quad n \in \mathbb{Z},\, \theta \in \mathbb{S}^1.$$

Then

$$(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m) = \begin{cases} (-1)^m e^{-n^2}/m^2, & m \neq 0, \\ \pi^2 e^{-n^2}/6, & m = 0, \end{cases}$$

and hence

$$\sigma_{nk} = \begin{cases} (-1)^{n-k} e^{-n^2}/(n-k)^2, & k \neq n, \\ \pi^2 e^{-n^2}/6, & k = n. \end{cases}$$

The 7×7 matrix $A_\sigma = (\sigma_{nk})_{-3\le k,n\le 3}$ is given numerically by

$$\begin{pmatrix} 0.0002 & -0.0001 & 0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\ -0.0183 & 0.0301 & -0.0183 & 0.0046 & -0.0020 & 0.0011 & -0.0007 \\ 0.0920 & -0.3679 & 0.6051 & -0.3679 & 0.0920 & -0.0409 & 0.0230 \\ -0.1111 & 0.2500 & -1.0000 & 1.6449 & -1.0000 & 0.2500 & -0.1111 \\ 0.0230 & -0.0409 & 0.0920 & -0.3679 & 0.6051 & -0.3679 & 0.0920 \\ -0.0007 & 0.0011 & -0.0020 & 0.0046 & -0.0183 & 0.0301 & -0.0183 \\ 0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0001 & 0.0002 \end{pmatrix}.$$

Remark 23.9. Almost diagonalization of wavelet multipliers using Weyl–Heisenberg frames can be found in [51].

Exercises

1. Let $\sigma \in L^2(\mathbb{S}^1)$. Prove that for all a in $L^2(\mathbb{Z})$,

$$T_{\sigma,\mathbb{Z}}a = \mathcal{F}_{\mathbb{S}^1}\sigma\mathcal{F}_{\mathbb{Z}}a,$$

where $T_{\sigma,\mathbb{Z}}$ is the pseudo-differential operator on $\mathbb{Z}$ with symbol σ.

2. Let $\sigma \in L^2(\mathbb{Z})$. Prove that for all f in $L^2(\mathbb{S}^1)$,

$$T_{\sigma,\mathbb{S}^1}f = \mathcal{F}_{\mathbb{Z}}\sigma\mathcal{F}_{\mathbb{S}^1}f,$$

where $T_{\sigma,\mathbb{S}^1}$ is the pseudo-differential operator on $\mathbb{S}^1$ with symbol σ.

3. Prove that $L^1(\mathbb{Z})$ is dense in $L^p(\mathbb{Z})$ for $1 \le p < \infty$.
4. Is $L^1(\mathbb{Z})$ dense in $L^\infty(\mathbb{Z})$? Explain your answer.

Bibliography

[1] M. S. Agranovich, Spectral properties of elliptic pseudodifferential operators on a closed curve, (Russian) *Funktsional. Anal. i Prilozhen.* **13** (1979), 54–56.

[2] M. S. Agranovich, Elliptic pseudo-differential operators on a closed curve, (Russian) *Trudy Moskov. Mat. Obshch.* **47** (1984), 22–67, 246.

[3] B. A. Amosov, On the theory of pseudodifferential operators on the circle, (Russian) *Uspekhi Mat. Nauk.* **43** (1988), 169–170; Translation in *Russian Math. Surveys* **43** (1988), 197–198.

[4] R. Beals, *Advanced Mathematical Analysis*, Springer-Verlag, 1973.

[5] R. N. Bracewell, *The Fourier Transform and its Applcations*, Third Edition, McGraw-Hill, 2000.

[6] L. Carleson, On convergence and growth of partial sums of Fourier series, *Acta Math.* **116** (1966), 135–157.

[7] L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, 1995.

[8] R. R. Coifman, Y. Meyer and V. Wickerhauser, Numerical Harmonic Analysis, in *Essays on Fourier Analysis in Honor of Elias M. Stein,* Editors: C. Fefferman, R. Fefferman and S. Wainger, Princeton University Press, 1995, 162–174.

[9] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, 1992.

[10] R. E. Edwards, *Fourier Series, A Modern Introduction*, Volume 1, Springer-Verlag, 1979.

[11] R. E. Edwards, *Fourier Series, A Modern Introduction*, Volume 2, Springer-Verlag, 1982.

[12] J. B. J. Fourier, *Théorie Analytique de la Chaleur*, Chez Firmin Didot, 1822.

[13] M. W. Frazier, *An Introduction to Wavelets Through Linear Algebra*, Springer, 1999.

[14] A. Friedman, *Foundations of Modern Analysis*, Holt, Rinehart and Winston, 1970.

M.W. Wong, *Discrete Fourier Analysis*, Pseudo-Differential Operators 5,
DOI 10.1007/978-3-0348-0116-4, © Springer Basel AG 2011

[15] I. Gohberg, S. Goldberg and N. Krupnik, *Traces and Determinants of Linear Operators,* Birkhäuser, 2000.

[16] G. H. Hardy and W. W. Rogosinski, *Fourier Series*, Dover, 1999.

[17] I. N. Herstein, *Topics in Algebra*, Second Edition, Xerox, 1975.

[18] L. Hörmander, Estimates for translation invariant linear operatos in L^p spaces, *Acta. Math.* **104** (1960), 93–139.

[19] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, I, Springer-Verlag, 1983.

[20] R. A. Hunt, On the convergence of Fourier series, in *Orthogonal Expansions and their Continuous Analogues*, Editor: D. T. Haimo, Southern Illinois University Press, 1968, 235–255.

[21] A. N. Kolmogorov, Une série de Fourier–Lebesgue divergent presque partout, *Fund. Math.* **4** (1923), 324–328.

[22] A. N. Kolmogorov, Une série de Fourier–Lebesgue divergent partout, *Comptes Rendus* **183** (1926), 1327–1328.

[23] P. D. Lax, *Functional Analysis*, Wiley, 2002.

[24] P. D. Lax, *Linear Algebra and its Applications*, Second Edition, Wiley, 2007.

[25] S. Molahajloo, Pseudo-differential operators on $\mathbb{Z}$, in *Pseudo-Differential Operators: Complex Analysis and Partial Differential Equations*, Editors: B.-W. Schulze and M. W. Wong, Birkhäuser, 2010, 213–221.

[26] S. Molahajloo and M. W. Wong, Pseudo-differential operators on $\mathbb{S}^1$, in *New Developments in Pseudo-Differential Operators*, Editors: L. Rodino and M. W. Wong, Birkhäuser, 2008, 297–306.

[27] J. E. Moyal, Quantum mechanics as a statistical theory, *Proc. Cambridge Phil. Soc.* **45** (1949), 99–124.

[28] D. Mumford, Mathematics belongs in a liberal education, in *Arts & Humanities in Higher Education*, Sage Publications, 2006, 21–32.

[29] L. B. Pierce, *Discrete Analogues in Harmonic Analysis*, Ph.D. Dissertation, Princeton University, 2009.

[30] M. Ruzhansky and V. Turunen, On the Fourier analysis of operators on the torus, in *Modern Trends in Pseudo-Differential Operators,* Editors: J. Toft, M. W. Wong and H. Zhu, Birkhäuser, 2007, 87–105.

[31] M. Ruzhansky and V. Turunen, *Pseudo-Differential Operators and Symmetries*, Birkhäuser, 2010.

[32] J. Saranen and G. Vainikko, *Periodic Integral and Pseudo-Differential Equations with Numerical Approximation,* Springer-Verlag, 2002.

[33] M. Schechter, *Principles of Functional Analysis,* Second Edition, American Mathematical Society, 2002

[34] R. T. Seeley, *An Introduction to Fourier Series and Integrals*, Dover, 2006.

[35] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[36] E. M. Stein, *Topics in Harmonic Analysis Related to Littlewood–Paley Theory*, Third Printing with Corrections, Princeton University Press, 1985.

[37] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, 1993.

[38] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[39] G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Revised Edition, Wellesley-Cambridge Press, 1997.

[40] E. C. Titchmarsh, *The Theory of Functions,* Second Edition, Oxford University Press, 1939.

[41] E. C. Titchmarsh, A contribution to the theory of Fourier transforms, *Proc. London Math. Soc* **23** (1925), 279–289.

[42] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals,* Third Edition, Chelsea, 1986.

[43] J. S. Walker, *Fourier Analysis*, Oxford University Press, 1988.

[44] J. S. Walker, *Fast Fourier Transforms*, Second Edition, CRC Press, 1996.

[45] J. S. Walker, *A Primer on Wavelets and their Scientific Applications*, Chapman & Hall/CRC, 1999.

[46] N. J. Weiss, Multipliers on compact Lie groups, *Proc. Nat. Acad. Sci. U.S.A.* **68** (1971), 930–931.

[47] N. J. Weiss, L^p estimates for bi-invariant operators on compact Lie groups, *Amer. J. Math.* **94** (1972), 103–118.

[48] M. W. Wong, *Weyl Transforms*, Springer, 1998.

[49] M. W. Wong, *An Introduction to Pseudo-Differential Operators*, Second Edition, World Scientific, 1999.

[50] M. W. Wong, *Wavelet Transforms and Localization Operators*, Birkhäuser, 2002.

[51] M. W. Wong and H. Zhu, Matrix representations and numerical computations of wavelet multipliers, in *Wavelets, Multiscale Systems and Hypercomplex Analysis*, Editor: D. Alpay, Birkhäuser, 2006, 173–182.

[52] N. Young, *An Introduction to Hilbert Space*, Cambridge University Press, 1988.

[53] A. Zygmund, *Trigonometric Series*, Third Edition, Volumes I & II Combined, Cambridge University Press, 2002.

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