D 100611 **(Pages : 3)** **Name..................................**

Reg. No...............................

SIXTH SEMESTER UG (CBCSS-UG) DEGREE EXAMINATION, MARCH 2024

Mathematics

MTS 6B 10—REAL ANALYSIS

(2019 Admissions onwards)

Time : Two Hours and a Half Maximum Marks : 80

Section A

Questions 1—15. *Answer any number of questions.*
Each carry 2 *marks. Maximum marks* 20.

1. State discontinuity criterion. Hence show that the signum function is not continuous at $x = 0$.
2. State maximum-minimum theroem.
3. Show that $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, \infty)$ where $a > 0$.
4. Define Riemann integral of a function f on an integral $[a, b]$.
5. If f and g are in R$[a, b]$ and if $f(x) \leq g(x)$ for all x in $[a, b]$ then show that $\int_a^b f \leq \int_a^b g$.
6. State Lebesgue's integrability criterion.
7. If f and g belong to R$[a, b]$ then the product fg belongs to R $[a, b]$.
8. Show that $\lim \frac{\sin(nx+n)}{n} = 0$ for $x \in \mathbb{R}$.
9. Discuss the uniform convergence of $f_n(x) = \frac{x}{n}$ on A = [0, 1].
10. Evaluate $\lim\left(e^{-nx}\right)$ for $x \in \mathbb{R}, x \geq 0$.
11. Define absolute convergence of series of functions.
12. Evaluate $\int_{-\infty}^{0} e^x \, dx$.
13. Find the principal value of $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$.

Turn over

14. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

15. Define Beta fucntion. St B (p, q) = B (q, p).

Section B

Questions 16—23. *Answer any number of questions.*
Each carry 5 *marks. Maximum marks* 35.

16. Let $A = \{x \in \mathbb{R} | x > 0\}$. Define h on A by $h(x) = 0$ if $x \in A$ is irrational and $h(x) = \frac{1}{n}$ if $x \in A$ is rational with $x = \frac{m}{n}, m, n \in \mathbb{N}$ have no common factor except 1. Then show that h is continuous at every irrational number in A and discontinuous at every rational number in A.

17. Let I be an interval and $f : I \to \mathbb{R}$ be a continuous function on I then show that $f(I)$ is an interval.

18. If $f \in R\,[a, b]$ then show that f is bounded on $[a, b]$.

19. Show that if $\phi : [a, b] \to R$ is a step function then $\phi \in R[a, b]$.

20. Evaluate $\lim \frac{x^2 + nx}{n}, x \in R$. Is the convergence uniform on $\mathbb{R}$?

21. Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$. Then show that (f_n) converges uniformly on A to a bounded function f iff for each $\varepsilon > 0$ there is a number $H(\varepsilon)$ in $\mathbb{N}$ such that for all $m, n \geq H(\varepsilon)$ then $\|f_m - f_n\|_A \leq \varepsilon$.

22. Discuss the convergence of $\int_0^\infty \frac{\sin^2 x}{x^2} dx$.

23. Define Beta function and show that $\forall p > 0, q > 0, B(p, q) = 2 \int_0^{\pi/2} \sin^{2p-1} \theta \cos^{2p-1} \theta \, d\theta$.

Section C

Questions 24—27. *Answer any* **two** *questions.*
Each carry 10 *marks.*

24. (a) Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$ and they are bounded on A then their product fg is also uniformly continuous.

(b) Show that $f(x) = \sqrt{x}$ is uniformly continuous on $[a, \infty)$ where $a > 0$.

25. Suppose f and g are in R $[a, b]$. Then

    (a) if $k \in \mathrm{R}$, show that $kf \in \mathcal{R}[a,b]$ and $\int_a^b kf = k\int_a^b f$.

    (b) $f + g \in \mathcal{R}[a,b]$ and $\int_a^b f + g = \int_a^b f + \int_a^b g$.

26. Discuss the pointwise and uniform convergence of :

    (a) $f_n(x) = \dfrac{\sin(nx + n)}{n}$ for $x \in \mathbb{R}$.

    (b) $g_n(x) = \dfrac{x^2 + nx}{n}$ for $x \in \mathbb{R}$.

27. Show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

(2 × 10 = 20 marks)

C 40602 **(Pages : 3)** **Name..**

Reg. No.....................................

SIXTH SEMESTER U.G. DEGREE EXAMINATION, MARCH 2023

(CBCSS–UG)

Mathematics

MTS 6B 10—REAL ANALYSIS

(2019 Admission onwards)

Time : Two Hours and a Half Maximum : 80 Marks

Section A

Answer any number of questions.
Each question carries 2 marks.
Maximum marks 25.

1. State sequential criterian for continuity.
2. Show that the sine function is continuous on $\mathbb{R}$.
3. Define Lipchitz function. If $f : A \to \mathbb{R}$ is a Lipschitz function then show that f is uniformly continuous on A.
4. Define tagged partition.
5. Show that every constant function on $[a,b]$ is in $\mathbb{R}[a,b]$.
6. State Cauchy's criterion for Riemann integrability.
7. Let F, G be differentiable on $[a,b]$ and let $f = F'$ and $g = G'$ belong to $\mathbb{R}[a,b]$, then show that

   $\int_a^b f\,G = [FG]_a^b - \int_a^b F\,g$.
8. Show that $\lim\left(\frac{x}{n}\right) = 0$ for $x \in \mathbb{R}$.
9. Define uniform convergence of a sequence of functions.
10. State bounded convergence theorem.
11. State Weirstrass M-test for the uniform convergence of series of functions.
12. Evaluate $\int_1^\infty \frac{dx}{x^2+1}$.

Turn over

13. Find the principal value of $\int_{-2}^{3} \frac{dx}{(x-1)^3}$.

14. Discuss the absolute convergence of $\int_{0}^{\infty} \frac{\sin x}{n+1}\, dx$ for $n\pi \le x \le (n+1)\pi$, $n = 0, 1, 2, \ldots$

15. If $\int_{0}^{b} \frac{dx}{1+ax} = \frac{1}{a} \ln(1+ab)$. Evaluate $\int_{0}^{b} \frac{xdx}{(1+ax)^2}$.

Section B

Questions 16–23, *answer any number of questions.*
Each question carries 5 *marks.*
Maximum marks 35.

16. State and prove Boundedness theorem for continuous function.

17. Show that $f(x) = \frac{1}{1+x^2}, x \in \mathbb{R}$ is uniformly continuous in $\mathbb{R}$.

18. State and prove Squeeze theorem for Riemann integrable functions.

19. If $f \in \mathbb{R}[a,b]$ and f is continuous at a point $c \in [a,b]$. Then show that the indefinite integral $F(z) = \int_{a}^{z} f$ for $z \in [a,b]$ is differentiable at c and $F'(c) = f(c)$.

20. Show that a sequence (f_n) of bounded functions on $A \subset \mathbb{R}$ converges uniformly on A to f iff $\| f_n - f \| n \to 0$.

21. Discuss the convergence of $f_n(x) = \frac{x^n}{n+x^n}, x \ge 0$. Is the convergence uniform on $[0, \infty]$.

22. Evaluate $\int_{-1}^{1} \frac{dx}{x^2 - 1}$.

23. Show that ~~r~~ $q \in \mathbb{R}, \int_{1}^{\infty} x^q e^{-x}\, dx$ converges.

Section C

Questions 24–27, *answer any* **two** *questions.*
Each question carries 10 *marks.*

24. State and prove Maximum Minimum Theorem.

25. State and prove Cauchy's criterion of Riemann integrability.

26. Let (f_n) be a sequence of functions in $\mathbb{R}[a,b]$ and suppose that (f_n) converges uniformly on $[a,b]$ to f. Then show that $f \in \mathbb{R}[a,b]$.

27. Show that $\int_0^\infty \frac{\sin x}{x} dx$ exists and converges to a finite real value and that this integral does not converge absolutely.

(2 × 10 = 20 marks)

C 20645 **(Pages : 3)** **Name.......................................**

Reg. No...................................

SIXTH SEMESTER U.G. DEGREE EXAMINATION, MARCH 2022

(CBCSS–UG)

Mathematics

MTS 6B 10—REAL ANALYSIS

(2019 Admissions)

Time : Two Hours and a Half Maximum : 80 Marks

Section A

Answer at least **ten** *questions.*
Each question carries 3 *marks.*
All questions can be attended.
Overall Ceiling 30.

1. Define continuity of a function. Show that the constant function $f(x) = b$ is continuous on $\mathbb{R}$.

2. State Boundedness theorem. Is boundedness of the interval, a necessary condition in the theorem ? Justify with an example.

3. If $f : A \to IR$ is uniformly continuous on $A \subseteq \mathbb{R}$ and (x_n) is a Cauchy sequence in A. Then show that $f(x_n)$ is a Caychy sequence in $\mathbb{R}$.

4. Define Riemann sum of a function $f : [a, b] \to \mathbb{R}$.

5. Suppose f and g are in $\mathbb{R}[a, b]$ then show that $f + g$ is in $\mathbb{R}[a, b]$.

6. State squeeze theorem for Riemann integrable functions.

7. If f belong to $\mathbb{R}[a, b]$, then show that its absolute value $|f|$ is in $\mathbb{R}[a, b]$.

8. Define pointwise convergence of a sequence (f_n) of functions.

9. Discuss the uniform convergence of $f_n(x) = x^n$ on (–1,1].

10. If $h_n(x) = 2nxe^{-nx^2}$ for $x \in [0,1], n \in \mathbb{N}$ and $h(x) = 0$ for all $x \in [0,1]$, then show that :

$$\lim \int_0^1 h_n(x)\,dx \neq \int_0^1 h(x)\,dx.$$

11. State Cauchy criteria for uniform convergence series of functions.

Turn over

12. Evaluate $\int_{-1}^{0} \frac{dx}{\sqrt[3]{x}}$.

13. What is Cauchy principle value. Find the principal value of $\int_{-1}^{1} \frac{dx}{x}$.

14. State Leibniz rule for differentiation of Ramann integrals.

15. State that $\Gamma(p+1) = p\Gamma p$ for $p > 0$.

(10 × 3 = 30 marks)

Section B

Answer at least **five** *questions.*
Each question carries 6 *marks.*
All questions can be attended.
Overall Ceiling 30.

16. Show that the Dirichlet's function :

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$ is not continuous at any point of $\mathbb{R}$.

17. State and prove Bolzano intermediate value theorem.

18. Show that the following functions are not uniformly continuous on the given sets :

    (a) $f(x) = x^2$ on $A = [0, \infty]$.

    (b) $g(x) = \sin \frac{1}{x}$ on $B = (0, \infty)$.

19. If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then show that $f \in \mathbb{R}[a,b]$.

20. Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$. Then show that f is continuous on A.

21. Let $f_n : [0,1] \to \mathrm{IR}$ be defined for $n \geq 2$ by :

$$f_n(x) = \begin{cases} n^2 x & , 0 \leq x \leq \frac{1}{n} \\ -n^2(x - 2/n), & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & , \frac{2}{n} \leq x \leq 1. \end{cases}$$

Show that the limit function is Riemann integrable. Verify whether $\lim \int_0^1 f_n(x) = \int_0^1 f(x)\,dx$.

22. Given $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy = \pi$, find the value of $\int_0^\infty e^{-x^2}\, dx = \sqrt{\frac{\pi}{2}}$.

23. Show that $\forall p > 0, q > 0$ $B(p,q) = \frac{\Gamma p\, \Gamma q}{\Gamma (p+q)}$.

(5 × 6 = 30 marks)

Section C

Answer any **two** *questions.*
Each question carries 10 *marks.*

24. State and prove Location of roots theorem.

25. State and prove Additivity theorem.

26. Evaluate (a) $\lim \frac{x^n}{1+x^n}$ for $x \in \mathbb{R}, x \geq 0$. (b) $\lim \frac{\sin nx}{1+nx}$ for $x \in \mathbb{R}, x \geq 0$.

    Discuss about their uniform convergence.

27. (a) Show that $\forall\, q > -1, \int_0^1 x^q e^{-x} dx$ converges.

    (b) Show that $\forall\, q \leq -1, \int_0^1 x^q e^{-x} dx$ diverges.

(2 × 10 = 20 marks)