D 110208 **(Pages : 3)** **Name.......................................**

Reg. No..................................

FIFTH SEMESTER (CBCSS—UG) DEGREE EXAMINATION NOVEMBER 2024

Mathematics

MTS 5B 05—ABSTRACT ALGEBRA

(2020 Admission onwards)

Time : Two Hours and a Half Maximum : 80 Marks

Section A

Answer any number of questions.
Each question carries 2 marks.
Ceiling is 25.

1. Let n be a positive integer. Prove that the congruence class $[a]_n$ has a multiplicative inverse in $\mathbb{Z}_n$ if and only if $(a, n) = 1$.

2. Make multiplication table for $\mathbb{Z}_6$.

3. Find the order of the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 2 & 1 \end{pmatrix}$.

4. Let G be a nonempty set with an associative binary operation in which the equations $ax = b$ and $xa = b$ have solutions for all $a, b \in G$. Prove that G is a group.

5. Let G be group. Prove that G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

6. Prove that any group of prime order is cyclic.

7. In $GL_2(\mathbb{R})$, find the order of $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

8. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic.

9. Let G be a cyclic group. If G is infinite, prove that $G \cong \mathbb{Z}$.

Turn over

10. Prove that the set of all even permutations of S_n is a subgroup of S_n.

11. Let $\phi : G_1 \to G_2$ be group homomorphism, with $K = \ker(\phi)$. Prove that K is a subgroup of G_1 such that $gKg^{-1} \in K$ for all $k \in K$ and $g \in G_1$.

12. Let $G = \mathbb{Z}_{12}$ and $H = \langle | 4 | \rangle$. Find all cosets of H.

13. State First isomorphism theorem.

14. Let G be a group. Prove that $\text{Aut}(G)$ is a group under composition of functions.

15. Prove that any subring of a field is an integral domain.

Section B

Answer any number of questions.
Each question carries 5 *marks.*
Ceiling is 35.

16. State and prove Euler theorem.

17. On $\mathbb{R}^2$, define $(a_1, a_2) \sim (b_1, b_2)$ if $a_1^2 + a_2^2 = b_1^2 + b_2^2$. Check that this defines an equivalence relation. What are the equivalence classes ?

18. Prove that the units of $\mathbb{Z}_8$ forms a group under multiplication of congruences.

19. Let G be a group with identity element e, and let H be a subset of G. Prove that H is a subgroup of G if and only if the following conditions hold :

    (a) $ab \in H$ for all $a, b \in H$;

    (b) $e \in H$; and

    (iii) $a^{-1} \in H$ for all $a \in H$.

20. Let G be a group, and let H and K be subgroups of G. If $h^{-1}kh \in K$ for all $h \in H$ and $k \in K$, Prove that HK is a subgroup of G.

21. Prove that every subgroup of a cyclic group is cyclic.

22. State and prove fundamental theorem of homomorphism.

23. Let G be a group with normal subgroups H, K such that $HK = G$ and $H \cap K = \{e\}$. Prove that $G \cong H \times K$.

Section C

Answer any **two** *questions.*
Each question carries 10 *marks.*
Maximum 20 *marks.*

24. a) Prove that every permutation in S_n can be written as a product of disjoint cycles

b) Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 7 & 6 & 3 & 8 & 1 & 4 \end{pmatrix}$ be a permutation in S_8. Express σ as a product of disjoint cycles.

25. a) Let $\phi : G_1 \to G_2$ be an isomorphism of groups. Prove that ϕ preserves following structural properties :

(i) If a has order n in G_1, then $\phi(a)$ has order n in G_2,

(ii) If G_1 is abelian, then so is G_2,

(iii) If G_1 is cyclic, then so is G_2.

b) Prove that $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

26. Let H be a subgroup of the group G. Prove that the following conditions are equivalent.

a) H is a normal subgroup of G;

b) $aH = Ha$ for all $a \in G$;

c) for all $a, b \in G$, $ab \in H$ is the set theoretic product $(aH)(bH)$;

d) for all $a, b \in G$, $ab^{-1} \in H$ if and only if $a^{-1}b \in H$.

27. State and prove second isomorphism theorem.

(2 × 10 = 20 marks)

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D 50665 **(Pages : 3)** Name.....................................

Reg. No.................................

FIFTH SEMESTER (CBCSS—UG) DEGREE EXAMINATION

NOVEMBER 2023

Mathematics

MTS 5B 05—ABSTRACT ALGEBRA

(2020 Admission onwards)

Time : Two Hours and a Half Maximum : 80 Marks

Section A

Answer any number of questions.
Each question carries 2 marks.
Ceiling is 25.

1. Make multiplication table for $\mathbb{Z}_7$.

2. State and prove Fermat theorem.

3. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}$ be permutation in S_7.

   Find $\sigma\tau$ and $\tau\sigma$.

4. State and prove cancellation property for groups.

5. Is $\mathbb{Z}_8^x$ cyclic ? Justify.

6. Let H be a subgroup of the group G. For $a, b \in$ G, define $a \sim b$ if $ab^{-1} \in$ H. Prove that $\sim$ is an equivalence relation.

7. Find HK in $\mathbb{Z}_{16}^x$, if $H = \langle [3] \rangle$ and $K = \langle [5] \rangle$.

8. Let G_1 and G_2 be groups, and let $\phi : G_1 \to G_2$ be a function such that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G_1$. Prove that ϕ is one to one if and only if $\phi(x) = e$ implies $x = e$, for all $x \in G_1$.

Turn over

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9. Let G be a group, and let $a, b \in$ G be elements such that $ab = ba$. If the orders of a and b are relatively prime, prove that o (ab) = o (a) o (b).

10. Let $\phi : G_1 \to G_2$ be a group homomorphism, with K = ker ϕ. Prove that K is a subgroup of G_1.

11. Let $\phi : G_1 \to G_2$ be an onto homomorphism. If H_1 is normal in G_1, prove that ϕ (H_1) is normal in G_2.

12. Let G = $\mathbb{Z}_{24}$ and H = $\langle [3] \rangle$. Find all cosets of H.

13. State second isomorphism theorem.

14. Prove that Aut $(\mathbb{Z}_n) \cong \mathbb{Z}_n^{\times}$.

15. If D is an integral domain, prove that D $[x]$ is an integral domain.

Section B

Answer any number of questions.

Each question carries 5 *marks.*

Ceiling is 35.

16. Let n be a positive integer. Prove that :

 (a) The congruence class $[a]_n$ has a multiplicative inverse in $\mathbb{Z}_n$ if and only if $(a, n) = 1$.

 (b) A non zero element of $\mathbb{Z}_n$ is either has a multiplicative inverse or is a divisor of zero.

17. (a) Let $\sigma \in S_n$ be written as a product of disjoint cycles, prove that the order of σ is the least common multiple of the lengths of its cycles.

 (b) Find the order of $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 3 & 2 & 1 \end{pmatrix}$.

18. Let G be a group and let H be a subset of G. Prove that H is a subgroup of G if and only if H is nonempty and $ab^{-1} \in$ H for all $a, b \in$ H.

19. Let G_1 and G_2 be groups. Prove that the direct product $G_1 \times G_2$ is a group under the operation defined for all $(a_1, a_2), (b_1, b_2) \in G_1 \times G_2$ by $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2)$.

20. If m and n are positive integers such that gcd $(m, n) = 1$, prove that $\mathbb{Z}_{mn}$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$.

21. Give the subgroup diagram of $\mathbb{Z}_{12}$.

22. State and prove fundamental homomorphism theorem.

23. Let G be a group. Prove that Aut (G) is a group under composition of functions, and Inn (G) is a normal subgroup of Aut (G).

Section C

Answer any **two** *questions.*
Each question carries 10 *marks.*
Maximum 20 *marks.*

24. If permutation written as a product of transpositions in two ways, prove that the number of transpositions is either even in both cases or odd in both cases.

25. (a) State and prove Lagrange theorem.

    (b) Prove that any group of prime order is cyclic.

26. State and prove Cayley theorem.

27. State and prove second isomorphism theorem.

(2 × 10 = 20 marks)

D 30568 **(Pages : 3)** Name.......................................

Reg. No..................................

FIFTH SEMESTER (CBCSS–UG) DEGREE EXAMINATION NOVEMBER 2022

Mathematics

MTS 5B 05—ABSTRACT ALGEBRA

(2020 Admission onwards)

Time : Two Hours and a Half Maximum : 80 Marks

Section A

Answer any number of questions.
Each question carries 2 marks.
Ceiling is 25.

1. Write addition and multiplication tables for $\mathbb{Z}_4$.

2. Check whether the relation on defined by $a \sim b$ if $n \mid (a - b)$, where n is a positive integer is an equivalence relation.

3. Consider the following permutations in S_7 :

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}.$$

Compute $\sigma\tau$ and $\tau\sigma$.

4. Show that cancellation property holds in a group G.

5. Find all cyclic subgroups of the group $\mathbb{Z}_6$.

6. Find the order of the element $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ in $GL_2(R)$.

7. Give addition table for $\mathbb{Z}_2 \times \mathbb{Z}_2$.

8. Show that composite of two group isomorphisms is a group isomorphism.

9. Give the subgroup diagrams of $\mathbb{Z}_{24}$.

Turn over

10. Find the order of the permutation (1, 2, 5) (2, 3, 4) (5, 6).

11. Let $G = \mathbb{Z}_{12}$, and let H be the subgroup $4\mathbb{Z}_{12}$. Find all cosets of H.

12. Define normal subgroup of a group G. Give an example.

13. Compute the factor group $\dfrac{\mathbb{Z}_6 \times \mathbb{Z}_4}{\langle(2, 2)\rangle}$.

14. Define commutative ring. Give an example.

15. Define Integral Domain. Give an example.

Section B

Answer any number of questions.
Each question carries 5 marks. Ceiling is 35.

16. If $(a, n) = 1$, then show that $a^{\phi(n)} \equiv 1 \pmod{n}$.

17. Let G be a group and let H be a subset of G. Then show that H is a subgroup of G if and only if H is non-empty and $ab^{-1} \in \mathrm{H}$ for all $a, b \in \mathrm{H}$.

18. Let G be a finite cyclic group with n elements. Show that $\mathrm{G} \cong \mathrm{Z}_n$.

19. Let $\phi : \mathrm{G}_1 \to \mathrm{G}_2$ be a group homomorphism with $\mathrm{Ker}\ \phi = \{x \in \mathrm{G}_1 : \phi(x) = e\}$. Show that ϕ is one to one if and only if $\mathrm{Ker}\ \phi = \{e\}$.

20. Let G be a group, and let $a, b \in \mathrm{G}$ be elements such that $ab = ba$. If the orders of a and b are relatively prime, then prove that $0(ab) = 0(a)\,0(b)$.

21. Show that any subring of a field is an integral domain.

22. Let G be an abelian group, and let n be any positive integer. Show that the function $\phi : \mathrm{G}_1 \to \mathrm{G}_2$ defined by $\phi(x) = x^n$ is a homomorphism.

23. State and prove Fundamental Homomorphism Theorem.

Section C

Answer any **two** *questions.*
Each question carries 10 *marks.*
Maximum 20 *marks.*

24. Show that the inverse of a group isomorphism is a group isomorphism.

25. Show that every sub-group of a cyclic group is cyclic.

26. Let H be a sub-group of the finite group G. Show that the order of H is a divisor of order of G.

27. State and prove First Isomorphism Theorem.