The Existence and Continuity of Utility Functions: A New Proof

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ABSTRACT

It is well known that the existence of a countable order dense subset is necessary and sufficient for a preference order to be representable by a utility function, and that this condition is also sufficient for the utility function to be continuous with respect to the order topology. While the modern proof of the first part of this result is based on a theorem of Cantor on ordered sets, the proof of continuity is usually based on a theorem of Debreu in real analysis. This paper seeks to eliminate this appeal to real analysis, and show that the proof of continuity requires only the order structure of the reals and does not need any metric or algebraic properties of the reals. We also show that any continuous preference ordering on a separable topological space with an at most countable number of connected components is representable by a continuous utility function thereby relaxing the usual assumption that the space be connected.

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A theorem of Cantor that the existence of a countable order dense subset is necessary and sufficient for an ordered set to be homeomorphically embeddable in the reals completely settles the question of the existence of a utility function representing a preference ordering (see Richter,1966). It does not seem to be realized that a closely related order type theorem is sufficient to establish the continuity of the utility function with respect to the order topology. Instead, proofs of continuity have relied on a theorem of Debreu in real analysis (see Debreu,1964 and Fishburn,1970). Unfortunately, the use of this theorem and particularly its method of proof introduce the algebraic and metrical properties of the reals into a problem which is concerned exclusively with order structure. Our proof eliminates this dependence on real analysis and displays the logical structure of the problem more clearly.

Since the utility function is continuous with respect to the order topology, continuity with respect to any other topology is equivalent to the continuity of the preference ordering itself (i.e. the order topology should be a coarsening of the original topology). In particular, it has been known that any continuous preference order on a separable connected topological space has a continuous utility representation (Debreu,1964). We show that the assumption of connectedness can be relaxed to the extent that the space may have an at most countable number of connected components.

We shall throughout assume that the preference order is always a strict total order. Techniques for extending the analysis to preorders and to partial orders are well known (Fishburn,1970). To ask whether a utility function exists is equivalent to asking whether the ordered set S is <u>similar</u> to a subset of the reals (R). (Two sets, A and B are said to be similar if there exists a one- to-one order preserving correspondence between them.) Two similar sets are said to have the same order type. A utility function exists if and only if the ordered set has the same order type as a subset of the reals.

<u>Theorem I</u> : A preference ordering on a set S is representable by a utility function which is continuous in the order topology if under this ordering S contains a countable order dense subset, i.e., a subset D of elements d_1, d_2... such that the following condition is satisfied :

If $x < y$ $(x, y \varepsilon S)$ then:

$x \varepsilon D$, or $y \varepsilon D$, or $d \varepsilon D$ where $x < d < y$.

(Order denseness is closely related to but not the same as denseness with respect to the order topology. If the open interval (x, y) is nonempty, topological denseness guarantees an element of D lying in this interval; order denseness is a weaker requirement in this case as it guarantees only an element of D lying in the closed interval [x, y]. If the open interval (x, y) is empty, order denseness guarantees that either x or y belongs to D; topological denseness guarantees nothing.)

Proof

The proof is based on the following order type theorem (Hausdorff, 1957 p63):

Every unbounded continuous set for which there exists a countable set that is dense in it has order type λ (the order type of the reals).

This theorem implies that we need only prove that we can embed the set S in an unbounded continuous set S^*, and find a countable set D^* which is dense in S^*. S^* would then be similar to the reals by the above theorem; S being a subset of S^* would then be similar to a subset of the reals. This would establish the existence of a utility function.

An unbounded set is one in which there is no first element and no last element i.e. given any x there exists a $y < x$ and a $z > x$. To embed S in an unbounded set we add some artificial elements if necessary. If there exists an element min such that there is no $x < min$ then we add the artificial elements $A^{min}(q)$ where q ranges over all the negative reals ($q<0$). We specify that $A^{min}(q)$ is greater than $A^{min}(p)$ for $q > p$ and is less than x for all other x. Similarly, if there exists a maximum element we add artificial elements $A^{max}(q)$ where q ranges over all the positive reals ($q>0$). We specify that $A^{max}(q)$ is less than $A^{max}(p)$ for $q < p$ and is greater than x for all other x. It is easily verified that with these additional points S becomes unbounded.

To define a continuous set, we need to consider partitions of a set :

$S = P + Q$

where $P < Q$ (every element of P is less than every element of Q).

Four cases can arise :

P has a last element and Q has a first element JUMP

P has a last element and Q has no first element CUT

P has no last element and Q has a first element CUT

P has no last element and Q has no first element GAP

A set is said to be continuous if there are no jumps and no gaps; in other words, in a continuous set every partition is a cut.

We must now eliminate all jumps and gaps that may exist in S. We shall first show that, under the conditions of the theorem, there can be only a countable number of jumps. If x is the last element of P and y is the first element of Q in a jump, then there is no element between x and y. The countable order denseness condition guarantees that either x or y belongs to the countable set D. This means that the number of such jumps is at most countable. For each such jump, we now add the artificial elements $A^{PQ}(q)$ where q ranges over the open unit interval : $0<q<1$. We specify that $A^{PQ}(q) < Q$, $P < A^{PQ}(q)$ and $A^{PQ}(p) < A^{PQ}(q)$ if $p < q$. It is readily verified that with the addition of these elements, all jumps are removed. Next we eliminate all gaps by adding an artificial element A_{PQ} whenever the partition $S = P + Q$ is a gap. We specify that $P < A_{PQ} < Q$.

Let S^* be the set obtained from S by adding the points $A^{min}(q)$, $A^{max}(q)$, $A^{PQ}(q)$ and A_{PQ}. As seen above S^* is an unbounded, continuous set containing S. Now consider the set D^* obtained from D by adding the points $A^{min}(q)$, $A^{max}(q)$ and $A^{PQ}(q)$ for all <u>rational</u> values of q. D^* is an at most countable union of countable sets and is therefore countable. It is easy to verify that D^* is dense in S^* (a subset B is dense in an ordered set A if for any $x,y \in A$ ($x < y$) there exists a $w \in B$ such that $x < w < y$).

We have thus shown that S^* is similar to the real numbers. Since the similarity map preserves the order relationship, it preserves the order topologies also; the mapping is thus a homeomorphism with respect to these topologies. The utility function U is the restriction of this map to S; U is, therefore, a homeomorphism (and <u>a fortiori</u> continuous) between S and a subset U(S) of R with respect to the subset topologies on S and U(S) induced from the topologies on S^* and R. To show that U is continuous from S to R, we must show that these subset topologies are the same as the order topologies on S and U(S). (Given an arbitrary subset B of an ordered set A, B itself is an ordered set; the order topology on B is not, in general, the same as the subset topology induced on B from the order topology on A.) A subbase for the order topology on S^* consists of the sets

$F^*(\mu) = \{ x \in S^* \mid x < \mu \}$ $\mu \in S^*$, and

$G^*(\mu) = \{ x \in S^* \mid x > \mu \}$ $\mu \in S^*$.

Similarly, a subbase for the order topology on S consists of the sets

$F(\mu) = \{ x \in S \mid x < \mu \}$ $\mu \in S$, and

$G(\mu) = \{ x \in S \mid x > \mu \}$ $\mu \in S$.

A subbase for the subset topology on S consists of the sets $S \cap F^*(\mu)$ and $S \cap G^*(\mu)$. If $\mu \in S$, $F(\mu) = S \cap F^*(\mu)$ and $G(\mu) = S \cap F^*(\mu)$. This proves that the subset topology is a refinement of the order topology (this is obviously true in general). We must now consider $\mu \notin S$. There are four possibilities:

1. $\mu = A^{min}(q)$. $S \cap F^*(\mu) = \phi$ and $S \cap G^*(\mu) = S$, both open.

2. $\mu = A^{max}(q)$. $S \cap F^*(\mu) = S$ and $S \cap G^*(\mu) = \phi$, both open.

3. $\mu = A^{PQ}(q)$. $S \cap F^*(\mu) = F(y)$ and $S \cap G^*(\mu) = G(x)$ where x is the last element of P and y is the first element of Q.

4. $\mu = A_{PQ}$. $S \cap F^*(\mu) = \bigcup_{x \in Q} F(x)$ and

$$S \cap G^*(\mu) = \bigcup_{x \in P} G(x)$$

are both unions of open sets and, therefore, open.

The result which we have just proved is of independent interest and can be stated in more general terms :

<u>Lemma</u> : Let A be a subset of the ordered set S and A^c be the complement of A. If the maximal intervals of S contained in A^c are either open or are degenerate (consist of only a single point), then the order topology on A coincides with the subset topology induced on A from S.

In the course of proving his corollary, Debreu proves a special case of this lemma though his results and proofs are couched in the language of real analysis rather than that of ordered sets. His proof does, however, carry over to the general case, and is essentially the same as our proof above.

The proof of continuity is now easy. The order and subset topologies on S are identical; since U is a homeomorphism from S^* to R these two topologies are identical on U(S) also. U is continuous from S endowed with the order topology to U(S) endowed with the subset topology, and therefore from S to R. ■

<u>Corollary</u> : In all cases, whenever a utility function exists, a bounded utility function exists with the same continuity properties; <u>a fortiori</u>, it is never necessary to introduce the extended real line (with $\pm\infty$ adjoined).

<u>Proof</u> : The open unit interval (0,1) is an unbounded, continuous set with a countable dense subset (viz., the rationals between 0 and 1), and is, therefore, by the order type theorem, similar to the reals. If U is a real valued utility function, and V is the similarity mapping from R to (0,1), then the composition V o U is a utility function taking values in (0,1); it is continuous if U is. Since R is similar to (0,1), the extended real line (with $\pm\infty$ adjoined) is similar to the closed unit interval [0,1]. Any utility function into the extended reals can be replaced by one into [0,1] with the same continuity properties. ■

Since the utility function is always continuous in the order topology, it is continuous with respect to any other finer topology, i.e. any topology in which the intervals $\{ x \mid \alpha < x < \beta \}$ are open.

When the set S is endowed with a topological structure and this topology is a refinement of the order topology, we say that the preference ordering is continuous. In some topological spaces, continuity of preferences implies that the countable order denseness condition is automatically satisfied; this makes the existence theorems simpler in those cases. Two well known examples are the separable connected spaces and the spaces with a countable base. The following theorem generalizes these results.

Theorem II: If S is a topological space and preferences are continuous, then a continuous utility function exists if either of the following conditions is satisfied:

a) S is a separable space with an at most countable number of connected components; or

b) S has a countable base of open sets.

Proof : Since the order topology is a coarsening of the topology of S, the order topology will also satisfy conditions (a) and (b) if the original topology does so.

Assume that S satisfies condition (a). Let D_1 be a countable set which is dense in S with respect to the order topology. Now consider all pairs of points x,y ε S, x < y such that no element of S lies between x and y. If x belongs to a component C, then x must be the last (or largest) element of C. This is because the open set $\{z \varepsilon C \mid z > x\}$ is the same as the closed set $\{z \varepsilon C \mid z \geq y\}$ and must be null as C is connected. Since there are only a countable number of components, there are only a countable number of such x's : we add these to D_1 to obtain the countable set D. It is seen that D is a countable order dense subset of S. The existence of a continuous utility function follows from Theorem I.

Now consider condition (b). Let O_j be a countable base of open sets for the order topology. By picking one point from each of the O_j, we have a countable subset D_1 which is dense with respect to the order topology. Once again we shall prove that there are only a countable number of pairs of points x,y ε S, x < y such that no element of S lies between x and y. Given any such pair, the set $\{z \mid z \leq x\} = \{z \mid z < y\}$ is an open set and is the union of some of the O_j; there must, therefore, be an O_j which is ≤ x and contains x; x is the largest element of O_j. We can thus associate a distinct O_j to each such x; since the O_j are countable, the x's must also be countable. Once again, we add the x's to D_1 to get the countable order dense set D, and use Theorem I.■

We now show that Debreu's main theorem on which he bases all continuity properties can itself be proved as a corollary of our theorems without invoking any algebraic or metric properties of the reals.

Theorem III (Debreu's main theorem): If S is a subset of $\overline{R}$ (the extended reals), there is an increasing function g from S to $\overline{R}$ such that all the maximal intervals in the complement of g(S) are either degenerate or open.

Proof : Since the extended real line is similar to the closed unit interval, it is sufficient to consider the reals instead of the extended reals in proving the above theorem. Since R (endowed with the order topology) has a countable base, so does any subset S of R (S is endowed with the subset topology). Since the order topology on S is a coarsening of the subset topology, we can invoke Theorem II to show that there exists a continuous utility function from S to R. An examination of the proof of Theorem I shows that the utility function constructed therein has the property required of g. ■

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