CALCULUS

and Analytic Geometry

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Preface

This Second Edition incorporates improvements in exposition, several new and clearer proofs, expansion of some problem sets, a review of Analytic Trigonometry, a few remarkable airbrush figures, and bonuses on pages previously only partially used. A quick evaluation of the alterations may be made by comparing old and new pages 3–4, 54–56, 76–77, 118, 179–186, 323–329, 352 (figure), 474–475, but all changes are too numerous to list.

The First Edition anticipated the trend in secondary and college mathematics so well that an extensive survey of users revealed little need for disturbing the content or order of the book, In fact, it was found that most of the suggested improvements could be made within the present format and without altering the compact size of the book. Thus a casual examination may not reveal the extent of the revision, but, as a matter of interest, the manuscript for it outweighed the book itself.

I deeply appreciate the help of Professors Billy J. Attebery and Marian Brashears of the University of Arkansas, Irving Drooyan, Los Angeles Pierce College, M. L. Madison, Colorado State University, Gary Mouck, Santa Barbara City College, Karl Stromberg, University of Oregon, and others who took the trouble to volunteer criticism or responded to requests for it. I hold no delusions of perfection and continue to seek advice from both students and faculty.

The preface of the First Edition is reproduced below, except for the apologetic justification of set notation and terminology at this level. In the six years since the First Edition went to press, elementary set theory has become so common that its omission would be untenable.

There is clear evidence that some students whose records qualify them for a first course in calculus are not ready for a rigorous treatment of the subject. It is gratifying, however, that an increasing number of students are impatient with vague ideas and half-truths, and that a generation of college instructors has emerged with sufficient training and judgment to present material within reach of most students, but also dedicated to elevating the potential of more gifted students. Democratic education must be continued to provide many technicians who have absorbed enough intuitive background for routine applications, but in the present international scientific marathon it is imperative that extra effort be extended toward providing earlier and better training for those students who have possibilities of making original contributions in their fields.

Since most colleges cannot section their students according to ability, this book was written in such a way that various types of students may be taught in the same class. On page 55, for example, the inequalities

$$1 - x^2 < \frac{\sin x}{x} < 1 \quad \text{for} \quad 0 < |x| < \frac{\pi}{2}$$

replace the usual ones with little extra effort. An instructor may now point out that "as x gets small, then $\cdots$" and in most classes it might be well to do so, but there is the intriguing possibility that an additional moment or two devoted to showing "for $\epsilon > 0$, then $\delta = \sqrt{\epsilon} \cdots$" might pay dividends.

Many students resent an instructor's insistence upon a standard above the texts, but readily accept a less accurate one if they are not frustrated by a patchwork of required and omitted portions. For this reason much of the rigorous development is placed in appendices with full confidence that a student capable of appreciating such work is also capable of interlacing it properly. An instructor may thus relax to fairly intuitive presentations in class, if this seems desirable, but can help his better students to higher goals by encouraging them to read critically and to discuss fundamental concepts with him. All students will have the opportunity of seeing the utility of powerful underlying principles, whereas the averse ones need not be subjected long, if at all, to ideas beyond their comprehension.

Even should the happy situation arise of a whole class of serious students, it might be well to omit the pertinent appendices until after Chapter 6. This procedure would fulfill the early service obligation to a companion physics course and give an overall picture of the main features of calculus. With this rough picture as a guide, the student would be in a better position to appreciate the value and significance of Appendices A1–A4.

The definite integral is undoubtedly the most erudite concept of calculus, but even so there seems to be an obsession to move it ever nearer the beginning of the course. It is argued that this shift is dictated by the needs of a concurrent physics course. Experience has shown, however, that reasonable physicists make no such demands, whereas others would not be satisfied unless definite integrals (even curvilinear integrals) were taken in high school. Consequently, in this book students are given time to polish their algebraic manipulations and equation-graph concepts, and to have transcendental functions in hand before tackling definite integrals late in the first or early in the second semester. If forced against his better judgment, an experienced instructor may work in an early exposure to definite integrals by postponing portions of Chapters 1, 2, and 4 and all of Chapters 3 and 5 until after selected problems of Chapter 6 have been worked.

Vector analysis is a handy tool which has either been sadly neglected or glorified beyond its due by making analytic geometry and calculus its slaves. With the plethora of new ideas that cannot be avoided in the early portion of the course, it seemed prudent to allow the students the comfort of their familiarity with plane rectangular coordinates for any new work on standard graphs and to delay introducing vectors until Chapter 8. Even here the third dimension is postponed by using vectors in the plane for curvilinear motion, parametric equations, rotation of axes, and polar coordinates. Now with vector ideas and notation entrenched, their power and economy as an aid to space considerations is capitalized upon in succeeding chapters.

This book was written with the numerous potential engineers and scientists more in mind than the small group of mathematics majors. The rigor and set theory included here are far surpassed in an introduction to analysis course and thus serves the mathematics major as a preview and only slight acceleration. Most undergraduate engineering programs, on the other hand, are so crowded that mathematics courses are confined to the first two years and hence students in these programs are forever handicapped unless they see some careful reasoning in their calculus course. Vector analysis is presented from the directed line segment point of view as used and understood by engineers rather than the linear space approach more satisfying to mathematicians. Also, the present tendency to subdue the traditional first course in differential equations is followed by including a short chapter to provide further manipulative skill in integration, introduce some of the standard techniques in finding so called solutions, and give a glimpse of approximations so important in modern high speed computing. It is hoped that students who continue their mathematical studies will not be spoiled for the solid and practical work in store for them when they are ready for an honorable differential equations course.

It is inevitable that the first half of the book is richer in theory than the second half. Once a fairly firm foundation has been laid, much of the superstructure follows without ostensive honor to the underlying theory. If a student has developed reasonable thought patterns, then he will be able to fill in many details for himself (or at least see there are gaps), but if he has not, there seems little point in plaguing him further. To be sure, Jacobians could have been developed for the sake of less intuitive discussions of $\rho d\rho d\theta$, upper and lower limits for some proofs or neater proofs,* functions of bounded

* In particular for L'Hospital's Rules, see A. E. Taylor, *American Mathematical Monthly*, Vol. 59 (1952), pp. 20–24.

variation for arc length, Stieltjes integrals for work and other concepts, or a rigorous Fubini theorem, but the effort would seem wasted on all students except those who will continue anyway.

Finally, this book steers a path between the terse theorem-proof listing of bare essentials and the bulky down-to-the-student-level books that try to usurp the role of the instructor by a wordy and chatty style. It is hoped that most students can, and will, read the book, but it is also assumed that each college course has an instructor who will augment the text according to the needs of his current class and motivate the work in a more effective and spontaneous manner than an impersonal author could hope for. If the book were used for self-study, then Chapter 1 would seem to start rather abruptly, as would the notion of the limit of a function. Lives there an instructor, however, who does not set the stage for his course by a ten or fifteen minute talk the first class period, or who does not generate limits and discontinuities before the very eyes of his students?

My former co-author and publisher did me the great favor of granting free use of material from *Analytic Geometry and Calculus*, The Macmillan Company, 1946, by John F. Randolph and Mark Kac. Professors Melvin Henriksen and Warren Stenberg went far beyond their assignment of a critical reading of the manuscript and offered many constructive criticisms, but I asked them to forgive my not following all of their suggestions. Professor Hewitt Kenyon helped greatly in version after version of the appendices. I thank (and exonerate) each of the following for criticism of at least one chapter and an independent set of its answers: Theodore A. Bick, Gordon Branche, Mrs. Martha Burton, David Burton, Michael Lodato, David F. Neu, Patrick S. O'Neill, William Pitt, Vemuri Sarma, Earl R. Willard, and Ronald Winkleman.

John F. Randolph

Contents

CHAPTER I

Rectangular Coordinates

The material of this chapter has the twofold purpose of sharpening algebraic manipulation and laying a foundation of analytic geometry, on which calculus calls heavily for illustration and interpretation. Work with inequalities reviews the algebraic laws hidden behind such terms as "transpose" and "cross multiply," but inequalities are indispensable for basic understanding of limits. The beginning of set theory is given even though the amount included is now taught in many high schools. Set theory is a relatively new discipline in elementary mathematics, but its clarifying and unifying features assure it a prominent and lasting place. The notions and notations of set theory reappear throughout the book wherever it is advantageous to use them.

At the risk of seeming to de-emphasize other topics, we recommend that the definition of a function and functional notation (Sec. 8) be given especially careful study. Also, the axiom stated in Sec. 4 might seem to have little importance as it appears here, but this axiom will be used repeatedly.

1. Inequalities and Absolute Values

The product of two numbers is zero if and only if at least one of the numbers is zero. Thus

$$(x+2)(3x-5)=0 \tag{1}$$

if $x=-2$ or else $x=\frac{5}{3}$ and there is no other number satisfying (1).

Only real numbers will be considered in this book, and the properties of real numbers sufficient for arithmetic and algebra involving equations will be assumed. Some work on inequalities will now be given.

To state that x represents a negative number, we write $x<0$ or $0>x$. Hence, $1/x<0$ if and only if $x<0$. Also $y>0$ or $0<y$ means that y represents a positive number.

DEFINITION. *The symbolism $s<t$ will be used if and only if s and t represent numbers such that $s-t<0$. Alternatively $t>s$ if and only if $t-s>0$. The symbol "$<$" is read "less than" and "$>$" is read "greater than".*

THEOREM 1.1. *Let x, y, and a represent numbers.*

1. *Then $x+a<y+a$ if and only if $x<y$.*

2. *In case $a > 0$, then $ax < ay$ if and only if $x < y$.*
3. *In case $a < 0$, then $ax > ay$ if and only if $x < y$.*

PROOF of 3 (As an illustration of a method of proof, we shall prove only 3). First take $a < 0$ and $x < y$. By definition, $x < y$ means that $x - y < 0$. Now a and $x - y$ are both negative so their product is positive; that is, $a(x - y) > 0$. Thus, $ax - ay > 0$ which by the alternative in the definition means that $ax > ay$, as we wished to show.

For the "only if" portion of 3 above, take $a < 0$ and $ax > ay$. Then $1/a < 0$ and $ay < ax$. Hence, by the first part of the proof we have $(1/a)ay > (1/a)ax$. Thus, $y > x$ so that $x < y$, as we wished to prove.

Notice that the above theorem gives rules for inequalities analogous to the rules for equations referred to as "transposition" and "cross multiplication." One must be careful, however, to reverse the sense of inequality if and only if both sides of an inequality are multiplied by a negative number. For example, if

$$-\tfrac{2}{3}x + 3 < 5,$$

we "transpose 3 to the other side" by adding -3 to both sides:

$$-\tfrac{2}{3}x + 3 - 3 < 5 - 3; \quad \text{i.e., } -\tfrac{2}{3}x < 2.$$

Now multiply both sides by $-\frac{3}{2}$ (notice the change of the sense of inequality)

$$(-\tfrac{3}{2})(-\tfrac{2}{3}x) > (-\tfrac{3}{2})2; \quad \text{i.e., } x > -3.$$

The inequality $(-\frac{2}{3})x + 3 < 5$ is said to have solution $x > -3$.

The statement "a, b, and x are numbers such that $a < x$ and $x < b$" is written $a < x < b$.

Example 1. Solve the inequalities

$$2 < -\tfrac{3}{2} - 2x < 5. \tag{2}$$

Solution. Consider the two inequalities $2 < -\frac{3}{2} - 2x$ and $-\frac{3}{2} - 2x < 5$ separately. From the first of these $2x < -\frac{3}{2} - 2 = -\frac{7}{2}$ and therefore $x < -\frac{7}{4}$, but from the second $-\frac{3}{2} - 5 < 2x$ and therefore $-\frac{13}{4} < x$. Thus, both inequalities in (2) will hold if and only if x is a number such that $x < -\frac{7}{4}$ and also $-\frac{13}{4} < x$. The solution of (2) is therefore written as

$$-\tfrac{13}{4} < x < -\tfrac{7}{4}.$$

Example 2. Solve the inequality $3x^2 + x - 10 > 0$.

Solution. Since† $3x^2 + x - 10 \equiv (3x - 5)(x + 2)$ the inequality may be

† The symbol "$\equiv$" is read "identically equal" and means, as used here, that both sides have the same value for each value of x. In contrast "$=$" signifies a conditional equality. For example $3x^2 + x - 10 \not\equiv -(x + 2)$ since $3x^2 + x - 10 = -(x + 2)$ if and only if x is either $\frac{4}{3}$ or -2.

written as $(3x - 5)(x + 2) > 0$. Either these two factors must both be positive or else both must be negative so two cases are to be considered.

CASE 1. *Both factors positive.* Hence, consider separately

$$3x - 5 > 0 \quad \text{and} \quad x + 2 > 0.$$

From the first $x > \frac{5}{3}$ and from the second $x > -2$. Thus, a number x will make both factors positive if and only if x is such that both $x > \frac{5}{3}$ and also $x > -2$. Thus, both factors are positive if and only if $x > \frac{5}{3}$.

CASE 2. *Both factors negative.* Hence, consider separately

$$3x - 5 < 0 \quad \text{and} \quad x + 2 < 0.$$

In this case it must be that both $x < \frac{5}{3}$ and $x < -2$. But x satisfies both of these inequalities if and only if $x < -2$.

Thus, a number x will satisfy the given inequality if either $x > \frac{5}{3}$ or else if $x < -2$. The answer is: *Either $x > \frac{5}{3}$ or else $x < -2$.*

Example 3. Solve the inequality $2x^2 - x - 15 < 0$.

Solution. $2x^2 - x - 15 \equiv (x - 3)(2x + 5) < 0$. Since the product of two numbers is negative if and only if one is negative and the other positive, we consider two cases as follows:

CASE 1	*or* CASE 2
$x - 3 > 0$ and $2x + 5 < 0.$	$x - 3 < 0$ and $2x + 5 > 0.$
$x > 3$ and $x < -\frac{5}{2}.$	$x < 3$ and $x > -\frac{5}{2}.$
Impossible, since no number is both greater than 3 **and** less than $-\frac{5}{2}$.	Both of these inequalities hold if and only if x is such that $-\frac{5}{2} < x < 3$.

Thus, the answer is: "The given inequality holds if and only if $-\frac{5}{2} < x < 3$."

DEFINITION. *The absolute value $|u|$ of a number u is defined by*

$$|u| = \begin{cases} u \text{ if } u \geq 0 \\ -u \text{ if } u < 0. \end{cases} \tag{3}$$

For example, $|3| = 3$ and $|-3| = -(-3) = 3$.

THEOREM 1.2. *For u any number*

$$-|u| \leq u \leq |u|. \tag{4}$$

PROOF. If $u \geq 0$ then (top line of (3)), $-|u| = -u \leq 0 \leq u = |u|$, but if $u < 0$ then (bottom line of (3)), $-|u| = -(-u) = u < 0 < -u = |u|$. Hence in both cases $-|u| \leq u \leq |u|$ so that (4) is established.

Example 4. If $|x - 10| < 0.5$ show that $9.5 < x < 10.5$.

Solution. By using (4) with $u = x - 10$, then

$$-0.5 < -|x - 10| \leq x - 10 \leq |x - 10| < 0.5$$

so that $-0.5 < x - 10 < 0.5$. Now add 10 to each term.

Example 5. Solve the equation $|x + 4| = 1$.

Solution. $x + 4 = \pm 1$ so that either $x = -3$ or $x = -5$.

Note: If $|x - 10| < 0.5$ we *cannot* set $x - 10 < \pm 0.5$. Why?

2. Linear Coordinate System

On a line label some point 0 and label some other point 1. The first of these points is called the **origin**, and the second is called the **unit point**. The distance between these points is now taken as the **unit length**. A point P_1 on the line is said to **precede** a point P_2 on the line if the direction from P_1 to P_2 is the same as the direction from the origin to the unit point.

Now with each point P on the line associate a number x, and with each number x associate a point P on the line, in such a way that:

1. The distance (in terms of the unit length) between P and the origin is $|x|$ units, and
2. P precedes the origin if and only if $x < 0$.

The number associated with a point is called the **coordinate** of the point. With x a number, the point having coordinate x will be referred to as "the point x."

In Fig. 2.1 the point x precedes the origin, so the number x is negative. Consequently, $|x| = -x$; that is, the actual distance between the origin and

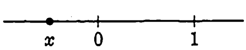


Figure 2.1

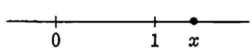


Figure 2.2

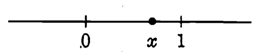


Figure 2.3

this point x is $-x$ units. Also, in this case, the distance between the point x and the unit point is

$$1 + |x| = 1 - x \text{ units.}$$

In Fig. 2.2 both the origin and the unit point precede the point x. In particular, $x > 0$ and the distance between the origin and the point x is $|x| = x$ units. Also, the distance between the unit point and the point x is $x - 1$ units.

In Fig. 2.3 the point x is between the origin and the unit point. Again, the distance between the origin and the point x is x units, but the distance between the point x and the unit point is $1 - x$ units.

It should be seen that a point x_1 precedes a point x_2 if and only if the numbers x_1 and x_2 are such that $x_1 < x_2$. Also, with x_1 and x_2 numbers such that $x_1 < x_2$, the distance between the point x_1 and the point x_2 is $(x_2 - x_1)$ units; that is:

To find the distance between two points on the line subtract the lesser coordinate from the greater coordinate.

For example, the distance between the point 2 and the point -3 is $2 - (-3) = 5$ units.

Since $x_2 - x_1 = -(x_1 - x_2)$ it follows that:

The distance between a point x_1 and a point x_2 is $|x_2 - x_1|$ units.

Example. With a and b numbers such that $a < b$, find the coordinate of the point which is two-fifths of the way from the point a to the point b.

Solution. The length of the interval with end points a and b is $b - a$ units. Thus, from point a proceed toward b the distance $\frac{2}{5}(b - a)$ units and reach the point whose coordinate is

$$a + \tfrac{2}{5}(b - a) = \frac{3a + 2b}{5}.$$

As a different problem notice that the point two-fifths of the way from point b to point a has coordinate

$$b - \tfrac{2}{5}(b - a) = \frac{2a + 3b}{5}.$$

PROBLEMS

1. Solve each of the inequalities:

a. $2x - \frac{3}{4} < 0$.

b. $4 - \dfrac{2x}{3} \geq 0$.

c. $\dfrac{2 - x}{3} + 3 < 0$.

d. $\dfrac{x}{4} + \dfrac{3 - x}{6} > 0$.

e. $\dfrac{x + 1}{4} \geq 0$.

f. $0.3x \leq 0.8$.

2. What conditions must x satisfy if:

a. $x^2 + x - 12 = 0$?

b. $x^2 + x - 12 > 0$?

c. $x^2 + x - 12 < 0$?

d. $2x^2 - 5x - 12 > 0$?

e. $12 + 5x - 2x^2 > 0$?

f. $x^2 - 4 \geq 0$?

3. Solve the inequalities:

a. $\dfrac{2x + 1}{x - 3} > 0$.

b. $\dfrac{2x + 1}{x - 3} \leq 0$.

c. $\dfrac{2x + 1}{x + 4} < 0$.

d. $\dfrac{1}{x} + 1 \geq 0$.

e. $\dfrac{1}{x + 1} > 0$.

f. $\dfrac{1}{x} < 1$.

4. Find conditions x must satisfy whenever:

a. $|x - 10| < 1$. c. $|2x - 3| < 1$.

b. $|x + 10| < 0.5$. d. $|2x + 5| < 0.4$.

5. Prove each of the following

a. If $x_1 < x_2$ then $x_1 < \frac{x_1 + x_2}{2} < x_2$ and $x_1 < \frac{2x_1 + x_2}{3} < x_2$.

b. $\frac{|x| + x}{2} = \begin{cases} x & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0. \end{cases}$

c. $|x_1 - x_2| = \begin{cases} x_1 - x_2 & \text{if} \quad x_1 \geq x_2 \\ x_2 - x_1 & \text{if} \quad x_1 < x_2. \end{cases}$

d. $\frac{a + b + |a - b|}{2}$ = the larger of a and b.

6. Find the coordinate of the point which is:

a. Two-thirds of the way from point -1 to point 5.
b. Two-thirds of the way from point 5 to point -1.
c. Such that the point 5 is two-thirds of the way from the point -1 to this point.
d. Such that the point -1 is two-thirds of the way from the point 5 to this point.
e. Twenty four-twenty fifths of the way from the point 1492 to the point 1965.

7. In each of the following, solve the equation, check the answers, and give an interpretation.

a. $|x + 1| = 2|x - 5|$. c. $4|2 - x| = 5|x + 3|$.

b. $3|x + 1| = 2|x - 5|$. d. $9|x - a| = 10|x - b|$.

3. Intervals, Half-lines, and Linear Motion

Let a and b be numbers such that $a < b$ and consider the point a and the point b.

The set† of all points actually between point a and point b is called the **open interval** with end points a and b. This open interval is represented by $I(a,b)$. Thus, a point x lies on $I(a,b)$ if and only if the number x is such that $a < x < b$.

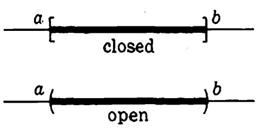


Figure 3.1

The set of all points whose coordinates satisfy $a < x$ is called the **open half-line** with lower end point a and is represented by $I(a,\infty)$. The open half-line with upper end point b is represented by $I(-\infty,b)$ and is the collection of all points whose coordinates satisfy $x < b$.

If $a < b$, notice that the set common to both $I(a,\infty)$ and $I(-\infty,b)$ is

† The word "collection" may be used instead of "set."

I(a,b), since a point x belongs to both of these half-lines if and only if $a < x < b$.

The set of all points x such that $a \leqslant x \leqslant b$ is said to be a **closed interval** and is represented by I$[a, b]$. Thus a closed interval includes both of its end points and all points between them.

An interval I$[a, b)$ which includes its lower end point, but not its upper end point, is said to be closed on the left and open on the right. In a similar way an interval I$(a, b]$, open on the left and closed on the right, is defined.

Figure 3.2

The intervals I(a,b), I$[a,b]$, I$[a,b)$, and I$(a,b]$ all have the same end points and thus the same length; namely, $b - a$ units. *For each of these intervals the mid-point has coordinate*

$$\frac{a+b}{2}. \tag{1}$$

For the distance between the mid-point and either end of any of these intervals is $(b - a)/2$ units, and thus the coordinate of the mid-point may be obtained either as

$$a + \frac{1}{2}(b - a) = \frac{a+b}{2} \quad \text{or as} \quad b - \frac{1}{2}(b - a) = \frac{a+b}{2}.$$

Many problems are concerned with, or may be interpreted as, the motion of a particle on a line. Say, for example, that a particle moves on a line according to the law

$$s = t^2 - 5t + 3, \qquad 0 \leq t \leq 10. \tag{2}$$

This means that some instant (probably the beginning of an experiment) has been selected as zero time, and for the next 10 units of time (seconds, minutes, months) a particle moves on the line in such a way that t time units after zero time, the position of the particle on the line is obtained by squaring t, subtracting 5 times t, and adding 3. Presumably, the experiment stops when $t = 10$ or some other law applies from then on. Thus

$$\begin{aligned}
& s = 0^2 - 5(0) + 3 = 3 && \text{when} \quad t = 0,\\
& s = 1^2 - 5(1) + 3 = -1 && \text{when} \quad t = 1,\\
& s = 2^2 - 5(2) + 3 = -3 && \text{when} \quad t = 2,\\
& s = 3^2 - 5(3) + 3 = -3 && \text{when} \quad t = 3,\\
& s = 4^2 - 5(4) + 3 = -1 && \text{when} \quad t = 4,\\
\text{etc., up to} \quad & s = 10^2 - 5(10) + 3 = 53 && \text{when} \quad t = 10.
\end{aligned}$$

One way of visualizing the motion of the particle is to mark its position on the coordinate system at several different times, as in Fig. 3.3. This motion is illustrated more realistically by Fig. 3.4. The location of the exact place at which the particle changes direction and the determination of the velocity of the particle at any designated times are problems of calculus that are

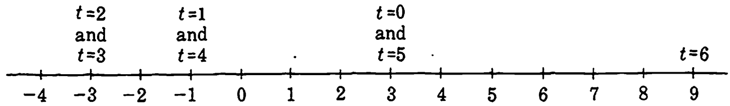


Figure 3.3

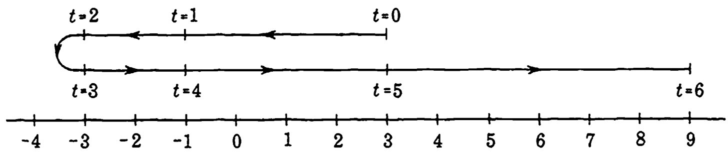


Figure 3.4

considered later. Assuming that a particle is moving on a line according to a law which gives the coordinate s of the particle at each time t in a certain range, we give the following definition.

DEFINITION. *The average velocity of the particle from time t_1 to time t_2 is defined to be*

$$\frac{(s \text{ at } t_2) - (s \text{ at } t_1)}{t_2 - t_1} \quad (\textit{linear units})/(\textit{time unit}).$$

For the particle moving according to the law (2),

$$\text{when } t_1 = 1, \text{ then } s = 1^2 - 5(1) + 3 = -1,$$

$$\text{when } t_2 = 1.1, \text{ then } s = (1.1)^2 - 5(1.1) + 3 = -1.29$$

and then during the time interval from $t_1 = 1$ to $t_2 = 1.1$ the average velocity is

$$\frac{-1.29 - (-1)}{1.1 - 1} = \frac{-0.29}{0.1} = -2.9 \quad \text{(linear units)/(time unit).}$$

The minus sign in -2.9 means that at time $t_2 = 1.1$ the position of the particle precedes its position at time $t_1 = 1$.

PROBLEMS

1. Find the coordinates of the points which divide the given interval into n subintervals of equal length.

   a. $I(-1,5)$, $n = 6$.
   b. $I(-2,10)$, $n = 6$.
   c. $I(-3,12)$, $n = 10$.
   d. $I(3,12)$, $n = 10$.
   e. $I(-12,-3)$, $n = 100$.
   f. $I(a,b)$ with $a < b$, $n = 100$.

2. Given that a particle moves on a line according to the law

$$s = 3 + 6t - t^2, \qquad 0 \le t \le 10.$$

a. Locate the particle when $t = 0$, $t = 2$, $t = 4$, and $t = 7$.
b. At what times will the coordinate of the particle be 3?
c. At what times will the coordinate of the particle be -3?
d. Does the particle ever have coordinate 13?
e. During what time intervals is the particle on I[3,11]?
f. Represent the motion of the particle by a diagram.
g. Find the average velocity of the particle in the time intervals between $t_1 = 5$ and $t_2 = 5.1$; between $t_1 = 5$ and $t_2 = 4.9$; between $t_1 = 5$ and $t_2 = 5 + h$ where $h \neq 0$.

3. A ball is thrown into the air and t seconds later it is

$$s = 6 + 65t - 16t^2 \text{ ft}$$

in the air; the law holding until the ball returns to earth.
a. Find the height of the ball when $t = 0, 1, 2, 3,$ and 4 sec.
b. Find the average velocity of the ball during the time interval between $t_1 = 2$ sec and $t_2 = 2.1$ sec; between $t_1 = 2$ sec and $t_2 = 1.9$ sec; between $t_1 = 3$ sec and $t_2 = 3.01$ sec; between t_1 sec and $(t_1 + h)$ sec.

4. Find the average velocity during the time interval from t_1 to $t_1 + h$ for each of the following laws of motion:

a. $s = 2t^3 - 4t + 5$.
b. $s = t + \dfrac{1}{t}$, $t > 0$.
c. $s = \sqrt{3t + 4}$.
d. $s = 3t - \sqrt{t^2 + 1}$.
e. $s = 5/t^2$, $t > 0$.
f. $s = 10/\sqrt{t^2 + 1}$.

4. Sets of Numbers and Sets of Points

A set A and a set B are said to be equal, and $A = B$ is written, if each element of A is also an element of B and each element of B is also an element of A; that is, if and only if A and B are merely different names for the same set. $A \neq B$ means there is at least one element in one of the sets which is not in the other.

It has become quite standard to use

$$\{x \mid \text{statement about } x\} \tag{1}$$

to mean, "The set of all entities which when substituted for x makes the statement true." Also (1) is read, "The set of all x for which the statement about x is true." Thus, the set R defined by

$$R = \{x \mid x \text{ is a rational number}\} \tag{2}$$

is "The set of all rational numbers" and hence R contains all those numbers and only those numbers which can be written as the ratio of two integers.

The set

$$(3) \qquad S = \{x \mid x \leq 0, \text{ or else both } x > 0 \text{ and } x^2 < 2\}$$

consists of all negative numbers and zero together with all those positive numbers whose squares are less than 2. For example, -1, -2.5, -200 are in S and also 1, 1.4, 1.41, and 1.414 are in S.

Notice that the sets

$$(4) \qquad T = \{x \mid -1 \leq x < 3\} \quad \text{and} \quad T' = \{x \mid -1 \leq x \leq 3\}$$

both have a smallest number, T has no largest number, but 3 is the largest number in T'.

A set A of numbers is said to be **bounded above** if there is a number b such that whenever x is in A, then $x \leq b$. Such a number b is said to be an **upper bound** of A. Hence, if b is an upper bound of A, then any number greater than b is also an upper bound of A. Similar definitions may be made for **bounded below** and **lower bound.**

Considering the sets R, S, T, and T' defined above; R is neither bounded above nor bounded below; S is bounded above, for example, by 2 and also by 1.415; T and T' are both bounded above and 3 is the smallest upper bound of both T and T'.

A statement about x need not be true of any number x. For example, "$x < 1$ and $x > 2$" does not hold for any number x. Thus,

$$E = \{x \mid x < 1 \text{ and } x > 2\}$$

is the **empty** set. The phrase "C is a non-empty set of numbers" means that the statement which defines C is true for at least one number.

In addition to the usual properties of numbers (which are either stated as axioms or taken as intuitive), we state the following axiom.

AXIOM. *If A is a set of numbers which is non-empty and bounded above and if*

$$B = \{x \mid x \textit{ is an upper bound of } A\},$$

then in B there is a smallest number; that is, every non-empty set of numbers which is bounded above has a smallest upper bound. Also, every non-empty set of numbers which is bounded below has a greatest lower bound.

Knowing that $\sqrt{2}$ means the positive number whose square is 2, we see that $\sqrt{2}$ is not in the set S defined in (3), but that $\sqrt{2}$ is the least upper bound of this set S.

Any set A of numbers may be visualized as a set of points on a line by putting a point in the point set if and only if the coordinate of the point is a number in A.

Given two sets A and B, then the **union** of these sets is defined to be the set consisting of all those elements which belong either to A or to B or to both. The union of sets A and B is denoted by

$$A \cup B. \tag{5}$$

For example, if A is the set of points on the closed interval joining -1 to 1, and if B is the set of points of the closed interval from 0 to 2; that is, if

$$A = \mathrm{I}[-1,1] \quad \text{and} \quad B = \mathrm{I}[0,2], \tag{6}$$

then the union $A \cup B$ is the set of points of the closed interval from -1 to 2:

$$A \cup B = \mathrm{I}[-1,2].$$

The **intersection** of two sets A and B is defined to be the set consisting of all elements which belong both to A and to B. The intersection of sets A and B is denoted by

$$A \cap B. \tag{7}$$

Thus, for the particular sets A and B given in (6),

$$A \cap B = \mathrm{I}[0,1],$$

since a number x belongs to both of these sets if and only if x satisfies $0 \le x \le 1$.

Notice that an element belongs to $A \cup B$ if and only if this element belongs to A **or** to B, or to both, but belongs to $A \cap B$ if and only if it belongs to A **and** to B. Thus, in the definition of union the key word is "or," but for the intersection the key word is "and."

If two sets A and B have no common elements at all, then a convenient descriptive expression is, "The intersection of these sets is empty."

Thus, given $A = \{x \mid -1 \le x < 1\}$ and $B = \{x \mid 2 < x < 3\}$, then

$$A \cap B \text{ is empty.}$$

As another example, $\mathrm{I}[-1,1) \cap \mathrm{I}[1,3]$ is also empty.

PROBLEMS

1. For each of the following sets, tell whether the set is bounded above, and if it is bounded above, find its least upper bound; also, tell whether it is bounded below, and if it is, find its greatest lower bound.

   a. $\{x \mid x^2 < 9\}$.
   b. $\{x \mid x^2 < 4\}$.
   c. $\{x \mid x^3 < 27\}$.
   d. $\{x \mid \sqrt[3]{x} < 2\}$.
   e. $\{x \mid \sqrt{x} < 2\}$.
   f. $\{x \mid \sqrt{x} > 3\}$.
   g. $\{x \mid \sqrt[3]{x} > 3\}$.
   h. $\{x \mid x = \sin \alpha \text{ for some angle } \alpha\}$.
   i. $\{x \mid x = \sec^2 \alpha \text{ for some angle } \alpha\}$.
   j. $\{x \mid x = \log t \text{ for some number } t > 0\}$.

2. For each of the following pairs of sets A and B, find the union $A \cup B$ and the intersection $A \cap B$.
   a. $A = \{x \mid -3 < x \leq 4\}$, $B = \{x \mid 2 \leq x < 10\}$.
   b. $A = \{x \mid -3 < x \leq 4\}$, $B = \{x \mid x > 2\}$.
   c. $A = \{x \mid -2 < x \leq 3\}$, $B = \{x \mid 3 < x \leq 4\}$.
   d. $A = \{x \mid 4 \leq x^2\}$, $B = \{x \mid x \leq 1\}$.
   e. $A = \{x \mid x^3 < 27\}$, $B = \{x \mid x^2 < 25\}$.

3. Find:
   a. $\{x \mid x^2 \leq 9\} \cap \{x \mid -1 < x < 4\}$.
   b. $\{x \mid x^3 \geq 27\} \cap \{x \mid -1 < x < 4\}$.
   c. $\{x \mid \sqrt{x} < 1.4\} \cup I[-2,1]$.
   d. $\{x \mid \sqrt{x} \geq 1.4\} \cup I[-2,2]$.
   e. $\{x \mid x^3 < 8\} \cap \{x \mid x^2 \geq 4\}$.
   f. $\{x \mid x^3 > 8\} \cap \{x \mid x^2 \leq 4\}$.
   g. $\{x \mid \sqrt{x} > \frac{1}{2}\} \cap \{x \mid \sqrt[3]{x} < \frac{1}{2}\}$.
   h. $\{x \mid \sqrt{x} > 2\} \cap \{x \mid \sqrt[3]{x} \leq 2\}$.
   i. $\{x \mid x = \sin t \text{ for some number } t\} \cup \{x \mid -2 < x < 1\}$.
   j. $\{x \mid x = 2 \sin \alpha \text{ for some angle } \alpha\} \cup \{x \mid -3 < x < 1\}$.
   k. $\{x \mid x = \sin (2\alpha) \text{ for some angle } \alpha\} \cup \{x \mid -2 < x < 1\}$.

5. Plane Rectangular Coordinates

In a plane draw two perpendicular lines. On each of these lines take a linear coordinate system with the point of intersection as the origin of each system. With x a number, the point on one of these lines that would be called "the point x" if only one line were involved will now be called "the point $(x,0)$." This line is called the **axis of abscissas** and is usually represented as horizontal with the point $(1,0)$ to the right of the origin $(0,0)$. The other line is called the **axis of ordinates**, and there is an analogous correspondence between points of this line and pairs of numbers $(0,y)$, the first of which is 0.

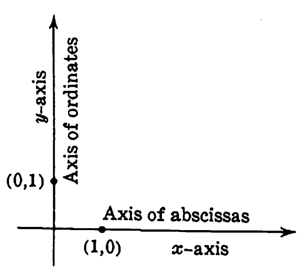


Figure 5.1

A pair of numbers with one of them selected as the first is called an **ordered pair** of numbers. The ordered pair of numbers "x first and y second" is denoted by (x,y). For example, the pair of numbers 2 and 3 may be used in either the ordered pair (2,3) or the different ordered pair (3,2).

A one-to-one correspondence between points of the plane and ordered pairs of numbers is established by using (x,y) to label a point if and only if the projections of this point on the axis of abscissas and the axis of ordinates are the points $(x,0)$ and $(0,y)$, respectively. With x and y numbers, the point labeled by the ordered pair (x,y) is said to have **abscissa** x, **ordinate** y; and together, x and y in this order are called the **coordinates** of the point.

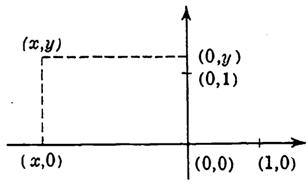


Figure 5.2

The point (x,y) illustrated in Fig. 5.2 has abscissa negative, since the point $(x,0)$ precedes the origin on the axis of abscissas. The ordinate of this point is positive.

Example. Find the coordinates of the point which is two-thirds of the way from the point $(5,1)$ to the point $(-2,9)$.

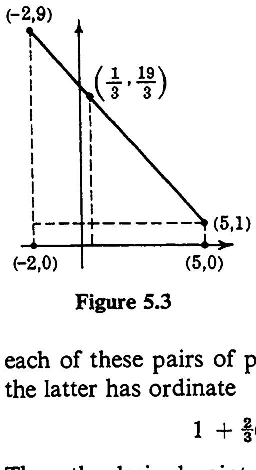


Figure 5.3

Solution. The line segment joining the points $(5,1)$ and $(-2,9)$ projected into the axis of abscissas is the interval joining the points $(5,0)$ and $(-2,0)$. By similar triangles (see Fig. 5.3) the desired point has the same abscissa as the point two-thirds of the way from the point $(5,0)$ to the point $(-2,0)$, and this abscissa is

$$5 - \frac{2}{3}\left(5 - (-2)\right) = \frac{15 - 14}{3} = \frac{1}{3}.$$

In a similar way, the points $(5,1)$ and $(-2,9)$ project onto the points $(0,1)$ and $(0,9)$ of the axis of ordinates and, for each of these pairs of points, the point two-thirds of the way from the former to the latter has ordinate

$$1 + \tfrac{2}{3}(9 - 1) = \tfrac{19}{3}.$$

Thus, the desired point is $(\frac{1}{3},\frac{19}{3})$.

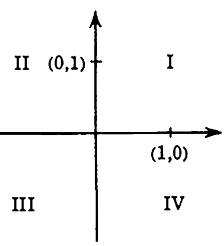


Figure 5.4

In Fig. 5.4 the regions marked I, II, III, and IV are called the first, second, third, and fourth **quadrants**, respectively. To be more specific, the first quadrant is $\{(x,y) \mid x > 0 \text{ and } y > 0\}$ with similar definitions, using actual inequalities, for the other three quadrants.

The axis of abscissas is also referred to as "the x-axis" and the axis of ordinates as "the y-axis."

THEOREM 5. *The line segment having end points* (x_1,y_1) *and* (x_2,y_2) *has mid-point*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

PROOF. The points $(x_1,0)$ and $(x_2,0)$ are the projections of the points (x_1,y_1) and (x_2,y_2) into the axis of abscissas. The point midway between these projections has (see (1) of Sec. 3) abscissa

$$\frac{x_1 + x_2}{2}$$

which is therefore the abscissa of the point midway between points (x_1,y_1) and (x_2,y_2). By projecting into the axis of ordinates the desired mid-point is seen to have ordinate

$$\frac{y_1 + y_2}{2}.$$

6. Slope and Equations of a Line

A line segment which is not parallel to the y-axis and has end points (x_1,y_1) and (x_2,y_2) is said to have **slope** m where

$$(1) \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

or equivalently

$$m = \frac{y_1 - y_2}{x_1 - x_2}.$$

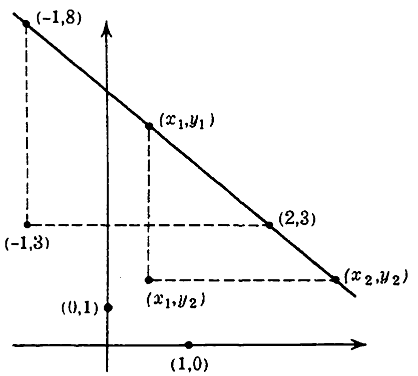


Figure 6

Notice that the difference of the ordinates is divided by the difference *in the same order* of the abscissas. Since the line segment is not parallel to the y-axis, the abscissas x_1 and x_2 are not equal, so the denominator in (1) is not zero.

Two line segments, neither parallel to the y-axis, which have at least one point in common, are on the same line if and only if both segments have the same slope. As an illustration, in Fig. 6

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{8 - 3}{-1 - 2} = \frac{3 - 8}{2 - (-1)} = \frac{-5}{3}.$$

Thus, if any one segment on a line has slope m, then all segments of this line have the same slope m.

A line not parallel to the y-axis is said to have **slope m** *where m is the slope of any line segment of the line.* For lines parallel to the y-axis, the notion of slope is not defined.

THEOREM 6.1. *Let points* (x_1,y_1) *and* (x_2,y_2) *be given with* $x_1 \neq x_2$. *Then a point* (x,y) *is on the line through these points if and only if*

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \tag{2}$$

PROOF. Since $x_1 \neq x_2$ the line is not parallel to the y-axis and has slope $(y_2 - y_1)/(x_2 - x_1)$. In case $x = x_1$ the point (x,y) is on the line if and only if $y = y_1$ and (2) also holds; both sides being zero. In case $x \neq x_1$, the point (x,y) is on the line if and only if the slope of the line segment with end points (x,y) and (x_1,y_1) is the same as the slope of the line segment with end points (x_1,y_1) and (x_2,y_2); that is, if and only if

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

and this equation is equivalent to (2) if $x \neq x_1$.

COROLLARY 1. *With* x_1, y_1, *and* m *given numbers, a point* (x,y) *lies on the line through the point* (x_1,y_1) *with slope* m *if and only if*

$$y - y_1 = m(x - x_1). \tag{3}$$

For with (x_2,y_2) a point on the line other than the point (x_1,y_1), then $x_2 \neq x_1$, $m = (y_2 - y_1)/(x_2 - x_1)$, and (3) follows from (2).

COROLLARY 2. *With* b *and* m *given numbers, a point* (x,y) *lies on the line through the point* $(0,b)$ *with slope* m *if and only if*

$$y = mx + b. \tag{4}$$

For in (3) set $x_1 = 0$ and $y_1 = b$ to obtain $y - b = mx$ which is equivalent to (4).

Let a line and an equation involving x and y be given. The equation is said to be **an equation** of the line if each point (x,y) whose coordinates satisfy the equation lies on the line and the coordinates of each point on the line satisfy the equation.

It is usual to refer to (2) as a **two-point** equation of a line, to (3) as a **point-slope** equation of a line, and to (4) as the **slope-y-intercept** equation of a line.

THEOREM 6.2. *If* A, B, *and* C *are numbers with* A *and* B *not both zero, then*

$$Ax + By + C = 0 \tag{5}$$

is an equation of a line. Also, any line has an equation in the form (5).

PROOF. First let A, B, and C be given numbers with A and B not both zero. In case $B = 0$, then $A \neq 0$ and numbers x and y satisfy (5) if and only if

$x = -C/A$; that is, if and only if the point (x,y) lies on the line perpendicular to the x-axis at the point $(-C/A,0)$. In case $B \neq 0$, then numbers x and y satisfy (5) if and only if

$$y = -\frac{A}{B}x - \frac{C}{B};$$

that is (compare this equation with (4)), if and only if the point (x,y) is on the line through the point $(0,-C/B)$ with slope $m = -A/B$.

Next, consider any line in the plane. In case the line is parallel to the y-axis, it cuts the x-axis at a point $(a,0)$ and a point (x,y) lies on the line if and only if $x = a$, which may be written

$$1 \cdot x + 0 \cdot y - a = 0,$$

and this is in the form (5) with $A = 1$, $B = 0$, and $C = -a$. In case the line is not parallel to the y-axis, it has a slope m and cuts the y-axis at a point $(0,b)$. In this case a point (x,y) lies on the line if and only if $y = mx + b$, and this equation may be written as

$$mx - 1 \cdot y + b = 0,$$

which is in the form (5) with $A = m$, $B = -1$, and $C = b$.

Equation (5) is called a **general equation** of a line.

It should be seen that:

Two lines neither of which is parallel to the y-axis are parallel to each other if and only if they have the same slope.

Example. Find a general equation of the line which passes through the point $(-1,5)$ and is parallel to the line having equation $6x + 8y - 7 = 0$.

Solution. The given equation is equivalent to

$$y = -\tfrac{3}{4}x + \tfrac{7}{8},$$

showing (from (4)) that the second line has slope $m = -\frac{3}{4}$. Hence, the parallel line through the point $(-1,5)$ also has slope $m = -\frac{3}{4}$ and (from (3)) has

$$y - 5 = -\tfrac{3}{4}(x + 1)$$

as an equation. This equation is equivalent to

$$3x + 4y - 17 = 0;$$

an equation of the line in general form with $A = 3$, $B = 4$, and $C = -17$.

PROBLEMS

1. On a plane rectangular coordinate system locate:
   a. The points $(-1,0)$, $(0,-1)$, $(3,-2)$, $(3,2)$, and $(\frac{1}{2},1)$.
   b. The point with abscissa -4 and ordinate $\frac{1}{2}$.
   c. The point with abscissa -2 and ordinate the square of the abscissa.
   d. The point with ordinate 3 and abscissa minus the ordinate.
   e. The points each with abscissa -5.
   f. The points each with ordinate half of the abscissa.

2. In the fourth quadrant select a point and label it (x,y), thus determining a number x and number y.
   a. Is x positive or negative? Is y positive or negative?
   b. With x and y determined above, locate the points
   $$(-x,y), \quad (x,-y), \quad (y,x), \quad \text{and} \quad (-y,x).$$

3. Find the ordered pair representing the point midway between the points:
   a. $(-3,2)$, $(5,2)$.
   b. $(-3,2)$, $(5,1)$.
   c. $(10,5)$, $(-10,8)$.
   d. $(1492,0)$, $(1962,0)$.

4. Find the coordinates of the point which is:
   a. Two-thirds of the way from the point $(-3,2)$ to the point $(9,8)$.
   b. Two-thirds of the way from the point $(9,8)$ to the point $(-3,2)$.
   c. Seven-tenths of the way from the point $(4,5)$ to the point $(-6,10)$.
   d. Reached by going from the point $(2,-4)$ to the point $(-3,5)$ and then proceeding an equal distance along the line joining these points.

5. Find an equation of the line which passes through the points:
   a. $(-5,1)$ and $(2,3)$.
   b. $(-5,1)$ and $(-5,27)$.
   c. $(-5,1)$ and $(2,1)$.
   d. $(-1,2)$ and $(1,-2)$.
   e. $(3,5)$ and $(3.01,4.98)$.
   f. $(-4,2)$ and $(-3.996,2.004)$.

6. Find a general equation of the line which:
   a. Passes through the point $(-1,4)$ with slope -2.
   b. Has no slope and passes through the point $(-5,4)$.
   c. Has slope zero and passes through the point $(-5,4)$.
   d. Passes through the point $(2,-3)$ and is parallel to the line having $2x + y = 6$ as an equation.

7. Let a and b be numbers neither of which is zero. Show that a line passes through the points $(a,0)$ and $(0,b)$ if and only if an equation of the line is
$$\frac{x}{a} + \frac{y}{b} = 1.$$

8. a. Show that $x^3 - x^2y + 4x^2 + 2x - 2y + 8 = 0$ is an equation of a line.
   b. Do the same for $-y^3 + 2(x + 2)y^2 + 6x - 3y + 12 = 0$.

9. Find the slope and y-intercept of the line whose equation is:
   a. $2x - 4y + 10 = 0$.
   b. $2(x - 1) = 3(y + 2)$.
   c. $x = 3y + 5$.
   d. $x \sin 30° + y \cos 30° = 5$.

7. Sets and Ordered Pairs

A set is said to be **well defined** whenever there is a criterion which determines whether a given entity does or does not belong to the set. Thus, the set of all ordered pairs of numbers such that each pair has its second number the square of its first number is a well-defined set. For example (2,4) is an element of this set, as is $(-2,4)$; but (3,8) is not an element of this set. We shall use

$$\{(x,y) \mid y = x^2\} \tag{1}$$

to describe this particular set of ordered pairs.

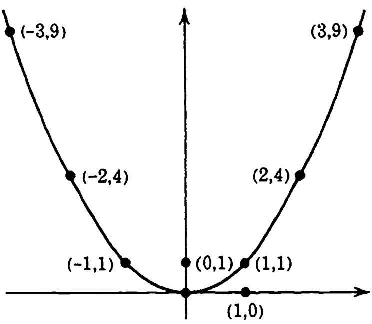


Figure 7.1

This is an illustration of the standard notation

$$\{(x,y) \mid \text{statement about } x \text{ and } y\}$$

to mean, "The set of all (x,y) for which the statement about x and y is true." By the **graph** of a set A of ordered pairs of numbers is meant the set of all those points of the plane each of whose points is represented by an ordered pair in A. Thus, the set described by (1) has the curve of Fig. 7.1 as its graph. A set of ordered pairs of numbers and its graph may be thought of interchangeably. Thus, we may speak of "The line $\{(x,y) \mid 2y = x\}$" meaning, of course, "The line consisting of the set of all points each with abscissa twice its ordinate." Also,

$$\{(x,y) \mid 2y \geq x\}$$

is the half-plane consisting of the set of all points (see Fig. 7.2) on and above

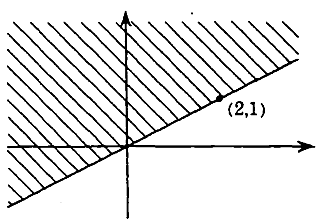


Figure 7.2

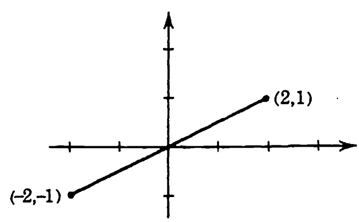


Figure 7.3

the line $\{(x,y) \mid 2y = x\}$. The line segment $\{(x,y) \mid 2y = x \text{ and } |x| \leq 2\}$ joins the points $(-2,-1)$ and (2,1) as shown in Fig. 7.3.

The letters x and y have been used in describing sets of ordered pairs, but any two symbols whatever may be used. For example

$$\{(s,t) \mid 2t = s\} = \{(x,y) \mid 2y = x\}$$

since each gives the same criterion for testing whether any given ordered pair of numbers belongs to the set.

Example. Show that $\{(x,y) \mid 0 \leq x \leq 2 \text{ and } 0 \leq y \leq \frac{1}{2}x\}$ may also be expressed in the form $\{(x,y) \mid 0 \leq y \leq 1 \text{ and } 2y \leq x \leq 2\}$.

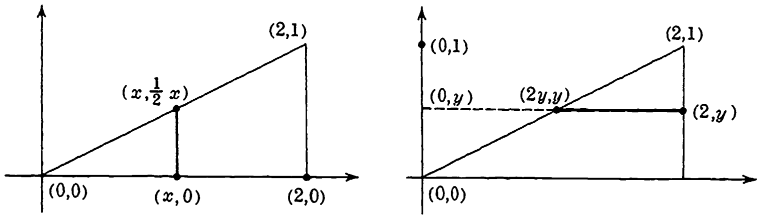


Figure 7.4

Solution. Considering the first expression, let x be a specific number such that $0 \leq x \leq 2$, and at the point $(x,0)$ erect a vertical line segment of length $x/2$. The totality of all points on such segments is the interior and edges of the triangle with vertices (0,0), (2,0), and (2,1).

Considering the second set, let y be a specific number such that $0 \leq y \leq 1$, note the points $(2y,y)$ and $(2,y)$, and join these points by a line segment. The totality of points on such segments is the same triangle and its interior as before. Thus, the two expressions describe the same set of ordered pairs.

8. Functions

Given a point on the curve of Fig. 8.1, there may be a different point with the same abscissa; in particular, there are three points with abscissa zero. On the other hand, Fig. 8.2 indicates a set of points such that whatever point

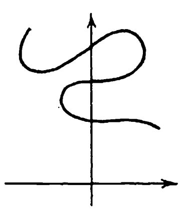


Figure 8.1

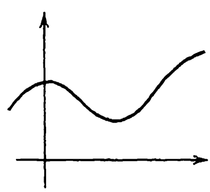


Figure 8.2

is taken in the set, then no other point in the set has the same abscissa; that is, in the corresponding set of ordered pairs of numbers no two ordered pairs have the same first number.

DEFINITION. *A set of ordered pairs is said to be a* **function** *if no two distinct ordered pairs in the set have the same first element. The set of first elements of a function is called the* **domain** *(of definition) of the function, and the set of second elements is called the* **range** *of the function.*

A letter or symbol may be used in referring to a function. Thus, upon using f for a function, then f is a set of ordered pairs such that: "*If* (a,b) *and* (a,c) *are in the set, then* $b = c$." Also, for f a function and x in the domain of f, then the ordered pair in f with first element x is denoted by $(x,f(x))$ and $f(x)$ is read "f at x."

A specific example of a function is the set of ordered pairs

$$(-1,2),\quad (0,3),\quad (1,3),\quad \text{and}\quad (2,4).$$

The domain of this function consists of the set of numbers $-1, 0, 1, 2$, whereas the set of numbers 2, 3, 4 constitutes the range. Upon denoting this set of ordered pairs by g, then

$$g(-1) = 2,\quad g(0) = 3,\quad g(1) = 3,\quad \text{and}\quad g(2) = 4,$$

but $g(3)$, for example, is not defined, nor is $g(1.5)$.

The set $(-1,2)$, $(-1,3)$, $(0,3)$, $(1,4)$ of ordered pairs is not a function, since two of these have the same first element, namely -1.

Another example of a function is

$$f = \{(s,t) \mid t = 3s^2 - 4\}.$$

This function has domain the set of all numbers and given any number x, then $(x,3x^2 - 4)$ is an ordered pair in f and

$$f(x) = 3x^2 - 4.$$

Thus, for this function $f(2) = 8$, $f(2.1) = 9.23$, and

$$\frac{f(2.1) - f(2)}{2.1 - 2} = \frac{9.23 - 8}{0.1} = 12.3.$$

Hence, the points $(2,8)$ and $(2.1,9.23)$ are in the graph of f, and the line joining these points has slope 12.3. Also, for x a number and $h \neq 0$, the points $(x,3x^2 - 4)$ and $(x + h,3(x + h)^2 - 4)$ are on the graph of f and the line joining these points has slope m where

$$m = \frac{[3(x + h)^2 - 4] - [3x^2 - 4]}{(x + h) - x} = \frac{6xh + 3h^2}{h} = 6x + 3h.$$

Given a function (and its domain), a symbol used to represent an arbitary element of the domain is called the **independent variable**, and a symbol used to represent an arbitrary element in the range is called the **dependent variable.** Thus, for a function f established by telling what its domain is and a rule for finding $f(x)$ whenever x is in that domain, it is customary to set

$$y = f(x)$$

and to call x the independent variable and y the dependent variable.

Moreover, a symbol used to represent an arbitrary element of a set (even though no function is mentioned) is called a **variable.** A symbol which is to be assigned one and only one value throughout a discussion is called a **constant.** Upon seeing

$$ax + by = c,$$

x and y are considered to be variables and a, b, and c constants.

9. Some Special Functions

For later use we review some common functions.

A. ABSOLUTE-VALUE FUNCTION. The set of ordered pairs

$$\{(x,y) \mid y = |x|\}$$

is a function whose domain consists of all numbers, but whose range is the set of non-negative numbers. The graph of this function is the two half-lines shown in Fig. 9.1.

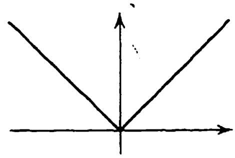


Figure 9.1

The absolute-value function may be used in the definition of other functions. For example, let

$$f = \left\{(x,y) \mid y = \frac{x + |x|}{2}\right\}.$$

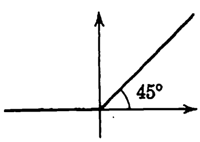


Figure 9.2

The graph of this function f is the negative x-axis together with the "45° half-line" shown in Fig. 9.2.

B. SQUARE-ROOT FUNCTION. *For x a non-negative number, $\sqrt{x}$ means the non-negative number whose square is x.* Thus, $\sqrt{4} = 2$ and **not** -2. Hence,

$$\{(x,y) \mid x \geq 0 \quad \text{and } y = \sqrt{x}\}$$

is a function whose domain and range both consist of the non-negative numbers. Fig. 9.3 indicates the graph of this function.

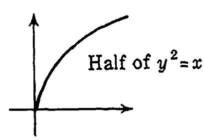


Figure 9.3

It may seem unnatural, but we emphasize that

$$\sqrt{x^2} \neq x \quad \text{if} \quad x < 0,$$

since for $x < 0$, then $\sqrt{x^2}$ is positive but x is negative; so $\sqrt{x^2}$ and x cannot be equal if $x < 0$. It does, however, follow that

$$\sqrt{x^2} = |x|$$

whether x is positive, negative, or zero.

Notice that if $|x| \leq |y|$, then $|x|\,|x| \leq |x|\,|y| \leq |y|\,|y|$ and $x^2 \leq y^2$. If, however, $|x| > |y|$, then $|x| > 0$, $|y| \geq 0$ and $|x|\,|x| > |x|\,|y| \geq |y|\,|y|$ so that $x^2 > y^2$. Thus:

$$\textit{if } x^2 \leq y^2, \quad \textit{then} \quad |x| \leq |y|.$$

This fact will be used in establishing the inequality

$$(1) \qquad |a + b| \leq |a| + |b|$$

for a and b any numbers whatever. To see this, note that $ab \leq |a|\,|b|$. $2ab \leq 2|a|\,|b|$ and, by adding $a^2 + b^2$ to both sides

$$a^2 + 2ab + b^2 \leq |a|^2 + 2|a|\,|b| + |b|^2,$$

$$(a + b)^2 \leq (|a| + |b|)^2 \quad \text{and hence}$$

$$|a + b| \leq \big|\,|a| + |b|\,\big| \quad \text{as noted above.}$$

Since $\big|\,|a| + |b|\,\big| = |a| + |b|$ the inequality (1) is seen to hold.

C. GREATEST-INTEGER FUNCTION. For x a number, the notation $[x]$ is sometimes used to mean the greatest integer less than or equal to x, or briefly *the greatest integer in* x. For example,

$$[2.5] = 2,\ [-2.5] = -3,\ [\pi] = 3,\ [-\pi] = -4,\ -[\pi] = -3.$$

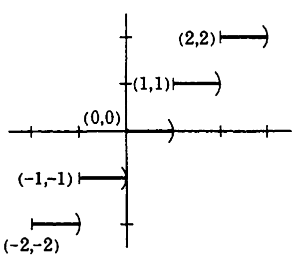


Figure 9.4

Also,

$$\begin{aligned} &\text{if} \quad 2 \leq x < 3, \quad \text{then} \quad [x] = 2;\\ &\text{if} \quad 1 \leq x < 2, \quad \text{then} \quad [x] = 1;\\ &\text{if} \quad 0 \leq x < 1, \quad \text{then} \quad [x] = 0;\\ &\text{if} \quad -1 \leq x < 0, \quad \text{then} \quad [x] = -1. \end{aligned}$$

Thus, the set

$$\{(x,y) \mid y = [x]\}$$

is a function whose domain consists of all numbers but whose range consists of all integers. The graph of this function (see Fig. 9.4) is a "stair-step" of unit intervals which are closed on the left and open on the right. Notice, for example, that $[2] = 2$ whereas $[1.9999] = 1$, and if e is any positive number (no matter how small), then $[2]$ and $[2 - e]$ differ by at least 1.

The set $\{(x,y) \mid y = x - [x]\}$ is a function whose range is $\{y \mid 0 \leq y < 1\}$. In particular among all the values of this function there is no largest value. See Fig. 9.5.

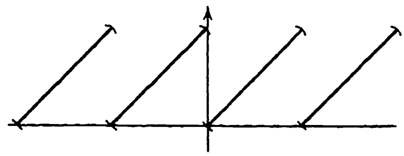


Figure 9.5

D. SINE AND COSINE FUNCTIONS. For x a number, then $\sin x$ means "The sine of an angle of x radians." (See page 578.) Thus,

$$S = \{(x,y) \mid y = \sin x\}$$

is a function whose domain consists of all numbers and whose range is given as $\{y \mid -1 \leq y \leq 1\}$. This function is "the sine function." The cosine function:

$$C = \{(x,y) \mid y = \cos x\}$$

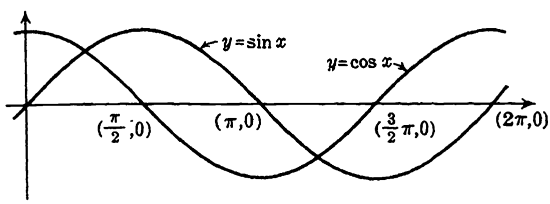


Figure 9.6

has the same domain and range as the sine function. The graphs of the sine and cosine functions are indicated in Fig. 9.6.

As in trigonometry, for A and B numbers, then (see page 575)

$$\sin(A + B) \equiv \sin A \cos B + \cos A \sin B,$$

$$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B,$$

and

$$\sin(A + B) - \sin(A - B) \equiv 2 \cos A \sin B.$$

Hence, with b, x, and h any numbers whatever, then upon setting

$$b(x + h) = A + B$$

and

$$bx = A - B$$

it follows that $A = bx + \frac{1}{2}bh$ and $B = \frac{1}{2}bh$. Upon substituting these relations for $A + B$, $A - B$, A, and B above, the result is

$$\text{(2)} \qquad \sin b(x + h) - \sin bx \equiv 2 \cos (bx + \tfrac{1}{2}bh) \sin \tfrac{1}{2}bh$$

which is an identity that will be used later.

E. ANOTHER SINE RELATION. Let f be the function defined by

$$f = \left\{(x,y) \mid x \neq 0 \quad \text{and} \quad y = \sin^2 \frac{1}{x}\right\}$$

This function has $\{x \mid x \neq 0\}$ for domain and $\{y \mid 0 \leq y \leq 1\}$ for range. Since $\sin A = 0$ if and only if $A = n\pi$ for some integer n, it follows for $x \neq 0$ that

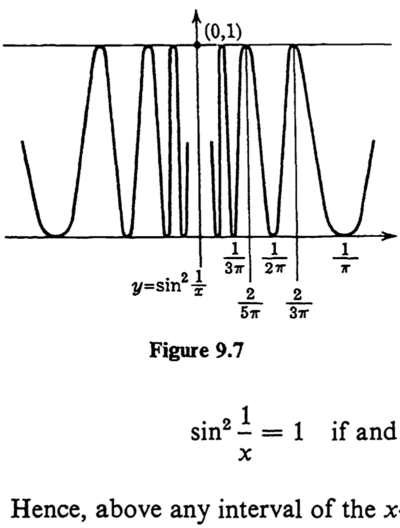


Figure 9.7

$$\sin^2 \frac{1}{x} = 0$$

$$\text{if and only if} \quad x = \frac{1}{n\pi}, n \neq 0$$

Since $\sin A = 1$ or $\sin A = -1$ if and only if $A = (2n + 1)\frac{\pi}{2}$ for some integer n, it follows for $x \neq 0$ that

$$\sin^2 \frac{1}{x} = 1 \quad \text{if and only if} \quad x = \frac{2}{(2n + 1)\pi}.$$

Hence, above any interval of the x-axis, which includes the origin, the graph of f (indicated in Fig. 9.7) oscillates infinitely many times.

PROBLEMS

1. Sketch each of the following sets:
   a. $\{(x,y) \mid y = |x + 2|\}$.
   b. $\{(x,y) \mid x = |y|\}$.
   c. $\{(x,y) \mid y + |x| = 1\}$.
   d. $\{(x,y) \mid |y| - x = 1\}$.
   e. $\{(x,y) \mid |x| + |y| = 1\}$.
   f. $\{(x,y) \mid |x| - |y| = 1\}$.
   g. $\{(x,y) \mid |y - x| = 1\}$.
   h. $\{(x,y) \mid |y - x| = 0\}$.

2. Which of the sets in Prob. 1 are functions? In each case where the set is a function, give the domain and range.

3. In Prob. 1 replace absolute value notation | | by the greatest integer notation [ ] and sketch the resulting sets.

4. Let $f = \left\{(x,y) \mid y = x - \frac{1}{x}\right\}$. Find:

a. $f(\frac{3}{4}), f(\frac{4}{3}), f(-\frac{1}{2}), f(-2)$.

b. $f(2 + \frac{1}{2}) - f(2)$.

c. $\dfrac{f(3.1) - f(3)}{3.1 - 3}$

d. $\{f(1) + f(\frac{3}{2}) + f(2) + f(\frac{5}{2})\}\frac{1}{2}$.

e. $\{f(\frac{3}{2}) + f(2) + f(\frac{5}{2}) + f(3)\}\frac{1}{2}$.

f. $\dfrac{f(3 + h) - f(3)}{(3 + h) - 3}$ for $h \neq 0$.

5. Let x and h be any numbers with $h \neq 0$. For each of the following definitions of a function f, obtain the given expression for the slope m of the line joining the points $(x, f(x))$ and $(x + h, f(x + h))$.

a. $f = \{(t,s) \mid s = 3t^2 - 4t + 5\}$; $m = 6x + 3h - 4$.

b. $f = \left\{(t,s) \mid t \neq 0, s = \frac{1}{t}\right\}$; $m = \dfrac{-1}{x(x + h)}$, for $x + h \neq 0, x \neq 0$.

c. $f = \{(t,s) \mid t > 0, s = \sqrt{t}\}$; $m = \dfrac{1}{\sqrt{x + h} + \sqrt{x}}$, $x > 0, x + h > 0$.

d. $f = \{(t,s) \mid s = \sin t\}$; $m = \cos(x + \frac{1}{2}h)\dfrac{\sin \frac{1}{2}h}{\frac{1}{2}h}$.

6. Relying mainly on $|a \pm b| \leq |a| + |b|$, then for c and d such that:

a. $|c - 2| < \frac{1}{2}$ and $|d - 2| < \frac{1}{3}$, show that $|c - d| < \frac{5}{6}$.
Hint: $|c - d| = |(c - 2) - (d - 2)| \leq |c - 2| + |d - 2|$.

b. $|c + 2| < \frac{1}{2}$ and $|d + 2| < \frac{1}{3}$, show that $|c - d| < \frac{5}{6}$.

c. $|c - d| < \frac{1}{2}$ and $|c + 2| < \frac{1}{3}$, show that $|d + 2| < \frac{5}{6}$.

d. $|c| < \frac{1}{2}$ and $|d| < \frac{1}{3}$, show that $|c + d| < \frac{5}{6}$ and $|c - d| < \frac{5}{6}$.

e. $|c - 2| < \frac{1}{2}$, show that $\frac{3}{2} < c < \frac{5}{2}$. [Recall: $-|u| \leq u \leq |u|$]

f. $|c - d| < \frac{1}{2}$, show that $d - \frac{1}{2} < c < d + \frac{1}{2}$.

7. Let f and g be functions such that:

if $0 < |x - 2| < 0.1$, then $|f(x) - 3| < 0.25$ and

if $0 < |x - 2| < 0.3$, then $|g(x) - 4| < 0.2$.

Show that if $0 < |x - 2| < 0.1$, then

a. $|f(x) + g(x) - 7| < 0.45$.

b. $|f(x)g(x) - 12| < 1.65$.

10. Distance Formula, Circles

The distance between the origin (0,0) and the point (0,1) will now be taken the same as the distance between the origin (0,0) and the point (1,0) and will be called the **unit distance**, i.e., the same unit will be used on both axes. The distance between the points $(-2,1)$ and $(4,9)$ is therefore 10 units. For (see Fig. 10.1) the segment joining these points is the hypotenuse of a right triangle with sides 6 units and 8 units long and, by the Pythagorean Theorem, $\sqrt{6^2 + 8^2} = 10$ units is the length of the hypotenuse.

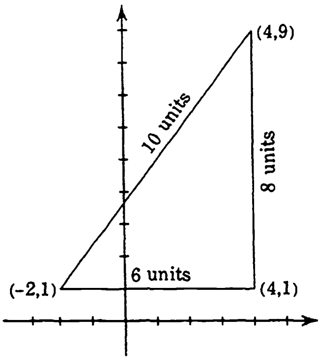


Figure 10.1

With x, y, x_1, and y_1 numbers, it follows that:

The distance between the points (x,y) *and* (x_1,y_1) *is*

$$(1) \qquad \sqrt{(x - x_1)^2 + (y - y_1)^2} \text{ units.}$$

For, as illustrated in Fig. 10.2, there is a right triangle with the segment joining the points (x,y) and (x_1,y_1) as hypotenuse and sides $|x - x_1|$ units and $|y - y_1|$ units long. Hence, this hypotenuse has length $\sqrt{|x - x_1|^2 + |y - y_1|^2}$ units. But $|x - x_1|^2 = (x - x_1)^2$ and $|y - y_1|^2 = (y - y_1)^2$, and hence the distance may be written as in (1).

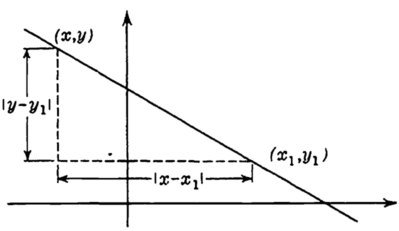


Figure 10.2

Example. Show that the triangle with vertices $(1,-2)$, $(4,1)$, and $(-1,3)$ is an isosceles triangle. Find the area of this triangle.

Solution. The lengths of the sides of this triangle are

$$\sqrt{(1-4)^2 + (-2-1)^2} = \sqrt{3^2 + 3^2} = 3\sqrt{2} \text{ units,}$$

$$\sqrt{(1+1)^2 + (-2-3)^2} = \sqrt{2^2 + 5^2} = \sqrt{29} \text{ units,}$$

$$\sqrt{(4+1)^2 + (1-3)^2} = \sqrt{5^2 + 2^2} = \sqrt{29} \text{ units,}$$

and since two of these distances are equal, the triangle is isosceles with equal sides meeting at the point $(-1,3)$. The altitude from this point to the side joining the points $(1,-2)$ and $(4,1)$ is perpendicular to that side at its mid-point $\left(\frac{1+4}{2}, \frac{-2+1}{2}\right) = \left(\frac{5}{2}, -\frac{1}{2}\right)$. Thus, the altitude h is

$$h = \sqrt{(-1 - \tfrac{5}{2})^2 + (3 + \tfrac{1}{2})^2} = \sqrt{(\tfrac{7}{2})^2 + (\tfrac{7}{2})^2} = \tfrac{7}{2}\sqrt{2} \text{ units.}$$

The base is, as computed above, $3\sqrt{2}$ units long, so the area is

$$\tfrac{1}{2}(3\sqrt{2})(\tfrac{7}{2}\sqrt{2}) = \tfrac{21}{2} \text{ units}^2.$$

With h and k numbers and r a positive number, the set of points

(2) $$\{(x,y) \mid (x-h)^2 + (y-k)^2 = r^2\}$$

is a circle with center the point (h,k) and radius r.

For a point (x,y) is on the circle with center at the point (h,k) and radius r if and only if $\sqrt{(x-h)^2 + (y-k)^2} = r$ and, thus, if and only if $(x-h)^2 + (y-k)^2 = r^2$.

Example. Show that

$$\{(x,y) \mid 2x^2 - 8x + 2y^2 + 12y = 1\}$$

is a circle and find the center and radius of the circle.

Solution. First write the equation in the form

$$2(x^2 - 4x \qquad) + 2(y^2 + 6y \qquad) = 1$$

where the space is left for "completing the square":

$$2(x^2 - 4x + 2^2) + 2(y^2 + 6y + 3^2) = 1 + 2\cdot 4 + 2\cdot 9 = 27,$$

and this is equivalent to $(x-2)^2 + (y+3)^2 = \frac{27}{2}$. Thus,

$$\{(x,y) \mid 2x^2 - 8x + 2y^2 + 12y = 1\} = \{(x,y) \mid (x-2)^2 + (y+3)^2 = \tfrac{27}{2}\}$$

and from this second form, the set is a circle with center the point $(2,-3)$ and radius $\sqrt{\frac{27}{2}}$ units.

A set A is said to be a **subset** of a set B and the notation

$$A \subset B$$

is used if every element of A is also an element of B. For example,

$$\{x \mid |x - \tfrac{3}{2}| < \tfrac{1}{2}\} \subset \{x \mid |x - 2| < 1\}.$$

Example. Show that

$$\{(x,y) \mid |x - \tfrac{3}{4}| < 1 \text{ and } |y - 2| < \tfrac{1}{3}\} \subset \{(x,y) \mid x^2 + y^2 < 9\}.$$

Solution. The first set is all points inside the rectangle with corners $(-\frac{1}{4},\frac{5}{3})$, $(\frac{7}{4},\frac{5}{3})$, $(\frac{7}{4},\frac{7}{3})$ and $(-\frac{1}{4},\frac{7}{3})$, whereas the second set is the interior of the circle with center at the origin and radius 3. Geometrically, the point $(\frac{7}{4},\frac{7}{3})$ is farthest from the origin and its distance to the origin is

$$\sqrt{\left(\frac{7}{4}\right)^2 + \left(\frac{7}{3}\right)^2} = \sqrt{\frac{49}{16} + \frac{49}{9}}$$
$$= \sqrt{\frac{(49)(25)}{144}} = \frac{35}{12} < 3,$$

so the whole rectangle lies inside the circle.

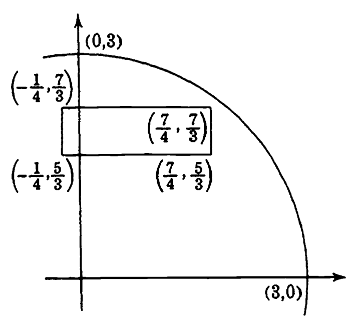


Figure 10.3

Algebraically, for (x,y) in the first set

$$-\tfrac{1}{4} < x < \tfrac{7}{4} \quad \text{and} \quad \tfrac{5}{3} < y < \tfrac{7}{3}$$

so that $0 \leq x^2 < \frac{49}{16}$, $\frac{25}{9} < y^2 < \frac{49}{9}$ and hence

$$\tfrac{25}{9} < x^2 + y^2 < \tfrac{49}{16} + \tfrac{49}{9} < 9.$$

PROBLEMS

1. Find the distance between each of the following pairs of points:

a. $(6,3)$, $(2,1)$.
b. $(-6,1)$, $(2,5)$.
c. $(-2,7)$, $(3,-5)$.
d. $(-2,-5)$, $(-8,-3)$.

2. Show that the three given points are vertices of a right triangle. Find the area of the triangle.

a. $(2,1)$, $(4,2)$, $(-1,7)$.
b. $(2,1)$, $(4,2)$, $(2,6)$.
c. $(-1,-2)$, $(4,2)$, $(6,-\frac{1}{2})$.
d. $(-0.5,3.5)$, $(1,0.5)$, $(2,1)$.

3. Express in the form (2) the set which is the circle having:

a. Center $(-2,3)$ and radius 2.
b. Center $(-2,3)$ and passing through $(1,-1)$.
c. Center $(12,-5)$ and passing through the origin.
d. Radius h and center (h,k).
e. Radius r, tangent to both axes, and center in the second quadrant.
f. The points $(-2,4)$ and $(3,9)$ as end points of a diameter.

4. Describe each of the sets:

a. $\{(x,y) \mid x^2 - 4x + y^2 + 6y = 3\}$.
b. $\{(x,y) \mid x^2 + 6x + y^2 - 2y = -1\}$.
c. $\{(x,y) \mid 4x^2 + 4y^2 - 4x + 16y + 13 = 0\}$.
d. $\{(x,y) \mid 4x^2 + 4y^2 - 24x + 12y = 55\}$.
e. $\{(x,y) \mid x^2 + y^2 - 2x + 4y + 5 = 0\}$.
f. $\{(x,y) \mid 9x^2 + 9y^2 - 72x + 3y + 145 = 0\}$.

5. Describe each of the sets:

a. $\{(x,y) \mid x^2 + y^2 + 6x - 8y < 0\}$.
b. $\{(x,y) \mid x^2 + y^2 + 8x + 10y > -32\}$.
c. $\{(x,y) \mid x^2 + y^2 < 25\} \cap \{(x,y) \mid x^2 + y^2 < 10x\}$.
d. $\{(x,y) \mid x^2 + y^2 < 25\} \cap \{(x,y) \mid \sqrt{2}(x^2 + y^2) < 10(x + y)\}$.
e. $\{(x,y) \mid x^2 + y^2 < 25\} \cup \{(x,y) \mid \sqrt{2}(x^2 + y^2) < 10(x + y)\}$.
f. $\{(x,y) \mid x^2 + y^2 \geq 25\} \cup \{(x,y) \mid x^2 + y^2 \geq 10x\}$.
g. $\{(x,y) \mid x^2 + y^2 < 100\} \cap \{(x,y) \mid x^2 + y^2 < 8y - 6x\}$.
h. $\{(x,y) \mid x^2 + y^2 \leq 100\} \cap \{(x,y) \mid x^2 + y^2 \geq 8y - 6x\}$.

6. Establish each of the following:
   a. $\{(x,y) \mid -1 < x < 3 \text{ and } -4 < y < 1\} \subset \{(x,y) \mid x^2 + y^2 < 5^2\}$.
   b. $\{(x,y) \mid (x - 3)^2 + (y - 4)^2 < 5^2\} \subset \{(x,y) \mid x^2 + y^2 < 10^2\}$.
   c. $\{(x,y) \mid |x - 2| < 1 \text{ and } y = x/2\} \subset \{(x,y) \mid |x - 2| < 1 \text{ and } |y - 1| \le \frac{1}{2}\}$.
   d. $\{(x,y) \mid |x - 2| < 1 \text{ and } y = x/2\} \subset \{(x,y) \mid x^2 + y^2 - 4x - 2y < 1\}$.
7. With $A \neq 0$, D, E, and F given numbers show that
   a. $\{(x,y) \mid Ax^2 + Ay^2 + Dx + Ey + F = 0\}$
   $$= \left\{(x,y) \,\middle|\, \left(x + \frac{D}{2A}\right)^2 + \left(y + \frac{E}{2A}\right)^2 = \frac{D^2 + E^2 - 4AF}{4A^2}\right\}$$
   b. Show that the set in a:
      1. Is empty if $D^2 + E^2 - 4AF < 0$.
      2. Consists of a single point if $D^2 + E^2 - 4AF = 0$.
      3. Is a circle if $D^2 + E^2 - 4AF > 0$.

11. Properties in the Large

An elementary way of plotting a graph is to obtain specific points on the graph and then join these points by a curve. This method is essentially one of approximation; the closer together the points on the graph are taken, the more trust one is likely to place on the form of the graph between and near these points. Using this method in plotting

$$\{(x,y) \mid x^2 + y^2 - 6x - 7 = 0\}, \tag{1}$$

the equation is first solved for y:

$$y = \pm\sqrt{7 + 6x - x^2}$$

and then a corresponding table made:

x	$\cdots$	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	$\cdots$
y	no values			0	± 2.6	± 3.5	± 3.9	± 4	± 3.9	± 3.5	± 2.6	0	no values		

Upon plotting the points $(-1,0)$, $(0,2.6)$, $(0,-2.6)$, $\cdots$, they seem to arrange themselves in a circular array, but with only this to go on one could not say with confidence that (1) is indeed a circle with center $(3,0)$ and radius 4.

Sometimes properties in the large (i.e., overall properties) of a graph may not be revealed even by a great many points on the graph. Some such properties are discussed below for graphs of sets of ordered pairs defined by means of equations.

A. SYMMETRY. Two points are said to be **symmetric to a line** if the line is the perpendicular bisector of the segment joining the points. A graph is

symmetric to a line if all points of the graph occur in pairs symmetric to the line; the line itself is called an **axis of symmetry** of the graph.

If all points of a graph can be arranged in pairs so that the line segments joining such pairs are all bisected by the same point, then the graph is said to be **symmetric to that point**; the point itself is said to be a **point of symmetry** of the graph.

A graph defined by means of an equation may be tested for symmetry to the coordinate axes or the origin by noting: *If an equivalent*† *equation is obtained upon replacing*

(i) *y by $-y$, then the graph is symmetric to the x-axis.*
(ii) *x by $-x$, then the graph is symmetric to the y-axis.*
(iii) *x by $-x$ and y by $-y$, then the graph is symmetric to the origin.*

For example, since $x^2 - xy = 1$ is equivalent to $(-x)^2 - (-x)(-y) = 1$, but is not equivalent to either $x^2 - x(-y) = 1$ or to $(-x)^2 - (-x)y = 1$, then

$$\{(x,y) \mid x^2 - xy = 1\}$$

is symmetric to the origin, but is not symmetric to either axis.

From the equivalence of the equation involved, it follows that:

$$\begin{aligned}\{(x,y) \mid x^2y^2 = x^2 - 1\} &= \{(x,y) \mid x^2(-y)^2 = x^2 - 1\} \\ &= \{(x,y) \mid (-x)^2y^2 = (-x)^2 - 1\} \\ &= \{(x,y) \mid (-x)^2(-y)^2 = (-x)^2 - 1\}.\end{aligned}$$

Consequently, this set of points is symmetric to the x-axis, to the y-axis, and to the origin.

B. EXTENT. The **x-extent** and **y-extent** of a graph are

$$\{x \mid \text{there is a } y \text{ such that the point } (x,y) \text{ is on the graph}\}, \quad \text{and}$$

$$\{y \mid \text{there is an } x \text{ such that the point } (x,y) \text{ is on the graph}\},$$

respectively. Thus, a number x_0 is in the x-extent of a graph if and only if the projection of the graph on the x-axis contains the point $(x_0,0)$.

For example, note that

$$\{(x,y) \mid x^2y^2 = x^2 - 1\} = \left\{(x,y) \mid y = \pm\sqrt{\frac{x^2 - 1}{x^2}}\right\}.$$

Thus, if a point (x_0,y_0) is in this set, then one observation is that $x_0 \neq 0$ but furthermore $x_0^2 - 1 \geq 0$. Hence, for this graph the x-extent is

$$\{x \mid \text{either } x \geq 1 \quad \text{or else} \quad x \leq -1\}.$$

† Two equations in x and y are said to be "equivalent" if whenever an ordered pair of numbers satisfies either equation it also satisfies the other.

This same set may also be expressed as

$$\left\{(x,y) \mid x = \frac{\pm 1}{\sqrt{1 - y^2}}\right\}.$$

Hence, the restriction is now $1 - y^2 > 0$ and the y-extent is

$$\{y \mid -1 < y < 1\}.$$

The general procedure is to solve the equation for y in terms of x, and then to note what restrictions on x are necessary to ensure that:

(i) *No denominator is zero.*

(ii) *No negative number occurs under a square root sign (or any radical sign with even index).*

Next, solve for x in terms of y and note restrictions on y to ensure (i) and (ii).

C. HORIZONTAL AND VERTICAL ASYMPTOTE. The set

$$\left\{(x,y) \mid y = \frac{2x + 10}{x}\right\}$$

contains the points (10,3), (100,2.1), (1000,2.01), (10000,2.001), and for e any positive number

$$2 < \frac{2x + 10}{x} = 2 + \frac{10}{x} < 2 + e \quad \text{if} \quad x > \frac{10}{e}.$$

The fact that "$(2x + 10)/x$ may be made to differ from 2 to within any stated accuracy merely by choosing x sufficiently large" is denoted by

$$\lim_{x\to\infty} \frac{2x + 10}{x} = 2.$$

DEFINITION. *For f a function and L a number, we write*

(2) $$\lim_{x\to\infty} f(x) = L$$

if to each positive number e there corresponds a positive number G such that

$$\textit{whenever } x > G, \quad \textit{then } |f(x) - L| < e.$$

If, however, for each $e > 0$ there is a number $G > 0$ such that

$$\textit{whenever } x < -G, \quad \textit{then } |f(x) - L| < e$$

we write

$$\lim_{x\to-\infty} f(x) = L.$$

The symbolism $x \to \infty$ is read, according to preference, "x becomes positively infinite" or "x approaches positive infinity" or "x increases without bound." For the specific functions we meet for some time, it will be intuitively evident whether the limits exist as $x \to \infty$ or $x \to -\infty$. For example,

$$\lim_{x\to\infty} \frac{x^2 - 1}{x^2} = 1, \quad \lim_{x\to\infty} \sqrt{\frac{x^2 - 1}{x^2}} = 1, \text{ and } \lim_{x\to-\infty} \sqrt{\frac{x^2 - 1}{x^2}} = 1$$

but $\lim_{x\to\infty}(x - x^{-1})$ does not exist nor does $\lim_{x\to\infty}\sin x$ although

$$\lim_{x\to\infty}\frac{\sin x}{x} \text{ exists and is equal to } 0.$$

Also, there is no sense in writing $\lim_{y\to\infty}\sqrt{1-y^2}$ since whenever $y > 1$, then $1 - y^2 < 0$ and $\sqrt{1-y^2}$ is not a real number.

DEFINITION. *Given a graph, if there is a function f and a number L such that the graph of f is a part (or all) of the given graph and if*

$$\lim_{x\to\infty} f(x) = L,$$

then the line $\{(x,y) \mid y = L\}$ is said to be a **horizontal asymptote** *in the positive direction of the given graph. By replacing $x \to \infty$ by $x \to -\infty$, the definition of a horizontal asymptote in the negative direction is obtained.*

For example, $\{(x,y) \mid x^2y^2 = x^2 - 1\}$ contains

$$\left\{(x,y) \mid y = \sqrt{\frac{x^2-1}{x^2}}\right\} \text{ whereas } \lim_{x\to\infty}\sqrt{\frac{x^2-1}{x^2}} = 1 \text{ and } \lim_{x\to-\infty}\sqrt{\frac{x^2-1}{x^2}} = 1,$$

so that the line $\{(x,y) \mid y = 1\}$ is a horizontal asymptote in both directions. Similarly, the line $\{(x,y) \mid y = -1\}$ is also a horizontal asymptote in both directions.

DEFINITION. *Given a graph, if there is a function g and a number L such that the given graph contains*

$$\{(x,y) \mid x = g(y)\} \quad \textit{and if} \quad \lim_{y\to\infty} g(y) = L,$$

then the line represented by $\{(x,y) \mid x = L\}$ is said to be a **vertical asymptote** *(in the positive direction) of the given graph.*

For example, $\{(x,y) \mid x^2y^2 = x^2 - 1\}$ has no vertical asymptote, since (as noted earlier) the y-extent is $\{y \mid -1 < y < 1\}$.

Example. Sketch $\{(x,y) \mid x^2y^2 = x^2 - 1\}$.

Solution. Had we not already done so, we would now find the graph to:

1. Be symmetric to the x-axis, the y-axis, and the origin.
2. Have x-extent $= \{x \mid x \geq 1 \text{ or } x \leq -1\}$, y-extent $= \{y \mid -1 < y < 1\}$.
3. Have lines $\{(x,y) \mid y = \pm 1\}$ as horizontal asymptotes in both directions, but no vertical asymptote.

By setting $x = 1$ in the equation, the point $(1,0)$ is the point on the graph with smallest positive abscissa. With this information, we can hardly fail to sketch fairly accurately the portion of the graph in the first quadrant and then the portions in the other quadrants by using the symmetry properties.

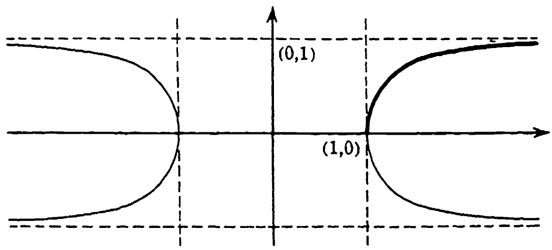


Figure 11.1

PROBLEMS

Discuss the properties in the large given above for the graph and sketch $\{(x,y) \mid E(x,y)\}$ where $E(x,y)$ is replaced by the equation:

1. $xy - 2y = 1$.
2. $xy - 2x = 1$.
3. $x^2y = 1$.
4. $x^2y^2 = y^2 - 1$.
5. $x^2y - 2y = 3$.
6. $4xy - 3y = 3x - 2$.
7. $y = \sin^2 x$.
8. $y = 2^x$.
9. $|x| + |y| = 1$.
10. $x^{2/3} + y^{2/3} = 1$.
11. $2x^2 + 5y^2 = 10$.
12. $9x^2 + 16y^2 = 144$.
13. $9x^2 + 16y^2 = 1$.
14. $\dfrac{x^2}{1} + \dfrac{y^2}{5^2} = 1$.

D. GEOMETRIC ADDITION. The set $\{(x,y) \mid x^2 - xy = 1\}$ is also the set

$$(1) \qquad \left\{(x,y) \mid y = x - \frac{1}{x}\right\}.$$

First $\{(x,y) \mid y = x\}$ and $\{(x,y \mid y = -1/x\}$ are easily drawn, and then from these two the desired set (1) may be obtained by so-called "Geometric Addition" as shown in Fig. 11.2. For example, the point P of the desired graph was obtained by taking the distance RS and locating P such that $PQ = RS$ with P below Q since S is below the x-axis.

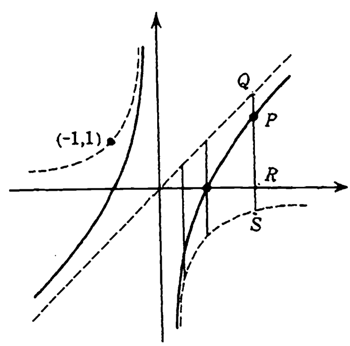


Figure 11.2

Figure 11.3 illustrates how geometric addition of the sine curve (dotted) and the line $\{(x,y) \mid y = x/2\}$ were used in sketching

$$\left\{(x,y) \mid y = \frac{x}{2} + \sin x\right\}.$$

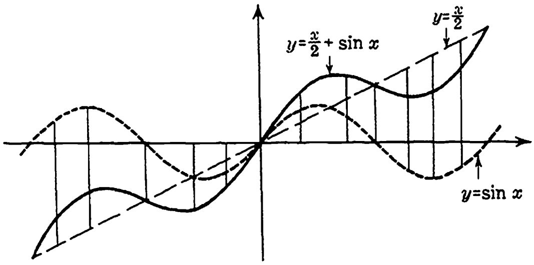


Figure 11.3

E. OBLIQUE ASYMPTOTES. A graph may have asymptotes which are neither horizontal nor vertical.

DEFINITION. *Let a graph be defined by means of an equation. If there are numbers m and c and a function f such that the given graph contains (or is)*

$$(2) \qquad \{(x,y) \mid y = mx + c + f(x)\} \quad \textit{and if} \quad \lim_{x\to\infty} f(x) = 0,$$

then the line $\{(x,y) \mid y = mx + c\}$ *is said to be an asymptote in the positive direction. A similar definition may be given for an asymptote in the negative direction.*

For example, $\{(x,y) \mid x^2 - 2xy + 2x - 1 = 0\}$ is also

$$\left\{(x,y) \mid y = \frac{1}{2}x + 1 - \frac{1}{2x}\right\}$$

and, hence, the line $\{(x,y) \mid y = \frac{1}{2}x + 1\}$ is an asymptote. The graph may be drawn by using geometric addition.

THEOREM 11. *For* $a > 0$ *and* $b > 0$ *the graphs*

$$(3) \qquad \left\{(x,y) \,\middle|\, \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right\} \quad \textit{and} \quad \left\{(x,y) \,\middle|\, \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1\right\}$$

each has the lines

$$\left\{(x,y) \mid y = \pm\frac{b}{a}x\right\}$$

as asymptotes in both directions.

PROOF. By solving the first equation in (3) for y, the given graph contains $\left\{(x,y) \mid y = \frac{b}{a}\sqrt{x^2 - a^2}\right\}$, and this $\left(\text{by adding and subtracting } \frac{b}{a}x\right)$ may be written as

$$\left\{(x,y) \mid y = \frac{b}{a}x + \frac{b}{a}(\sqrt{x^2 - a^2} - x)\right\}$$

$$= \left\{(x,y) \mid y = \frac{b}{a}x + \frac{b}{a}(\sqrt{x^2 - a^2} - x)\frac{\sqrt{x^2 - a^2} + x}{\sqrt{x^2 - a^2} + x}\right\}$$

$$= \left\{(x,y) \mid y = \frac{b}{a}x + \frac{b}{a}\frac{(x^2 - a^2) - x^2}{\sqrt{x^2 - a^2} + x}\right\}$$

$$= \left\{(x,y) \mid y = \frac{b}{a}x + \frac{-ab}{\sqrt{x^2 - a^2} + x}\right\}$$

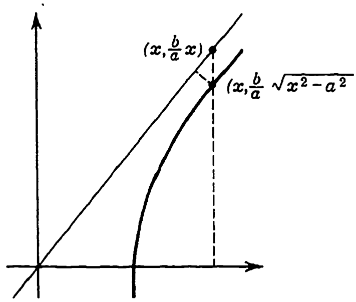


Figure 11.4

which is in the form (2) with $m = b/a$, $c = 0$ and

$$f(x) = \frac{-ab}{\sqrt{x^2 - a^2} + x}.$$

The denominator can be made as large as we please (and the absolute value of the fraction as small as we please) merely by choosing x sufficiently large. Consequently $\lim_{x\to\infty} f(x) = 0$ and, as defined above, the given graph has the line $\{(x,y) \mid y = bx/a\}$ as an asymptote in the positive direction. Since the given graph is symmetric to both the x-axis and to the y-axis, the further asymptotic properties of the graph hold.

In a similar way, the second set in (3) may be shown to have the same asymptotes.

Example. Sketch $\{(x,y) \mid 3x^2 - 4y^2 = 12\}$.

Solution. A. The graph is symmetric to the x-axis, to the y-axis, and, hence, also to the origin.

B. Since the graph is also

$$\left\{(x,y) \mid y = \pm\frac{\sqrt{3}}{2}\sqrt{x^2 - 4}\right\} \quad \text{or} \quad \left\{(x,y) \mid x = \pm\frac{2}{\sqrt{3}}\sqrt{y^2 + 3}\right\},$$

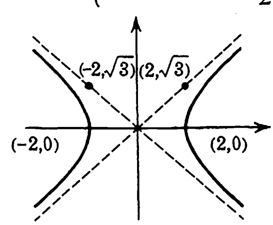


Figure 11.5

we have x-extent $= \{x \mid x \geq 2 \text{ or } x \leq -2\}$, but there is no limitation on the y-extent.

C. There are no horizontal or vertical asymptotes.

D. Geometric addition is not readily applicable.

E. The graph is also

$$\left\{(x,y) \,\middle|\, \frac{x^2}{4} - \frac{y^2}{3} = 1\right\},$$

which is in the form (3), and thus the lines

$$\left\{(x,y) \mid y = \pm\frac{\sqrt{3}}{2}x\right\}$$

are asymptotes in both directions.

Upon checking that the points (2,0) and $(-2,0)$ are on the graph, we obtain Fig. 11.5.

PROBLEMS

1. Use geometric addition to sketch each of the following sets:

a. $\left\{(x,y) \mid y = x + \frac{1}{x^2}\right\}$.

b. $\left\{(x,y) \mid x = y + \frac{1}{y^2}\right\}$.

c. $\{(x,y) \mid x^2y + x^2 + 1 = 0\}$.

d. $\left\{(x,y) \mid y = \frac{x}{2} + \cos x\right\}$.

e. $\{(x,y) \mid xy - y - x^2 + x = 1\}$.

f. $\{(x,y) \mid xy + x^2 - x = 1\}$.

g. $\{(x,y) \mid y = \sin x + \cos x\}$.

h. $\{(x,y) \mid y = 3 + \sin 2x\}$.

2. Each of the following sets has at least one oblique asymptote. Find the asymptotes and sketch the graphs.

a. $\left\{(x,y) \,\middle|\, \frac{x^2}{4} - \frac{y^2}{6} = 1\right\}$.

b. $\left\{(x,y) \,\middle|\, \frac{x^2}{6} - \frac{y^2}{4} = 1\right\}$.

c. $\left\{(x,y) \,\middle|\, \frac{y^2}{4} - \frac{x^2}{6} = 1\right\}$.

d. $\{(x,y) \mid 9x^2 - 16y^2 = 144\}$.

e. $\{(x,y) \mid 9x^2 - 16y^2 = 1\}$.

f. $\left\{(x,y) \,\middle|\, \frac{x^2}{3^2} + \frac{y^2}{3^2(1 - 2^2)} = 1\right\}$.

3. Each of the parts below helps in working the next part. Show that

a. $(a - b)(a^2 + ab + b^2) = a^3 - b^3$.

b. $(a^{1/3} - b^{1/3})(a^{2/3} + a^{1/3}b^{1/3} + b^{2/3}) = a - b$.

c. $[(1 + x^3)^{1/3} - x][(1 + x^3)^{2/3} + (1 + x^3)^{1/3}x + x^2] = 1$.

d. $\{(x,y) \mid y^3 - x^3 = 1\} = \{(x,y) \mid y = x + [(1 + x^3)^{1/3} - x]\}$

$$= \left\{(x,y) \mid y = x + \frac{1}{(1 + x^3)^{2/3} + (1 + x^3)^{1/3}x + x^2}\right\}.$$

e. The set $\{(x,y) \mid y^3 - x^3 = 1\}$ has the line $\{(x,y) \mid y = x\}$ as an oblique asymptote in both directions.

f. The set $\{(x,y) \mid x^3 + y^3 = 1\}$ has the line $\{(x,y) \mid y = -x\}$ as an oblique asymptote in both directions.

12. Translation of Coordinates

Let h and k be numbers, and plot the points (h,k), $(h+1,k)$, and $(h,k+1)$. Now establish a new coordinate system with these points, respectively, as the new origin, unit point on the new axis of abscissas, and unit point on the new axis of ordinates. A point in the plane now has coordinates (x,y) relative to the original axes, but (X,Y) relative to the new axes where

(1) $$X = x - h, \quad Y = y - k.$$

As a check, note that if $X = 0$ and $Y = 0$, then $x = h$ and $y = k$. Also, if $X = 1$ and $Y = 0$, then $x = h + 1$ and $y = k$.

Translation of axes may be used to simplify graphing.

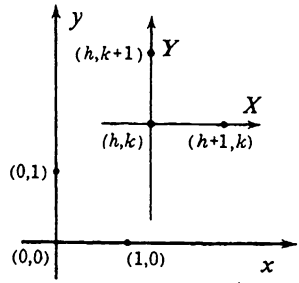


Figure 12.1

Example 1. Sketch the set

(2) $$\{(x,y) \mid 36x^2 - 16y^2 - 108x - 64y = 127\}$$

by first translating axes so the new origin is at the point $(\frac{3}{2}, -2)$.

Solution. In (1) substitute $h = \frac{3}{2}$ and $k = -2$ to obtain $X = x - \frac{3}{2}$ and $Y = y - (-2) = y + 2$, so that in the equation of (2) substitute

(3) $$x = X + \tfrac{3}{2} \quad \text{and} \quad y = Y - 2.$$

The result of this substitution, with simplifications, is

$$36(X + \tfrac{3}{2})^2 - 16(Y - 2)^2 - 108(X + \tfrac{3}{2}) - 64(Y - 2) = 127,$$

$$36X^2 + 108X + 81 - 16Y^2 + 64Y - 64 - 108X - 162 - 64Y + 128 = 127,$$

$$36X^2 - 16Y^2 = 144,$$

$$\frac{X^2}{4} - \frac{Y^2}{9} = 1.$$

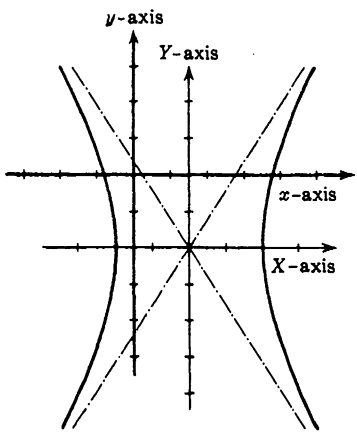


Figure 12.2

The set (2) plotted relative to the xy-system will be the same as the set

(4) $$\left\{(X,Y) \,\middle|\, \frac{X^2}{4} - \frac{Y^2}{9} = 1\right\}$$

plotted relative to the XY-system. The graph is symmetric to the X-axis, the Y-axis, and to the XY-origin, and thus is symmetric to the lines

$$\{(x,y) \mid x = \tfrac{3}{2}\} \quad \text{and} \quad \{(x,y) \mid y = -2\}$$

and to the xy-point $(\frac{3}{2}, -2)$. Also, X-extent $= \{X \mid X \leq -2 \text{ or } X \geq 2\}$, so that

$$x\text{-extent} = \{x \mid x - \tfrac{3}{2} \leq -2 \text{ or } x - \tfrac{3}{2} \geq 2\} = \{x \mid x \leq -\tfrac{1}{2} \text{ or } x \geq \tfrac{7}{2}\}.$$

There is no restriction on the Y-extent, and thus no restriction on the y-extent.

The graph has oblique asymptotes which, in the proper coordinate system, are the lines $\{(X, Y) \mid 2Y = \pm 3X\}$ or

$$\{(x,y) \mid 2(y + 2) = \pm 3(x - \tfrac{3}{2})\} = \{(x,y) \mid 6x - 4y = 17 \quad \text{or} \quad 6x + 4y = 1\}.$$

Two methods for determining a suitable translation are given below.

FIRST METHOD

Example 2. Determine the translation used in Example 1 to simplify the equation in (2).

Solution. Rewrite the equation as

$$36(x^2 - \tfrac{108}{36}x) - 16(y^2 + 4y) = 127,$$

$$36(x^2 - 3x + \qquad) - 16(y^2 + 4y + \qquad) = 127$$

where enough space was left for "completing the square":

$$36(x^2 - 3x + \tfrac{9}{4}) - 16(y^2 + 4y + 4) = 127 + 81 - 64,$$

$$36(x - \tfrac{3}{2})^2 - 16(y + 2)^2 = 144.$$

This equation looks simpler upon setting

$$X = x - \tfrac{3}{2} \quad \text{and} \quad Y = y + 2,$$

which are equivalent to the equations (3) used in Example 1.

SECOND METHOD

Example 3. Sketch $\{(x,y) \mid xy + 4x - y = 5\}$ by first making a translation of axes.

Solution. Let h and k be numbers, as yet unspecified, and in the equation $xy + 4x - y = 5$ substitute

(5) $$x = X + h \quad \text{and} \quad y = Y + k.$$

The resulting equation and a rearrangement are

$$(X + h)(Y + k) + 4(X + h) - (Y + k) = 5 \quad \text{and}$$

(6) $$XY + (k + 4)X + (h - 1)Y = 5 - hk - 4h + k.$$

Upon selecting $k = -4$ and $h = 1$, the coefficients of X and Y are both zero, and $5 - hk - 4h + k = 5 - (1)(-4) - 4(1) + (-4) = 1$ so that the equations of (5) become

$$x = X + 1, \quad y = Y - 4$$

and (6) becomes $XY = 1$. Thus, the new origin $X = 0$, $Y = 0$ is at the point where $x = 0 + 1 = 1$ and $y = 0 - 4 = -4$. Hence,

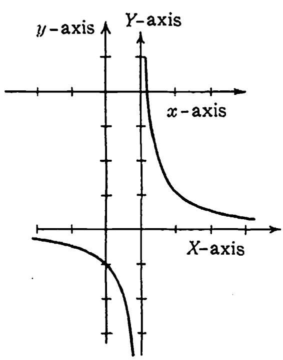


Figure 12.3

$$\{(x,y) \mid xy + 4x - y = 5\} \quad \text{and} \quad \{(X,Y) \mid XY = 1\}$$

are the same; it being understood that the first is drawn relative to the original system and the second relative to the new system. Hence, in terms of the original system, the graph is symmetric to the point $(1, -4)$ and has the lines

$$\{(x,y) \mid y = -4\} \quad \text{and} \quad \{(x,y) \mid x = 1\}$$

as horizontal and vertical asymptotes.

PROBLEMS

1. Sketch each of the following after first making a translation of axes using the given values of h and k.
   a. $\{(x,y) \mid 2xy - 4x + 3y = 6\}$, $h = -\frac{3}{2}$, $k = 2$.
   b. $\{(x,y) \mid x - y^2 - 6y = 14\}$, $h = 5$, $k = -3$.
   c. $\{(x,y) \mid y = 3 + 32(2^x)\}$, $h = -5$, $k = 3$.
   d. $\{(x,y) \mid y = 1 + \sin(2x - 5)\}$, $h = 2.5$, $k = 1$.
   e. $\{(x,y) \mid y = ax^2 + bx + c\}$, $h = -\dfrac{b}{2a}$, $k = -\dfrac{b^2 - 4ac}{4a}$.
2. Use the first method to determine a translation which simplifies each of the following, then sketch the graph.
   a. $\{(x,y) \mid 3x^2 + 4y^2 + 6x - 16y + 7 = 0\}$.
   b. $\{(x,y) \mid 3x^2 - 4y^2 + 6x - 16y = 17\}$.
   c. $\{(x,y) \mid 4x^2 + 6y^2 - 12x + 36y + 63 = 0\}$.
   d. $\{(x,y) \mid 12x^2 + 18y^2 - 60x + 24y + 89 = 0\}$.
3. Transform each of the following so the resulting XY-equation will have no X-term or Y-term. In each case give the xy-coordinates of the XY-origin.
   a. $\{(x,y) \mid 2xy + 3x - 4y = -6\}$.
   b. $\{(x,y) \mid xy - 3x + 5y = 17\}$.
   c. $\{(x,y) \mid (x - 1)(y + 2) + 2x = 3\}$.
   d. $\left\{(x,y) \mid y = 3 + \dfrac{4}{x + 2}\right\}$.

13. Conic Sections

Let e be a positive number. In the plane select a point F and a line l not passing through F.

DEFINITION. *The point set*
$\{P \mid P$ *is a point in the plane such that the distance from* P *to* F *is* e *times the perpendicular distance from* P *to* $l\}$.
is a ***conic section*** *with* ***eccentricity*** *e,* ***focus*** *F, and* ***directrix*** *l. In case*:
1. $e = 1$ *the conic is called a* ***parabola***.
2. $0 < e < 1$ *the conic is called an* ***ellipse***.
3. $1 < e$ *the conic is called a* ***hyperbola***.

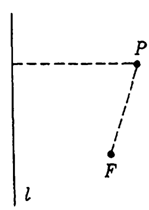


Figure 13

Simple expressions for sets which are the various conic sections may be obtained by judicious choices of F and l. Such selections of F and l are discussed first for the parabola in Sec. 14, and then for the ellipse and hyperbola together in Sec. 15.

14. Parabola

In this case $e = 1$. Let p be a number different from zero; hence, p may be positive or negative, but $p \neq 0$. Select the point $(p,0)$ for F and select the line $\{(x,y)|x = -p\}$ for l. The distance from a point (x,y) to F is $\sqrt{(x-p)^2 + y^2}$ units, whereas the perpendicular distance from a point (x,y) to l is $|x - (-p)| = |x + p|$ units. Since $e = 1$, a point (x,y) is on the parabola with focus F and directrix l if and only if $\sqrt{(x-p)^2 + y^2} = |x + p|$. But

$$\begin{aligned}
&\{(x,y) \mid \sqrt{(x-p)^2 + y^2} = |x + p|\} \\
&= \{(x,y) \mid (x-p)^2 + y^2 = (x+p)^2\} \\
&= \{(x,y) \mid x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2\} \\
&= \{(x,y) \mid y^2 = 4px\}.
\end{aligned}$$

It therefore follows that

(1) $$\{(x,y) \mid y^2 = 4px\}, \quad \text{with } p \neq 0,$$

is the parabola with focus $(p,0)$ and directrix the line $\{(x,y) \mid x = -p\}$.

Example 1. Show that $\{(x,y) \mid 20x + 3y^2 = 0\}$ is a parabola. Find the focus and directrix of this parabola.

Solution. $\{(x,y) \mid 20x + 3y^2 = 0\} = \{(x,y) \mid y^2 = -\frac{20}{3}x\} = \{(x,y) \mid y^2 = 4(-\frac{5}{3})x\}$ and this last expression is in the form (1) with $p = -\frac{5}{3}$. Thus, the graph is a parabola with focus $(-\frac{5}{3},0)$ and directrix the line $\{(x,y) \mid x = \frac{5}{3}\}$. Notice that the points $(0,0)$, $(-\frac{5}{3},\frac{10}{3})$, and $(-\frac{5}{3},-\frac{10}{3})$ are on the parabola.

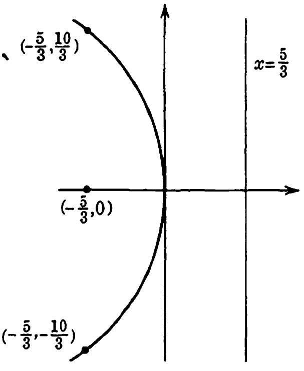


Figure 14.1

For a parabola with focus F and directrix l, let V be the mid-point of the line segment from F perpendicular to l. V is equidistant from F and l and is, therefore, on the parabola. The point V is called the **vertex** of the parabola.

With the distance from F to l as radius, draw a circle with center F and note the points P_1 and P_2 where this circle cuts the line through F parallel to l.

P_1 and P_2 are each equidistant from F and l, so both are on the parabola. The line segment between P_1 and P_2 is called the **right focal chord** of the parabola.

For the parabola (1), the vertex is (0,0) and the ends of the right focal chord are $(p,2p)$ and $(p,-2p)$, so that the length of the right focal chord is $4\,|p|$ units. Knowing the general shape of a parabola, the position of the vertex and ends of the right focal chord enable one to sketch the graph near the vertex with sufficient accuracy for most purposes.

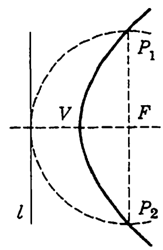


Figure 14.2

Example 2. Show that $\{(x,y) \mid 3y^2 + 12y - 20x + 42 = 0\}$ is a parabola and find its vertex, focus, ends of the right focal chord, and its directrix.

Solution. Toward obtaining a suitable translation we write the equation involved as

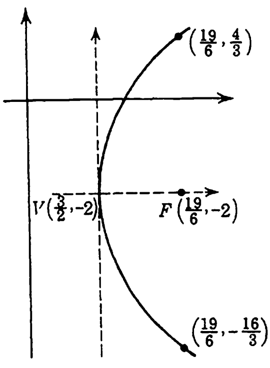


Figure 14.3

$$
\begin{aligned}
3(y^2 + 4y \qquad) &= 20x - 42,\\
3(y^2 + 4y + 4) &= 20x - 42 + 12 = 20x - 30\\
3(y + 2)^2 &= 20(x - \tfrac{3}{2}).
\end{aligned}
$$

Hence, from the translation determined by

$$(2) \qquad X = x - \tfrac{3}{2}, \quad Y = y + 2,$$

we see that the graph is

$$\{(X, Y) \mid 3Y^2 = 20X\} = \{(X, Y) \mid Y^2 = \tfrac{20}{3}X\} = \{(X, Y) \mid Y^2 = 4(\tfrac{5}{3})X\}.$$

Upon comparing this last form with (1), the graph is seen to be a parabola with $p = \frac{5}{3}$. The required information about this parabola is found in terms of XY-coordinates and then changed to the original xy-coordinates by means of (2):

	XY-coordinates	xy-coordinates
vertex	$(0,0)$	$(\frac{3}{2}, -2)$
focus	$(\frac{5}{3}, 0)$	$(\frac{19}{6}, -2)$
ends of right focal chord	$(\frac{5}{3}, \frac{10}{3})$, $(\frac{5}{3}, -\frac{10}{3})$	$(\frac{19}{6}, \frac{4}{3})$, $(\frac{19}{6}, -\frac{16}{3})$
directrix	$\{(X, Y) \mid X = -\frac{5}{3}\}$	$\{(x,y) \mid x = -\frac{1}{6}\}$

By analogy, it follows that

$$(3) \qquad \{(x,y) \mid x^2 = 4py\}$$

is a parabola with vertex (0,0), focus $(0,p)$, ends of right focal chord $(-2p,p)$ and $(2p,p)$, and directrix the line $\{(x,y) \mid y = -p\}$.

The line containing the vertex and focus of a parabola is called the **axis** of the parabola.

PROBLEMS

1. Sketch the following parabolas, giving in each case the vertex, focus, ends of the right focal chord, and the directrix.

   a. $\{(x,y) \mid y^2 = x\}$.
   b. $\{(x,y) \mid y^2 = -6x\}$.
   c. $\{(x,y) \mid x^2 = 4y\}$.
   d. $\{(x,y) \mid 2x^2 + 9y = 0\}$.
   e. $\{(x,y) \mid x^2 + 2x - 4y = 1\}$.
   f. $\{(x,y) \mid y^2 + 2y + 2x - 1 = 0\}$.
   g. $\{(x,y) \mid y^2 - 8 = 8(x + 1)\}$.
   h. $\{(x,y) \mid 2x^2 + 2x + 3y - 1 = 0\}$.
   i. $\{(x,y) \mid (x - 1)^2 - 2x - 8(y + 2) = 5\}$.

2. Determine A, C, D, E, and F so that

$$\{(x,y) \mid Ax^2 + Cy^2 + Dx + Ey + F = 0\}$$

   will† be a parabola with axis parallel to one of the coordinate axes and having:

   a. Vertex (0,0), focus (2,0).
   b. Vertex (0,0), focus $(0,-2)$.
   c. Vertex (1,2), focus (1,5).
   d. Vertex (2,2), focus $(-1,2)$.
   e. Directrix $\{(x,y) \mid y = 5\}$, focus (2,1).
   f. Directrix $\{(x,y) \mid x = 10\}$, vertex (9,4).
   g. x-extent $= \{x \mid x \geq \frac{8}{3}\}$, focus $(\frac{10}{3}, -1)$.

3. Let a be a positive number, and consider a parabola with right focal chord a units long. Show that the line through the focus of this parabola and making an angle θ with the axis of the parabola intersects the parabola in two points whose distance apart is $a \csc^2 \theta$ units. [Hint: Select a coordinate system with origin at the vertex and x-axis the axis of the parabola.]

4. Take a parabola with its vertex at the origin and its focus F on the x-axis. Draw a line through F and any point P on the parabola. From P draw the perpendicular to the directrix and label the foot of this perpendicular A. Draw AF and let M be the point where this line cuts the y-axis. Prove that angle FPM = angle MPA. Also, prove that the line through P and M contains no point of the parabola other than P.

5. Given that a ball thrown straight up is s ft above the ground t sec after being thrown, where $s = 128t - 16t^2$. Show that $\{(t,s) \mid s = 128t - 16t^2\}$ is a parabola and find its vertex.

† The letter B has been omitted, since later we will discuss equations of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

6. Find a number δ such that:
   a. $\{(x,y) \mid |x - 1| < \delta \text{ and } y = x^2\} \subset \{(x,y) \mid |x - 1| < \delta \text{ and } |y - 1| < 0.25\}$.
   b. $\{(x,y) \mid |x - 2| < \delta \text{ and } y = x^2\} \subset \{(x,y) \mid |x - 2| < \delta \text{ and } |y - 4| < 0.1\}$.
   c. $\{(x,y) \mid |x + 3| < \delta \text{ and } y = x^2\} \subset \{(x,y) \mid |x + 3| < \delta \text{ and } |y - 9| < 0.1\}$.
   d. Let ϵ be a positive number such that $0 < \epsilon < 9$. Determine $\delta > 0$ such that $\{(x,y) \mid |x - 3| < \delta \text{ and } y = x^2\} \subset \{(x,y) \mid |x - 3| < \delta \text{ and } |y - 9| < \epsilon\}$.
7. Sketch $\{(x,y) \mid 0 \le x \le 2 \text{ and } 0 \le y \le x^2\}$ and show that $\{(x,y) \mid 0 \le y \le 4 \text{ and } \sqrt{y} \le x \le 2\}$ is the same.

15. Ellipse and Hyperbola

According to the definition (see Sec. 13) an ellipse or a hyperbola is a conic with eccentricity $e \neq 1$, $e > 0$. We now restrict e to be such that $e > 0$, but $e \neq 1$. Let a be a positive number. Thus, $ae \neq a/e$; in fact

$$ae < \frac{a}{e} \quad \text{if} \quad 0 < e < 1$$

$$ae > \frac{a}{e} \quad \text{if} \quad 1 < e.$$

Select the point $(ae,0)$ for the focus F and the line $\{(x,y) \mid x = a/e\}$ for the directrix l. The distance from a point (x,y) to the focus is $\sqrt{(x - ae)^2 + y^2}$ units, and the perpendicular distance from a point (x,y) to l is $|x - a/e|$ units. Thus, a point (x,y) is on the conic with focus $(ae,0)$ and directrix the line $\{(x,y) \mid x = a/e\}$ if and only if

$$\sqrt{(x - ae)^2 + y^2} = e|x - a/e|.$$

This conic is therefore the set expressed in any of the following ways:

$$\begin{aligned}
&\{(x,y) \mid \sqrt{(x - ae)^2 + y^2} = e|x - a/e|\} \\
&\qquad = \{(x,y) \mid (x - ae)^2 + y^2 = e^2(x - a/e)^2\} \\
&\qquad = \{(x,y) \mid x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2\} \\
&\qquad = \{(x,y) \mid x^2(1 - e^2) + y^2 = a^2(1 - e^2)\} \\
(1) &\qquad = \left\{(x,y) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1\right\}.
\end{aligned}$$

With the same number e as above ($e > 0$, but $e \neq 1$) and the same number $a > 0$ as above, consider the conic with focus $(-ae,0)$ and directrix the line $\{(x,y) \mid x = -a/e\}$. This conic is

$$\{(x,y) \mid \sqrt{(x + ae)^2 + y^2} = e|x + a/e|\}$$

which also simplifies to (1). These results may be stated as:

For $e > 0$ but $e \neq 1$ and for $a > 0$ the conic

$$\left\{(x,y) \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1\right\} \tag{2}$$

has the point $(ae,0)$ and the line $\{(x,y) \mid x = a/e\}$ as one focus and directrix, but also has the point $(-ae,0)$ and the line $\{(x,y) \mid x = -a/e\}$ as another focus and directrix.

Graphical illustrations of this statement and other pertinent facts appear on page 47 for easy reference while working the problems.

By analogy

$$\left\{(x,y) \;\middle|\; \frac{y^2}{a^2} + \frac{x^2}{a^2(1-e^2)} = 1\right\} \tag{3}$$

is a conic with foci $(0,ae)$ and $(0,-ae)$ and with respective directrices the lines $\{(x,y) \mid y = a/e\}$ and $\{(x,y) \mid y = -a/e\}$.

Example 1. For the sets

$$\{(x,y) \mid 4x^2 = 60 - 3y^2\} \quad \text{and} \quad \{(x,y) \mid 4x^2 = 60 + 3y^2\}$$

show that one is an ellipse and the other a hyperbola. In each case find the foci and the directrices.

Solution (for the first set). This set may be written in turn as

$$\{(x,y) \mid 4x^2 + 3y^2 = 60\} = \left\{(x,y) \;\middle|\; \frac{x^2}{15} + \frac{y^2}{20} = 1\right\}.$$

This looks, at first glance, to be in the form (2), but if we try to set $a^2 = 15$ and $a^2(1 - e^2) = 20$ we obtain $15(1 - e^2) = 20$, $1 - e^2 = \frac{4}{3}$, and $e^2 = -\frac{1}{3}$, which is impossible. However, from (3) upon setting $a^2 = 20$ and $a^2(1 - e^2) = 15$ we have $20(1 - e^2) = 15$, $(1 - e^2) = \frac{3}{4}$, $e^2 = \frac{1}{4}$, and $e = \frac{1}{2}$, since $e > 0$. Since $0 < e < 1$ the first graph is an ellipse. Since $a = 2\sqrt{5}$ and $e = \frac{1}{2}$ (and the form (3) is applicable) the foci and directrices are given as:

$$(0,\sqrt{5}),\ (0,-\sqrt{5});\ \{(x,y) \mid y = 4\sqrt{5}\}, \quad \text{and} \quad \{(x,y) \mid y = -4\sqrt{5}\}.$$

Solution (for the second set). This set may be written as

$$\left\{(x,y) \;\middle|\; \frac{x^2}{15} - \frac{y^2}{20} = 1\right\} \tag{4}$$

which, because of the minus sign, may not look like either (2) or (3). It is, however, in the form (2), for upon setting $a^2 = 15$ and $a^2(1 - e^2) = -20$ we have $15(1 - e^2) = -20$, $1 - e^2 = -\frac{4}{3}$, $e^2 = \frac{7}{3}$, and thus $e = \sqrt{\frac{7}{3}}$. Since $1 < e$ the graph is a hyperbola. Since $a = \sqrt{15}$ (and the form (2) is applicable) this hyperbola has foci and directrices given by

$$(\sqrt{35},0),\ (-\sqrt{35},0);\ \{(x,y) \mid x = 3\sqrt{\tfrac{5}{7}}\},\ \{(x,y) \mid x = -3\sqrt{\tfrac{5}{7}}\}.$$

Also note from (4), and the previous discussion of oblique asymptotes (see Sec. 11E, pg. 34), this hyperbola has the lines

$$\{(x,y) \mid \sqrt{3}y = \pm 2x\}$$

as oblique asymptotes.

The line containing the two foci of an ellipse or hyperbola is called the **principal axis** of the conic, and the mid-point of the segment joining the foci is called the **center** of the conic. From the typical form (2), it is seen (by substituting $y = 0$ in the equation and solving for x) that an ellipse or hyperbola cuts the principal axis in the two points $(a,0)$ and $(-a,0)$; these points are called the **vertices** of the conic. The line segment between the vertices is called:

The **major axis** when the conic is an ellipse.

The **transverse axis** when the conic is a hyperbola.

For an ellipse ($0 < e < 1$ so $1 - e^2 > 0$), it follows from (2) that an ellipse cuts the line perpendicular to the major axis in the two points $(0,a\sqrt{1 - e^2})$ and $(0,-a\sqrt{1 - e^2})$; the segment between these points is called the **minor axis** of the ellipse.

A hyperbola has two asymptotes since (2) may be written as

$$\left\{(x,y) \,\middle|\, \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1\right\}$$

where both denominators are positive since $e > 1$ and $e^2 - 1 > 0$. In this form it is seen (from Theorem 11 p. 34) that the graph has two asymptotes given by

$$\{(x,y) \mid y = \pm\sqrt{e^2 - 1}x\}.$$

Also, the hyperbola does not cut the line perpendicular to the transverse axis at the center; in fact from the above form

$$x\text{-extent} = \{(x,y) \mid x \leq -a \quad \text{or} \quad x \geq a\}.$$

Example 2. Find the parts defined above for the conic

$$\{(x,y) \mid 9x^2 - 16y^2 - 18x - 64y = 199\}.$$

Solution. We first determine that under the translation

$$X = x - 1, \quad Y = y + 2$$

the conic is also the graph of

$$\left\{(X,Y) \,\middle|\, \frac{X^2}{16} + \frac{Y^2}{-9} = 1\right\}.$$

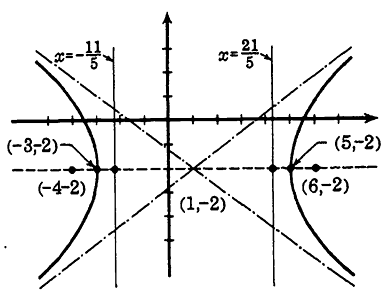


Figure 15

Thus, $a^2 = 16$, $a^2(1 - e^2) = -9$, $1 - e^2 = -\frac{9}{16}$, and $e = \frac{5}{4}$. Since $1 < e$ the graph is a hyperbola. Now $ae = 5$ and $a/e = \frac{16}{5}$. The desired information in terms of the XY-system and then changed to the xy-system is:

	XY	xy
center	$(0,0)$	$(1,-2)$
foci	$(\pm 5,0)$	$(6,-2), (-4,-2)$
directrices	$\{(X,Y) \mid X = \pm\frac{16}{5}\}$	$\{(x,y) \mid x = \frac{21}{5} \text{ or } x = -\frac{11}{5}\}$
vertices	$(\pm 4,0)$	$(5,-2), (-3,-2)$
asymptotes	$\{(X,Y) \mid Y = \pm\frac{3}{4}X\}$	$\{(x,y) \mid 3x - 4y = 11 \text{ or } 3x + 4y = -5\}$

Now that set notation has instilled the concept of sets and their geometric representations, the use of set notation will be subdued. Unless there is danger of being misunderstood, a mathematician would not write $\{(x,y) \mid 3x - 4y = 11\}$, but only $3x - 4y = 11$ and even say "The line $3x - 4y = 11$." He means, of course, "The line consisting of the set of all points in the coordinate plane whose coordinates satisfy the equation." We will continue to use set notation to introduce some new concepts or when we want to be unequivocal about what is meant.

PROBLEMS

1. Find the eccentricity, center, foci, directrices, vertices, the ends of the minor axis in case of an ellipse, and the asymptotes in case of a hyperbola.

a. $\dfrac{x^2}{6^2} - \dfrac{y^2}{8^2} = 1.$

b. $\dfrac{x^2}{8^2} - \dfrac{y^2}{6^2} = 1.$

c. $\dfrac{x^2}{8^2} + \dfrac{y^2}{10^2} = 1.$

d. $\dfrac{x^2}{10^2} + \dfrac{y^2}{8^2} = 1.$

e. $\dfrac{(x-2)^2}{5^2} + \dfrac{(y+3)^2}{4^2} = 1.$

f. $\dfrac{(x-2)^2}{9} - \dfrac{(y+3)^2}{16} = 1.$

g. $16y^2 - 9x^2 + 36x + 128y + 76 = 0.$

h. $25x^2 + 16y^2 + 50x + 64y = 311.$

2. Find numbers h, k, $a > 0$ and $e > 0$ such that

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2(1-e^2)} = 1 \quad \text{or} \quad \frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{a^2(1-e^2)} = 1$$

represents a conic with

a. Center $(0,0)$, focus $(\frac{3}{4},0)$, and vertex $(1,0)$.

b. Center $(0,0)$, focus $(\frac{4}{3},0)$, and vertex $(1,0)$.

c. Center $(2, -1)$, vertex $(2,4)$, end of minor axis $(-1, -1)$.

d. Center $(-2,3)$, vertex $(-2,5)$, directrix $\{(x,y) \mid y = 4\}$.

e. Asymptotes $\{(x,y) \mid 4x - 3y = 17 \text{ or } 4x + 3y = -1\}$, vertex $(-1,-3)$.

f. Asymptotes $\{(x,y) \mid 3y = \pm 4x\}$, vertex $(8,0)$.

3. For $0 < e < 1$ and $a > 0$, let (x,y) be a point such that its distance to the point $(ae,0)$ plus its distance to the point $(-ae,0)$ is $2a$ units. Prove the point (x,y) is on the ellipse with foci $(\pm ae,0)$ and vertices $(\pm a,0)$.

4. For $1 < e$ and $a > 0$, let (x,y) be a point such that the difference of its distances to the points $(\pm ae,0)$ is in absolute value $2a$ units. Prove the point (x,y) is on the hyperbola with foci $(\pm ae,0)$ and vertices $(\pm a,0)$.

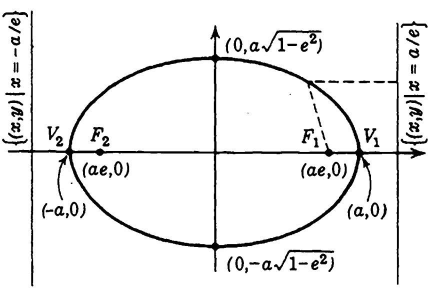
5. A rod with ends called A and B has a point C marked on it. The rod is moved so A stays on the x-axis and B on the y-axis. Show that C describes a circle if $AC = BC$ and an ellipse if $AC \neq BC$.

6. Show that $\{(x,y) \mid 1 \leq x \leq 2 \text{ and } -\sqrt{x^2 - 1} \leq y \leq \sqrt{x^2 - 1}\}$ is exactly the same set as $\{(x,y) \mid -\sqrt{3} \leq y \leq \sqrt{3} \text{ and } \sqrt{y^2 + 1} \leq x \leq 2\}$.

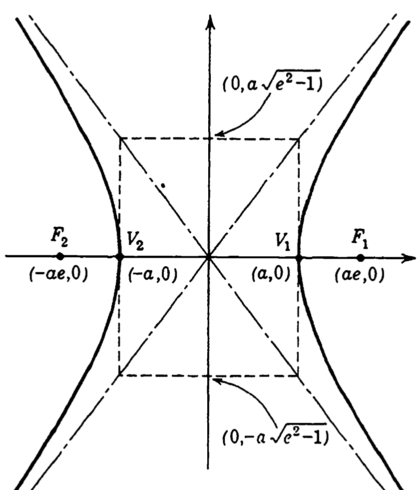
7. The eccentricities and lengths of the semi-major axes of the orbits of the planets Neptune and Pluto are:

	e	a
Neptune	0.0082	$2{,}793.5 \times 10^6$ mi
Pluto	0.25	$3{,}680 \times 10^6$ mi

Show that Pluto is sometimes closer than Neptune to the sun. [Note: It will be in 1989. See the *Encyclopaedia Britannica* (under Pluto).]



Ellipse $0<e<1$



Hyperbola $e>1$

CHAPTER 2

Limits and Derivatives

The most striking difference between calculus on one hand and algebra or geometry on the other is the degree of dependence upon the notion of a limit. Algebra and geometry are not devoid of limits (e.g., the sum of a geometric series with common ratio in absolute value less than 1 or the area of a circle), but limits play a minor role here; whereas, in calculus limits are basic to and permeate the whole subject.

Calculus is traditionally divided into two main parts; one primarily concerned with derivatives of functions (introduced in this chapter), the other with integrals of functions (Chap. 6). This chapter, then, begins the study of calculus.

16. Limit of a Function

For c a number, L a number, ϵ a positive number, and δ a positive number, then the set

$$\{(x,y) \mid \text{both } |x - c| < \delta \text{ and } |y - L| < \epsilon\}$$

is a rectangle with center (c,L), width 2δ units, and altitude 2ϵ units. The set

$$\{(x,y) \mid 0 < |x - c| < \delta \text{ and } |y - L| < \epsilon\}$$

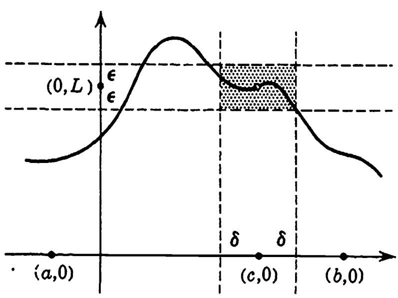


Figure 16.1

differs from the above rectangle only by not containing the vertical segment of length 2ϵ units through the center.

The function f whose graph is represented in Fig. 16.1 is such that $f(x)$ is defined at least for all x satisfying $a \leq x < c$ or $c < x \leq b$ and $f(c)$ may or may not be defined. For x close to c the value of $f(x)$ is close to L; that is, given any positive number ϵ there is a positive number δ (as illustrated in Fig. 16.1) such that

$$\{(x,y) \mid 0 < |x - c| < \delta \text{ and } y = f(x)\} \subset \{(x,y) \mid 0 < |x - c| < \delta \text{ and } |y - L| < \epsilon\}.$$

DEFINITION. *Given a function f, a number c, and a number L, the function f is said to have* **limit L at c**:

1. *If for each number* $\delta > 0$ *the set* $\{x \mid 0 < |x - c| < \delta\}$ *contains numbers in the domain of f, and*
2. *If given any positive number* ϵ *no matter how small, there is a positive number* δ *such that*

$$\{(x,y) \mid 0 < |x - c| < \delta \text{ and } y = f(x)\} \subset \{(x,y) \mid 0 < |x - c| < \delta \text{ and } |y - L| < \epsilon\};$$

that is, whenever $0 < |x - c| < \delta$, *then* $|f(x) - L| < \epsilon$.

Example 1. Show that the function f has limit 2 at $c = 1$, where

$$f(x) = \frac{1 - x}{1 - \sqrt{x}}, \text{ for } x > 0 \text{ and } x \neq 1.$$

Solution. 1. Notice that $f(1)$ is not defined, but $f(x)$ is defined if $0 \leq x < 1$ or $1 < x$. Also, notice that if $0 \leq x < 1$ or $1 < x$, then

$$\frac{1 - x}{1 - \sqrt{x}} = \frac{1 - x}{1 - \sqrt{x}} \cdot \frac{1 + \sqrt{x}}{1 + \sqrt{x}}$$

$$= \frac{(1 - x)(1 + \sqrt{x})}{1 - x} = 1 + \sqrt{x}.$$

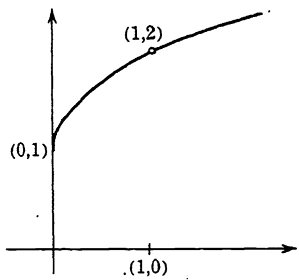


Figure 16.2

Hence,
$f = \{(x,y) \mid 0 \leq x < 1 \text{ or } 1 < x \text{ and } y = 1 + \sqrt{x}\}$
and this was used to obtain Fig. 16.2.

2. Let ϵ be any number such that $0 < \epsilon < 1$. We first find the point of f with ordinate $2 + \epsilon$:

$$2 + \epsilon = 1 + \sqrt{x}, \quad \sqrt{x} = 1 + \epsilon, \quad x = (1 + \epsilon)^2,$$

so that $((1 + \epsilon)^2, 2 + \epsilon)$ is the point. In the same way the point on the graph with ordinate $2 - \epsilon$ is the point $((1 - \epsilon)^2, 2 - \epsilon)$. Now $(1 - \epsilon)^2 < 1 < (1 + \epsilon)^2$ and if either $(1 - \epsilon)^2 < x < 1$ or $1 < x < (1 + \epsilon)^2$, then $2 - \epsilon < f(x) < 2 + \epsilon$. We thus take for δ the smaller of

$$(1 + \epsilon)^2 - 1 = 2\epsilon + \epsilon^2 \quad \text{and} \quad 1 - (1 - \epsilon)^2 = 2\epsilon - \epsilon^2$$

and hence take $\delta = 2\epsilon - \epsilon^2$ (which is positive since $0 < \epsilon < 1$) and have that

$$\left\{(x,y) \,\middle|\, 0 < |x - 1| < \delta \quad \text{and} \quad y = \frac{1 - x}{1 - \sqrt{x}}\right\} \subset \{(x,y) \mid 0 < |x - 1| < \delta \quad \text{and} \quad |y - 2| < \epsilon\};$$

that is,

$$\text{if} \quad 0 < |x - 1| < \delta, \quad \text{then} \quad |f(x) - 2| < \epsilon.$$

Consequently, from the definition, 2 is a limit of f at $c = 1$.

A very simple limit to prove, but one used repeatedly, is

$$\lim_{x \to c} x = c.$$

With f defined by $f(x) = x$ we are to show that "Given an $\epsilon > 0$ there exists a δ such that if $0 < |x - c| < \delta$ then $|f(x) - c| < \epsilon$." Merely choose $\delta = \epsilon$.

THEOREM 16. *Let f be a function and c a number such that f has a limit at c, and let L be this limit. Then f has no other limit at c.*

PROOF. Assume there is a second limit L_1 of f at c. Then $L_1 \neq L$ so $|L - L_1| > 0$ and $\frac{1}{2}|L - L_1| > 0$. Choose $\epsilon = \frac{1}{2}|L - L_1|$ and corresponding to this particular positive number ϵ let $\delta > 0$ be such that

$$\text{if} \quad 0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \epsilon$$

and let $\delta_1 > 0$ be such that

$$\text{if} \quad 0 < |x - c| < \delta_1, \quad \text{then} \quad |f(x) - L_1| < \epsilon.$$

Let x^* be such that both $0 < |x^* - c| < \delta$ and $0 < |x^* - c| < \delta_1$ and note that both $|f(x^*) - L| < \epsilon$ and $|f(x^*) - L_1| < \epsilon$. Hence,

$$|L - L_1| \leq |L - f(x^*)| + |f(x^*) - L_1| < 2\epsilon = |L - L_1|$$

which says that $|L - L_1| < |L - L_1|$ and this cannot be true. The assumption, "There is a second limit of f at c" is thus incorrect, and the theorem is proved.

NOTATION AND TERMINOLOGY. *Following convention, we shall write*

$$L = \lim_{x \to c} f(x) \tag{1}$$

to mean "The limit at c of this function f exists and is L." Also, we write

$$\lim_{x \to c} f(x) \quad \textit{does not exist}$$

to mean "For this function f and this number c there is no number L which satisfies the condition 1 and 2 of the above definition."

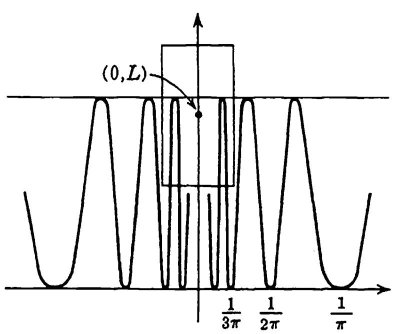


Figure 16.3

It is customary to read (1) as "L is the limit of $f(x)$ as x approaches c." Also, a common alternative to (1) is $f(x) \to L$ as $x \to c$.

Example 2. For $f = \{(x,y) \mid y = \sin^2(1/x)\}$ prove that

$$\lim_{x \to 0} f(x) \quad \text{does not exist.}$$

Solution. Let L be an arbitrary number. (The point $(0,L)$ is pictured in Fig. 16.3 as if L were about 0.9.) We shall prove that this function f does not have limit L at $c = 0$ by showing "There is a number $\epsilon > 0$ such that for δ an arbitrary positive number

$$\{(x,y) \mid 0 < |x - 0| < \delta \quad \text{and} \quad y = \sin^2(1/x)\} \tag{1}$$

is not a subset of

$$\{(x,y) \mid 0 < |x - 0| < \delta \quad \text{and} \quad |y - L| < \epsilon\}.\text{"} \tag{2}$$

No matter how small $\delta > 0$ is taken, the graph (1) will have points on the x-axis and points one unit above the x-axis. Hence, for any positive number $\epsilon < \frac{1}{2}$ there are points of (1) not in (2).

17. Limit Theorem

Given two functions f and g, the **sum** of f and g is the function defined by

$$f + g = \{(x,y) \mid y = f(x) + g(x)\}.$$

Hence, the domain of $f + g$ is the intersection of the domain of f and the domain of g. The **product** of f and g is the function

$$f \cdot g = \{(x,y) \mid y = f(x)g(x)\}$$

and the quotient function f/g is defined in a similar way, but note that the domain of f/g is the intersection of the domains of f and g diminished by those numbers x for which $g(x) = 0$.

THEOREM 17. *Let f and g be functions whose limits at c exist:*

$$\lim_{x\to c} f(x) = L_1 \quad \textit{and} \quad \lim_{x\to c} g(x) = L_2,$$

and for every number $\delta > 0$ the domain of f, the domain of g, and $\{x \mid 0 < |x - c| < \delta\}$ have numbers in common. Then the limits at c of the sum and product functions exist and

$$\text{I.} \qquad \lim_{x\to c} [f(x) \pm g(x)] = L_1 \pm L_2,$$

$$\text{II.} \qquad \lim_{x\to c} [f(x) \cdot g(x)] = L_1 \cdot L_2.$$

For $L_2 \neq 0$ the limit at c of the quotient function exists and

$$\text{III.} \qquad \lim_{x\to c} \frac{1}{g(x)} = \frac{1}{L_2} \quad \textit{and} \quad \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

Also, for $L_2 > 0$ and n a positive integer

$$\text{IV.} \qquad \lim_{x\to c} \sqrt[n]{g(x)} = \sqrt[n]{L_2}.$$

For k a constant then from I *and* II (*by setting* $f(x) = k$)

V. $\qquad \lim\limits_{x\to c} [k + g(x)] = k + L_2$ *and* $\lim\limits_{x\to c} [k\, g(x)] = k\, L_2.$

PROOF of I. Let ϵ be an arbitrary positive number. Since it is given that f has limit L_1 at c, choose $\delta_1 > 0$ corresponding to $\epsilon/2$ such that

$$\text{if} \quad 0 < |x - c| < \delta_1, \quad \text{then} \quad |f(x) - L_1| < \epsilon/2$$

and, since g has limit L_2 at c, choose $\delta_2 > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta_2, \quad \text{then} \quad |g(x) - L_2| < \epsilon/2.$$

Let δ be the smaller of δ_1 and δ_2. Hence, $\delta > 0$ and for x a number such that $0 < |x - c| < \delta$, it follows that both $|f(x) - L_1| < \epsilon/2$ and $|g(x) - L_2| < \epsilon/2$ so that

$$\begin{aligned} |f(x) + g(x) - (L_1 + L_2)| &= |(f(x) - L_1) + (g(x) - L_2)| \\ &\le |f(x) - L_1| + |g(x) - L_2| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Hence, corresponding to the arbitrary positive number ϵ, the existence of a positive number δ has been shown such that

$$\text{if} \quad 0 < |x - c| < \delta, \quad \text{then} \quad |f(x) + g(x) - (L_1 + L_2)| < \epsilon.$$

Hence, the function $f + g$ has limit $L_1 + L_2$ at c. In a similar way, the function $f - g$ has limit $L_1 - L_2$ at c.

This proof was given here to demonstrate the method of proof. Proofs of II–V are given in Appendix A1, but the results will be used whenever convenient. As an example, we establish the existence and obtain the value of $\lim\limits_{x\to 3} (5 + \sqrt{x^2 + 4^2})$. The usual procedure is to write

$$\begin{aligned} \lim_{x\to 3} (5 + \sqrt{x^2 + 4^2}) &= 5 + \lim_{x\to 3} \sqrt{x^2 + 4^2} = 5 + \sqrt{\lim_{x\to 3} (x^2 + 4^2)} \\ &= 5 + \sqrt{\left(\lim_{x\to 3} x\right)^2 + 4^2} = 5 + \sqrt{\left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} x\right) + 4^2} \\ &= 5 + \sqrt{3^2 + 4^2} = 10, \end{aligned}$$

not knowing the existence of any, except the last of these limits as it is written, but then mentally to reverse the steps to justify the existence of each limit and equality by using I, II, III, IV, or V.

The use of III is restricted to quotient functions whose denominator function has limit different from zero. If, however, both numerator and denominator functions have limit 0, it may be possible to determine the limit.

Example. Show that $\lim_{x\to 3} \dfrac{5 - \sqrt{x^2 + 4^2}}{x^2 - 9} = -\dfrac{1}{10}$.

Solution. Both numerator and denominator functions have limit 0. However,

$$\begin{aligned}\frac{5 - \sqrt{x^2 + 4^2}}{x^2 - 9} &= \frac{5 - \sqrt{x^2 + 4^2}}{x^2 - 9}\,\frac{5 + \sqrt{x^2 + 4^2}}{5 + \sqrt{x^2 + 4^2}}\\ &= \frac{9 - x^2}{(x^2 - 9)(5 + \sqrt{x^2 + 4^2})}\\ &= \frac{-1}{5 + \sqrt{x^2 + 4^2}}, \quad \text{for } x \neq \pm 3.\end{aligned}$$

In this form the denominator has limit 10 at 3 so that

$$\lim_{x\to 3} \frac{5 - \sqrt{x^2 + 4^2}}{x^2 - 9} = \lim_{x\to 3} \frac{-1}{5 + \sqrt{x^2 + 4^2}} = -\frac{1}{10}.$$

PROBLEMS

1. Using Theorem 17 find each of the following limits:

a. $\lim_{x\to 2} (3x^3 - 4x^2 + 2)$.

b. $\lim_{x\to -2} (5x^3 - 2x + 3)$.

c. $\lim_{x\to -3} \dfrac{x^2 - 4x + 1}{3x - 4}$.

d. $\lim_{x\to 0} \dfrac{x^2 - 4x + 1}{3x - 4}$.

e. $\lim_{t\to 5} (2t - 6/t)$.

f. $\lim_{h\to 5} (h + 6h^2 - 4)$.

g. $\lim_{h\to 0} \dfrac{6 + \sqrt{h^2 - 3h + 4}}{2 - h}$.

h. $\lim_{x\to 2} \dfrac{\sqrt{5x^3 + 3x + 3}}{x^2 - 5}$.

2. Establish each of the following:

a. $\lim_{x\to 1} \dfrac{1 - x^2}{2 - \sqrt{x^2 + 3}} = 4$.

b. $\lim_{x\to -1} \dfrac{1 + x}{2 - \sqrt{x^2 + 3}} = 2$.

c. $\lim_{x\to +1} \dfrac{1 - x}{2 - \sqrt{x^2 + 3}} = 2$.

d. $\lim_{x\to 2} \dfrac{\sqrt{x + 7} - 3}{\sqrt{x + 2} - 2} = \dfrac{2}{3}$.

e. $\lim_{h\to 0} \dfrac{\sqrt{6 + h} - \sqrt{6}}{h} = \dfrac{1}{2\sqrt{6}}$.

f. $\lim_{h\to 0} \dfrac{\sqrt[3]{6 + h} - \sqrt[3]{6}}{h} = \dfrac{1}{3(\sqrt[3]{6})^2}$.

g. $\lim_{x\to -1} \dfrac{3 - \sqrt{x^2 + x + 9}}{x^3 + 1} = \dfrac{1}{18}$.

3. Use Theorem 17 to establish each of the following.

a. $\lim_{x\to 1} \dfrac{1 - x^2}{2 + \sqrt{x^2 + 3}} = 0.$

b. $\lim_{x\to 2} \dfrac{x - 2}{x^2 - \sqrt{14 + x}} = \dfrac{8}{31}.$

c. $\lim_{x\to 1} \dfrac{x - 2}{x^2 - \sqrt{14 + x}} = \dfrac{1}{\sqrt{15} - 1}$

d. $\lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{1}{2 + h} - \dfrac{1}{2}\right) = -\dfrac{1}{4}.$

e. $\lim_{h\to -1} \dfrac{1}{h}\left(\dfrac{1}{2 + h} - \dfrac{1}{2}\right) = -\dfrac{1}{2}.$

f. $\lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{1}{\sqrt{2 + h}} - \dfrac{1}{\sqrt{2}}\right) = -\dfrac{1}{4\sqrt{2}}.$

4. By using the definition of a limit prove that the function

a. $f = \{(x,y) \mid 2x + 3y = 15\}$ has limit $\frac{11}{3}$ at 2.

b. $f = \{(x,y) \mid y = \sqrt{x}\}$ has limit 2 at 4.

c. $f = \{(x,y) \mid y$ is the greatest integer less than $x\}$ has no limit at 2, but has limit 2 at 2.1 and has limit 1 at 1.9.

18. Limits of Trigonometric Functions

By inscribing and circumscribing sequences of polygons, Archimedes' (c. 250 B.C.) "Method of Exhaustion" led him to conclude that the circumferences c and C, and the areas k and K of circles or radii r and R are such that

$$\frac{c}{2r} = \frac{C}{2R} = \frac{k}{r^2} = \frac{K}{R^2}.$$

This constant ratio is denoted by π. The method of exhaustion is the genesis of the concept of definite integrals (Ch. 6). By means of definite integrals it can be proved, without using the next theorem, that the areas of two sectors of a circle are to each other as their central angles. It then follows that a circular sector of radius r units and central angle α radians has

(1) $$\text{area} = \frac{\alpha}{2} r^2 \text{ units}^2.$$

(see page 578). This fact is used in proving the next theorem.

THEOREM 18. *With angles measured in radians*

I. $$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{1 - \cos x}{x} = 0,$$

II. $$\lim_{x\to 0} \sin x = 0, \quad \text{and } \lim_{x\to 0} \cos x = 1.$$

PROOF. Let x be a number such that $0 < x < \pi/2$ and construct the angle of x radians in standard position. Let $P(c,s)$ be the point where the terminal side of this angle cuts the circle with center at the origin and radius 1. As illustrated in Fig. 18, sector OBC has radius c and thus has area $\frac{1}{2}c^2x$. Sector OAP has area $\frac{1}{2}1^2x = \frac{1}{2}x$. Triangle OAP has base 1 and altitude s. Since area sector $OBC <$ area $\triangle\ OAP <$ area sector OAP, then

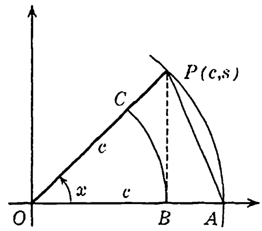


Figure 18

$$\frac{1}{2}c^2x < \frac{1}{2}s < \frac{1}{2}x; \quad \text{i.e. } c^2x < s < x.$$

Hence $s^2 < x^2$ and thus $1 - c^2 = s^2 < x^2$ so that $1 - x^2 < c^2$. Therefore $(1 - x^2)x < s < x$. Since $s = \sin x$, then upon dividing by $x > 0$

$$1 - x^2 < \frac{\sin x}{x} < 1 \quad \text{for} \quad 0 < x < \frac{\pi}{2}.$$

But $(-x)^2 = x^2$ and $\dfrac{\sin(-x)}{-x} = \dfrac{-\sin x}{-x} = \dfrac{\sin x}{x}$ so that

$$1 - x^2 < \frac{\sin x}{x} < 1 \quad \text{for} \quad 0 < |x| < \frac{\pi}{2}. \tag{2}$$

Thus $1 = \lim\limits_{x\to 0}(1 - x^2) \le \lim\limits_{x\to 0}\dfrac{\sin x}{x} \le 1$ so the first limit in I. is seen to hold.†
Consequently

$$\lim_{x\to 0}\sin x = \lim_{x\to 0} x\frac{\sin x}{x} = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right) = 0\cdot 1 = 0$$

which is the first limit of II. Since $0 < \cos x = \sqrt{1 - \sin^2 x}$ for $0 < |x| < \pi/2$, the second limit of II. follows.

Since $\sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$ we have

$$\lim_{x\to 0}\frac{1 - \cos x}{x} = \lim_{x\to 0}\frac{\sin^2 x}{x(1 + \cos x)}$$
$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{1 + \cos x}\right) = 1\cdot\frac{0}{2} = 0$$

which is the second limit of I. and finishes the proof.

Example 1. $\displaystyle \lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\frac{1}{\cos x}\right) = 1\cdot\frac{1}{1} = 1.$

† To apply the definition of a limit directly, choose $0 < \epsilon < \pi/2$ arbitrarily and set $\delta = \sqrt{\epsilon}$. Hence, from (2), whenever $0 < |x| < \delta$ then

$$1 - \epsilon < \frac{\sin x}{x} < 1 < 1 + \epsilon; \text{ i.e. } \left|\frac{\sin x}{x} - 1\right| < \epsilon.$$

Merely by a change of notation, II may be written as

$$\lim_{h\to 0} \sin h = 0 \quad \text{and} \quad \lim_{h\to 0} \cos h = 1.$$

Example 2. Prove that for x any given number, then

$$\lim_{h\to 0} \cos (x + h) = \cos x.$$

Solution. We write the following (then check existence in reverse order)

$$\begin{aligned}\lim_{h\to 0} \cos (x + h) &= \lim_{h\to 0} (\cos x \cos h - \sin x \sin h)\\ &= \lim_{h\to 0} (\cos x \cos h) - \lim_{h\to 0} (\sin x \sin h)\\ &= \cos x \lim_{h\to 0} \cos h - \sin x \lim_{h\to 0} \sin h\\ &= (\cos x)(1) - (\sin x)(0) = \cos x.\end{aligned}$$

19. Composition Functions

Let f and u be functions and let F be the function

$$F = \{(x,y) \mid y = f(u(x))\}.$$

The function F is called the **composition** of f upon u. Notice that the domain of F is

$$\{x \mid x \text{ is in the domain of } u \text{ and } u(x) \text{ is in the domain of } f\}.$$

Thus, for x in the domain of F, then not only is x in the domain of u, but $u(x)$ is in the domain of f and

$$F(x) = f(u(x)).$$

For example if f and u are the functions

$$f = \left\{(t,y) \mid y = \frac{\sin t}{t}\right\} \quad \text{and} \quad u = \{(x,t) \mid t = x^2 - 1\}$$

then the composition function F of f upon u is

$$F = \left\{(x,y) \mid y = \frac{\sin (x^2 - 1)}{x^2 - 1}\right\} \quad \text{with domain} \quad \{x \mid x \neq \pm 1\}.$$

Hence, for x in the domain of this function F, then

$$F(x) = \frac{\sin (x^2 - 1)}{x^2 - 1} \quad \text{for} \quad x \neq \pm 1.$$

THEOREM 19. *With c, a, and L numbers, let f and u be functions such that*

$$\lim_{t\to a} f(t) = L,$$

let u be a function whose range, except possibly for $u(c)$, is in the domain of f and is such that

(1) $$u(x) \neq a \quad \textit{for} \quad x \neq c, \quad \textit{but}$$
$$\lim_{x\to c} u(x) = a.$$

Then the composition function of f upon u also has limit L at c:

(2) $$\lim_{x\to c} f(u(x)) = L.$$

This theorem is proved in Appendix A1. The following example illustrates its use.

Example. Find $\lim\limits_{x\to 1} \dfrac{\sin (x^2 - 1)}{x^2 - 1}$

Solution. Make the identifications

$$f(t) = \frac{\sin t}{t}, \quad u(x) = x^2 - 1 \quad \text{and} \quad c = 1.$$

Since $\lim\limits_{x\to 1} (x^2 - 1) = 0$, think of $a = 0$. Then from

$$\lim_{t\to 0} \frac{\sin t}{t} = 1$$

it follows from Theorem 19 that also

$$\lim_{x\to 1} \frac{\sin (x^2 - 1)}{x^2 - 1} = 1.$$

By a similar procedure, merely involving changes of letters, it should be seen that

$$\lim_{y\to 2} \frac{\sin (y^2 - 4)}{y^2 - 4} = 1, \quad \lim_{z\to -1} \frac{\sin (z + 1)}{z + 1} = 1, \quad \text{and} \quad \lim_{h\to 0} \frac{\sin (h/2)}{h/2} = 1.$$

PROBLEMS

1. Find each of the following limits:

a. $\lim\limits_{h\to 0} \dfrac{\sin 3h}{h} = 3.$

b. $\lim\limits_{h\to 0} \dfrac{3 \sin 5h}{h} = 15.$

c. $\lim\limits_{h\to 0} \dfrac{\sin^2 h}{h} = 0.$

d. $\lim\limits_{h\to 0} \dfrac{\sin^2 h}{h^2} = 1.$

e. $\lim\limits_{h\to 0} h \cot h = 1.$

f. $\lim\limits_{h\to 0} \dfrac{\sin^2 h - 2 \sin h}{h} = -2.$

g. $\lim\limits_{h\to 0} \dfrac{1 - \cos h}{h^2} = \dfrac{1}{2}.$

h. $\lim\limits_{h\to 0} \dfrac{\tan^2 h}{h \sin h} = 1.$

i. $\lim\limits_{h\to 0} \dfrac{\sin (h/2)}{\tan h} = \dfrac{1}{2}.$

j. $\lim\limits_{h\to 0} \dfrac{\sqrt{1 + \sin h} - 1}{h} = \dfrac{1}{2}.$

2. In each of the following, find the limit at c of the composition function of f upon u.

a. $f(t) = \sqrt{t+3}$, $u(x) = \dfrac{\sin x}{x}$; $c = 0$. $\left[\text{Hint: } \lim\limits_{x\to 0}\sqrt{\dfrac{\sin x}{x} + 3}.\right]$

b. $f(t) = \dfrac{\sin t}{t}$, $u(x) = 3x^2 + 6x - 9$; $c = 1$.

c. $f(t) = \dfrac{5t+4}{t-2}$, $u(x) = \dfrac{2\sin x - 3}{\sin x - 1}$; $c = 0$.

d. $f(t) = \dfrac{\sin 2t}{\tan t}$, $u(x) = 3x - 1$; $c = \frac{1}{3}$.

e. $f(t) = 3t - 2$, $u(x) = \dfrac{\sin 2x}{\tan x}$; $c = 0$.

3. In each of the following show that $\lim\limits_{x\to c} u(x)$ does not exist, but that the limit exists at c of the composition function of f upon u. (Note: This shows that the conditions of Theorem 19 are sufficient but not necessary.)

a. $f(t) = \dfrac{3t+4}{2t-6}$, $u(x) = \dfrac{3x+5}{x-3}$; $c = 3$.

b. $f(t) = \dfrac{5t+3}{1+t}$, $u(x) = \dfrac{\sin x}{1-\sin x}$; $c = \dfrac{\pi}{2}$.

c. $f(t) = \dfrac{2t-5}{4+t}$, $u(x) = \dfrac{3+\cos x}{\sin x}$; $c = 0$.

4. In terms of an appropriate composition function, justify each of the following:

a. $\lim\limits_{x\to 2} \dfrac{\sin x(x-2)}{x(x-2)} = 1$ and $\lim\limits_{x\to 0} \dfrac{\sin x(x-2)}{2(x-2)} = 0$.

b. $\lim\limits_{x\to -3} \dfrac{\sin\{(x+3)\pi/4\}}{x+3} = \dfrac{\pi}{4}$ and $\lim\limits_{x\to 3} \dfrac{\sin\{(x+3)\pi/4\}}{x+3} = -\dfrac{1}{6}$.

c. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+\sin x} - 1}{x} = \dfrac{1}{2}$ and $\lim\limits_{x\to 0} \dfrac{(1+\sin x)^2 - 1}{x} = 2$.

d. $\lim\limits_{h\to 0} \dfrac{1-\cos 2h}{h} = 0$ and $\lim\limits_{h\to 0} \dfrac{1-\cos 2h}{h^2} = 2$.

e. $\lim\limits_{x\to 1} \dfrac{x-1}{x} = 0$ and $\lim\limits_{x\to 1} \dfrac{x}{x-1}$ does not exist.

20. Continuous Functions

DEFINITION. *A function f is said to be continuous at c if*

$$\lim_{x\to c} f(x) = f(c). \tag{1}$$

Thus, a function f is continuous at c if the three questions:

1. Is $f(c)$ defined, i.e., is c in the domain of f?
2. Does $\lim_{x\to c} f(x)$ exist?
3. Does $\lim_{x\to c} f(x) = f(c)$?

all have affirmative answers, but if any of the answers is "No" then f is not continuous at c. Turn to page 83 for illustrations of "No" answers.

Since $\lim_{x\to c} x = c$, it follows that f is continuous at c if and only if

$$\lim_{x\to c} f(x) = f\left(\lim_{x\to c} x\right), \tag{2}$$

i.e., if and only if "f and $\lim_{x\to c}$ may be interchanged."

The definition of a limit may be combined with the above definition to yield:

A function f is continuous at c provided c is in the domain of f and in addition, corresponding to an arbitrary positive number ϵ there is a positive number δ such that

$$\textit{whenever} \quad |x - c| < \delta, \quad \textit{then} \quad |f(x) - f(c)| < \epsilon.$$

For example, the function f defined by $f(x) = 2|x|$ is continuous at c for every number c, since for $\epsilon > 0$ then whenever $|x - c| < \epsilon/2$ it follows that

$$\big|\, 2|x| - 2|c| \,\big| = 2 \,\big|\, |x| - |c| \,\big| \le 2|x - c| < 2\frac{\epsilon}{2} = \epsilon.$$

The greatest-integer function

$$[\;] = \{(x,y) \mid y \text{ is the greatest integer less than or equal to } x\}$$

is continuous at c if c is not an integer, but is not continuous at c if c is an integer.

The following theorem gives conditions under which a limit and a continuous function may be interchanged. For a proof see Appendix A1.

THEOREM 20.1. *For c and a numbers and for f and u functions such that*

$$\lim_{x\to c} u(x) = a$$

and such that f is continuous at a, then

$$\lim_{x\to c} f(u(x)) = f(a) = f\left(\lim_{x\to c} u(x)\right).$$

Stated loosely, "If, in a composition function, the outside function is continuous, then the limit and the outside function may be interchanged, provided the inside function has a limit."

Since $0 < (\sin x)/x < 1$ for $0 < |x| < \pi/2$ and since the square root function is continuous at 1, we may write

$$\lim_{x\to 0} \sqrt{\frac{\sin x}{x}} = \sqrt{\lim_{x\to 0} \frac{\sin x}{x}} = \sqrt{1} = 1,$$

but for the greatest-integer function [ ] we have

$$0 = \lim_{x\to 0} \left[\frac{\sin x}{x}\right] \neq \left[\lim_{x\to 0} \frac{\sin x}{x}\right] = 1.$$

Note that a function f is continuous at x if

$$\lim_{h\to 0} f(x+h) = f\left(\lim_{h\to 0} (x+h)\right) = f(x). \tag{1}$$

Thus, the cosine function is continuous at x for each number x (see Example 2 of Sec. 18) and in the same way the continuity of the sine function may be demonstrated.

THEOREM 20.2. *Given a number x and functions f and g continuous at x, then the sum function $s = f + g$ and the product function $p = fg$ are continuous at x, and provided $g(x) \neq 0$ the quotient function $q = f/g$ is continuous at x.*

PROOF. We write, checking limits and equalities in progress,

$$\begin{aligned}
s(x) &= f(x) + g(x) && \text{(from the definition of the function } s\text{)}\\
&= \lim_{h\to 0} f(x+h) + \lim_{h\to 0} g(x+h) && \text{(since } f \text{ and } g \text{ are continuous at } x\text{)}\\
&= \lim_{h\to 0} [f(x+h) + g(x+h)] && \text{(by Theorem 17 I)}\\
&= \lim_{h\to 0} s(x+h) && \text{(by the definition of the function } s\text{),}
\end{aligned}$$

which shows that the sum function is continuous at x.

In the same way (but using Theorem 17 II and 17 III, respectively) the product and quotient functions are continuous at x, provided that $g(x) \neq 0$, for the quotient function q.

Hence, for f continuous at x, the function $f^2 = f \cdot f$ is continuous at x, then $f^3 = f^2 \cdot f$ is continuous at x, and for n a positive integer $f^n = f^{n-1} \cdot f$ is continuous at x.

DEFINITION. *A function which is continuous at each number x in its domain is said to be a* **continuous function.**

Since the function $\{(x,y) \mid y = x\}$ is continuous it follows, for n a positive integer and for a any constant, that the function $\{(x,y) \mid y = ax^n\}$ is continuous. Now with $a_0, a_1, a_2, \cdots, a_n$ given numbers, the functions

$$\{(x,y) \mid y = a_0\},\ \{(x,y) \mid y = a_0 + a_1x\},\ \{(x,y) \mid y = a_0 + a_1x + a_2x^2\}, \cdots$$

$$\{(x,y) \mid y = a_0 + a_1x + \cdots + a_nx^n\}$$

is each in turn the sum of two continuous functions; that is, *any polynomial function is a continuous function.*

Let p be a positive integer, let n be any integer, and let f be the function $f = \{(x,y) \mid y = x^{n/p}\} = \{(x,y) \mid y = (\sqrt[p]{x})^n\}$; i.e.

$$f(x) = (\sqrt[p]{x})^n \tag{2}$$

whenever x is such that $\sqrt[p]{x}$ and $(\sqrt[p]{x})^n$ are real. For example, $\sqrt{-1}$ and $\sqrt[p]{-1}$ for p even are not real. Also "$x = 0$ and n negative" is excluded.

With $x_0 > 0$ and n, as well as p, positive

$$\lim_{x \to x_0} \sqrt[p]{x} = \sqrt[p]{x_0} \text{ from IV, page 51 and } \lim_{x \to x_0} (\sqrt[p]{x})^n = (\sqrt[p]{x_0})^n$$

by repeated use of II, page 51. This proves the first case of the following theorem and the other cases should be checked.

THEOREM 20.3. *With p a positive integer, the function defined by (2) is continuous at x in each of the cases:*

(i) $x > 0$ and $n > 0$.

(ii) x any number, $n > 0$, p odd.

(iii) $x \neq 0$, $n < 0$, p odd.

THEOREM 20.4. *Let x_0 be a number and f a function such that:*

1. *x_0 is in an open interval of the domain of f,*
2. *f is continuous at x_0, and*
3. $f(x_0) > 0$.

Then there are numbers x_1 and x_2 such that $x_1 < x_0 < x_2$ and such that f is positive on the open interval $I(x_1,x_2)$.

PROOF. From 1 above select $\delta_1 > 0$ so the interval $I(x_0 - \delta_1, x_0 + \delta_1)$ is in the domain of f. Corresponding to $\epsilon = (\frac{1}{2})f(x_0)$, which is positive by 3 above, let $\delta > 0$ be a number $\leq \delta_1$ and such that if x is any number satisfying

$$|x - x_0| < \delta \quad \text{then} \quad |f(x) - f(x_0)| < \tfrac{1}{2}f(x_0); \qquad \text{that is,}$$

$$f(x_0) - \tfrac{1}{2}f(x_0) < f(x) < f(x_0) + \tfrac{1}{2}f(x_0).$$

Hence, from the left hand inequality alone, f is positive on $I(x_0 - \delta, x_0 + \delta)$ which is the domain of f' since $\delta \le \delta_1$. By selecting $x_1 = x_0 - \delta$ and $x_2 = x_0 + \delta$ we have numbers as stated by the theorem to exist.

21. Tangents

With f a given function, let x and $x + h$ with $h \neq 0$ be in the domain of f. Then the points

$$(x,f(x)) \quad \text{and} \quad (x + h, f(x + h))$$

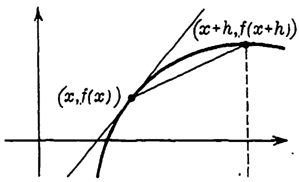


Figure 21

are on the graph of f and the line joining these points has

$$\text{slope} = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}.$$

DEFINITION. *Let f be a function and let x be a number in the domain of f. If whenever $|h|$ is sufficiently small the number $x + h$ is also in the domain of f and if*

$$\lim_{h\to 0} \frac{f(x + h) - f(x)}{h} \quad \textit{exists},$$

then the line with the value of this limit as slope and passing through the point $(x,f(x))$ is said to be **tangent** *to the graph of f at the point $(x,f(x))$.*

Example 1. Show that the graph of $y = \sqrt{x}$ has a tangent at the point of the graph having abscissa 2.

Solution. The point in question is $(2, \sqrt{2})$. Also, if $-2 < h < 0$ or $0 < h$, the point $(2 + h, \sqrt{2 + h})$ is on the graph. Moreover, by methods used previously

$$\lim_{h\to 0} \frac{\sqrt{2 + h} - \sqrt{2}}{h} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

Since this limit exists, the graph has a tangent at the designated point and this tangent has slope $m = \sqrt{2}/4$.

Notice that the line tangent to the graph of Example 1 at the point $(2,\sqrt{2})$ has slope $\sqrt{2}/4$, and thus has equation

$$y - \sqrt{2} = (\sqrt{2}/4)(x - 2).$$

Example 2. For the function $f = \{(x,y) \mid y = |x|\}$ show that the graph does not have a tangent at the origin.

Solution. Since

$$\frac{|0 + h| - |0|}{h} = \begin{cases} h/h = 1 & \text{if } h > 0 \\ -h/h = -1 & \text{if } h < 0, \end{cases}$$

the limit as h approaches zero does not exist.

THEOREM 21. *If the graph of a function has a tangent at a point, then the function is continuous at the abscissa of the point.*

PROOF. Let f be a function and x a number such that the tangent to the graph exists at the point $(x, f(x))$. Consequently

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exists. Let m be the value of this limit. Guided by

$$f(x+h)=f(x+h)-f(x)+f(x)=\frac{f(x+h)-f(x)}{h}h+f(x)$$

we write, knowing the existence of each limit as we write it,

$$\begin{aligned}f(x)=m\cdot 0+f(x)&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\lim_{h\to 0}h+f(x)\\&=\lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\cdot h\right\}+f(x)\\&=\lim_{h\to 0}\{f(x+h)-f(x)+f(x)\}=\lim_{h\to 0}f(x+h).\end{aligned}$$

But the fact that $\lim_{h\to 0}f(x+h)=f(x)$ means (see (1) of Sec. 20) that f is continuous at x, as we wished to prove.

In establishing the existence of a tangent to the graph of a function, and obtaining the slope of the tangent when it exists, it is only necessary to evaluate a limit of the form

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}. \tag{1}$$

Example 3. Evaluate (1) given that $f(x) = x^3 - 4$.

Solution. For x any number, $f(x) = x^3 - 4$. For x and h any numbers, then $x + h$ is a number and $f(x + h) = (x + h)^3 - 4$. Thus, for x any number and $h \neq 0$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{\{(x+h)^3-4\}-\{x^3-4\}}{h}\\&=\frac{\{x^3+3x^2h+3xh^2+h^3-4\}-\{x^3-4\}}{h}\\&=\frac{3x^2h+3xh^2+h^3}{h}=3x^2+3xh+h^2,\quad\text{for}\quad h\neq 0.\end{aligned}$$

Consequently, for this particular function and any number x,

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2.$$

PROBLEMS

1. Find an equation of the line tangent to the graph of each of the following equations at the points of the graph with the given properties:

   a. $y = \sqrt{x+1}$; point with $x = 1$, point with $x = 0$.
   b. $y = 1/x$; point with $x = 1$, point with $y = 2$.
   c. $y = 3x^2$; point with $x = 2$, point with $x = -2$.
   d. $y = 3x^2 - 4$; point with $x = 2$, point with $x = 0$.
   e. $y = 3x^2 - 2x$; point with $x = 2$, point with $x = \frac{1}{3}$.
   f. $y = 3x^2$; points with $y = 12$.
   g. $y = 3x^2 - 2x$; points with $y = 8$.

2. For each of the following functions f find the limit

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

   when it exists, and find the domain of numbers x for which it exists.

   a. $f(x) = 2x + 1$.
   b. $f(x) = x^2 - 1$.
   c. $f(x) = (x^2 - 1)^{-1}$, $|x| \neq 1$.
   d. $f(x) = \sqrt{x+3}$, $x \geq -3$.
   e. $f(x) = (x+3)^{-1/2}$, $x > -3$.
   f. $f(x) = x\sqrt{x+3}$, $x \geq -3$.
   g. $f(x) = x(x+3)^{-1/2}$, $x > -3$.
   h. $f(x) = \sqrt{x^2+3}$.
   i. $f(x) = \frac{1}{2}x^2 + 3x - 4$.
   j. $f(x) = x + \sqrt{x+1}$, $x \geq -1$.
   k. $f(x) = \sqrt{x^2-4} + \sqrt{x-1}$, $x \geq 2$.

3. Sketch $\{(x,y) \mid x > 0 \text{ and } y = x^2\}$ and $\{(x,y) \mid x > 0 \text{ and } y = \sqrt{x}\}$. Let (a,b) be a point on the first graph and find the slope of the tangent to this graph at this point. With the same numbers a and b, notice that the point (b,a) is on the other graph and now find the slope of the tangent to this graph at this point. Show that the two slopes are reciprocals of each other.

22. Velocity

A car going 50 miles in 2 hours is said to average 25 mi/hr. A particle moving along a coordinate line and having coordinate s_1 at time t_1 and coordinate s_2 at time t_2, with $t_2 \neq t_1$, is said to have

$$\text{average velocity } \frac{s_2 - s_1}{t_2 - t_1} = \frac{s_1 - s_2}{t_1 - t_2} \text{ (coord. units)/(time unit).}$$

For example, if the particle has coordinate 1 when $t = \frac{1}{3}$ and coordinate $\frac{1}{2}$ when $t = \frac{2}{3}$, then the average velocity during the time interval between $t = \frac{1}{3}$ and $t = \frac{2}{3}$ is

$$\frac{1 - \frac{1}{2}}{\frac{1}{3} - \frac{2}{3}} = -\frac{3}{2} \text{ (coord. units)/(time unit).}$$

DEFINITION. *Given a law of linear motion as a function s and given a number t in the domain of s, then for $h \neq 0$ and $t + h$ in the domain of s,*

$$(1) \qquad \frac{s(t + h) - s(t)}{h} \textit{ (coord. units)/(time unit)}$$

is the ***average velocity*** *during the time interval between time t and time $t + h$. Also, the limit (assumed to exist)*

$$(2) \qquad \lim_{h \to 0} \frac{s(t + h) - s(t)}{h} \textit{ (coord. units)/(time unit)}$$

is the ***instantaneous velocity*** *at time t.*

Example. Let distance be measured in feet and time in minutes and let the law of linear motion be

$$s(t) = \cos \pi t.$$

For t any number and $h \neq 0$ show that the average velocity between time t minutes and time $t + h$ minutes is

$$(3) \qquad -\frac{2}{h} \sin \pi \left(t + \frac{h}{2}\right) \sin \left(\frac{\pi}{2} h\right) \text{ ft/min}$$

and that the velocity at time t minutes is

$$(4) \qquad -\pi \sin \pi t \text{ ft/min.}$$

Solution. The average velocity, directly from (1), is

$$(5) \qquad \frac{s(t + h) - s(t)}{(t + h) - t} = \frac{\cos \pi(t + h) - \cos \pi t}{h} \text{ ft/min.}$$

This expression may be changed by using the trigonometric identity

$$\cos (\alpha + \beta) - \cos (\alpha - \beta) = -2 \sin \alpha \sin \beta.$$

Upon setting $\alpha + \beta = \pi(t + h)$ and $\alpha - \beta = \pi t$, it follows that $2\alpha = \pi(2t + h)$ and $2\beta = \pi h$ so the right side of (5) may be replaced by (3), giving the average velocity as

$$\frac{s(t + h) - s(t)}{h} = -\frac{2}{h} \sin \pi \left(t + \frac{h}{2}\right) \sin \left(\frac{\pi}{2} h\right) \text{ ft/min.}$$

To find the instantaneous velocity, write

$$\lim_{h\to 0} \frac{s(t+h)-s(t)}{h} = \lim_{h\to 0}\left\{-\frac{2}{h}\sin \pi\left(t+\frac{h}{2}\right)\sin\left(\frac{\pi}{2}h\right)\right\}$$

$$= -2\left\{\lim_{h\to 0}\sin \pi\left(t+\frac{h}{2}\right)\right\}\left\{\lim_{h\to 0}\frac{\sin\frac{\pi}{2}h}{h}\right\}$$

$$= -2\sin\left\{\lim_{h\to 0}\pi\left(t+\frac{h}{2}\right)\right\}\lim_{h\to 0}\left\{\frac{\pi}{2}\frac{\sin\frac{\pi}{2}h}{\frac{\pi}{2}h}\right\}$$

$$= -2\{\sin \pi t\}\left\{\frac{\pi}{2}\right\} = -\pi\sin \pi t \text{ ft/min.}$$

23. Derived Function

In obtaining tangents to curves and velocities in linear motions, an essential feature is the determination of such limits as

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad\text{and}\quad \lim_{h\to 0}\frac{s(t+h)-s(t)}{h}.$$

Later on it will be seen that limits of this form will have other applications. The study of such limits without specific interpretations of the functions involved is more inclusive than a development restricted by a physical or geometric meaning at each step.

DEFINITION. *Given a function f, the function f' defined by*

$$(1)\qquad f' = \left\{(x,y) \mid y = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right\}$$

is called the ***derived function*** *of f. For x in the domain of f', the number $f'(x)$ is called the* ***derivative*** *of f at x, and*

$$(1')\qquad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

In particular, if x is in the domain of f$'$, then certainly x is in the domain of f (since $f(x)$ must be defined), but moreover the limit in (1$'$) must exist. Hence, the domain of f' is a subset of (and may be all of) the domain of f.

Thus, for a function f, a number x is in the domain of the derived function f' if and only if

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ exists.}$$

Also, if x is in the domain of f', then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Example. Given that f is the function defined by

$$f(x) = \sqrt{x+3} \quad \text{for} \quad x \geq -3,$$

find the derived function f'. Also, determine the domain of f'.

Solution. $f' = \left\{(x,y) \mid y = \lim_{h\to 0} \dfrac{\sqrt{x+h+3} - \sqrt{x+3}}{h}\right\}$

$$= \left\{(x,y) \mid y = \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \cdot \frac{\sqrt{x+h+3} + \sqrt{x+3}}{\sqrt{x+h+3} + \sqrt{x+3}}\right\}$$

$$= \left\{(x,y) \mid y = \lim_{h\to 0} \frac{(x+h+3-(x+3))}{h(\sqrt{x+h+3} + \sqrt{x+3})}\right\}$$

$$= \left\{(x,y) \mid y = \lim_{h\to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}}\right\}$$

$$= \left\{(x,y) \mid y = \frac{1}{2\sqrt{x+3}}\right\}.$$

Thus, the domain of f is $\{x \mid x \geq -3\}$, whereas the domain of the derived function f' is $\{x \mid x > -3\}$.

THEOREM 23. *Given a function f, if x is in the domain of the derived function f', then f is continuous at x.*

PROOF. We must show that $\lim_{h\to 0} f(x+h) = f(x)$ for x in the domain of f'. We write

$$\lim_{h\to 0} f(x+h) = \lim_{h\to 0} \left\{\frac{f(x+h) - f(x)}{h} h + f(x)\right\}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} h + f(x) = f'(x)\cdot 0 + f(x) = f(x)$$

and then retrace all steps in reverse order to make sure that each limit exists and each equality holds.

PROBLEMS

1. Given, for t any number satisfying $0 \leq t \leq 2$ that t sec after a body is thrown in the air it will be $s(t) = 32t - 16t^2$ ft above the ground.

   a. Find the average velocity of the body during the time interval between $\frac{1}{2}$ and $\frac{3}{4}$ sec; between $\frac{5}{4}$ and $\frac{3}{2}$ sec; between $\frac{3}{4}$ and $\frac{5}{4}$ sec.

   b. For $0 < t < 2$ find the velocity of the body t sec after it is thrown.

c. Show that t sec after the body is thrown its velocity will be >0 if $0 < t < 1$, will be 0 if $t = 1$, and will be <0 if $1 < t < 2$.

d. How high above the ground is the body when its velocity is zero?

2. Each of the following represents a law of linear motion for $0 \le t \le 10$. In each case find the velocity at time t where

a. $s(t) = 6t - 4$.
b. $s(t) = 4 - 6t$.
c. $s(t) = (t + 1)^{-1}$.
d. $s(t) = \sqrt{t + 4}$.
e. $s(t) = (20 - t)^{-1/2}$
f. $s(t) = \sin \pi t$.

3. For each of the following definitions of the function f, find $f'(x)$.

a. $f(x) = x^2$.
b. $f(x) = x^3$.
c. $f(x) = x^4$.
d. $f(x) = x^{1/2}$.
e. $f(x) = x^{1/3}$.
f. $f(x) = x^{1/4}$.
g. $f(x) = x + x^3$.
h. $f(x) = x + \sqrt{x}$.
i. $f(x) = x + \sqrt{x + 3}$
j. $f(x) = x + 3 + \sqrt{x}$.
k. $f(x) = x\sqrt{x + 3}$.
l. $f(x) = (x + 3)\sqrt{x}$.

24. Derivative Theorems

For f a given function and x in the domain of the derived function f', then

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}.$$

There are several alternative notations for $f'(x)$, one of which is

$$D_x f(x).$$

The symbolism $D_x f(x)$ is usually read "The derivative of $f(x)$ with respect to x," but should be understood to indicate "The derivative of f at x"; that is,

$$D_x f(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}.$$

In the example of Sec. 23 it was shown that:

$$\text{If } f(x) = \sqrt{x + 3} \text{ for } x \ge -3, \text{ then } f'(x) = \frac{1}{2\sqrt{x + 3}} \text{ for } x > -3.$$

This result in the D-notation is written as

$$D_x\sqrt{x + 3} = \frac{1}{2\sqrt{x + 3}}, \quad \text{for } x > -3.$$

Also, the results of Probs. 3(a) and (b) above may be written

$$D_x x^2 = 2x \quad \text{and} \quad D_x x^3 = 3x^2.$$

THEOREM 24.1. *With n a positive integer and f the function defined by $f(x) = x^n$, then $f'(x) = nx^{n-1}$; that is,*

$$D_x x^n = nx^{n-1}. \tag{1}$$

PROOF. From the definition of a derivative

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + \dfrac{n(n-1)}{1\cdot 2}x^{n-2}h^2 + \cdots + h^n\right) - x^n}{h}$$

$$= \lim_{h\to 0} \left(nx^{n-1} + \frac{n(n-1)}{1\cdot 2}x^{n-2}h + \cdots + h^{n-1}\right)$$

$$= nx^{n-1}.$$

The formula (1) may be used even if $n = 1$ provided $x \neq 0$

(2) $$D_x x = 1,$$

since $1 \cdot x^{1-1} = 1 \cdot x^0 = 1$ if $x \neq 0$, and $\lim_{h\to 0} \dfrac{(x+h) - x}{h} = \lim_{h\to 0} \dfrac{h}{h} = 1.$

THEOREM 24.2. *With c a constant and f the function defined by $f(x) = c$, then $f'(x) = 0$; that is,*

(3) $$D_x c = 0.$$

PROOF. Since both $f(x) = c$ and $f(x+h) = c$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

For use in the following proof, let $a \neq 0$ be constant and note that

$$\lim_{h\to 0} \frac{\sin ah}{h} = a \lim_{h\to 0} \frac{\sin ah}{ah} = a \cdot 1 = a \quad \text{and}$$

$$\lim_{h\to 0} \frac{1 - \cos ah}{h} = a \lim_{h\to 0} \frac{1 - \cos ah}{ah} = a \cdot 0 = 0,$$

from I in Theorem 18, page 54.

THEOREM 24.3. *With a and b constants, then*

(4) $$D_x \sin(ax + b) = a\cos(ax + b) \text{ and}$$

(5) $$D_x \cos(ax + b) = -a\sin(ax + b).$$

PROOF. To obtain (4) we use the definition of a derivative and hence want to evaluate the limit as $h \to 0$ of the quotient

$$\frac{\sin[a(x+h) + b] - \sin(ax+b)}{h} = \frac{\sin(ax + ah + b) - \sin(ax+b)}{h}.$$

From the formula $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, this quotient may be written, with $\alpha = ax + b$ and $\beta = ah$, as

$$\frac{1}{h}[\sin (ax + b) \cos ah + \cos (ax + b) \sin ah - \sin (ax + b)]$$
$$= \cos (ax + b) \frac{\sin ah}{h} - \sin (ax + b) \frac{1 - \cos ah}{h}.$$

This specific form was obtained so the limits immediately above the statement of the theorem could be used to show that

$$\begin{aligned} D_x \sin (ax + b) &= \cos (ax + b) \lim_{h\to 0} \frac{\sin ah}{h} - \sin (ax + b) \lim_{h\to 0} \frac{1 - \cos ah}{h} \\ &= [\cos (ax + b)] \cdot a - [\sin (ax + b)] \cdot 0 \\ &= a \cos (ax + b). \end{aligned}$$

Formula (5) may be obtained without going all the way back to the definition of the derivative. Since $\cos \alpha = \sin (\pi/2 - \alpha)$, then

$$\begin{aligned} D_x \cos (ax + b) &= D_x \sin \left[\frac{\pi}{2} - (ax + b)\right] \\ &= D_x \sin \left[-ax + \left(\frac{\pi}{2} - b\right)\right]. \end{aligned}$$

Now we use (4) with a replaced by $-a$ and b replaced by $\pi/2 - b$:

$$\begin{aligned} D_x \cos (ax + b) &= -a \cos \left[-ax + \left(\frac{\pi}{2} - b\right)\right] \\ &= -a \cos \left[\frac{\pi}{2} - (ax + b)\right] \\ &= -a \sin (ax + b) \end{aligned}$$

since $\cos (\pi/2 - \alpha) = \sin \alpha$.

THEOREM 24.4. *Let u and v be given functions, let s and p be their sum and product functions defined for each x in the common domain of u and v by $s(x) = u(x) + v(x)$ and $p(x) = u(x) \cdot v(x)$. Also, let*

$$A = \{x \mid \text{both } u'(x) \text{ and } v'(x) \text{ exist}\}.$$

Then, for x in A both $s'(x)$ and $p'(x)$ exist, $s'(x) = u'(x) + v'(x)$ and $p'(x) = u(x)v'(x) + u'(x)v(x)$; that is,

$$D_x[u(x) + v(x)] = D_xu(x) + D_xv(x) \quad \text{and} \tag{6}$$

$$D_x[u(x) \cdot v(x)] = u(x)D_xv(x) + v(x)D_xu(x). \tag{7}$$

PROOF. Let x be in A. Then u and v are both continuous at x (see Theorem 23, since $u'(x)$ and $v'(x)$ both exist, so that x is in an open interval common to the domain of u and v. Hence we write (then check in reverse order)

$$\begin{aligned}
s'(x) &= \lim_{h\to 0} \frac{s(x+h) - s(x)}{h} \\
&= \lim_{h\to 0} \left\{\frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h}\right\} \\
&= \lim_{h\to 0} \left\{\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right\} \\
&= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} = u'(x) + v'(x)
\end{aligned}$$

which, in different notation, is (6). Toward establishing (7), first notice that for x and $x + h$ in the domain of p, then

$$\begin{aligned}
p(x+h) - p(x) &= u(x+h)v(x+h) - u(x)v(x) \\
&= u(x+h)v(x+h) - u(x+h)v(x) \\
&\qquad + u(x+h)v(x) - u(x)v(x) \\
&= u(x+h)[v(x+h) - v(x)] + v(x)[u(x+h) - u(x)].
\end{aligned}$$

With this relation as a guide, and with x in the set A, we write the following, but then check each limit and equality by starting at the bottom and working up:

$$\begin{aligned}
p'(x) &= \lim_{h\to 0} \frac{p(x+h) - p(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \left\{\frac{u(x+h)v(x+h) - u(x+h)v(x) + v(x)u(x+h) - v(x)u(x)}{h}\right\} \\
&= \lim_{h\to 0} \left\{u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h}\right\} \\
&= \lim_{h\to 0} u(x+h) \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + v(x) \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\
&= u(x)v'(x) + v(x)u'(x).
\end{aligned}$$

This result, in different notation, is (7).

Example 1. $D_x(x^2 + \sin 3x) = D_x x^2 + D_x \sin 3x$ by (6)

$= 2x + 3\cos 3x$ by (1) and (4)

Example 2. $D_x(x^2 \sin 3x) = x^2 D_x \sin 3x + \sin 3x D_x x^2$ by (7)

$$= x^2(3 \cos 3x) + (\sin 3x)2x$$
$$= 3x^2 \cos 3x + 2x \sin 3x.$$

Example 3. Find $D_x \cos^2 (3x + 4)$.

Solution. In Theorem 24.4 the functions u and v need not be different. From (7) with both $u(x) = \cos (3x + 4)$ and $v(x) = \cos (3x + 4)$, then

$$\begin{aligned} D_x \cos^2 (3x + 4) &= \cos (3x + 4) D_x \cos (3x + 4) + \cos (3x + 4) D_x \cos (3x + 4) \\ &= 2[\cos (3x + 4)][-3 \sin (3x + 4)] \\ &= -6 \cos (3x + 4) \sin (3x + 4). \end{aligned}$$

In (6) and (7) with u the constant function $u(x) = c$, then

$$D_x[c + v(x)] = D_x c + D_x v(x) = 0 + D_x v(x), \quad \text{and}$$
$$D_x[cv(x)] = c D_x v(x) + v(x) D_x c = c D_x v(x) + v(x) \cdot 0.$$

These results, although combinations of previous results, are listed as formulas:

(8) $$\boldsymbol{D_x[v(x) + c] = D_x v(x)}$$

(9) $$\boldsymbol{D_x cv(x) = c D_x v(x).}$$

Hence, $D_x[u(x) - v(x)] = D_x[u(x) + (-1)v(x)] = D_x u(x) + D_x(-1)v(x)$

$$= D_x u(x) + (-1) D_x v(x) \qquad \text{by (9)}$$

and, therefore, as a companion to (6) we have

(10) $$\boldsymbol{D_x[u(x) - v(x)] = D_x u(x) - D_x v(x).}$$

Example 4. $D_x \left\{ \frac{2}{a^2} x \sin ax + \frac{2}{a^3} \cos ax - \frac{1}{a} x^2 \cos ax \right\}$

$$\begin{aligned} &= D_x \frac{2}{a^2} x \sin ax + D_x \frac{2}{a^3} \cos ax - D_x \frac{1}{a} x^2 \cos ax \\ &= \frac{2}{a^2} D_x x \sin ax + \frac{2}{a^3} D_x \cos ax - \frac{1}{a} D_x x^2 \cos ax \\ &= \frac{2}{a^2} [x D_x \sin ax + \sin ax\, D_x x] + \frac{2}{a^3} (-a \sin ax) \\ &\qquad - \frac{1}{a} [x^2 D_x \cos ax + \cos ax D_x x^2] \\ &= \frac{2}{a^2} [x(a \cos ax) + (\sin ax)(1)] - \frac{2}{a^2} \sin ax \\ &\qquad - \frac{1}{a} [x^2(-a \sin ax) + (\cos ax)2x] \\ &= \frac{2}{a} x \cos ax + \frac{2}{a^2} \sin ax - \frac{2}{a^2} \sin ax + x^2 \sin ax - \frac{2}{a} x \cos ax \\ &= x^2 \sin ax. \end{aligned}$$

PROBLEMS

1. Use Formulas 1–10 to obtain each of the derivatives:

a. $D_x(5x^4 - 6x^2 + 3)$.

b. $D_x(x^2 + 2 \sin x)$.

c. $D_x(2x^2 \sin x)$.

d. $D_x(\frac{3}{5} - \frac{1}{2}x^2 + \frac{1}{4}x^4)$.

e. $D_x(x^3 + 2)^2$.

f. $D_x \sin^2 (4x + 3)$.

g. $D_x(\sin x - x \cos x)$.

h. $D_x(x \sin x + \cos x)$.

2. Letters other than x may be used for the independent variable. Find

a. $D_t \sin \pi t$.

b. $D_s(s^3 - 3s^2 + 4)$.

c. $D_u(\frac{1}{4}u^4 - \frac{1}{2}u^2 + \frac{3}{5})$.

d. $D_t(t^3/3)$.

e. $D_v(v \sin v)$.

f. $D_\xi \sin^2 \xi$.

3. Establish each of the following:

a. $D_x \left\{\dfrac{1}{a^2} \sin ax - \dfrac{1}{a} x \cos ax\right\} = x \sin ax.$

b. $D_x \left\{\dfrac{2}{a^2} x \cos ax - \dfrac{2}{a^3} \sin ax + \dfrac{1}{a} x^2 \sin ax\right\} = x^2 \cos ax.$

c. $D_x \left\{\dfrac{1}{4} x^2 - \dfrac{1}{4a} x \sin 2ax - \dfrac{1}{8a^2} \cos 2ax\right\} = \dfrac{x}{2} (1 - \cos 2ax) = x \sin^2 ax.$

25. Power Formulas

In Sec. 24, under the specific condition that n is a positive integer, we derived the formula

$$D_x x^n = n\, x^{n-1}. \tag{1}$$

We shall now write this formula with a different exponent as

$$D_x x^p = p\, x^{p-1} \tag{2}$$

in order to show that it also holds either:

(i) *If p is a negative integer and $x \neq 0$, or*

(ii) *If $p = 1/n$ where n is a positive integer (provided $x > 0$ whenever n is even).*

To establish (2) under condition (i), let $p = -n$ and $x \neq 0$ where n is a positive integer. Then

$$D_x x^p = D_x x^{-n} = \lim_{h\to 0} \frac{(x+h)^{-n} - x^{-n}}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^n} - \frac{1}{x^n}\right] = \lim_{h\to 0} \frac{1}{h} \cdot \frac{x^n - (x+h)^n}{(x+h)^n x^n}$$
$$= \lim_{h\to 0} \frac{1}{(x+h)^n x^n} \lim_{h\to 0} \frac{x^n - (x+h)^n}{h} = \frac{1}{x^n x^n} \lim_{h\to 0}\left[-\frac{(x+h)^n - x^n}{h}\right]$$
$$= \frac{(-1)}{x^{2n}} \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \frac{-1}{x^{2n}} D_x x^n = \frac{-1}{x^{2n}} n x^{n-1} \qquad \text{by (1)}$$
$$= -n x^{-n-1} = p x^{p-1} \qquad \text{since } p = -n.$$

Toward establishing (ii) let n be a positive integer and check the algebraic identity

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}$$

where there are n terms on the right. Next, write this identity in the form

$$\text{(3)} \qquad \frac{a-b}{a^n - b^n} = \frac{1}{a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}}.$$

With x and $x + h$ neither zero (and both positive if n is even) set $a = (x+h)^{1/n}$ and $b = x^{1/n}$ so $a^n = x + h$, $b^n = x$, and (3) becomes

$$\frac{(x+h)^{1/n} - x^{1/n}}{(x+h) - x}$$
$$= \frac{1}{(x+h)^{\frac{n-1}{n}} + (x+h)^{\frac{n-2}{n}} x^{\frac{1}{n}} + \cdots + (x+h)^{\frac{1}{n}} x^{\frac{n-2}{n}} + x^{\frac{n-1}{n}}}.$$

The limit as $h \to 0$ of each term in the denominator on the right is $x^{(n-1)/n}$ (and there are n of them) so the limit as $h \to 0$ of the left exists; that is,

$$D_x x^{1/n} = \lim_{h\to 0} \frac{(x+h)^{1/n} - x^{1/n}}{h} = \frac{1}{n x^{(n-1)/n}} = \frac{1}{n} x^{(1/n)-1}$$

which is (2) with $p = 1/n$.

Example. $D_x \sqrt{x} \sin x = D_x[x^{1/2} \sin x] = x^{1/2} D_x \sin x + \sin x D_x x^{1/2}$
$$= x^{1/2} \cos x + (\sin x)(\tfrac{1}{2}x^{-1/2})$$
$$= \sqrt{x} \cos x + \frac{1}{2\sqrt{x}} \sin x.$$

The product formula

$$(4) \qquad D_x u(x)v(x) = u(x)D_x v(x) + v(x)D_x u(x)$$

holds even if $v(x) = u(x)$ and in this special case we have

$$D_x u(x)u(x) = u(x)D_x u(x) + u(x)D_x u(x); \quad \text{that is,}$$

$$(5) \qquad D_x u^2(x) = 2u(x)D_x u(x).$$

Now by setting $v(x) = u^2(x)$ in (4) the result is

$$D_x u(x)u^2(x) = u(x)D_x u^2(x) + u^2(x)D_x u(x),$$

$$D_x u^3(x) = u(x) \cdot 2u(x)D_x u(x) + u^2(x)D_x u(x) \qquad \text{(from (5))}$$

$$= 3u^2(x)D_x u(x).$$

By continuing in this way it may be shown that

$$(6) \qquad \boldsymbol{D_x u^n(x) = nu^{n-1}(x)D_x u(x)} \quad \text{for any positive integer } n,$$

which is another in the family of power formulas.

Example. $D_x \sin^4 2x = 4 \sin^3 2x D_x \sin 2x$
$= 4 \sin^3 2x(2 \cos 2x)$
$= 8 \sin^3 2x \cos 2x.$

Example. $D_x(x^3 + 2x + 1)^5 = 5(x^3 + 2x + 1)^4 D_x(x^3 + 2x + 1)$
$= 5(x^3 + 2x + 1)^4(3x^2 + 2).$

PROBLEMS

1. Find each of the following derivatives:

a. $D_x \sqrt{x} \cos x.$

b. $D_x x^{-2} \sin x.$

c. $D_x x^{-2} \cos 2x.$

d. $D_x \dfrac{\sin 2x}{x^2}.$

e. $D_x(\sqrt[3]{x} + \sqrt{x}).$

f. $D_x(\sqrt[3]{x}\sqrt{x}).$

g. $D_x(\frac{1}{4}x^4 + 4\sqrt[4]{x}).$

h. $D_x \dfrac{x^5 - 4x^2 + 1}{x^2}.$

2. Use formula (6) to make the first step in finding:

a. $D_x \cos^4 2x.$

b. $D_x \sin^6 (3x + 4).$

c. $D_x \sin^5 (4x + 2).$

d. $D_x(x^2 + \sin x)^3.$

e. $D_x(2x^3 + 7x + 1)^5.$

f. $D_x(\sqrt{x} + 3\sqrt[3]{x} + 5x)^3.$

g. $D_x(x^2 \sin x)^4.$

h. $D_x(\sin 2x + \cos 3x)^4.$

3. Find each of the following derivatives:

a. $D_x(x \sin^3 2x)$.
b. $D_x(x \sin 2x)^3$.
c. $D_x(x^2 + \cos x)^3$.
d. $D_x(x^2 + \cos^3 x)$.
e. $D_x(\sqrt{x} \sin x)^2$.
f. $D_x x(4x - 5)^5$.
g. $D_x(3x - 6)^7 2x^2$.
h. $D_x[(x^2 + 1)^4 - (x + 3)^2]$.

4. Start with formula (4) and derive the formula

$$D_x u(x)v(x)w(x) = u(x)v(x)D_x w(x) + u(x)w(x)D_x v(x) + v(x)w(x)D_x u(x).$$

Use this formula to find:

a. $D_x(x \sin x \cos 2x)$.
b. $D_x(x^2 \sin 2x \cos 3x)$.
c. $D_x(\sqrt{x} \sin 5x \cos^2 x)$.
d. $D_x(x^3 \sin^2 x \cos^4 2x)$.

26. The Chain Rule

The defining form and an equivalent form of the definition of $f'(c)$ are

$$(1) \qquad f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = \lim_{y \to c} \frac{f(y) - f(c)}{y - c}.$$

The second form is used below and in the proof of Theorem 26.

The defining relation (page 56) of the composition function F of two given functions f and u is

$$(2) \qquad F(x) = f(u(x)).$$

Among the derivative formulas the one which seems to be used most is called the **Chain Rule** and relates the derivative F' with the derivatives f' and u'. The formal manipulation

$$\lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \to 0} \frac{f(u(x + h)) - f(u(x))}{h}$$
$$= \lim_{h \to 0} \left\{ \frac{f(u(x + h)) - f(u(x))}{u(x + h) - u(x)} \cdot \frac{u(x + h) - u(x)}{h} \right\}$$

indicates that the formula is (by using the second form of (1))

$$(3) \qquad F'(x) = f'(u(x))u'(x).$$

Unfortunately it is possible, even though $h \neq 0$, for $u(x + h) - u(x)$ to be zero and thus a zero denominator might occur in the above manipulation. The proof of the following theorem avoids zero denominators, but otherwise is essentially the above manipulation. In the proof, an auxiliary function g is defined. After reading the proof, one can see how someone's hindsight led him to define this function g.

THEOREM 26. *If x_0 is such that both $u'(x_0)$ and $f'(u(x_0))$ exist, then $F'(x_0)$ also exists and*

$$F'(x_0) = f'(u(x_0))u'(x_0). \tag{4}$$

PROOF. We first define an auxiliary function g by setting, for y in the domain of f,

$$g(y) = \begin{cases} \dfrac{f(y) - f(u(x_0))}{y - u(x_0)} & \text{for } y \neq u(x_0) \\ f'(u(x_0)) & \text{for } y = u(x_0). \end{cases} \tag{5}$$

From the upper portion of this definition we have

$$f(y) - f(u(x_0)) = g(y)(y - u(x_0)) \tag{6}$$

for $y \neq u(x_0)$, but this equation also holds if $y = u(x_0)$ since both sides are then zero. Also from (5), g is continuous at $u(x_0)$ since (from the upper portion)

$$\lim_{y \to u(x_0)} g(y) = \lim_{y \to u(x_0)} \frac{f(y) - f(u(x_0))}{y - u(x_0)} = f'(u(x_0)) = g(u(x_0)),$$

the last equation coming from the lower portion of (5). This continuity of g at $u(x_0)$ and the continuity of u at x_0 (since $u'(x_0)$ exists) shows that

$$\lim_{x \to x_0} g(u(x)) = g(u(x_0)) = f'(u(x_0)).$$

With $x \neq x_0$ we now replace y in (6) by $u(x)$, then divide by $x - x_0 \neq 0$, and see that

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{f(u(x)) - f(u(x_0))}{x - x_0} = g(u(x))\frac{u(x) - u(x_0)}{x - x_0}.$$

Finally as $x \to x_0$ the last term approached $f'(u(x_0))u'(x_0)$ so that $F'(x_0)$ exists and (4) holds.

Since x_0 was any point for which $u'(x_0)$ and $f'(u(x_0))$ exist, (4) and (3) are equivalent.

In the D-notation a formula which is interpreted to have the same meaning as (3) is

$$D_x f(u(x)) = D_u f(u) \cdot D_x u \tag{3'}$$

wherein it is understood that whenever specific functions f and u are given, then after taking $D_u f(u)$ the u in the result is to be replaced by $u(x)$.

Example 1. Show that $D_x \sin x^2 = 2x \cos x^2$.

Solution. Consider f the function defined by $f(u) = \sin u$ so $D_u f(u) = \cos u$. Then from (3′)

$$\left.\begin{aligned} D_x \sin x^2 &= D_u \sin u D_x u \\ &= (\cos u) D_x u \end{aligned}\right\} \text{wherein it is understood that } u \text{ is the function defined by } u(x) = x^2$$
$$= (\cos x^2) D_x x^2 = 2x \cos x^2.$$

Example 2. Show that $D_x \sin^2 x = 2 \sin x \cos x$.

Solution. Consider f the function defined by $f(u) = u^2$. Then from (3′)

$$\left.\begin{aligned} D_x \sin^2 x &= D_u u^2 D_x u \\ &= 2u D_x u \end{aligned}\right\} \text{wherein } u \text{ is the function defined by } u(x) = \sin x$$
$$= 2 \sin x D_x \sin x = 2 \sin x \cos x.$$

Example 3. Show that $D_x \sin^2 x^2 = 4x \sin x^2 \cos x^2$.

Solution. First consider f temporarily as the function defined by $f(u) = u^2$:

$$\left.\begin{aligned} D_x \sin^2 x^2 &= D_u u^2 \cdot D_x u \\ &= 2u D_x u \end{aligned}\right\} \text{wherein } u \text{ is the function defined by } u(x) = \sin x^2$$
$$= 2 \sin x^2 D_x \sin x^2.$$

The differentiation is not complete so now discard the original use of f and this time think of f as the function defined by $f(u) = \sin u$. Hence, by continuing with the above result,

$$D_x \sin^2 x^2 = 2 \sin x^2 D_x \sin x^2$$
$$\left.\begin{aligned} &= 2 \sin x^2 [D_u \sin u \cdot D_x u] \\ &= 2 \sin x^2 [(\cos u) D_x u] \end{aligned}\right\} \text{wherein } u \text{ is the function defined by } u(x) = x^2$$
$$\begin{aligned} &= 2 \sin x^2 [(\cos x^2) D_x x^2] \\ &= 2 \sin x^2 [(\cos x^2)\, 2x] \\ &= 4x \sin x^2 \cos x^2. \end{aligned}$$

Example 4. Show that $D_x x^{3/2} = \frac{3}{2} x^{1/2}$, for $x > 0$.

Solution. Recalling that $x^{3/2} = (x^{1/2})^3$, think of f as defined by $f(u) = u^3$

$$\left.\begin{aligned} D_x x^{3/2} &= D_u u^3 D_x u \\ &= 3u^2 D_x u \end{aligned}\right\} \text{wherein } u \text{ is the function defined by } u(x) = x^{1/2}$$
$$\begin{aligned} &= 3(x^{1/2})^2 D_x x^{1/2} = 3x \cdot \tfrac{1}{2} x^{-1/2} \\ &= \tfrac{3}{2} x^{1/2}. \end{aligned}$$

With p and q integers and $r = p/q$, the method of Example 4 may be used to show that

$$D_x x^r = r x^{r-1}$$

thus extending the power formula (1) of Sec. 25 to any rational number r. This formula written as

$$D_u u^r = r u^{r-1} \tag{4}$$

may now be used in connection with (3′).

Example 5. Find $D_x(x^2 + 4)^{3/2}$.

Solution.

$$\begin{aligned} D_x(x^2 + 4)^{3/2} &= \left.\begin{aligned} &D_u u^{3/2} D_x u \\ &= \tfrac{3}{2} u^{1/2} D_x u \end{aligned}\right\} \\ &= \tfrac{3}{2}(x^2 + 4)^{1/2} D_x(x^2 + 4) \\ &= \tfrac{3}{2}\sqrt{x^2 + 4}(2x + 0) \\ &= 3x\sqrt{x^2 + 4}. \end{aligned}$$

PROBLEMS

1. Obtain each of the following:

a. $D_x \cos x^2 = -2x \sin x^2$.

b. $D_x \cos^2 x = -2 \cos x \sin x$.

c. $D_x \cos^3 x^2 = -6x \cos^2 x^2 \sin x^2$.

d. $D_x(x^2 + 1)^{5/2} = 5(\sqrt{x^2 + 1})^3 x$.

e. $D_x\sqrt{x^2 + 2x} = (x^2 + 2x)^{-1/2}(x + 1)$.

f. $D_x \sin (x^2 + 2x) = 2(x + 1) \cos (x^2 + 2x)$.

2. Find each of the following derivatives:

a. $D_x(3x^2 + 1)$.

b. $D_x(3x^2 + 1)^{3/2}$.

c. $D_x(x \sin x^2)^{4/3}$.

d. $D_x(x\sqrt{x + 1})^{1/3}$.

e. $D_x(x^2 + x \sin x)^{5/2}$.

f. $D_x(x^2 + x \sin x)^{3/2}$.

g. $D_x[(x^2 + 1)^3 + x]$.

h. $D_x[(x^2 + 1)^3 + x]^4$.

3. Obtain each of the following pairs of derivatives:

a. $D_x(x^3 + x^2 + x + 1) = 3x^2 + 2x + 1$, $D_x(3x^2 + 2x + 1) = 6x + 2$.

b. $D_x \sin x = \cos x$, $D_x \cos x = -\sin x$.

c. $D_x\sqrt{x + 4} = \frac{1}{2}(x + 4)^{-1/2}$, $D_x\frac{1}{2}(x + 4)^{-1/2} = -\frac{1}{4}(x + 4)^{-3/2}$.

d. $D_x \sin^2 x = \sin 2x$, $D_x \sin 2x = 2 \cos 2x$.

e. $D_x \sqrt{a^2 - x^2} = -x(a^2 - x^2)^{-1/2}$, $D_x[-x(a^2 - x^2)^{-1/2}] = \dfrac{-a^2}{(\sqrt{a^2 - x^2})^3}$.

4. Obtain each of the following pairs of derivatives and explain why each pair has the same answer.

a. $D_x(x - 2)^2$ and $D_x(x^2 - 4x)$.

b. $D_x \cos 2x$ and $D_x 2 \cos^2 x$.

c. $D_x(-\frac{1}{2} \cos x)$ and $D_x \sin^2 \dfrac{x}{2}$.

d. $D_x(\sin^4 x - \cos^4 x)$ and $D_x(\sin^2 x - \cos^2 x)$.

From the power formula (2) of Sec. 25, it follows that

$$D_v v^{-1} = -v^{-2} = \frac{-1}{v^2}.$$

and, therefore, by using (3′) of this section

$$D_x[v(x)]^{-1} = \frac{-1}{v^2(x)} D_x v(x). \tag{5}$$

Example. $D_x \sec x = D_x(\cos x)^{-1} = \dfrac{-1}{\cos^2 x} D_x \cos x = \dfrac{-1}{\cos^2 x}(-\sin x)$

$$= \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x.$$

By using (5) we also have

$$\begin{aligned} D_x \frac{u(x)}{v(x)} &= D_x[v(x)]^{-1}u(x) \\ &= [v(x)]^{-1}D_x u(x) + u(x)D_x[v(x)]^{-1} \\ &= \frac{1}{v(x)} D_x u(x) + u(x) \cdot \frac{-1}{v^2(x)} D_x v(x). \end{aligned}$$

By putting the right side over the common denominator the result is listed as the formula

$$D_x \frac{u(x)}{v(x)} = \frac{v(x)D_x u(x) - u(x)D_x v(x)}{v^2(x)} \tag{6}$$

for taking the derivative of the quotient of two functions.

5. Substitute into Formula (6) and obtain each of the following derivatives.

a. $D_x \dfrac{\sin x}{x}$.

b. $D_x \dfrac{x}{\sin x}$.

c. $D_x \dfrac{\sqrt{x+1}}{x}$.

d. $D_x \dfrac{x}{\sqrt{x+1}}$.

e. $D_x \dfrac{x + \cos x}{\sin x}$.

f. $D_x \dfrac{\sin x}{x + \cos x}$.

6. Establish each of the following:

a. $D_x \tan x = \sec^2 x$. b. $D_x \cot x = -\csc^2 x$. c. $D_x \csc x = -\csc x \cot x$.

7. Obtain each of the following pairs of derivatives and explain why each pair has the same answer.

a. $D_x \dfrac{x+1}{x}$ and $D_x\left(5+\dfrac{1}{x}\right)$.

b. $D_x \dfrac{x^2+1}{x}$ and $D_x \dfrac{x^2+5x+1}{x}$.

c. $D_x \dfrac{3x^2+1}{x^2+1}$ and $D_x \dfrac{-2}{x^2+1}$.

d. $D_x \dfrac{\sin 2x}{\cos x}$ and $D_x 2 \sin x$.

e. $D_x \dfrac{\cos 2x}{\sin^2 x}$ and $D_x \csc^2 x$.

f. $D_x \dfrac{\sin x(2+\sin x)}{(1+\sin x)^2}$ and $D_x \dfrac{-1}{(1+\sin x)^2}$.

8. Obtain each of the following pairs of derivatives:

a. $D_x \dfrac{\cos x}{x} = -\dfrac{x \sin x + \cos x}{x^2}$,

$$D_x\left[-\frac{x \sin x + \cos x}{x^2}\right] = \frac{2x \sin x + 2\cos x - x^2 \cos x}{x^3}.$$

b. $D_x \dfrac{x}{\cos x} = \dfrac{\cos x + x \sin x}{\cos^2 x}$,

$$D_x \frac{\cos x + x \sin x}{\cos^2 x} = \frac{x\cos^2 x + 2 \sin x \cos x + 2x \sin^2 x}{\cos^3 x}.$$

$$= \frac{x + \sin 2x + x \sin^2 x}{\cos^3 x}.$$

27. Second Derivatives

Given a function f, its derived function is defined by

$$f' = \left\{(x,y) \,\middle|\, y = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right\}.$$

If x is in the domain of f', then the limit exists and

$$D_x f(x) = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Now f' is a function and its derived function is denoted by f'' so that

$$f'' = \left\{(x,y) \,\middle|\, y = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}\right\}$$

If x is in the domain of f'', then the limit exists and

$$D_x f'(x) = f''(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}.$$

The function f'' is called the **second derived** function of f, and for x in the domain of f'' the value $f''(x)$ is called the **second derivative** of f at x.

An alternative notation for $f''(x)$ is $D_x^2 f(x)$ so that

$$D_x^2 f(x) = f''(x).$$

No new derivative formulas are necessary for finding second derivatives, since we merely "Take the derivative of the derivative."

Example.
$$\begin{aligned} D_x^2 \sin x^2 &= D_x[D_x \sin x^2] = D_x[\cos x^2 D_x x^2] \\ &= D_x[2x \cos x^2] = 2[x D_x \cos x^2 + \cos x^2 D_x x] \\ &= 2[x(-\sin x^2 D_x x^2) + \cos x^2] \\ &= 2[-(x \sin x^2)2x + \cos x^2] \\ &= 2[-2x^2 \sin x^2 + \cos x^2]. \end{aligned}$$

Geometric and physical interpretations of the second derivative will be given in the next chapter.

PROBLEMS

1. Find each of the following second derivatives.

a. $D_x^2\left(x + \dfrac{1}{x}\right)$.

b. $D_x^2\left(x^2 + \dfrac{1}{x^2}\right)$.

c. $D_x^2(\frac{1}{6}x^3 + \frac{1}{2}x^2)$.

d. $D_x^2 \sin(x^2 + 1)$.

e. $D_x^2 \dfrac{\sqrt{x+1}}{x}$.

f. $D_x^2 \dfrac{x}{\sqrt{x+1}}$.

g. $D_x^2\left(x + \dfrac{1}{x+1}\right)$.

h. $D_x^2(\sin x + \cos x)$.

i. $D_x^2[2(2x - \sin x \cos x) + \sin 2x]$.

2. For each of the following, find $D_x^2 f(x)$ and then evaluate the limit

$$\lim_{h\to 0} \frac{f(x + 2h) - 2f(x + h) + f(x)}{h^2}$$

and notice that the two are equal.

a. $f(x) = x^3$.

b. $f(x) = \dfrac{1}{x}$.

c. $f(x) = \sin x$.

3. Work Prob. 2 after replacing the limit there by $\displaystyle\lim_{h\to 0} \frac{f(x - h) - 2f(x) + f(x + h)}{h^2}$.

4. Let f be the function defined by $f(x) = x|x|$.

a. Show that if x is any number, then $f'(x)$ exists.

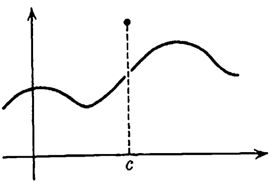
b. Show that if $x \neq 0$ then $f''(x)$ exists and show that $f''(0)$ does not exist.

c. Sketch the graphs of f, f', and f''.

5. For each of the following definitions of the function f, obtain the accompanying equations:
   a. $f(x) = \sin x, f''(x) = -f(x)$.
   b. $f(x) = a \sin x + b \cos x,\ D_x^2 f(x) = -f(x)$.
   c. $f(x) = a \sin x + b \cos x + x \sin x,\ D_x^2 f(x) = -f(x) + 2 \cos x$.
   d. $f(x) = ax^2 + b,\ xD_x^2 f(x) = D_x f(x)$.
   e. $f(x) = \dfrac{1}{x} + ax^2 + b,\ x^3 D_x^2 f(x) - x^2 D_x f(x) = 3$.

Examples to illustrate the three questions on page 59.

Example 1.



1. $f(c)$ defined? Yes.
2. $\lim_{x \to c} f(x)$ exist? Yes.
3. $\lim_{x \to c} f(x) = f(c)$? No.

Example 2.

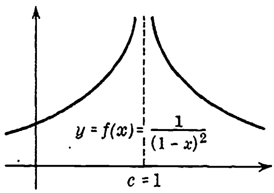
$$f(x) = \begin{cases} \sin^2 \dfrac{1}{x}, & x \neq 0 \\ 0.9, & x = 0 \end{cases}$$

See Fig. 16.3, page 50.

For $c = 0$.

1. $f(c)$ defined? Yes.
2. $\lim_{x \to c} f(x)$ exist? No.
3. Disregard.

Example 3.



1. $f(c)$ defined? No.
2. Need not ask (although the answer is "No")

(Note: Any number which is not rational (see page 9) is said to be irrational. In a more advanced course it is proved that between any two numbers there are both rational and irrational numbers.)

Problem. With f defined by $f(x) = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1. \text{ if } x \text{ is irrational} \end{cases}$

and with c any number, answer Questions 1 and 2 at the top of page 59.

CHAPTER 3

Applications of Derivatives

The derivative at x of a function f (as defined in Chapter 2 by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists) is a purely analytic notion. As already seen, however, derivatives may be interpreted geometrically in terms of slopes of tangents to curves or physically in terms of velocities of moving particles. In this chapter these interpretations and others are exploited in attacking practical problems.

The innocent appearing Mean Value Theorem for Derivatives (Sec. 32) should not be dismissed after its immediate application here; it will be the key to several developments at strategic points later on.

28. Equations of Tangents

Given a function f and a number x_1 in the domain of f', then the point $(x_1, f(x_1))$ is on the graph of f, the line tangent to this graph at this point has slope $m = f'(x_1)$ and an equation of this tangent line is

$$y - f(x_1) = f'(x_1)(x - x_1).$$

Example. Given $f = \{(x,y) \mid y = \sqrt{2x^2 + 1}\}$, find an equation of the line tangent to f at the point of the graph with abscissa 2.

Solution. First $f(2) = \sqrt{2 \cdot 2^2 + 1} = 3$ so that (2,3) is the desired point. Also

$$f'(x) = D_x\sqrt{2x^2 + 1} = \frac{4x}{2\sqrt{2x^2 + 1}} = \frac{2x}{\sqrt{2x^2 + 1}}$$

and, thus, $m = f'(2) = 4/\sqrt{2 \cdot 2^2 + 1} = \frac{4}{3}$ is the slope of the tangent line at the point (2,3). Hence, the tangent has equation

$$y - 3 = \tfrac{4}{3}(x - 2) \quad \text{or} \quad 4x - 3y + 1 = 0;$$

that is, $\{(x,y) \mid 4x - 3y + 1 = 0\}$ is a straight line which is tangent at the point (2,3) to $\{(x,y) \mid y = \sqrt{2x^2 + 1}\}$.

29. Solutions of Equations

DEFINITION. *Let f be a function and c and L numbers. If corresponding to each arbitrary positive number ϵ, there is a number $\delta > 0$ such that*

whenever $c < x < c + \delta$, then $|f(x) - L| < \epsilon$

we write

(1) $$\lim_{x \to c+} f(x) = L.$$

For example,

$$\lim_{x \to 0+} (1 + \sqrt{x}) = 1.$$

With $\epsilon > 0$ chosen arbitrarily, Fig. 29.1 illustrates a $\delta > 0$ such that

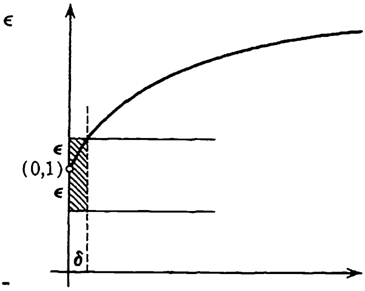


Figure 29.1

$$\{(x,y) \mid 0 < x < \delta, y = 1 + \sqrt{x}\} \subset \{(x,y) \mid 0 < x < \delta, |y - 1| < \epsilon\};$$

the meaning of the definition for this example. Also, we may let x approach $c = 0$ only through positive values (geometrically, only from the right).

The symbolism (1) is read "The **limit from the right** of f at c is equal to L." In case $L = f(c)$ the function is said to be **right continuous** at c. In a similar way the limit from the left, and left continuity, are defined.

A function f is said to be **continuous on a closed interval** I$[a,b]$ if f is continuous at x for $a < x < b$, right continuous at a, and left continuous at b.

THEOREM 29. *Let f be a function continuous on a closed interval* I$[a,b]$. *If $f(a)$ and $f(b)$ have opposite signs, then there is a number x^* such that $a < x^* < b$ and $f(x^*) = 0$.*

PROOF in case $f(a) < 0 < f(b)$. Let A be the set defined by

$$A = \{x \mid a \leq x \leq b \quad \text{and} \quad f(x) < 0\}.$$

The set A certainly contains the number a since $f(a) < 0$. Also, the set A is bounded above by b. Thus (see the axiom on p. 10) the set A has a least upper bound which we call x^*. We now show that $a < x^* < b$ and $f(x^*) = 0$.

Since $f(a) < 0$ and f is right continuous at a, there is a number $\underline{x} > a$ such that f is negative on I$[a,\underline{x})$. Hence $\underline{x} \leq x^*$. In the same way there is an $\bar{x}$ such that f is positive on I$(\bar{x},b]$. Hence $x^* \leq \bar{x}$. Thus $a < \underline{x} \leq x^* \leq \bar{x} < b$ so that

(2) $$a < x^* < b.$$

Now f is continuous at x^*. If $f(x^*)$ were > 0, then f would be positive on an interval I(x_1,x_2)

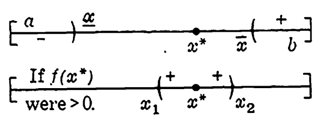


Figure 29.2

with $x_1 < x^* < x_2$ and this would mean that x^* is not the least upper bound of A, which is a contradiction. Consequently $f(x^*) \leq 0$. In a similar way it follows that $f(x^*)$ is not <0. Thus $f(x^*) = 0$ which, together with (2), is what we wished to prove.

Example. Show that $2x^4 - 9x^3 - x^2 + 25x + 11 = 0$ has a root between 3 and 4.

Solution. The function $f = \{(x,y) \mid y = 2x^4 - 9x^3 - x^2 + 25x + 11\}$ is continuous (see page 61), and hence it is sufficient to show that $f(3)$ and $f(4)$ have opposite signs. To review a synthetic process, sometimes given in algebra, note that

$$\begin{aligned} f(x) &= 2x^4 - 9x^3 - x^2 + 25x + 11 \\ &= \{[(2x - 9)x - 1]x + 25\}x + 11. \end{aligned}$$

Thus $f(3) = \{[(\mathbf{2} \cdot 3 - \mathbf{9})3 - \mathbf{1}]3 + \mathbf{25}\}3 + \mathbf{11}$. This computation may be arranged as

$$\begin{array}{rrrrr|l} 2 & -9 & -1 & 25 & 11 & \underline{3} \\ & 6 & -9 & -30 & -15 & \\ \hline 2 & -3 & -10 & -5 & -4 & \end{array} \quad \text{so that } f(3) = -4.$$

Similarily, to find $f(4)$ we write

$$\begin{array}{rrrrr|l} 2 & -9 & -1 & 25 & 11 & \underline{4} \\ & 8 & -4 & -20 & 20 & \\ \hline 2 & -1 & -5 & 5 & 31 & \end{array} \quad \text{so that } f(4) = 31.$$

The following intermediate value property will be used later.

COROLLARY. *Let f be a continuous function on* I$[a,b]$ *such that $f(a) \neq f(b)$ and let y^* be a number actually between $f(a)$ and $f(b)$. Then there is a number x^* such that $a < x^* < b$ and $f(x^*) = y^*$.*

PROOF. Let g be the function defined by $g(x) = f(x) - y^*$, note that g is continuous on I$[a,b]$ with $g(a)$ and $g(b)$ having opposite signs, apply Theorem 29 to obtain x^* such that $a < x^* < b$ and $g(x^*) = 0$, and finally note that $f(x^*) = y^*$.

In the next section it will be shown how derivatives may be used to aid in the solution of equations.

PROBLEMS

1. Find equations of lines tangent to $\{(x,y) \mid y = 2x^3 - 8x^2 + 5\}$:
   a. At the points $(0,5)$, $(1,-1)$, and $(-1,-5)$.
   b. At points of the graph where the slope is 0.
   c. At the point of the graph having abscissa 3.

2. Let $f(x) = x^3 + 3x^2 - 3x - 4$. In the example of Sec. 29 it was explained why, in the accompanying array, that $f(2) = 10$. Explain why, by continuing the array as shown that $f'(2) = 21$. By using this scheme for finding values of f and f', find an equation of the tangent to the graph of $y = f(x)$ at the points:

$$\begin{array}{cccc|l}
1 & 3 & -3 & -4 & \underline{2} \\
 & 2 & 10 & 14 & \\
\hline
1 & 5 & 7 & 10 = f(2) & \\
 & 2 & 14 & & \\
\hline
1 & 7 & 21 = f'(2) & &
\end{array}$$

a. $(1,f(1))$. b. $(-1,f(-1))$. c. $(-5,f(-5))$. d. $(5,f(5))$. e. $(-3,f(-3))$.

3. Find the equation of the tangent to each of the following graphs at the point of the graph indicated.

a. $y = \dfrac{2\sqrt{5x+1}}{x^2}$, point with abscissa 3.

b. $y = (x+1)\sqrt{x^2+4}$, point with abscissa $\frac{3}{2}$.

c. $y = x^2 - 5x + 5$, point with ordinate -1.

d. $y = \sin x + \cos x$, point with abscissa $\pi/4$.

e. $y = \sin x \cos 2x$, point with abscissa $\pi/3$.

4. Verify that between two points where each of the sets intersects the x-axis there is a point of the set where the tangent has slope 0; that is, where the tangent is parallel to the x-axis.

a. $\{(x,y) \mid y = x^3 - 3x\}$.
b. $\{(x,y) \mid y = \sin x\}$.
c. $\{(x,y) \mid y = \sin x - \cos x\}$.
d. $\{(x,y) \mid y = (x+1)(x-2)\}$.
e. $\{(x,y) \mid y = 2\sin x + \sin 2x\}$.
f. $\{(x,y) \mid y = \cos x + \cos 2x\}$.

5. Verify that between two points where the set intersects the x-axis there is no point where the tangent to the set has slope 0.

a. $\left\{(x,y) \mid y = \dfrac{x^2-1}{x}\right\}$. b. $\{(x,y) \mid y = \tan x\}$. c. $\{(x,y) \mid y = 1 - |x|\}$.

6. Prove: If f is a function and c is a number such that the left and right limits at c both exist and are equal, then the limit of f at c exists.

30. Newton's Method

In this section an intuitive discussion is given of Newton's method of starting with an approximation x_1 of a solution of an equation $f(x) = 0$ and successively obtaining numbers x_2, $x_3, \cdots$ which, under appropriate conditions, are better and better approximations of a solution.

Let f be a function whose derived function f' exists and is not zero wherever we wish to use it, and let f be such that $\{(x,y) \mid y = f(x)\}$ intersects the x-axis at a point $(x_0,0)$ so that $f(x_0) = 0$. Ordinarily the number x_0 will not be known exactly, but by some means (such as the one of Sec. 29) an approximation x_1 of x_0 can be found. Then the tangent to the graph at the point $(x_1, f(x_1))$ has equation

$$y - f(x_1) = f'(x_1)(x - x_1).$$

By setting $y = 0$ and solving for x, this tangent crosses the x-axis at the point $(x_2,0)$ where

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Now, by proceeding with x_2 as we did with x_1, we obtain

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}, \quad \text{then} \quad x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \text{ etc.,}$$

(1) $$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \text{for} \quad n > 2.$$

With a starting value x_1 given, or obtained by an intelligent guess, the numbers x_2, x_3, $\cdots$ are called **Newton iterates.**

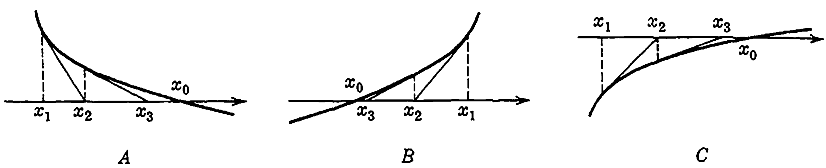


Figure 30.1

At this point in the book it is not feasible to give conditions and prove them sufficient to insure that for each positive number ϵ there is an integer N such that

$$\text{whenever} \quad n \geq N \quad \text{then} \quad |x_n - x_0| < \epsilon.$$

Such questions are considered in a course on numerical analysis. An examination of the curves of Fig. 30.1 should, however, indicate that if a curve continues "to bend" in the same direction on the interval joining the points $(x_1,0)$ and $(x_0,0)$, then the Newton iterates $x_2, x_3, \cdots$ seem to approach the solution x_0.

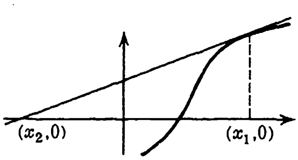


Figure 30.2

Figure 30.2 illustrates a situation in which x_2 is not a better approximation than x_1 to the desired solution.

Example 1. Given $f(x) = 2\cos x - x^2$ and the starting value $x_1 = \pi/3$ as an approximation of a root of $f(x) = 0$, obtain Newton iterates until two successive iterates agree to three significant figures.

Solution. First $f'(x) = -2(\sin x + x)$. By using the radian column of Table 2, we have $x_1 = \pi/3 = 1.0472$,

$$x_2 = 1.0472 - \frac{2(0.5000) - (1.0472)^2}{-2(0.8660 + 1.0472)} = 1.0219,$$

$$x_3 = 1.0219 - \frac{2(0.5217) - (1.0219)^2}{-2(0.8531 + 1.0219)} = 1.0219 - \frac{0.0009}{3.75}.$$

Since to three significant figures $x_2 = x_3 = 1.02$, we have proceeded as far as directed in this example. We followed the usual practice of allowing more figures in intermediate computations than are eventually retained.

Note: The above example stemmed from the problem of solving the equation $2 \cos x = x^2$. Solutions of this equation are the same as those of $f(x) = 0$, where $f(x) = 2 \cos x - x^2$. Easy substitutions gave $f(0) = 2$ and $f(\pi/2) = 0 - (\pi/2)^2$ which have opposite signs, so (by the continuity of f) a root is between $x = 0$ and $x = \pi/2$. We could have used $x_1 = \pi/2$. It is, however, highly desirable to have a close starting value so we sketched the graphs of

$$y = 2 \cos x \quad \text{and} \quad y = x^2.$$

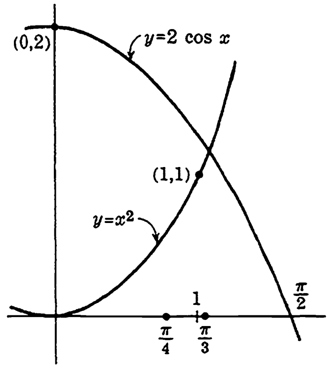


Figure 30.3

These graphs (Fig. 30.3) intersected at a point whose abscissa seemed closer to $\pi/3$ than to $\pi/2$, so we chose $x_1 = \pi/3$.

Example 2. For A a positive number and x_1 an approximation of $\sqrt{A}$, show that the Newton iterates for the function f defined by $f(x) = x^2 - A$ are

$$x_2 = \frac{1}{2}\left(x_1 + \frac{A}{x_1}\right), \quad x_3 = \frac{1}{2}\left(x_2 + \frac{A}{x_2}\right), \quad \text{etc.}$$

Let $A = 3$ and use $x_1 = 2$ as an approximation of $\sqrt{3}$. Compute the above Newton iterates for these values, continuing computation until two successive iterates agree to three decimal places and note the agreement with $\sqrt{3}$.

Solution. Since $f(x) = x^2 - A$, then $f'(x) = 2x$ so that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^2 - A}{2x_1} = \frac{2x_1^2 - x_1^2 + A}{2x_1} = \frac{1}{2}\left(x_1 + \frac{A}{x_1}\right).$$

By analogy the formula for x_3 in terms of x_2 is obtained, etc. Hence, with $A = 3$ and $x_1 = 2$, these iterates (with at most three decimal places kept) are

$$x_2 = \frac{1}{2}\left(2 + \frac{3}{2}\right) = 1.75,$$

$$x_3 = \frac{1}{2}\left(1.75 + \frac{3}{1.75}\right) = 0.8750 + 0.8571 = 1.7321,$$

$$x_4 = \frac{1}{2}\left(1.7321 + \frac{3}{1.7321}\right) = 0.8661 + 0.8660 = 1.7321.$$

Since the five decimal approximation of $\sqrt{3}$ is 1.73205, the above values for x_3 and x_4 are accurate approximations of $\sqrt{3}$ as far as they go.

In many practical problems the functions involved are not exact and an actual solution of an equation $f(x) = 0$ may be no more meaningful than a number x^* such that, for example, $|f(x^*)| < 5 \times 10^{-3}$. If x_n is a Newton iterate, then in finding x_{n+1} the value of $f(x_n)$ is computed so that at each stage the nearness to zero of the value of the function may be seen although the nearness of x_n to a solution of $f(x) = 0$ may not be gauged without further analysis, which we do not go into here.

If F and G are functions and c is a positive number, the imprecise directive "Solve $F(x) = G(x) \pm c$" is sometimes used to mean "Find a number x^* such that $|F(x^*) - G(x^*)| < c$." Newton's method applied to $f(x) = F(x) - G(x)$ is one means of attack. It may happen, even if c is relatively small, that a "solution" of $F(x) = G(x) \pm c$ is not a good approximation of a solution of $F(x) = G(x)$.

In spite of the indefiniteness shrouding the present discussion of Newton's method, this method is useful when applied with discretion. Conditions under which Newton iterates approximate a solution to any desired precision are given, and proved to be sufficient, in a course on numerical analysis.

PROBLEMS

1. For the equation $f(x) = 0$ obtain a starting value x_1 and compute Newton iterates until they agree to two decimal places.

   a. $f(x) = 3x^2 - 4x - 5$.
   b. $f(x) = x^3 - 12.65$.
   c. $f(x) = x - 1 + \sin x$.
   d. $f(x) = \dfrac{1}{x^2 + 4} - 3x + 4$.

2. "Solve"

   a. $x^3 + 45x = (12x^2 + 35) \pm 5 \times 10^{-2}$.
   b. $x^3 - 12x + 22 = 0 \pm 5 \times 10^{-2}$.
   c. $\cos x = x \pm 5 \times 10^{-3}$.
   d. $x^2 - 1 = \sin x \pm 5 \times 10^{-3}$.

3. Find an iterative process for approximating $\sqrt[3]{A}$.

4. Find an iterative process which does not involve division for approximating $1/A$. By starting with $x_1 = 0.1$ as an approximation of $\frac{1}{6}$ show that $x_4 = 0.1666$. (Note: In an electronic computer the equipment for performing division is more complicated than for multiplication. For this reason some electronic computers do not have the necessary equipment to perform the operation B/A directly, but the operator instructs the machine to compute $1/A$ by the process of this problem and then to multiply $1/A$ by B.) (Hint: Set $f(x) = 1/x - A$.)

31. Maxima and Minima

A function f is said to have **maximum** M if there is a number c such that $f(c) = M$ and $f(x) \leq M$ for all x in domain f. A value $f(x_0)$ is a **relative**

maximum if $f(x) \leq f(x_0)$ for all x in the intersection of domain f and some interval centered at x_0 (even if x_0 is an end-point of the domain of f). The **minimum** and a **relative minimum** of f are defined in a similar way.

Figure 31.1 shows a function with domain $I[a,b]$ having:
max. $f(b)$,
rel. max. $f(b)$, $f(x_1)$, and $f(x_3)$
min. $f(x_2)$,
rel. min. $f(a)$, $f(x_2)$, and $f(x_5)$.

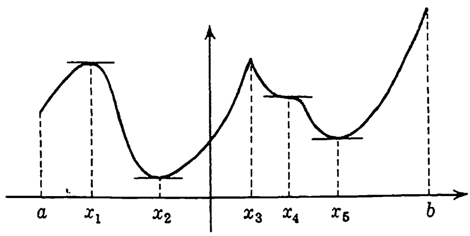


Figure 31.1

THEOREM 31.1. *Let f be a function and x_0 a number such that $f'(x_0)$ exists and $f'(x_0) \neq 0$. Then $f(x_0)$ is neither a relative maximum nor a relative minimum of f.*

PROOF. CASE 1. $f'(x_0) > 0$. Choose a number $\epsilon > 0$ such that $0 < f'(x_0) - \epsilon$. Let $\delta > 0$ be such that whenever $0 < |x - x_0| < \delta$ then

$$(1) \qquad -\epsilon < \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) < \epsilon.$$

Hence, from the left hand inequality only, whenever $x_0 - \delta < x < x_0$ or $x_0 < x < x_0 + \delta$, then

$$0 < f'(x_0) - \epsilon < \frac{f(x) - f(x_0)}{x - x_0}.$$

The numerator is negative if $-\delta < x - x_0 < 0$, but is positive if $0 < x - x_0 < \delta$. Thus $f(x_0)$ is neither a relative maximum nor a relative minimum of f.

CASE 2. $f'(x_0) < 0$. The proof is similar; by choosing $0 < \epsilon < -f'(x_0)$ and using the right hand inequality of (1).

DEFINITION. *A* **critical value** *for a function f on $I[a,b]$ is any value x_0, $a \leq x_0 \leq b$, such that either:*

$$x_0 = a,\ x_0 = b,\ f'(x_0) = 0, \text{ or } f' \textit{ does not exist at } x_0.$$

A logical deduction from Theorem 31.1 is:

COROLLARY. *On a closed interval $I[a,b]$, a relative maximum or relative minimum of a function f can occur only at a critical value for f on $I[a,b]$.*

Note that $f = \{(x,y) \mid y = x - (\text{the greatest integer} \leq x)\}$ has no maximum (even on $I[0,1]$), since each value of f is actually <1, but f has values greater than any given number <1.

The function $\{(x, y) \mid y = x + 1/x\}$ has neither a maximum nor a minimum on $I[-1, 1]$.

It is, however, true that:

THEOREM 31.2. *If a function f is continuous on a closed interval* I$[a,b]$, *then f has a maximum, and a minimum on* I$[a,b]$; *that is, there are numbers $\underset{\text{-}}{x}$ and $\bar{x}$ such that $a \le \underset{\text{-}}{x} \le b$, $a \le \bar{x} \le b$, and if $a \le x \le b$, then*

$$f(\underset{\text{-}}{x}) \le f(x) \le f(\bar{x}).$$

This theorem is proved in Appendix A2. In particular, Theorem 31.2 and the corollary to Theorem 31.1 imply the validity of the following test, which may be applied to determine the maximum and minimum values of a continuous function on a closed interval.

TEST I. *The maximum and minimum of a continuous function on a closed interval* I$[a,b]$ *may be obtained as follows:*

(i) *Find all critical values of f on* I$[a,b]$; *that is, note a and b, find all solutions of $f'(x) = 0$ which are in* I(a,b), *and find all values of x in* I(a,b) *which are not in the domain of f'.*

(ii) *Compute $f(x)$ for x each of the critical values and among the numbers obtained, the largest is the maximum and the smallest is the minimum of f on* I$[a,b]$.

Example 1. For f the function defined by

$$f(x) = x^{1/3}(x-3)^{2/3}, \quad -1 \le x \le 4$$

find all critical values. Also, find the maximum and the minimum of f on I$[-1,4]$.

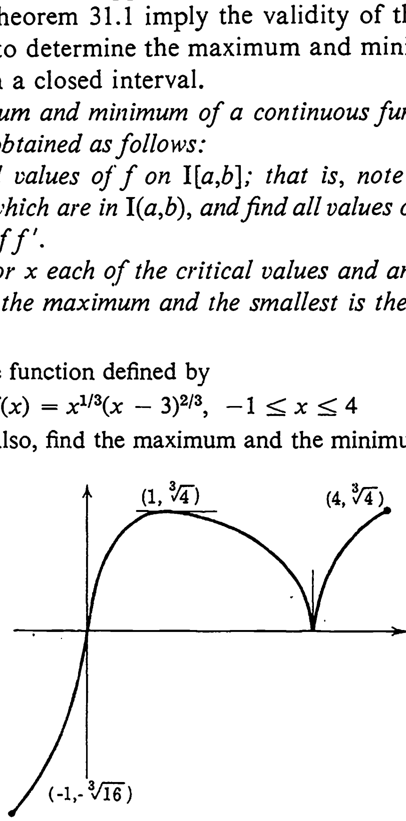


Figure 31.2

Solution. From derivative formulas developed earlier

$$f'(x) = \frac{2}{3}\frac{x^{1/3}}{(x-3)^{1/3}} + \frac{1}{3}\frac{(x-3)^{2/3}}{x^{2/3}} = \frac{x-1}{x^{2/3}(x-3)^{1/3}} \quad \text{for} \quad x \ne 0 \quad \text{and} \quad x \ne 3.$$

Thus, critical values of f on I$[-1,4]$ are:

$x = -1, \quad x = 4$	(end points of the interval)
$x = 1$	(since $f'(1) = 0$)
$x = 0, \quad \text{and} \quad x = 3$	(values on I$[-1,4]$ but not in the domain of f')

Now $f(-1) = (-1)^{1/3}(-4)^{2/3} = -\sqrt[3]{16}$, $f(4) = \sqrt[3]{4}$, $f(1) = 1^{1/3}(-2)^{2/3} = \sqrt[3]{4}$, $f(0), = 0$, and $f(3) = 0$. Thus, the minimum value of the function is $-\sqrt[3]{16}$

(occurring when $x = -1$), whereas the maximum value is $\sqrt[3]{4}$ (occurring both when $x = 1$ and when $x = 4$). Figure 31.2 is a graph of f.

The following examples illustrate applications of maxima and minima.

Example 2. A pan is to be made from a rectangle of tin 6 in. × 10 in. by cutting squares of side x in. from each corner and turning up the tin to form the sides and ends of the pan. Find the pan of largest volume which can be so made.

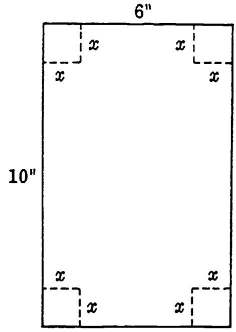


Figure 31.3

Solution. The pan will have length $10 - 2x$ in., breadth $6 - 2x$ in., where $0 \leq x \leq 3$, depth x in., and thus volume $V(x)$ in^3 where

$$\begin{aligned} V(x) &= (10 - 2x)(6 - 2x)x \\ &= 4(x^3 - 8x^2 + 15x), \quad 0 \leq x \leq 3. \end{aligned}$$

Since $V'(x) = 4(3x^2 - 16x + 15)$ exists for x on I(0,3], and on this interval the only solution of $V'(x) = 0$ is

$$x = \frac{8 - \sqrt{19}}{3},$$

we see that this value together with $x = 0$ and $x = 3$ are the only critical values. Since $V(0) = 0$ and $V(3) = 0$, whereas $V\left(\frac{8 - \sqrt{19}}{3}\right) > 0$, it follows that (to within slide rule accuracy) the values of x to use and the maximum volume are

$$x = \frac{8 - \sqrt{19}}{3} = 1.21 \quad \text{and} \quad V(1.21) = 32.8 \text{ in}^3.$$

Example 3. At 1230^h (Navy time) one ship is steaming north at 12 knots and is 25 nautical miles due south of a second ship steaming due west at 9 knots. Find when the ships are closest together and their distance apart at this time.

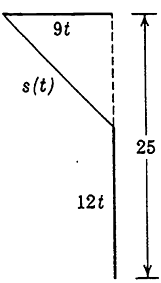


Figure 31.4

Solution. Starting to measure time t in hours from the described instant, the number $s(t)$ of miles apart, and an appropriate time interval are (see Fig. 31.4),

$$s(t) = \sqrt{(25 - 12t)^2 + (9t)^2}, \quad 0 \leq t \leq 25/12.$$

Since
$$\begin{aligned} s'(t) &= \frac{2(25 - 12t)(-12) + 2(9t)9}{2s(t)} \\ &= \frac{75(-4 + 3t)}{s(t)}, \quad 0 \leq t \leq \frac{25}{12}, \end{aligned}$$

we see that $s'(t) = 0$ if and only if $t = \frac{4}{3} = 1^h20^m$. The critical values are thus $t = 0$, $t = \frac{4}{3}$, and $t = \frac{25}{12}$. Since

$$s(0) = 25, \quad s(\tfrac{4}{3}) = 15, \quad \text{and} \quad s(\tfrac{25}{12}) = \tfrac{75}{4},$$

the closest approach occurs at $1230^h + 1^h20^m = 1350^h$ and this shortest distance is 15 nautical miles.

In solving such problems a sequence of steps to follow is:

1) Express the quantity to be maximized or minimized as a function. [The problem may be stated in terms of letters as constants, and it may be convenient to introduce new letters to express the given relations. Then, by using conditions of the problem, all except two of the introduced letters must be eliminated (one of them representing a function expressed in terms of the other as independent variable).]

2) For the independent variable, determine the significant interval whose end points are therefore critical values.

3) Determine additional critical values by examining the derivative of the function.

4) Since the largest and smallest values of the function (whichever is pertinent to the problem) occur at critical values, determine the desired maximum or minimum.

Example 4. Of all right circular cones inscribed in a sphere of radius R, find the one having largest volume.

Solution. For this problem R is a constant. A right circular cone of radius r and altitude h has volume V where

$$V = \tfrac{1}{3}\pi r^2 h.$$

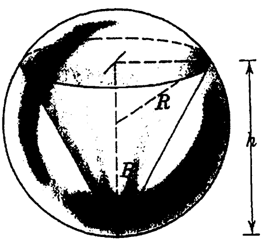


Figure 31.5

Since, however, the cone is inscribed in a sphere of radius R, then (see Fig. 31.5) r, h, and R are related by

$$r^2 = R^2 - (h - R)^2 = 2Rh - h^2.$$

Thus, the volume in terms of h and bounds for h are

$$V(h) = \tfrac{1}{3}\pi(2Rh - h^2)h, \quad R \leq h \leq 2R.$$

Since $V'(h) = \frac{1}{3}\pi(4Rh - 3h^2) = \frac{1}{3}\pi h(4R - 3h)$, the only additional critical value of h is $h = 4R/3$. A check shows that $V(R) = \frac{1}{3}\pi R^3$, $V(2R) = 0$, and $V(\frac{4}{3}R) = \frac{32}{81}\pi R^3$. Thus, the largest possible volume is $\frac{32}{81}\pi R^3$.

PROBLEMS

1. For each of the given functions, find all critical values and also the maximum and minimum of the function.
   a. $f = \{(x,y) \mid 0 \leq x \leq \pi \text{ and } y = \sin x\}$.
   b. $f = \{(x,y) \mid -\pi/4 \leq x \leq \pi/4 \text{ and } y = \sin x\}$.
   c. $f = \{(x,y) \mid -1 \leq x \leq 1 \text{ and } y = x^2\}$.
   d. $f = \{(x,y) \mid -1 \leq x \leq 1 \text{ and } y = x^3\}$.
   e. $f = \{(x,y) \mid 0 \leq x \leq 6 \text{ and } y = \sqrt{x}(x - 5)^{1/3}\}$.
   f. $f = \{(x,y) \mid 5 \leq x \leq 14 \text{ and } y = 3(6 - x)^{2/3}(x - 13)^{5/3}\}$
   g. $f = \{(x,y) \mid 0 \leq x \leq 2\pi \text{ and } y = \sin x + \cos x\}$.

h. $f = \{(x,y) \mid 0 \leq x \leq 2\pi \text{ and } y = \cos x - \sin x\}$.
i. $f = \{(x,y) \mid 0 \leq x \leq \pi \text{ and } y = \frac{1}{2}x + \sin x\}$.
j. $f = \{(x,y) \mid 0 \leq x \leq \pi \text{ and } y = x + \cos x\}$.

2. Work Example 2 with a sheet of tin (a) 12 in. × 18 in., (b) 12 in. × 12 in.

3. A box with a lid is to be made from a sheet of tin by cutting along the dotted lines as shown in Fig. Prob. 3. The tin is turned up to form the ends and sides and the flap turned over to form the lid. Find the dimensions of the box of largest volume which can be formed if the sheet is:

a. 12 in. × 12 in.

b. 14 in. × 30 in.; squares cut from a 14 in. side.

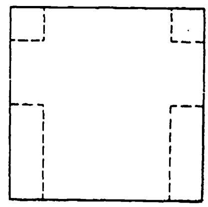


Figure Prob. 3

4. Of all right circular cylinders inscribed in a sphere of radius R, find the one of largest

✓a. Volume. b. Lateral Area. c. Total area.

5. Find the rectangle of greatest area which can be inscribed in:

a. A circle of radius 4. b. A semicircle of radius R.

c. An isosceles triangle of base 10 and altitude 10.

d. An isosceles triangle of base B and altitude H.

e. An isosceles trapezoid with bases 10 and 6 and altitude 8.

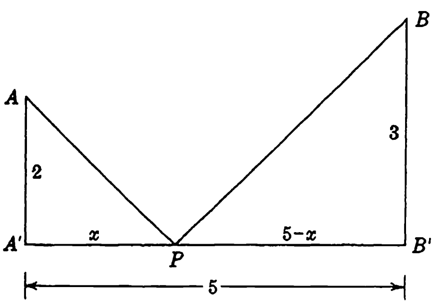


Figure Prob. 6

6. In Fig. Prob. 6, find the value of x, where $0 \leq x \leq 5$, which minimizes the given expression, and find the minimum of the expression.

a. $AP + PB$

b. $(AP)^2 + (PB)^2$

c. $(AP)^2 + 2(PB)^2$

d. $(AP)^2 - (PB)^2$

e. Area $AA'P$ + area $PB'B$.

f. Area APB.

7. Find the dimensions of the rectangle of perimeter 36 which will sweep out a volume as large as possible when revolved (a) about one of its sides; (b) about a line parallel to a side and one unit away.

8. The set $\{(x,y) \mid 0 \leq x \leq 1 \text{ and } x^2 \leq y \leq x\}$ is a region. Revolve this region:

a. About the x-axis and find the largest vertical cross section.

b. About the y-axis and find the largest horizontal cross section.

9. a. Sketch $\{(x,y) \mid x > 0 \text{ and } y = 1/x\}$ and the line whose equation is $y = mx$ $m > 0$. Find the point where these graphs intersect, draw the tangent to the curve at this point, and find the point where this tangent intersects the x-axis. With these two points and the origin as vertices of a triangle, find an expression for the area $A(m)$ of the triangle.
   b. Work Part a with $y = 1/x$ replaced $y = 1/x^2$.

32. A Mean Value Theorem

The theorems proved in this section have immediate applications in maximum and minimum considerations, but are used again in Chap. 4 and form the basis of much of the development in Chap. 12.

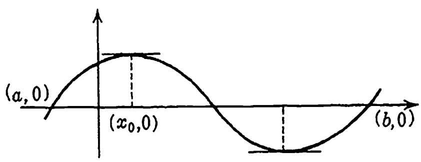


Figure 32.1

THEOREM 32.1 (Rolle's Theorem). *Let g be a function such that*

(i) *g is continuous on a closed interval* $\text{I}[a,b]$,
(ii) *g' exists on the open interval* $\text{I}(a,b)$, *and*
(iii) $g(a) = g(b) = 0$.

Then there is a number x_0 such that

(1) $$a < x_0 < b \quad \text{and} \quad g'(x_0) = 0.$$

PROOF. Since g is continuous on the closed interval $\text{I}[a,b]$, it has a maximum M and a minimum m on this interval. (See Theorem 31.2.) Consider three cases.

CASE 1. $M > 0$. Let x_0 be on $\text{I}[a,b]$ and $g(x_0) = M$. Then by (iii)

(2) $$a < x_0 < b.$$

Thus, $g'(x_0)$ exists because of (ii), but then $g'(x_0) = 0$ because of Theorem 31.1. Then (1) holds in this case.

CASE 2. $m < 0$. The proof is similar to the proof in Case 1.

CASE 3. $M \leq 0$ and $m \geq 0$. Since $m \leq M$ we have $M = m = 0$ so that g is constant on $\text{I}[a,b]$. Thus, $g'(x) = 0$ for $a < x < b$ and we select for x_0 any number in $\text{I}(a,b)$.

Since these cases include all possibilities, the proof is complete.

Rolle's Theorem is used in proving the Mean Value Theorem for derivatives, which is sometimes called the Law of the Mean for derivatives.

THEOREM 32.2 (Mean Value Theorem for Derivatives). *Let f be a function which satisfies the conditions:*

(i) *f is continuous on the closed interval* $\text{I}[a,b]$,
(ii) *f' exists on the open interval* $\text{I}(a,b)$.

Then there is a number ξ in the open interval I(a,b) *such that*

(3) $$f(b) = f(a) + (b - a)f'(\xi), \quad a < \xi < b.$$

PROOF. Let g be the function defined on I$[a,b]$ by

(4) $$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a), \quad a \leq x \leq b.$$

(Note: A way of remembering (4) is to write the equation of the chord joining $(a,f(a))$ to $(b,f(b))$ as

(5) $$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a),$$

and then subtracting this expression from $f(x)$ to see that $g(x)$ is the vertical distance (or its negative) between points of the curve and the chord.)

By substituting into (4), $g(a) = g(b) = 0$ and since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

$$a \leq x \leq b$$

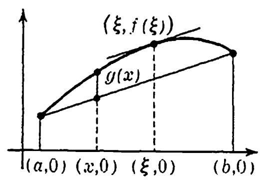


Figure 32.2

the conditions of Rolle's Theorem are satisfied. Hence there is a number ξ such that $a < \xi < b$ and $g'(\xi) = 0$; i.e.

$$0 = f'(\xi) - \frac{f(b) - f(a)}{b - a}$$

which is equivalent to (3).

A geometric interpretation of Theorem 32.2 is: *On a smooth arc there is a point where the tangent is parallel to the chord.*

DEFINITION. *A function f is said to be* **non-decreasing** *on* I$[a,b]$ *if whenever x_1 and x_2 are numbers such that $a \leq x_1 < x_2 \leq b$, then*

(6) $$f(x_1) \leq f(x_2).$$

By replacing (6) *by $f(x_1) \geq f(x_2)$ the definition of f being* **non-increasing** *is obtained.*†

THEOREM 32.3. *If f is a continuous function on* I$[a,b]$, *if f' exists on* I(a,b), *and*

(i) *if $f'(x) \geq 0$ for $a < x < b$, then f is non-decreasing on* I$[a,b]$, *but*
(ii) *if $f'(x) \leq 0$ for $a < x < b$, then f is non-increasing on* I$[a,b]$.

† Some books use "monotonically increasing" in place of "non-decreasing," and "monotonically decreasing" in place of "non-increasing."

PROOF. Assuming f satisfies the conditions (i), let x_1 and x_2 be numbers such that $a \leq x_1 < x_2 \leq b$. Then the conditions of Theorem 32.2 are satisfied on the interval $I[x_1,x_2]$. Thus, let x_0 be such that

(1) $\qquad x_1 < x_0 < x_2 \quad \text{and} \quad f(x_2) = f(x_1) + (x_2 - x_1)f'(x_0).$

Hence, $a < x_0 < b$ so that $f'(x_0) \geq 0$ whereas $x_2 - x_1 > 0$ and thus

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(x_0) \geq 0.$$

Consequently $f(x_1) \leq f(x_2)$, which states that f is non-decreasing on $I[a,b]$. In case (ii) all statements down to and including (1) hold, but now $(x_2 - x_1)f'(x_0) \leq 0$ so that $f(x_1) \geq f(x_2)$ and f is non-increasing on $I[a,b]$.

A paraphrase of Theorem 32.3 is the following test which may be used in connection with maximum-minimum problems. (See p. 92 for Test I.)

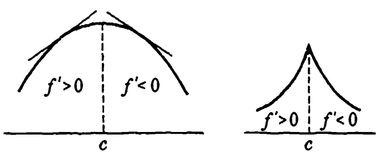


Figure 32.3

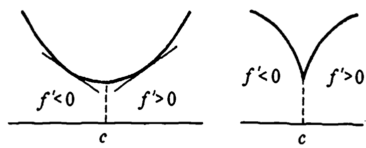


Figure 32.4

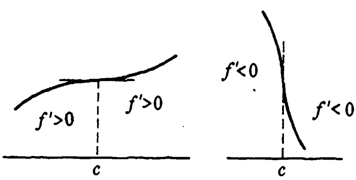


Figure 32.5

TEST II. *For f a continuous function and c in the domain of f, if $f'(c)$ does not exist or if $f'(c) = 0$, then the value of $f(c)$ is:*

(i) *A relative maximum (see Fig. 32.3) if there are numbers $x_1 < c < x_2$ such that $f' \geq 0$ on $I(x_1,c)$ but $f' \leq 0$ on $I(c,x_2)$.*

(ii) *A relative minimum (see Fig. 32.4) if $f' \leq 0$ on $I(x_1,c)$ but $f' \geq 0$ on $I(c,x_2)$.*

(iii) *Neither a relative maximum nor a relative minimum (see Fig. 32.5) if f' has the same sign in both $I(x_1,c)$ and $I(c,x_2)$.*

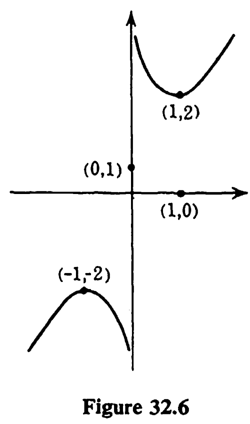


Figure 32.6

Example 1. Find the relative maximum and relative minimum of the function f defined by

$$f(x) = x + \frac{1}{x}, \quad x \neq 0,$$

Solution. Since $x = 0$ is not in the domain of f and since

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} \quad \text{for} \quad x \neq 0,$$

there is no critical value where the derivative fails to exist. The only critical values are solutions of $f'(x) = 0$; namely, $x = \pm 1$. Since

$f' > 0$ on $I(-2,-1)$, but $f' < 0$ on $I(-1,0)$, then $f(-1) = -2$ is a relative maximum,

$f' < 0$ on $I(0,1)$, but $f' > 0$ on $I(1,2)$, then $f(1) = 2$ is a relative minimum.

This example shows that a function may thus have a relative maximum less than a relative minimum.

Example 2. Given f defined by $f(x) = (x-1)^3(x-3)+1$, determine all relative maxima and minima.

Solution. $f'(x) = (x-1)^3 + (x-3)\,3(x-1)^2 = 2(x-1)^2(2x-5)$. Hence, $f'(x)$ exists for all x and $f'(x) = 0$ if either $x = 1$ or $x = \frac{5}{2}$. Moreover $(x-1)^2$ is never negative and

$$f'(x) < 0 \quad \text{if} \quad x < 1 \quad \text{or if} \quad 1 < x < \tfrac{5}{2},$$

$$f'(x) > 0 \quad \text{if} \quad x > \tfrac{5}{2}.$$

Hence, this function has no relative maximum and the only relative minimum (see Fig. 32.7) is $f(\frac{5}{2}) = -\frac{11}{16}$.

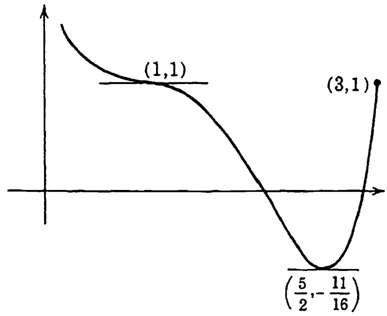


Figure 32.7

This example shows that a function may have a derivative equal to zero without having a maximum or minimum there.

A point of a curve [such as (1,1) of Fig. 32.7] where the tangent is horizontal with the curve above the tangent on one side and below on the other, is called a point of ***horizontal inflection.***

Example 3. A vertical post with square cross section 1 ft × 1 ft is set squarely on a cubical block 3 ft on a side. Supports from the floor to the sides of the post are desired. Find the inside length of the shortest possible support.

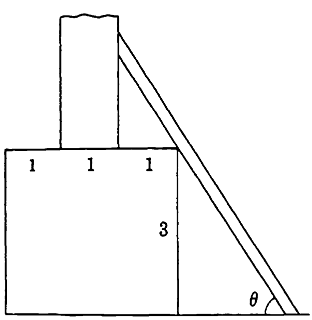


Figure 32.8

Solution. For a support making an angle θ with the floor (see Fig 32.8), the inside length $L(\theta)$ is given by

(7) $L(\theta) = 3\csc\theta + \sec\theta, \quad 0 < \theta < \pi/2.$

We first find (see page 80, Example and Prob. 6)

$$L'(\theta) = -3\csc\theta\cot\theta + \sec\theta\tan\theta$$

$$= \frac{-3\cos^3\theta + \sin^3\theta}{\sin^2\theta\cos^2\theta}, \quad 0 < \theta < \frac{\pi}{2}.$$

Hence, $L'(\theta) = 0$ only if the angle θ is such that $\sin^3\theta = 3\cos^3\theta$; that is,

(8) $$\tan\theta = 3^{1/3}, \quad 0 < \theta < \pi/2.$$

Moreover $L'(\theta)$ is negative for θ close to 0, but $L'(\theta)$ is positive for θ close to $\pi/2$, and hence the value of θ for which (8) holds will make (7) a minimum. From Fig. 32.9 in which $\tan\theta = 3^{1/3}$, the value of $\csc\theta$ and $\sec\theta$ may be read off to give the minimum length as

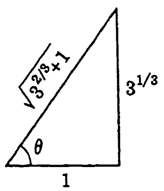


Figure 32.9

$$\frac{3\sqrt{3^{2/3}+1}}{3^{1/3}} + \sqrt{3^{2/3}+1} = (3^{2/3}+1)^{3/2}.$$

By slide rule computation the minimum length is 5.41 ft.

PROBLEMS

1. Find the relative maxima and minima of f, find the points of horizontal inflection of the graph, and sketch the graph.

a. $f = \{(x,y) \mid y = (x-1)^3(x-5)+1\}$.

b. $f = \{(x,y) \mid y = (x-1)^3(x-5)^2+1\}$.

c. $f = \{(x,y) \mid y = (x-1)(x-5)^3+1\}$.

d. $f = \{(x,y) \mid y = \sin^3 x\}$.

2. A farmer's barn lot is a rectangle 54 ft × 128 ft and is at the intersection of two perpendicular roads. Find the shortest straight path from one road to the other passing behind the lot.

3. In a gabled attic the roof has pitch $\frac{3}{4}$ and is 30 ft across. A storage space is to be made with horizontal ceiling and vertical side walls (ends and floor are already there). Find the breadth and height for maximum capacity.

4. Work Prob. 3 for a Quonset hut whose end is a semicircle of radius 15 ft.

5. Work Prob. 4 for a Quonset hut whose end is a semi-ellipse 30 ft broad and 10 ft high.

6. A pup tent is to shelter 36 ft^3; the ridge is to be 3 ft high and at least 6 ft long, but there are no ends to the tent and no canvas for a floor. Find the dimensions of the tent fulfilling these specifications that uses the least amount of canvas.

7. Given a right circular cone of base-radius r and altitude h. Show that the total area $A(x)$ of an inscribed right circular cylinder of radius x is given by

$$A(x) = 2\pi\left[hx + x^2\left(\frac{r-h}{r}\right)\right], \quad 0 < x < r.$$

Show that:

a. If $r < h$, then $A\left(\dfrac{hr}{2(h-r)}\right) = \dfrac{\pi h^2 r}{2(h-r)}$ is the maximum area.

b. If $r = h$, then $A(x) = 2\pi r x$ for $0 < x < r$.

c. If $r \geq h$, then there is (properly) no maximum area.

33. Points of Inflection

Given a function f, then f' is its derived function, and f'' is (as defined in Sec. 27) its second derived function. We may thus apply the Law of the Mean for derivatives (see Theorem 32.2) to the function f'; that is,

Let f be a function such that

(i) *f' is continuous on a closed interval* $\mathrm{I}[a,b]$ *and*

(ii) *f'' exists on the open interval* $\mathrm{I}(a,b)$,

and let c and d be any two distinct numbers on $\mathrm{I}[a,b]$. *Then between c and d there is a number η such that*

$$f'(d) = f'(c) + (d - c)f''(\eta)$$

This fact will be used presently. First, we give a definition in which an "arc" of a graph means "a connected portion of the graph."

DEFINITION. *An arc of a graph is said to be:*

(i) ***concave upward*** *if each point of the arc is above the tangent to any other point of the arc, but*

(ii) ***concave downward*** *if each point of the arc is below the tangent to any other point of the arc.*

As an illustration, for the graph in Fig. 33.1 the arc between S and T is concave upward, the arc between T and U is concave downward, and the arc between U and V is concave upward.

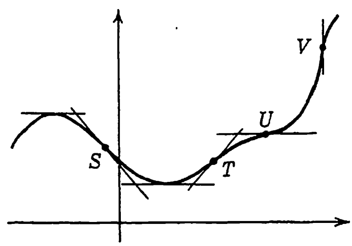


Figure 33.1

THEOREM 33.1. *For a function f, an arc of the graph of f is concave upward if throughout this arc $f'' > 0$, but is concave downward if everywhere on the arc $f'' < 0$.*

PROOF. Under the condition that $f''(x) > 0$ for $a \leq x \leq b$, we prove that the arc between $(a,f(a))$ and $(b,f(b))$ is concave upward.

Let x_1 be any number such that $a \leq x_1 \leq b$. The tangent to the arc at $(x_1,f(x_1))$ has equation

$$y = f(x_1) + f'(x_1)(x - x_1).$$

We shall show that every point of the arc is above this tangent, except the point $(x_1,f(x_1))$ of contact, by showing that

(1) $\quad f(x_2) > f(x_1) + f'(x_1)(x_2 - x_1)$ for $x_2 \neq x_1$, but $a \leq x_2 \leq b$.

To do so, we first use the Law of the Mean to obtain

(2) $\quad f(x_2) = f(x_1) + f'(x_3)(x_2 - x_1)$ with x_3 between x_1 and x_2.

Next we apply the Law of the Mean to the derived function f' to obtain

$$f'(x_3) = f'(x_1) + f''(x_4)(x_3 - x_1) \quad \text{with } x_4 \text{ between } x_1 \text{ and } x_3.$$

This expression for $f'(x_3)$ substituted into (2) yields

$$(3) \qquad f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1) + f''(x_4)(x_3 - x_1)(x_2 - x_1).$$

The last term in (3) is positive. For if $x_1 < x_2$ then $x_1 < x_3 < x_2$ so that $0 < x_3 - x_1$ and $0 < x_2 - x_1$, but if $x_2 < x_1$ then $x_2 < x_3 < x_1$ so that $x_3 - x_1 < 0$ and $x_2 - x_1 < 0$. In either case $(x_3 - x_1)(x_2 - x_1) > 0$. Thus the last term in (3) is positive since we are under the condition that $f''(x) > 0$ for all x between a and b.

Consequently, by dropping the last term from the equality (3) we obtain the inequality (1). Hence the arc is concave upward under the condition $f''(x) > 0$ for $a \le x \le b$.

The condition $f''(x) < 0$ for $a \le x \le b$ leads again to (3), but this time with the last term negative (as should be checked). Consequently the inequality in (1) is reversed, showing the arc to be concave downward.

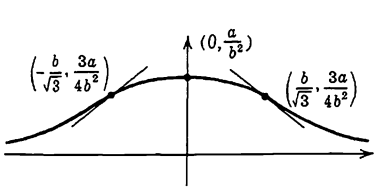


Figure 33.2

Example 1. With a and b positive constants, find the concave upward and concave downward portions of the graph of the function f defined by

$$f(x) = \frac{a}{x^2 + b^2}.$$

Solution. We first compute

$$f'(x) = \frac{-2ax}{(x^2 + b^2)^2} \quad \text{and} \quad f''(x) = 2a\frac{3x^2 - b^2}{(x^2 + b^2)^3},$$

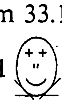
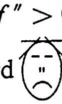
and note that $f''(x) > 0$ if $3x^2 - b^2 > 0$, but $f''(x) < 0$ if $3x^2 - b^2 < 0$. Thus,

$$f''(x) > 0 \text{ if either } x > b/\sqrt{3} \text{ or } x < -b/\sqrt{3}, \text{ but}$$

$$f''(x) < 0 \quad \text{if} \quad -b/\sqrt{3} < x < b/\sqrt{3}.$$

The concave upward portions are therefore separated from the concave downward portion by the points (see Fig. 33.2)

$$\left(-\frac{b}{\sqrt{3}}, \frac{3a}{4b^2}\right) \quad \text{and} \quad \left(\frac{b}{\sqrt{3}}, \frac{3b}{4b^2}\right).$$

From Theorem 33.1, a portion of the graph of a function where $f'' > 0$ is concave upward  but a portion where $f'' < 0$ is concave downward .

Thus, if x_1 is such that $f'(x_1) = 0$ and $f''(x_1) > 0$, then the point $(x_1, f(x_1))$ is the lowest point on a concave upward portion of the graph of f, whereas if $f'(x_1) = 0$ and $f''(x_1) < 0$ then the point $(x_1, f(x_1))$ is the highest point on a concave downward portion. See Fig. 33.3. We therefore have a third test for determining relative maxima and minima of a function. (For Test I see p. 92, and for Test II see p. 98.)

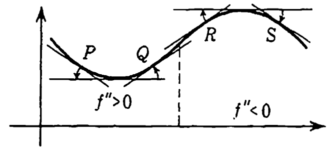


Figure 33.3

TEST III. *If f is a function and x_1 is a number such that $f'(x_1) = 0$ and $f''(x_1)$ exists, then the value $f(x_1)$ is:*

(i) *A relative minimum of f provided $f''(x_1) > 0$.*
(ii) *A relative maximum of f provided $f''(x_1) < 0$.*
(iii) *Not applicable if $f''(x_1) = 0$ or if $f''(x_1)$ does not exist.*

Example 2. Find the relative maxima and minima of the function f defined by

$$f(x) = 2\cos x - \cos 2x, \quad -\pi \le x \le \pi.$$

Solution. $f'(x) = -2\sin x + 2\sin 2x = -2\sin x + 4\sin x\cos x$

$$= 2\sin x(2\cos x - 1) = 0$$

if $x = -\pi, -\pi/3, 0, \pi/3$, or π. For Test III we also need

$$f''(x) = -2\cos x + 4\cos 2x.$$

Now $f''(-\pi) = 2 + 4 > 0$, so $f(-\pi) = -3$ is a relative minimum,
$f''(-\pi/3) = -1 - 2 < 0$, so $f(-\pi/3) = \frac{3}{2}$ is a relative maximum,
$f''(0) = -2 + 4 > 0$, so $f(0) = 1$ is a relative minimum,
$f''(\pi/3) = -1 - 2 < 0$, so $f(\pi/3) = \frac{3}{2}$ is a relative maximum,
$f''(\pi) = 2 + 4 > 0$, so $f(\pi) = -3$ is a relative minimum.

Example 3. In a homogeneous medium, light travels in a straight line at constant velocity depending upon the medium. Let the velocity of light in air be v_1 and in water be v_2. Show that a light ray from a source S in air to an object O under calm water will travel in the shortest time (as it does travel) if

$$\text{(5)} \qquad \frac{\sin \alpha_1}{v_1} = \frac{\sin \alpha_2}{v_2}$$

where α_1 and α_2 are the angles the ray in air and water makes, respectively, with the normal to the surface.

Solution. A ray from S to a point P on the surface and then to O, would do so in time $T(x)$ where (see Fig. 33.4 for the meaning of x, a, b, and c)

$$T(x) = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (c - x)^2}}{v_2}$$

Since $T'(x) = \dfrac{x}{v_1\sqrt{a^2 + x^2}} - \dfrac{c - x}{v_2\sqrt{b^2 + (c - x)^2}}$, this derivative is zero if and only if

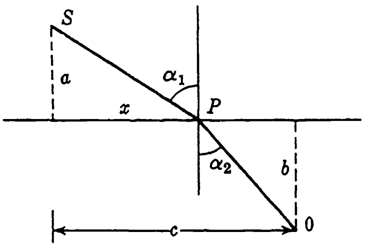


Figure 33.4

$$\text{(6)} \quad \frac{1}{v_1}\frac{x}{\sqrt{a^2 + x^2}} = \frac{1}{v_2}\frac{c - x}{\sqrt{b^2 + (c - x)^2}}.$$

Since $\sin \alpha_1 = x/\sqrt{a^2 + x^2}$ and $\sin \alpha_2 = (c - x)/\sqrt{b^2 + (c - x)^2}$, we see that if (6) holds, then (5) also holds.

There might be some doubt as to whether the value of x which satisfies (6) actually gives the minimum value of T. By taking the second derivative and simplifying:

$$T''(x) = \frac{1}{v_1}\frac{a^2}{(\sqrt{a^2 + x^2})^3} + \frac{1}{v_2}\frac{b^2}{(\sqrt{b^2 + (c - x)^2})^3}$$

which is positive for all x. Thus, the whole graph of T is concave upward so the only place where T' is zero [namely, when (6) is satisfied] does furnish the minimum value of T.

DEFINITION. *A point of a graph is said to be a* **point of inflection** *if at this point the tangent to the graph lies above an arc on one side and below an arc on the other side.*

Thus, in Fig. 33.1 the points S, T, U, and V are points of inflection. Also, the graph in Fig. 33.2 has two points of inflection.

For the graph of a function f, any arc where $f'' > 0$ is concave upward, and any portion where $f'' < 0$ is concave downward. Thus, at a point of inflection f'' can be neither positive nor negative. Consequently:

If $(x_0, f(x_0))$ is a point of inflection, then either $f''(x_0)$ exists and is zero or else $f''(x_0)$ does not exist.

It therefore follows that the abscissas of all points of inflection will be found among the solutions of $f''(x) = 0$ together with all values of x which are not in the domain of f''. Note, however, that even if x_0 is such that $f''(x_0) = 0$ it may possibly be that $(x_0, f(x_0))$ is not an inflection point. For example, the function f defined by

$$f(x) = (x - 1)^4$$

is such that $f''(x) = 12(x - 1)^2$ so $f''(1) = 0$, but $f''(x) > 0$ for $x \neq 1$ and the whole curve is concave upward with no point of inflection.

PROBLEMS

1. For the graphs of each of the following functions, find the portions concave upward, the portions concave downward, and find the points of inflection (if any):

a. $\{(x,y) \mid y = x^3 - 3x^2 + 2x - 4\}$.

b. $\{(x,y) \mid 0 \le x \le \pi, \quad y = \sin^2 x\}$.

c. $\{(x,y) \mid 0 \le x \le \pi, \quad y = 1 + \cos 2x\}$.

d. $\{(x,y) \mid -\pi/2 \le x \le \pi/2, \quad y = x^2/4 + \sin x\}$.

e. $\{(x,y) \mid y = x^{1/3}(x - 2)^{2/3}\}$.

f. $\{(x,y) \mid y = x^2 - 1/x\}$.

2. By using Test III, determine the relative maxima and minima for the function defined by

a. $f(x) = (x - 2)^2(x + 2)$.   c. $f(x) = \cos 3x - 3 \cos x, \quad -\pi \le x \le \pi$.

b. $f(x) = (x - 2)^2(x + 2)^2$.   d. $f(x) = 3 \cos x - 4 \cos^3 x, \quad -\pi \le x \le \pi$.

3. A box of square base and volume V is to be made from thin sheet metal. Find the dimensions for minimum:

a. Area if there is no lid.   b. Area if there is a lid.

c. Cost if there is no lid, bottom costs a¢/ft², sides b¢/ft².

d. Cost of lid c¢/ft² and otherwise as in Part c.

4. A sealed tin can is to be made in the shape of a right circular cylinder with volume V. Find the dimensions for:

a. Smallest total area.

b. The smallest amount of tin to buy if squares and rectangles of tin can be ordered of any size and each end is to be cut from a square (the corner portions being wasted). Give the size of the square and the rectangle to be ordered.

5. Figure Prob. 5 represents a straight high tension line passing at distance a and b from two factories F_1 and F_2, a transformer station T, and secondary power cables from T to F_1 and T to F_2. If the cost of the electric cable to F_1 is d_1\$/ft and to F_2 is d_2\$/ft, show that the minimum cost of cable is obtained by locating T in such a way that

$$d_1 \sin \alpha = d_2 \sin \beta.$$

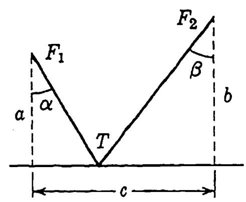


Figure Prob. 5

6. Does $\{(x,y) \mid y = x^3 - 3x^2 + 6x + 1\}$ have a tangent of minimum slope?

7. On $\{(x,y) \mid y = x^3\}$ find two points whose abscissas differ by 2 and the line joining them has slope as small as possible.

8. A window is to be designed in the form of a rectangle surmounted by a semicircle with diameter coinciding with the upper base of the rectangle. The perimeter of the window is to be s ft. Find the dimensions of the window that admits as much light as possible if:

a. The whole window is made of the same kind of glass.

b. The rectangle is clear glass, but the semicircle is colored glass admitting half as much light per square foot as the clear glass admits per square foot.

9. Prove that $\ln x \leq x - 1$ for all $x > 0$. (Hint: Find the maximum of $f(x) = \ln x - x + 1$.)

34. Simple Econometrics

In many problems some or all of the variables involved may take only integer values. Even under such circumstances it is sometimes possible to use calculus in such a way that much computation or mere routine work may be eliminated.

If the price of a good is increased, the demand for the good is (presumably) decreased and whether the profit is increased by higher price or decreased by smaller sale is a question an economist may be called upon to answer.

Example. An economist studying a certain hardware business finds that the wholesale price and overhead in handling n washing machines a month is

$$\$(110 + 39n)$$

and that n washing machines are sold a month at $\$p$ apiece where (this is the **demand law** in this case)

$$n = 50 - \tfrac{1}{2}p.$$

Find the retail price the economist should advise in order to yield the greatest profit.

Solution. Let $\$P(n)$ be the profit on n washing machines at $\$p$ apiece. Note that

$$\begin{aligned} P(n) &= np - (110 + 39n) \\ &= n(100 - 2n) - (110 + 39n) \qquad \text{(From (1))} \\ &= -2n^2 + 61n - 110, \end{aligned}$$

which is to be maximized. Since $P = \{(n,y) \mid y = -2n^2 + 61n - 110\}$ where n takes only integer values, consists of isolated points, but with

$$f = \{(x,y) \mid y = -2x^2 + 61x - 110\},$$

f is a curve passing through the isolated points of the set P, and we may find the highest of these isolated points by seeing which of them is nearest the level of the highest point of the curve. Since

$$f'(x) = -4x + 61 \quad \text{and} \quad f''(x) = -4$$

we see that the curve is concave downward with its highest point occurring where $x = \frac{61}{4}$. Since $15 < \frac{61}{4} < 16$, we compute

$$P(15) = 355 \quad \text{and} \quad P(16) = 354$$

and therefore the sale of 15 washing machines yields the greatest profit (which is

\$355) and to obtain the proper demand the price should be set [according to (1) with $n = 15$] at

$$\$p = \$(100 - 2.15) = \$70.$$

PROBLEMS

1. In a certain locality and type of soil it is found that if 20 orange trees are planted per acre, the yield will be 500 oranges per tree, and that the yield per tree is reduced by 15 for each additional tree per acre. What is the best number of trees to plant per acre?

2. The manager of a chain of stores finds that to buy and distribute x cans of tomatoes a day costs

$$65 + \frac{3}{10^2}x + \frac{25}{10^6}x^2 \text{ dollars.}$$

Also, by varying the price charged for the tomatoes he finds that if p cents is charged per can, then x cans a day are sold where

$$p = 23 - \frac{1}{400}x.$$

Find the number of cans he should buy per day and the price at which he should sell them.

3. The manufacturer of a certain article finds that if he makes x articles, $0 < x < 1000$ a day, he has:

First, fixed organizational cost of \$90 per day;

Second, unit production cost for each article of \$0.09; and

Third, maintenance, repairs, etc., $x^2/10^4$ dollars per day. Show that it costs $C(x)$ dollars to produce each article where

$$C(x) = \frac{90}{x} + 0.09 + \frac{x}{10^4}$$

and determine what number of articles per day makes the cost per article least.

4. A steel plant is capable of producing x tons, $0 \leq x \leq 8$, of low-grade steel per day and y tons of high-grade steel per day, where $y = (40 - 5x)/(10 - x)$. If the fixed market price of low-grade steel is half that of high-grade steel, show that about $5\frac{1}{2}$ tons of low-grade steel should be produced per day for maximum profit. Show that the given relation between tons of high- and low-grade steel steel is not a sensible one if $0 \leq x \leq 12$.

5. A manufacturer finds that he can sell x units per week of his product at p dollars per unit where $p = \frac{1}{3}(375 - 5x)$ and that the cost of producing x units per week is $500 + 15x + \frac{1}{5}x^2$ dollars.

   a. Find the number of units he should produce per week and the price he should charge per unit to make his profit as great as possible.

   b. A tax of \$5 per unit is later imposed on the manufacturer. How should he change his production and selling price to keep his profit as great as possible?

35. Rates

We have already seen that if s is a function and for each number t we let $s(t)$ be the coordinate at time t of a particle on a linear coordinate system, then $s'(t)$ is the velocity of the particle at time t. In many problems involving rates, a function s may be defined whose derivative gives pertinent information. The following examples illustrate two such problems.

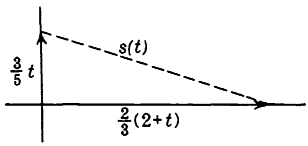


Figure 35.1

Example 1. A highway crosses a railroad track at right angles. A car traveling at the constant rate of 40 mi/hr goes through the intersection 2 min before the engine of a train traveling at the constant rate of 36 mi/hr goes through the intersection. At what rate are the car and engine separating 10 min after the train went through the intersection?

Solution. Just t min after the engine went through the intersection, the engine is $\frac{36}{60}t = \frac{3}{5}t$ mi down the track, the car is $\frac{40}{60}(2 + t) = \frac{2}{3}(2 + t)$ mi down the highway, the car and engine are

$$s(t) = \sqrt{(\tfrac{3}{5}t)^2 + (\tfrac{2}{3})^2(2 + t)^2} = \tfrac{1}{15}\sqrt{181t^2 + 400t + 400}\ \text{mi}$$

apart, and are separating

$$s'(t) = \frac{1}{15}\,\frac{2(181t + 200)}{2\sqrt{181t^2 + 400t + 400}}\ \text{mi/hr}.$$

Thus, the answer to the problem is $s'(10) = 201/(15)^2$ mi/min $= 53.6$ mi/hr.

Example 2. A man grasps a rope 6 ft above the ground. From his hand the rope goes straight up to a pulley 36 ft above the ground and then straight down to a weight resting on the ground. If the man, holding fast to the rope and keeping his hand 6 ft above ground, walks away at a constant rate of 10 ft/sec, how fast is the weight rising:

a. one sec after the man starts to walk?

b. When the weight is 20 ft above ground?

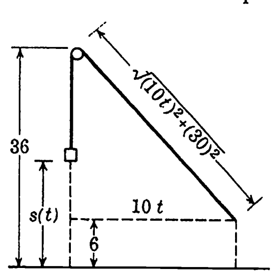


Figure 35.2

Solution. The length of the rope from the man's hand over the pulley and down to the ground is 66 ft. Thus (see Fig. 35.2), t sec after the man started walking he has moved $10t$ ft, the weight has moved $s(t)$ ft where

$$66 = \sqrt{(10t)^2 + (30)^2} + 36 - s(t); \quad \text{i.e.,}$$

$$s(t) = \sqrt{(10t)^2 + (30)^2} - 30$$

and the weight is moving

$$s'(t) = \frac{10t}{\sqrt{t^2 + 9}} \text{ ft/sec.}$$

The answer to Part a is therefore $s'(1) = \sqrt{10}$ ft/sec or approximately 3.16 ft/sec.

Part b is answered by first finding the value of t for which $s(t) = 20$ and then substituting this value in $s'(t)$. Since

$$20 = \sqrt{(10t)^2 + (30)^2} - 30$$

if and only if $100t^2 = 2500 - 900 = 1600$, i.e., if and only if $t = \pm 4$, we discard -4 and obtain the answer

$$s'(4) = \frac{40}{\sqrt{16 + 9}} = 8 \text{ ft/sec.}$$

PROBLEMS

1. Concentric rings are produced by a stone thrown into a calm lake. If the radius of a ring is increasing at the rate of 1.5 ft/sec, find the rates of increase of the circumference of the ring and the area of the ring when the radius of the ring is (a) 5 ft; (b) 10 ft.

2. A 20-ft ladder leans against a vertical wall. The lower end is pulled along the horizontal floor at a constant rate of 4 ft/sec and the top slides down the wall. Find the rate at which the top moves when:
   a. The lower end is 5 ft from the wall.
   b. The lower end is 16 ft from the wall.
   c. The upper end is 16 ft above the floor.
   d. For what time interval is the top moving faster than the bottom?

3. Two highways make an angle of 60° with each other. One car traveling 40 mi/hr goes through the intersection at 11:30 A.M., and another car traveling 48 mi/hr on the other highway goes through the intersection at 12 noon. Show that at t hr after noon the distance between the cars is

$$s(t) = 4\sqrt{124t^2 + 40t + 25} \quad \text{or} \quad S(t) = 4\sqrt{364t^2 + 160t + 25}$$

   according to the relative directions of the cars. In both cases find the rate at which the cars are separating at 1 P.M.

4. An airplane traveling level and going due north at 180 mi/hr passed over an airport just 10 min before a second airplane traveling at the same level going north 60° west at 240 mi/hr passed over the same airport. Show that t min after the second airplane passed over the airport, the airplanes were $s(t) = \sqrt{13t^2 + 60t + 900}$ miles apart. Find how fast the airplanes were separating when the second airplane passed over the airport, and also 5 min later.

5. A water tank is in the shape of a right circular cone of radius 3 ft and altitude 9 ft with axis vertical and vertex down. Water is flowing into the tank at the constant rate of 8 ft^3/min. Let $h(t)$ be the depth of water in the tank t min after the water begins to flow in. Find
   a. $h(t)$.
   b. How long it takes to fill the tank.
   c. How fast the water is rising when $t = 8$.
   d. How fast the water is rising when it is 6 ft deep.

6. Sand is being poured from a spout to the ground at the constant rate of 48 ft^3/min. The sand forms a conical pile with diameter of its base 3 times its altitude. Let $h(t)$ be the height of the pile t min after sand begins to flow.
   a. Find $h(t)$.
   b. How fast is the height increasing 8 min after the sand begins to flow?
   c. How long does it take to form a pile of height 4 ft?
   d. How fast is the height increasing when the pile is 4 ft high?

7. Two airplanes traveling at the same level are headed for the same airport, one traveling east at 180 mi/hr and the other north 30° east at 240 mi/hr. At a certain instant the first is 100 mi from the airport and the second is 50 mi from the airport. At this instant and 5 min later, find the distance between the airplanes and the rates at which they are approaching each other.

8. At 1 P.M. a ship going due north at 10 knots is 13 nautical miles due south of a ship going west at 15 knots. Is the distance between the ships increasing or decreasing at 1:15 P.M.; at 1:30 P.M.? Find the rates at these two times. At what time are the ships closest together?

9. Figure Prob. 9 represents a crank arm OA, which revolves with constant rate ω radians/sec and has length r in. The connecting rod AB has length L in. The piston B moves back and forth along the x-axis. Show that B has velocity

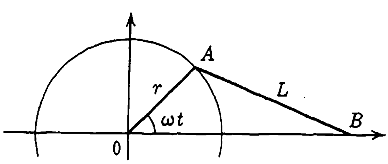


Figure Prob. 9

$$-\left\{r \sin \omega t + \frac{r^2 \sin \omega t \cos \omega t}{\sqrt{L^2 - r^2 \sin^2 \omega t}}\right\} \omega \text{ in./sec.}$$

36. Related Rates

Given the function f defined by $f(x) = 3x^2 - x$, then

$$f'(x) = 6x - 1 \quad \text{and also} \quad D_x f(x) = 6x - 1.$$

For example, $f'(2) = 6 \cdot 2 - 1 = 11$, but on the other hand, by common consent and usage, $D_x f(2)$ is not equal to 11, but the notation

$$D_x f(x)]_{x=2} = 6x - 1]_{x=2} = 6 \cdot 2 - 1 = 11$$

is used. Since $f(2) = 3 \cdot 2^2 - 2 = 10$, the meaning of $D_x f(2)$ is "the derivative of a constant" so that $D_x f(2) = D_x 10 = 0$.

In Sec. 35 rate problems were solved by defining pertinent functions explicitly in terms of time, such that the time-derivatives of these functions gave the desired results. In some situations it is, however, not feasible to express the functional relationship explicitly in terms of t.

Example 1. From a hemispherical bowl of radius 10 ft which is partly full of water, the water is flowing through a hole. At the instant the water is 6 ft deep, the rate of flow is 5ft³/min. Find the rate the surface of water is falling at this instant.

Solution. As illustrated in Fig. 36.1, when water stands h ft deep in the bowl, the volume is $V = \pi(10h^2 - \frac{1}{3}h^3)$ ft³, a formula which is given now and will be derived later. Thus, the physical problem is translated into mathematical terms as:

Given: 1) $V = \pi(10h^2 - \frac{1}{3}h^3)$

2) $D_t V]_{h=6} = -5$

To find: $D_t h]_{h=6}$.

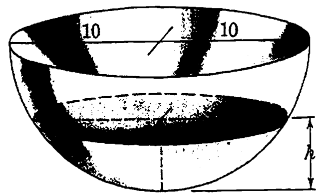


Figure 36.1

Although no explicit expression for the function h in terms of t is known, we may use the formula for the derivative of the composite function V to write

$$D_t V = D_h V \cdot D_t h = D_h[\pi(10h^2 - \tfrac{1}{3}h^3)] \cdot D_t h$$
$$= \pi(20h - h^2) \cdot D_t h.$$

The rate of change of V and the rate of change of h are related by this equation. The rate of change of V is known only at one instant (when $h = 6$) and the rate of change of h at this instant is required. Hence, from

$$-5 = D_t V]_{h=6} = \pi(20 \cdot 6 - 6^2) \cdot D_t h]_{h=6}, \quad \text{so that}$$

$$D_t h]_{h=6} = \frac{-5}{\pi(120 - 36)} = -\frac{5}{84\pi}.$$

Thus, to return to the physical problem, the surface is falling $5/(84\pi)$ ft/min.

In Example 1 the functions V and h cannot be expressed specifically in terms of t. If the problem had been more specific (and even less practical) by stating that water leaked out at the rate of 5 ft³/min whenever there was any water present, then V (and hence h) could have been expressed in terms of t.

Even though all functions in a problem are expressible specifically in terms of t, it may be less trouble not to so express the functions.

Example 2. A cruiser is steaming a straight track at 20 knots. An airplane, flying so low that its angle of elevation from the cruiser may be neglected, is going 200 knots along a straight line making an angle of 60° with the track of the cruiser.

A radar scope keeps a battery of guns on the cruiser pointed toward the airplane. At the instant the guns are perpendicular to the cruiser, the airplane is $\frac{1}{2}$ nautical miles away. Find the rate at which the guns are turning at this instant.

Solution. Referring to Fig. 36.2, and using the sine law, we are:

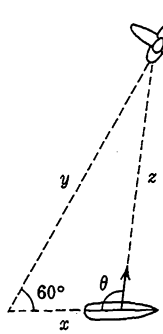


Figure 36.2

Given: $D_t x = 20,$

$$D_t y = -200,$$

$$(1) \qquad \frac{\sin \theta}{y} = \frac{\sin(\frac{2}{3}\pi - \theta)}{x},$$

$$z = \tfrac{1}{2} \quad \text{when} \quad \theta = \pi/2,$$

To find: $D_t\theta]_{\theta = \pi/2}$

After writing (1) as $x \sin\theta = y \sin(\frac{2}{3}\pi - \theta)$, we have

$$D_t[x \sin \theta] = D_t[y \sin(\tfrac{2}{3}\pi - \theta)],$$

$$x D_t \sin\theta + (\sin\theta) D_t x = y \cos(\tfrac{2}{3}\pi - \theta) D_t(\tfrac{2}{3}\pi - \theta) + \sin(\tfrac{2}{3}\pi - \theta) D_t y,$$

$$x \cos\theta D_t\theta + (\sin\theta) D_t x = -y \cos(\tfrac{2}{3}\pi - \theta) D_t\theta + \sin(\tfrac{2}{3}\pi - \theta) D_t y.$$

Now substitute $D_t x = 20$ and $D_t y = -200$ and solve for $D_t\theta$:

$$D_t\theta = \frac{-200 \sin(\frac{2}{3}\pi - \theta) - 20 \sin\theta}{x \cos\theta + y \cos(\frac{2}{3}\pi - \theta)}.$$

But at the instant when $\theta = \pi/2$, and only at this instant, $z = \frac{1}{2}$ so that at this instant $x = 1/(2\sqrt{3})$ and $y = 1/\sqrt{3}$. Hence,

$$D_t\theta]_{\theta=\pi/2} = \frac{-200 \sin(\pi/6) - 20 \sin(\pi/2)}{\frac{1}{2\sqrt{3}} \cos\frac{\pi}{2} + \frac{1}{\sqrt{3}} \cos\frac{\pi}{6}} = -240 \text{ rad/hr} = -4 \text{ rad/min}.$$

PROBLEMS

1. At a certain instant the area of a sphere is increasing at the rate of 3 in.^{2}/min. At what rate is the volume changing at the same instant?

2. The area of a right triangle is increasing at the rate of 2 in.^{2}/sec and one of the legs is increasing at the rate of $\frac{1}{2}$ in./sec. For what lengths of the other leg will that leg be increasing? Decreasing?

3. Water is being poured into a hemispherical bowl of radius 4 ft at the rate of 2 ft^3/min. Find the rate at which the level of the water is rising at the instant it is 2 ft deep.

4. The height of a right circular cylinder is increasing at the rate of 3 in./min, and the radius of the base is decreasing at the rate of 2 in./min. Find the rate at which the volume of the cylinder is changing.

5. A gas, obeying Boyle's law (i.e., pressure $\times$ volume $=$ constant), occupies a volume of 1,000 in.^{3} and is under pressure of 1 lb/in.^{2} If the gas is being compressed at the rate of 4 in.^{3}/min, find the rate at which the pressure is changing at the instant when the volume is 800 in.^{3} (The temperature of the gas remains unaltered during the process.)

6. A ladder 20 ft long leans against a vertical wall. The bottom of the ladder is being pulled away from the wall at the rate of 2 ft/sec. Find the rate at which the area of the right triangle bounded by the ladder, the wall, and the horizontal floor is changing when the top of the ladder is 12 ft above the floor.

7. A lamp post is 8 ft high. A stone is thrown vertically upward with initial velocity of 20 ft/sec from a point on the ground which is 100 ft from the bottom of the post. At what rate is the shadow of the stone moving on the ground when the stone is 4 ft above the ground if:

   a. The ground is level?

   b. The ground is inclined in such a way that the level of the point from which the stone is thrown is 10 ft below the level of the bottom of the post?

   Disregard air resistance and use $g = 32$ ft/sec.

8. Wheat pours at the rate of 10 ft^3/min from a spout to the floor forming a conical pile with radius of base always half the altitude. How fast is the height changing at the instant it reaches 5 ft?

9. A snowball is the shape of a sphere; as it starts rolling down a mountain, it is growing in size but remaining spherical. If the rate at which the volume is increasing is proportional to the surface, prove that the rate at which the radius is changing remains constant.

10. A cube has volume increasing at the rate of 300 in.^{3}/sec at the instant the edge is 25 in. How fast is the edge changing at this instant?

11. A boat with deck 25 ft above the harbor bed is anchored with 75 ft of rope which is kept taut in a straight line by a stiff breeze. Rope is then hauled in at the rate of 5 ft/min. How fast is the boat moving the instant 10 ft of rope have been hauled in?

12. A conical tank (vertex down) is 30 ft across the top and 10 ft deep. Water is flowing in at a constant rate, but is also leaking out at 0.5 ft^3/min. The instant the water is 5 ft deep the surface is rising at the rate of 1.5 ft/min. How fast is the water flowing in?

13. A baseball diamond is a square, 90 ft on each side. The instant a runner is halfway from home to first base he is going toward first base at 30 ft/sec. How is his distance from second base changing at this instant?

14. A bridge is 30 ft above and at right angles to the banks of a river in which the water is flowing 10 ft/sec. A man walking 5 ft/sec on the bridge sees a block of wood directly below him on the water. How fast are man and block separating 3 sec later?

37. Linear Acceleration

The first derivative is interpreted geometrically in terms of slope and physically in terms of velocity. In Sec. 33, the second derivative was interpreted geometrically in terms of concavity properties of a graph. We now investigate a physical interpretation of the second derivative.

Let a particle move on a coordinate line in such a way that at time t its position is $s(t)$. Then

$$v(t) = s'(t) \quad \text{linear units/time unit},$$

is its velocity at time t. The velocity may not be constant and

$$v(t+h) - v(t) \text{ velocity units}$$

is the change in velocity during the time interval from t to $t + h$. Also,

$$\frac{v(t+h) - v(t)}{h} \frac{\text{velocity units}}{\text{time unit}}$$

is the average change in velocity, or the average acceleration in the time interval from t to $t + h$. Furthermore,

$$v'(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h} \frac{\text{velocity units}}{\text{time unit}}$$

is defined to be **acceleration** at time t. The formalism

$$\frac{\text{velocity units}}{\text{time unit}} = \frac{\text{linear units/time unit}}{\text{time unit}} = \frac{\text{linear units}}{(\text{time unit})^2}$$

and the use of α for the acceleration function leads to

$$\alpha(t) = v'(t) = s''(t) \quad \text{linear units/(time unit)}^2.$$

As usually stated loosely, "The acceleration is the derivative of the velocity or the second derivative of the position."

Example. Describe the motion under the foot-minute law

$$s(t) = t^3 - 2t^2 - 4t + 3, \quad -2 \le t \le 3.5.$$

Solution. $v(t) = 3t^2 - 4t - 4 = (3t + 2)(t - 2)$, and

$$\alpha(t) = 6t - 4 = 2(3t - 2).$$

The beginning and end of the time interval and pertinent instants (where either $v(t) = 0$ or $\alpha(t) = 0$) are $t = -2, -\frac{2}{3}, \frac{2}{3}$, 2 and 3.5:

$$v(t) > 0 \quad \text{if either} \quad -2 \le t < -\tfrac{2}{3} \quad \text{or} \quad 2 < t \le 3.5,$$

$$v(t) < 0 \quad \text{if} \quad -\tfrac{2}{3} < t < 2,$$

$$\alpha(t) < 0 \quad \text{if} \quad 2 \le t < \tfrac{2}{3}, \quad \text{but} \quad \alpha(t) > 0 \quad \text{if} \quad \tfrac{2}{3} < t < 3.5.$$

Motion "starts" at $s(-2) = -5$ with initial velocity $v(-2) = 16$ ft/min, hence in the positive direction, and continues in this direction until $t = -\frac{2}{3}$ when, at

$s(-\frac{2}{3}) = \frac{121}{27}$, direction of motion is reversed and maintained until $t = 2$, when, at $s(2) = -5$, motion in the positive direction is resumed to continue throughout the remainder of the given time interval. Since $\alpha(t) < 0$ for $-2 \le t < \frac{2}{3}$ the velocity decreases through positive values to 0 to negative values until $\alpha(t) = 0$ when $t = \frac{2}{3}$, $v(\frac{2}{3}) = -\frac{16}{3}$ ft/min, and $s(\frac{2}{3}) = -\frac{7}{27}$. Velocity then increases throughout the remaining time interval (since $\alpha(t) > 0$ for $\frac{2}{3} < t < 3.5$); that is, velocity increases through negative values to 0 and then through positive values.

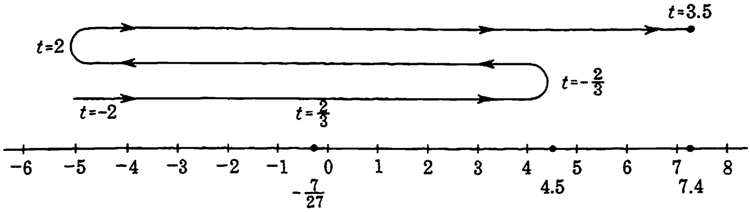


Figure 37

A realistic interpretation of the relation between velocity and acceleration can be made in terms of a car going forward ($v > 0$) or backward ($v < 0$) and either applying gas or the brakes. Thus,

$v > 0, \alpha > 0$ means forward and gas,
$v > 0, \alpha < 0$ means forward and brakes,
$v < 0, \alpha > 0$ means backward and brakes,
$v < 0, \alpha < 0$ means backward and gas.

Notice that $\alpha > 0$ means the driver's shoulders are being pressed back. In particular the brake is a "positive accelerator" in case the car is going backward.

By definition, **speed** is the absolute value of velocity.

PROBLEMS

1. Describe the motion given by the following laws:
   a. $s(t) = t^3 - 4t^2 - 3t + 10, \quad -5 \le t \le 5.$
   b. $s(t) = 4 + 4t + 2t^2 - t^3, \quad -3 \le t \le 3.$
   c. $s(t) = 3 \sin 2t, \quad 0 \le t \le \pi.$
   d. $s(t) = \frac{1}{2}(t - \sin t \cos t), \quad 0 \le t \le 2\pi.$

2. Given the position function s defined below, find the velocity function v and the acceleration function α. Solve the inequalities $v(t) > 0$, $v(t) < 0$, $\alpha(t) > 0$, and $\alpha(t) < 0$.
   a. $s(t) = 2t^3 - 3t^2 - 36t + 5, \quad -5 \le t \le 5.$
   b. $s(t) = 2t^3 - 9t^2 + 12t + 1, \quad t > 0.$
   c. $s(t) = \sin^2 t, \quad 0 \le t \le 2\pi.$
   d. $s(t) = \cos^2 t, \quad 0 \le t \le 2\pi.$

3. For each of the following laws of motion find the relative maxima, relative minima, the maximum, and the minimum of both the velocity and the acceleration.

   a. $s(t) = t^4 - 6t^2 + 5, \quad -2 \leq t \leq 3.$

   b. $s(t) = 6t^5 - 10t^4 + 20t^3 - 60t^2 - 160t + 101, \quad 0 \leq t \leq 3.$

   c. $s(t) = 3t - \sin 2t + \cos 2t, \quad 0 \leq t \leq \pi.$

   d. $s(t) = t^2 - \sin^2 t, \quad 0 \leq t \leq 2\pi.$

38. Simple Harmonic Motion

DEFINITION. *With $a \neq 0$, $b \neq 0$ and t_0 given numbers, a motion governed by either of the laws*

$$x(t) = a \cos b(t - t_0) \quad \text{or} \quad x(t) = a \sin b(t - t_0) \tag{1}$$

is said to be ***simple harmonic*** *with* ***amplitude*** $|a|$, ***period*** $2\pi/|b|$ *and* **phase constant** t_0.

Either law may be transformed into the other merely by changing the phase constant. For example,

$$\begin{aligned} a \sin b(t - t_0) &= a \cos [b(t - t_0) - \pi/2] \\ &= a \cos b[t - (t_0 + \pi/2b)]. \end{aligned}$$

The period is given as $2\pi/|b|$ since $\cos (A \pm 2\pi) = \cos A$, and hence

$$a \cos b\left[\left(t + \frac{2\pi}{|b|}\right) - t_0\right] = a \cos \left[b(t - t_0) + 2\pi \frac{b}{|b|}\right] = a \cos b(t - t_0)$$

because $b/|b| = \pm 1$ according to whether $b > 0$ or $b < 0$, respectively.

In studying the harmonic laws (1), it is convenient to make a translation to a new time-origin by setting

$$T = t - t_0$$

(so $T = 0$ when $t = t_0$) and obtaining, respectively,

$$x(T) = a \cos bT \quad \text{and} \quad x(T) = a \sin bT. \tag{2}$$

Example. Show that the motion governed by the law

$$s(t) = a_1 \cos bT + a_2 \sin bT$$

is simple harmonic and find its amplitude, period, and phase constant.

Solution. There is a number T_0 (see Fig. 38) such that

$$\cos T_0 = \frac{a_1}{\sqrt{a_1^2 + a_2^2}} \quad \text{and} \quad \sin T_0 = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}.$$

Thus
$$s(t) = \sqrt{a_1^2 + a_2^2}\left\{\frac{a_1}{\sqrt{a_1^2 + a_2^2}} \cos bT + \frac{a_2}{\sqrt{a_1^2 + a_2^2}} \sin bT\right\}$$

$$= \sqrt{a_1^2 + a_2^2}\,\{\cos T_0 \cos bT + \sin T_0 \sin bT\}$$

$$= \sqrt{a_1^2 + a_2^2}\, \cos (bT - T_0) = \sqrt{a_1^2 + a_2^2} \cos b\left(T - \frac{T_0}{b}\right).$$

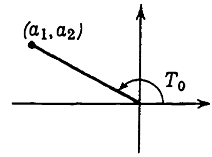


Figure 38

Hence, $s(t)$ is written in the form of the first law in (1) with amplitude $\sqrt{a_1^2 + a_2^2}$ period $2\pi/b$ and phase constant T_0/b.

For x_0 a constant the law of motion given as

$$x = x_0 + a \cos b(t - t_0)$$

is again simple harmonic, since the law may be written in the form (2) by setting $X = x - x_0$ and $T = t - t_0$ to obtain

$$X = a \cos bT.$$

PROBLEMS

1. For each of the following harmonic laws, find the amplitude, period, and phase constant.

   a. $x = 3 \cos (2t + 1)$.
   b. $x = 3 \sin (1 - 2t)$.
   c. $x = 5 \sin \pi(t - 1)$.
   d. $x = 10 \sin 120\pi t$.
   e. $x = 0.5 \sin \pi(120t - 0.5)$.
   f. $x = 6 \sin \pi(72 - 2t)$.

2. Show that each of the following may be written as a harmonic law. In each case find the amplitude, period, and the phase constant of smallest absolute value.

   a. $x = \cos t + \sin t$.
   b. $x = \cos 2t - \sin 2t$.
   c. $x = 3 \cos \pi t + 4 \sin \pi t$.
   d. $x = 12 \cos t - 5 \sin t$.
   e. $x = \cos^2 t$.
   f. $x = \sin^2 \pi t$.
   g. $x = \cos^2 t - \sin^2 t$.
   h. $x = 2 \sin 3t \cos 3t$.

3. For a particle moving according to either of the laws (1), show that each of the following is also a simple harmonic law.

   a. The velocity.
   b. The acceleration.
   c. The product of position and velocity.
   d. The square of the velocity.

4. Figure Prob. 4 represents a circular disc of radius R fastened to a shaft at a point a units from the center. The shaft turns uniformly, and the U-shaped mechanism is clamped so the prongs are always vertical, but the attached shank can move back and forth. Show that the shank moves harmonically.

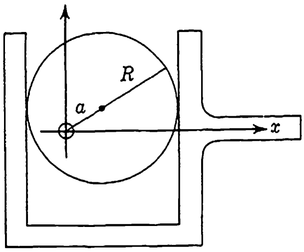
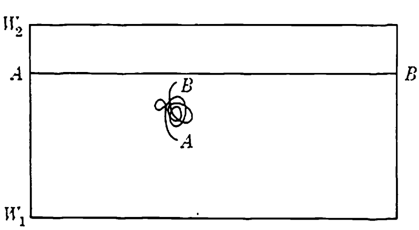


Figure Prob. 4

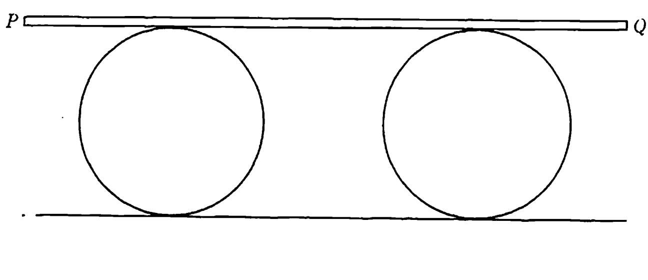
5. A point starts at the point $(R,0)$ and moves with uniform speed counter clockwise around a circle of radius R and center at the origin. Show that the projection of the point on the x-axis, or the y-axis, has harmonic motion.



The rectangle represents the floor of a room and AB an inelastic piece of string. The blob represents the same piece of string wadded up and thrown onto the floor. Prove there is at least one point of the string that is the same distance from the wall line W_1W_2 in both positions.

Visualize the figure below as the ends of two perfect circular-cylindrical logs on a smooth horizontal plane and PQ as the edge of a board. As the logs roll the board stays level. Can one of the logs be replaced by another with a non-circular cross section and the board still stay level as the logs roll?

Think before turning to page 618.



CHAPTER 4

Additional Concepts

In recorded history there is clear evidence that before the 17th century some men were pondering questions which later were considered as in the domain of calculus. Isaac Newton (1642–1727) in England and Gottfried Wilhelm Leibniz (1646–1716) in Germany are, however, regarded as independently founding systematic methods now classified as calculus. The underlying principles adopted and expanded by these two men are remarkably similar in content, but the terminology and notation they used are quite different. As the subject developed and was propagated over the world these two streams of influence merged, but neither subdued the other, so that now rudiments of both systems permeate the literature of mathematics and its applications. For example, $D_x f(x), f'(x)$, and $\dfrac{d f(x)}{dx}$ (see Sec. 41) are alternative notations for the derivative of a function f at x.

Although this chapter is primarily concerned with problems inverse to finding derivatives, a secondary purpose is to begin using alternative terms and symbols in connection with derivatives.

Do you know what "Fluxions" are? Had Newton's appellation persisted this book would have been entitled *Methods of Fluxions and Analytic Geometry.*

39. Derived Functions Equal

For c a constant and F a function let G be the function defined by

$$G(x) = F(x) + c.$$

Then $D_xG(x) = D_x[F(x) + c] = D_xF(x)$ so that $F' = G'$; that is:

If two functions differ at most by a constant then they both have the same derived function.

The converse theorem is also true; precisely:

THEOREM 39. *Let F and G be continuous on the closed interval* I$[a,b]$ *and such that F' and G' exist and are equal on the open interval* I(a,b); *that is,*

$$(1) \qquad F'(x) = G'(x) \quad \textit{for} \quad a < x < b.$$

Then F and G differ on I$[a,b]$ *at most by a constant; in particular,*

$$(2) \qquad F(x) - F(a) = G(x) - G(a) \quad \textit{for} \quad a \le x \le b,$$

$$(3) \qquad F(b) - F(a) = G(b) - G(a)$$

and there is a constant c such that

(4) $$F(x) = G(x) + c \quad for \quad a \leq x \leq b.$$

PROOF. Let f be the function defined by

$$f(x) = F(x) - G(x), \quad \text{for } a \leq x \leq b.$$

Now f is continuous on $I[a,b]$, f' exists on $I(a,b)$; and from (1)

(5) $$f'(x) = F'(x) - G'(x) = 0, \quad \text{for} \quad a < x < b.$$

Let x be a number such that $a \leq x \leq b$. Then f satisfies the conditions of the Mean Value Theorem (Theorem 32.2) on $I[a,x]$ so choose ξ_x such that $a < \xi_x < x$ and

(6) $$f(x) = f(a) + (x - a)f'(\xi_x).$$

Now $a < \xi_x < b$ so that $f'(\xi_x) = 0$ from (5) and thus for $a < x \leq b$

$$f(x) = f(a)$$

and then also for $x = a$. But $f(x) = F(x) - G(x)$ and $f(a) = F(a) - G(a)$ so that

$$F(x) - G(x) = F(a) - G(a) \quad \text{for} \quad a \leq x \leq b$$

which is equivalent to (2). Upon setting $x = b$ we have an equation equivalent to (3) and upon setting $F(a) - G(a) = c$ we obtain (4).

A geometric interpretation is obtained by taking two curves such that whenever a line perpendicular to the x-axis intersects one curve, it also intersects the other, and at the points of intersection the slopes of the tangents to the respective curves are equal. Then, according to Theorem 39, the points on the curve with a given abscissa are the same distance apart as for any other abscissa.

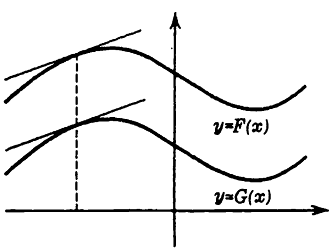


Figure 39

From Theorem 39 it also follows that if two particles move in such a way that they have the same velocity law, then the particles need not coincide, but their distance apart at one instant is the same as at any other instant.

40. Derivative Systems

Solving $x^2 = 4$ is an inverse operation to squaring a number, finding logarithms of a given base is inverse to raising this base to powers, and finding a number (or angle) whose sine is $\frac{1}{2}$ is inverse to taking the sine of a number (or angle). Consider now the inverse of finding the derived function of a given function. Thus, if

$$f(x) = x^2$$

then $D_x f(x) = 2x$, but inversely $D_x(\frac{1}{3}x^3) = \frac{1}{3}(3x^2) = x^2 = f(x)$. Furthermore, $D_x(\frac{1}{3}x^3 + 2) = f(x)$ and, from Theorem 39, if g is any function whose derived function is f, then there is a constant c such that

$$g(x) = \tfrac{1}{3}x^3 + c.$$

DEFINITION 40.1. *Given a function f, then any function whose derived function is f is said to be an* ***anti-derived function*** *of f or an* ***anti-derivative*** *of f.*

In this terminology, Theorem 39 may be stated as:

If f is a given function, then any two anti-derived functions of f differ by a constant.

Thus, if f has an anti-derived function, then there is a whole family of functions each of which is an anti-derived function of f and any member of the family may be obtained by adding a constant to any other member of the family.

In some situations, as described below, a function f is not given, but its derived function is given and a specific value of f is also given. A problem is then to find the function f itself. For example, let it be given that

(1) $$f'(x) = 2x \quad \text{for all } x \text{ and that } f(3) = 1.$$

Since $D_x x^2 = 2x$ it follows that there is a constant c such that $f(x) = x^2 + c$ for all x. In particular then $f(3) = 3^2 + c$. But it is given that $f(3) = 1$ so it is now known that $9 + c = 1$ and, hence, that $c = -8$. Consequently the function f given by

$$f(x) = x^2 - 8.$$

satisfies both conditions in (1) and is the only function satisfying these conditions.

DEFINITION 40.2. *The derived function of a function together with a specific value of the function is referred to as a* ***derivative system****, and the function satisfying both conditions is said to be the* ***solution*** *of the derivative system.*

Example 1. A particle in linear motion starts when $t = 0$ at the point 2 and thereafter has velocity law

$$v(t) = \sin 3t.$$

Find the position of the particle at any time $t > 0$.

Solution. As $s'(t) = v(t)$, an equivalent problem is: Solve the derivative system

$$s'(t) = \sin 3t, \quad s(0) = 2.$$

Noting that $D_t(-\frac{1}{3}\cos 3t) = -\frac{1}{3}(-3\sin 3t) = \sin 3t$ we have

$$s(t) = -\tfrac{1}{3}\cos 3t + c,\ s(0) = -\tfrac{1}{3}\cos 0 + c = -\tfrac{1}{3} + c = 2.$$

Hence, $c = \frac{7}{3}$ and the solution is $s(t) = -\frac{1}{3}\cos 3t + \frac{7}{3}$.

Example 2. A curve passes through the point (1,0) and at each point (x,y) of the curve the slope m of the tangent to the curve is given by $m = \cos^2 \pi x \sin \pi x$. Find an equation of the curve.

Solution. The problem is translated into: Solve the derivative system

$$D_x y = \cos^2 \pi x \sin \pi x, \quad y = 0 \quad \text{when} \quad x = 1. \tag{2}$$

Since $D_x \cos^3 \pi x = 3\cos^2 \pi x D_x \cos \pi x = -3\pi \cos^2 \pi x \sin \pi x$, it should be seen that

$$D_x\left(-\frac{1}{3\pi}\cos^3 \pi x\right) = \cos^2 \pi x \sin \pi x.$$

Hence, there is a constant c such that the desired equation is

$$y = -\frac{1}{3\pi}\cos^3 \pi x + c.$$

But $y = 0$ when $x = 1$ so that $0 = -\frac{1}{3}\pi \cos^3 \pi + c = \frac{1}{3}\pi + c$ and $c = -\frac{1}{3}\pi$. Hence,

$$y = -\frac{1}{3\pi}(1 + \cos^3 \pi x)$$

satisfies both conditions of (2) and is the desired equation.

Before attempting the problems, check each of the following:

If $f'(x) = ax^p$, then $f(x) = \dfrac{a}{p+1}x^{p+1} + c,\ p \neq -1.$

If $f'(x) = a\cos bx$, then $f(x) = \dfrac{a}{b}\sin bx + c,\ b \neq 0.$

If $f'(x) = a\sin bx$, then $f(x) = -\dfrac{a}{b}\cos bx + c,\ b \neq 0.$

If $f'(x) = \sin^p bx \cos bx$, then $f(x) = \dfrac{1}{b(p+1)}\sin^{p+1} bx + c,\ b \neq 0,\ p \neq -1.$

If $f'(x) = \cos^p bx \sin bx$, then $f(x) = \dfrac{-1}{b(p+1)}\cos^{p+1} bx + c,\ b \neq 0,\ p \neq -1.$

PROBLEMS

1. Solve each of the derivative systems:

a. $f'(x) = x,\ \ f(2) = 3.$

b. $f'(x) = x^2,\ \ f(-1) = 4.$

c. $f'(x) = \sqrt{x},\ \ x > 0;\ \ f(1) = 2.$

d. $f'(x) = x^3$, $f(2) = -3$.
e. $f'(x) = \sin 2x$, $f(\pi/3) = 1$.
f. $f'(x) = 3 \cos x$, $f(\pi/2) = 3$.
g. $f'(x) = x + \sin x$, $f(0) = 2$.
h. $f'(x) = \sin^2 x \cos x$, $f(\pi/2) = -1$.
i. $f'(x) = \cos^2 2x \sin 2x$, $f(0) = 4$.
j. $f'(x) = \sin 2x + \cos 3x$, $f(\pi/2) = 10$.

2. If $v(t)$ is the velocity and $s(t)$ is the coordinate of a particle, find $s(t)$.
a. $v(t) = 3t$, $s(2) = 4$.
b. $v(t) = 3t^2 + 4$, $s(0) = 0$.
c. $v(t) = 4 - 3t^2$, $s(1) = 0$.
d. $v(t) = (t - 1)^2$, $s(1) = 0$.
e. $v(t) = \sin 3t$, $s(\pi/9) = 2$.
f. $v(t) = \sin \frac{1}{3}t + \cos 2t$, $s(\pi) = 1$.
g. $v(t) = \cos \frac{2}{3}t + 5 \sin \frac{5}{3}t$, $s(0) = 1$.
h. $v(t) = 320 \cos^5 \frac{5}{6}t \sin \frac{5}{6}t$, $s(\pi) = 0$.

3. Determine the function f if the graph of f passes through the given point and at the point on the graph having abscissa x the tangent to the graph has slope m.
a. $(2,4)$, $m = 3x - 1$.
b. $(4,2)$, $m = 3\sqrt{x}$, $x > 0$.
c. $(0,0)$, $m = 1 + \sin x$.
d. $(2,4)$, $m = (3x - 1)(x + 1)$.
e. $(0,2)$, $m = \cos 2x - 2 \cos x$.
f. $(\pi,\frac{1}{3})$, $m = \sin 2x \sin x$.

4. Find the relative maxima and minima of the function which has the given specific value and the given derived function
a. $f(0) = 4$, $f'(x) = (x - 2)(x + 3)$.
b. $f(0) = 6$, $f'(x) = (x - 2)(x + 3)$.
c. $f(2) = 3$, $f'(x) = (x^3 - 1)/x^2$.
d. $f(3) = 4$, $f'(x) = D_x(x\sqrt{x + 1})$.
e. $f(3) = \frac{5}{2}$, $f'(x) = D_x(x/\sqrt{x + 1})$.
f. $f(0) = 2$, $f'(x) = D_x(x + \sin x)$.

5. In each of the following an acceleration law is given and also specific values of the velocity and position are given. Find the position law.
a. $\alpha(t) = -32.2$, $v(0) = 10$, $s(0) = 25$.
b. $\alpha(t) = -9 \sin 3t$, $v(0) = 5$, $s(0) = -5$.
c. $\alpha(t) = 9 \cos 3t - 4 \sin 2t$, $v(0) = 2$, $s(0) = 1$.
d. $\alpha(t) = -4 \sin 2\left(t - \dfrac{\pi}{3}\right)$, $v(0) = 1$, $s(0) = 4 - \dfrac{\sqrt{3}}{2}$.

41. Differentials

A function f of a single variable is a set of ordered pairs such that if (a,b) and (a,c) are in the set then $b = c$. A set F of ordered triples is, by definition, a function of two independent variables if whenever (p,q,r) and (p,q,s) are in the set F then $r = s$; that is, in the set F no two distinct ordered triples have the same ordered pair for their first two elements. Later on (Chap. 9) a more detailed study of functions of two independent variables will be made, but now we consider a special type of such functions.

Given a function f of a single variable, let F be the function of two independent variables defined by

$$F = \{(x,h,z) \mid z = f'(x)h\}.$$

Thus, the domain of F is

$$\{(x,h) \mid x \text{ is in the domain of } f'\}.$$

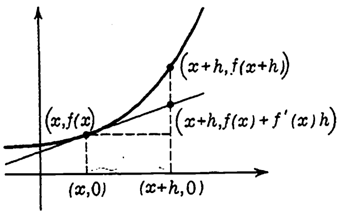


Figure 41

For x in the domain of f' and h any number, a geometric interpretation of the value $F(x,h) = f'(x)h$ may be obtained by noting that the points

$$(1) \qquad \big(x + h, f(x) + F(x,h)\big) \quad \text{and} \quad \big(x, f(x)\big)$$

both lie on the tangent to the graph of f at the point $(x, f(x))$. For if $h = 0$ then the points in (1) are the same, but if $h \neq 0$ the points are distinct and the line joining them has (see Fig. 41)

$$\text{slope} = \frac{[f(x) + f'(x)h] - f(x)}{(x + h) - x} = \frac{f'(x)h}{h} = f'(x).$$

The relation between the functions f and F is usually expressed in different notation and terminology which we now define.

DEFINITION 41. *Let f be a function of a single variable denoted by x. For x any number in the domain of f' and for dx denoting an arbitrary number, the product $f'(x)\,dx$ is represented by $df(x)$:*

$$(2) \qquad df(x) = f'(x)\,dx.$$

Under these circumstances dx is called a ***differential of x*** *and $df(x)$ is called the corresponding* ***differential of f at x.***

Thus, dx denotes a number and in the symbolism dx, the d and the x are inexorably bound together so that dx is **not** d times x and should never be read or even thought of as such. Also, $df(x)$ is ***not*** d times $f(x)$. Thus, for $dx \neq 0$

$$(1) \qquad \frac{df(x)}{dx} = f'(x)$$

and there should be no thought of cancelling the d's. We read (1) as "Differential f at x divided by differential x is equal to the derivative of f at x."

Thus, in addition to the previous notations $f'(x)$ and $D_x f(x)$ we now have another for the derivative of f at x:

$$\frac{df(x)}{dx} = f'(x) = D_x f(x)$$

each of which represents the value of

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

In particular $du(x) = D_x u(x)\,dx$ and $dv(x) = D_x v(x)\,dx$. Thus, by using the formula for the derivative of the product of two functions,

$$\begin{aligned} d[u(x)\,v(x)] &= D_x[u(x)\,v(x)]\,dx \\ &= [u(x)\,D_x v(x) + v(x)\,D_x u(x)]\,dx \\ &= u(x)\,D_x v(x)\,dx + v(x)\,D_x u(x)\,dx \\ &= u(x)\,dv(x) + v(x)\,du(x). \end{aligned}$$

It is customary, moreover, to further shrink this (and similar) formulas by leaving out any indication of the independent variable whenever possible and to write merely

$$duv = u\,dv + v\,du.$$

In differential notation, eight of the formulas previously discussed (with special cases of some of them) are:

1. $dx^r = rx^{r-1}\,dx$, $\quad r$ rational.
2. $dc = 0$, $\quad c$ constant.
3. $d(u + v) = du + dv$. $\qquad 3_1.\ d(u + c) = du.$
4. $duv = u\,dv + v\,du$. $\qquad 4_1.\ d\,cv = c\,dv.$
5. $d\dfrac{u}{v} = \dfrac{v\,du - u\,dv}{v^2}$ $\qquad 5_1.\ d\left(\dfrac{c}{v}\right) = -c\dfrac{dv}{v^2}.$
6. $df = \dfrac{df}{du}\,du = D_u f\,du$.
7. $d\sin u = (\cos u)\,du$.
8. $d\cos u = -(\sin u)\,du$.

Example 1.

$$\begin{aligned} d\sin(3x^2 - 4x + 1) &= d\sin u && (\text{where } u = 3x^2 - 4x + 1) \\ &= (\cos u)\,du && (\text{from } 7) \\ &= \cos(3x^2 - 4x + 1)\,d(3x^2 - 4x + 1) \\ &= [\cos(3x^2 - 4x + 1)][d3x^2 - d4x + d1] \\ &= [\cos(3x^2 - 4x + 1)][6x\,dx - 4\,dx + 0] \\ &= 2(3x - 2)[\cos(3x^2 - 4x + 1)]\,dx. \end{aligned}$$

Example 2.

$$\begin{aligned}
d\sin^3 x^2 &= du^3 && \text{(where } u = \sin x^2\text{)}\\
&= 3u^2\,du && \text{(from } \mathbf{1} \text{ with } x \text{ replaced by } u\text{)}\\
&= 3\sin^2 x^2\, d\sin x^2 &&\\
&= 3\sin^2 x^2\, d\sin u && \text{(previous } u \text{ is gone and now } u = x^2\text{)}\\
&= 3\sin^2 x^2(\cos u)\,du &&\\
&= 3\sin^2 x^2(\cos x^2)\,dx^2 && (u \text{ replaced by } x^2)\\
&= 6x\sin^2 x^2(\cos x^2)\,dx. &&
\end{aligned}$$

By definition, $d[df(x)]$ is called the **second differential** of f at x and is denoted by $d^2f(x)$:

$$d^2f(x) = d[df(x)].$$

Select a number for dx. The selection is arbitrary but once made dx is constant. Consequently

$$\begin{aligned}
d^2f(x) = d[df(x)] = d[f'(x)\,dx] &= d[f'(x)]\,dx \qquad (by\ \mathbf{4}_1)\\
= [D_x f'(x)]\,dx\,dx &= f''(x)(dx)^2.
\end{aligned}$$

Hence, for $dx \neq 0$ we have a differential notation for the second derivative of f at x:

$$\frac{d^2f(x)}{(dx)^2} = f''(x) = D_x^2 f(x).$$

Example 3. Find $d^2(x^2 + \sin x)$.

Solution. According to the above development

$$d^2(x^2 + \sin x) = [D_x^2(x^2 + \sin x)](dx)^2 = [D_x(2x + \cos x)](dx)^2 = (2 - \sin x)(dx)^2.$$

It may seem unduly confusing to have three different notations for the derivative of a function, but all three are so engrained in mathematical and physical literature that none of them may be neglected. Moreover, each has some manipulative or intuitive advantage over the other two, so that practice in all three must be continued. We mention, in passing, that some books denote first and second time-derivatives by $\dot{x}$ and $\ddot{x}$, so that $\dot{x}$ represents the velocity and $\ddot{x}$ the acceleration for a motion whose law expresses x in terms of t.

PROBLEMS

1. Given $f(x) = (x - 2)(x - 4)$, find $df(x)$ if

   a. $x = 2$ and $dx = 0.5$.
   b. $x = 3$ and $dx = 0.5$.
   c. $x = 4$ and $dx = 0.5$.
   d. $x = 2$ and $dx = -0.1$.
   e. $x = 3$ and $dx = -0.1$.
   f. $x = 4$ and $dx = -0.1$.

2. Find
   a. $d(x^3 - x + 3)$ at $x = 1, dx = 2$ and at $x = 2, dx = 0.5$.
   b. $d\sqrt{x^3 - x + 3}$ at $x = 2, dx = 1$ and at $x = 2, dx = 0.1$.
   c. $d \sin 2x$ at $x = \pi/3, dx = \pi/180$ and at $x = \pi/4, dx = \pi/180$.
   d. $d \cos 3x$ at $x = \pi/4, dx = 0.6$ and at $x = -\pi/4, dx = 0.6$.
3. Find each of the following differentials:
   a. $d[(2x + 1) \sin x]$.
   b. $d\left(\dfrac{2x + 1}{\sin x}\right)$.
   c. $d \sin^4 x^2$.
   d. $d \cos^2 x^4$,
   e. $d(\sin^2 x - \cos^2 x)$.
   f. $d\left(\dfrac{x}{2} + \dfrac{3}{x}\right)$.
   g. $d(\sin x \cos^2 x)$.
   h. $d(\sin x \cos x)^2$.
   i. $d(\sin x + \cos x)^2$
   j. $d\left(\dfrac{x^2 + 5x + 1}{x}\right)$.
   k. $d[\sin (\cos x)]$.
   l. $d[\sin^2 (\cos x)]$.
4. Find each of the following second differentials:
   a. $d^2(3x^3 + 4x^2 - 5x + 1)$.
   b. $d^2(3 \sin^3 x + 4 \sin^2 x - 5 \sin x + 1)$.
   c. $d^2\left(\dfrac{x^3 + 1}{x}\right)$.
   d. $d^2\left(\dfrac{x}{x^3 + 1}\right)$.
   e. $d^2\left(\dfrac{\sin^3 x + 1}{\sin x}\right)$.
   f. $d^2[(x + 1)^2 + 3x + 2]$.
5. Show that $df(x) = \lim\limits_{t \to 0} \dfrac{f(x + t\,dx) - f(x)}{t}$.

42. Differential Systems

For the function f defined by $f(x) = \sin (3x^2 + 1)$ we shall (whenever there seems to be no danger of misunderstanding in this and similar cases) speak of "The function y where

$$y = \sin (3x^2 + 1)\text{"}$$

and shall even follow the custom of saying "y is a function of x" or "The dependent variable y is a function of the independent variable x." Also, $u = 3x^2 + 1$ is a function of the independent variable x and $y = \sin u$ is a function of u. For this composite function of a function

$$\begin{aligned} dy &= \frac{dy}{du}\,du = (D_u \sin u)\,du = \cos u\,du \\ &= \cos (3x^2 + 1)\,d(3x^2 + 1) = \cos (3x^2 + 1)[6x\,dx + 0] \\ &= 6x \cos (3x^2 + 1)\,dx. \end{aligned}$$

Inversely, if we were given $dy = x \cos (3x + 1)\,dx$, we would see that

$$y = \sin (3x + 1) + c.$$

Previously we have solved derivative systems. There seem to be some manipulative advantages in translating such solutions into equivalent differential notation. Thus if we are given

$$dy = x \cos (3x^2 + 1)\, dx,$$

then each step below may be anticipated:

$$\begin{aligned} dy &= \tfrac{1}{6}[\cos (3x^2 + 1)]6x\, dx \\ &= \tfrac{1}{6} \cos (3x^2 + 1)\, d3x^2 \\ &= \tfrac{1}{6} \cos (3x^2 + 1)\, d(3x^2 + 1) \\ &= \tfrac{1}{6} \cos u\, du && \text{(where } u = 3x^2 + 1) \\ &= \tfrac{1}{6}\, d \sin u, && \text{(since } d \sin u = \cos u\, du) \\ y &= \tfrac{1}{6} \sin u + c = \tfrac{1}{6} \sin (3x^2 + 1) + c. \end{aligned}$$

We shall speak of $dy = f(x)\, dx$ as a **differential equation** and of

$$dy = f(x)\, dx, \quad y = a \quad \text{when} \quad x = b,$$

as a **differential system.** Also, any function y which satisfies $dy = f(x)\, dx$ is called an **anti-differential** of $f(x)\, dx$.

Example. Solve the differential system

$$dy = x^2 \sin^2 x^3 \cos x^3\, dx, \quad y = 2 \quad \text{when} \quad x = 0.$$

Solution. We write

$$\begin{aligned} dy &= \tfrac{1}{3} \sin^2 x^3 \cos x^3\, dx^3 && \text{(since } dx^3 = 3x^2\, dx) \\ &= \tfrac{1}{3} \sin^2 u \cos u\, du && \text{(where } u = x^3) \\ &= \tfrac{1}{3} \sin^2 u\, d \sin u && \text{(since } d \sin u = \cos u\, du) \\ &= \tfrac{1}{3} v^2\, dv && \text{(where } v = \sin u) \\ &= \tfrac{1}{9}\, dv^3, && \text{(since } dv^3 = 3v^2\, dv) \end{aligned}$$

$$y = \tfrac{1}{9} v^3 + c = \tfrac{1}{9} \sin^3 u + c = \tfrac{1}{9} \sin^3 x^3 + c.$$

But $y = 2$ when $x = 0$ so $2 = \frac{1}{9} \sin^3 0 + c = 0 + c$ and, hence, the solution is

$$y = 2 + \tfrac{1}{9} \sin^3 x^3.$$

PROBLEMS

1. Solve each of the following differential systems:
   a. $dy = x \sin (3x^2 + 1)\, dx, \quad y = 2 \quad \text{when} \quad x = 0.$
   b. $dy = x^2 \cos^2 x^3 \sin x^3\, dx, \quad y = 2 \quad \text{when} \quad x = 0.$
   c. $dy = x\sqrt{3x^2 + 1}\, dx, \quad y = 3 \quad \text{when} \quad x = -1.$
   d. $dy = x^2\sqrt{3x^3 + 1}\, dx, \quad y = -4 \quad \text{when} \quad x = 2.$

e. $dy = \dfrac{x}{\sqrt{3x^2+1}}\,dx, \quad y = 0$ when $x = 1$.

f. $dy = x(3x^2+1)\,dx, \quad y = 20$ when $x = 2$.

2. First find each of the four differentials, and then use the results to solve the four differential systems.

a. $d(\sin x - x\cos x)$.

b. $d(x - \sin x\cos x)$.

c. $d(x^{-1}\sqrt{x^2+4^2})$.

d. $d[(3x-6)(x+3)^{3/2}]$.

e. $dy = \dfrac{dx}{x^2\sqrt{x^2+4^2}}, \quad y = 2$ when $x = -3$.

f. $dy = \sin^2 x\,dx, \quad y = -\frac{1}{4}$ when $x = \pi/4$.

g. $dy = x\sin x\,dx, \quad y = 3$ when $x = \pi/2$.

h. $dy = x\sqrt{x+3}\,dx, \quad y = -\frac{1}{5}$ when $x = 1$.

3. Solve each of the following differential equations:

a. $dy = x^3\sqrt{x^4+3}\,dx$.

b. $dy = x^3\sin x^4\,dx$.

c. $dy = x^3(x^4+3)\,dx$.

d. $dy = (3x^2+4x+1)\sin(x^3+2x^2+x-6)\,dx$.

e. $dy = \left(1 - \dfrac{1}{x^2}\right)\cos\left(x + \dfrac{1}{x}\right)dx$.

f. $dy = \dfrac{1}{\sqrt{x}}\sin^2\sqrt{x}\cos\sqrt{x}\,dx$.

43. Increments

We have systematically used h to denote a number to be added to a value of the independent variable. The notation and terminology as defined below should also be known.

DEFINITION 43. *In connection with a function f, if the letter x is used to denote the independent variable, then Δx is used to denote an arbitrary number different from zero and $\Delta f(x)$ is used (see Fig. 43.1) for the corresponding difference*

(1) $$\Delta f(x) = f(x+\Delta x) - f(x).$$

Also, Δx is called an ***increment*** *of x and $\Delta f(x)$ is called the corresponding* ***increment*** *of f at x.*

Hence, Δx is a number and is **not** Δ times x. In particular

$$x\Delta x \neq \Delta x^2, \quad \frac{\Delta x}{x} \neq \Delta, \quad \text{and} \quad \frac{\Delta f(x)}{\Delta x} \neq \frac{f(x)}{x}.$$

If $f'(x)$ exists then, in this "increment" notation

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

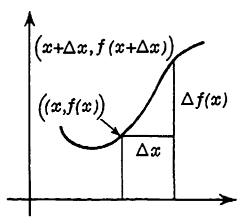


Figure 43.1

or, with even more condensation,

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}. \tag{2}$$

An abridged reading of (2) is "The derivative of a function is the limit approached by the ratio of the increment of the function to the increment of the independent variable as the increment of the independent variable approaches zero."

In connection with the equation $y = f(x)$, we speak of the differential of y and the increment of y and write

$$dy = f'(x)\,dx \quad \text{and} \quad \Delta y = f(x + \Delta x) - f(x).$$

Therefore, $f(x) + \Delta y = f(x + \Delta x)$ or, since $y = f(x)$,

$$y + \Delta y = f(x + \Delta x).$$

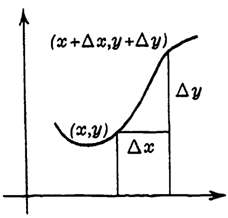


Figure 43.2

If, for example, $y = x^2 - x + 5$, then we write

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 - (x + \Delta x) + 5 \\ &= x^2 + 2x\Delta x + (\Delta x)^2 - x - \Delta x + 5 \\ &= (x^2 - x + 5) + 2x\Delta x + (\Delta x)^2 - \Delta x \\ &= y + (2x + \Delta x - 1)\Delta x. \end{aligned}$$

Hence, $\Delta y = (2x + \Delta x - 1)\Delta x$, $\dfrac{\Delta y}{\Delta x} = 2x + \Delta x - 1$, and

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 1) = 2x - 1.$$

With x denoting the independent variable, the symbols dx and Δx both represent arbitrary numbers (which may be the same or different), but the distinctive notations have come to betoken the way these arbitrary numbers are to be used; dx to multiply $f'(x)$, but Δx to be added to x and later allowed to approach zero. Figure 43.3 shows a geometric interpretation of dx, Δx, $df(x)$, and $\Delta f(x)$ in which all are positive with Δx smaller than dx. From Fig. 43.3 it should be visualized why

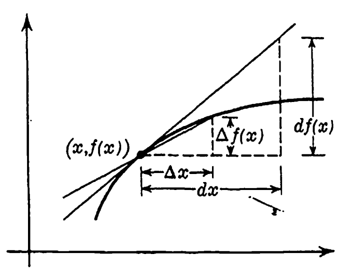


Figure 43.3

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}$$

$= \text{slope of tangent at } (x, f(x)).$

Under assumption of continuity and existence of derivatives, the law of the mean (Theorem 32.2) states that for x_1 and x_2 given numbers, there is a number ξ between x_1 and x_2 such that

$$f(x_2) = f(x_1) + f'(\xi)(x_2 - x_1). \tag{3}$$

By using increment notation there are equivalent relations which look quite different but are sometimes met and should be recognized. Let x and $\Delta x \neq 0$ be given numbers. Now think of $x_1 = x$ and $x_2 = x + \Delta x$ in the Law of the Mean. Then there is a number λ such that $0 < \lambda < 1$ and $\xi = x + \lambda\Delta x$, and now (3) takes the form

$$\begin{aligned} f(x + \Delta x) &= f(x) + f'(x + \lambda\,\Delta x)\Delta x, \quad \text{or} \\ f(x + \Delta x) - f(x) &= f'(x + \lambda\Delta x)\Delta x, \quad \text{or} \\ \Delta f(x) &= f'(x + \lambda\Delta x)\Delta x, \quad \text{or} \\ \frac{\Delta f(x)}{\Delta x} &= f'(x + \lambda\Delta x). \end{aligned} \tag{4}$$

44. Approximations by Differentials

In one sense, two quantities are approximations of each other if their difference has small absolute value, but if both quantities are themselves close to zero this is not a good criterion. For example, if $a = 0.001$ and $b = 0.01$, then $b - a = 0.009$, but b is ten times as large as a. Instead of this "difference criterion" the following "ratio criterion" is generally used:

If two quantities have ratio close to 1, *then these quantities are considered to be approximations of one another.*

Hence, x is an approximation of $\sin x$ for x small since

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

If x represents the independent variable, then dx and Δx are arbitrary and we may, if we wish, choose them equal. Hence, for f a function and x a number such that $f'(x) \neq 0$, then

$$\lim_{\Delta x = dx\to 0} \frac{\Delta f(x)}{df(x)} = \lim_{\Delta x = dx\to 0} \frac{\Delta f(x)}{f'(x)\,dx} = \frac{1}{f'(x)} \lim_{\Delta x = dx\to 0} \frac{\Delta f(x)}{\Delta x} = \frac{1}{f'(x)} f'(x) = 1.$$

Consequently, $\Delta f(x)$ and $df(x)$ are approximations of one another if $f'(x) \neq 0$ and $dx = \Delta x$ is sufficiently small. A common symbolism is:

$$\Delta f(x) \sim df(x); \quad \text{that is,} \quad f(x + dx) - f(x) \sim df(x), \quad \text{or}$$

$$f(x + dx) \sim f(x) + f'(x)\,dx \tag{1}$$

if dx is sufficiently small.†

† Natural questions are "What does sufficiently small mean?" and "How closely does $f(x) + f'(x)\,dx$ approximate $f(x + dx)$?" We postpone answering these questions until Chapter 13.

There are problems for which the answer is (or may be expressed in terms of) the increment of a function, but it may be that this increment is hard or tedious to compute and that the differential approximates the increment closely enough for the purpose and is easily computed.

Example 1. Find approximately the value of $(1.01)^{10} - (1.01)^5 - 5$.

Solution. First let $f(x) = x^{10} - x^5 - 5$ so that $f(1.01)$ is the desired answer but is tedious to compute. However, $f(1)$ is easily found and $f'(x) = 10x^9 - 5x^4$ so $f'(1)$ is also easily found. Thus, from (1) with $x = 1$ and $dx = 0.01$,

$$f(1.01) \sim f(1) + f'(1)(0.01) = (1^{10} - 1^5 - 5) + (10 \cdot 1^9 - 5 \cdot 1^4)(0.01) = -4.95.$$

Since $(1.01)^{10} - (1.01)^5 - 5$ is slightly greater than -4.947 the approximation -4.95 is within 0.01.

Example 2. Each edge of a cube is measured as 4 in. by an instrument accurate to within $\frac{1}{64}$ in. Find approximately how much the volume may differ from $4^3 = 64$ in^3.

Solution. Let $v(x) = x^3$. The volume is thus between $v(4 - \frac{1}{64})$ and $v(4 + \frac{1}{64})$ and the error in the volume is between $v(4 \pm \frac{1}{64}) - v(4)$. By one of the above relations $v(x + dx) - v(x) \sim dv(x) = 3x^2\,dx$ and we see, using $x = 4$ and $dx = \pm\frac{1}{64}$, that the error is approximately $3(4)^2(\pm\frac{1}{64}) = \pm\frac{3}{4}$ in^3.

Since the differential $df(x)$ exists if and only if $f'(x)$ exists, it is customary to define:

*If x is in the domain of f', then f is said to be **differentiable** at x.*

Notice that if f is differentiable at x, then (from only the continuity of f at x)

$$\lim_{\Delta x = dx \to 0} [\Delta f(x) - df(x)] = \lim_{dx \to 0} [f(x + dx) - f(x) - f'(x)\,dx]$$
$$= f(x) - f(x) - f'(x) \cdot 0 = 0.$$

This approach to zero is, however, so much "faster" than the approach of dx to zero that also

(2) $$\lim_{\Delta x = dx \to 0} \frac{\Delta f(x) - df(x)}{dx} = 0.$$

The fact that (2) holds follows from:

$$\lim_{dx \to 0} \left[\frac{f(x + dx) - f(x) - f'(x)\,dx}{dx}\right] = \lim_{dx \to 0} \left[\frac{f(x + dx) - f(x)}{dx} - f'(x)\right]$$
$$= f'(x) - f'(x) = 0.$$

PROBLEMS

1. Given $f(x) = x^2 - x + 1$, find $\Delta f(x)/df(x)$ for each of the following values of x and $\Delta x = dx$.

   a. $x = 4, \quad \Delta x = dx = 1$ c $x = 4, \quad \Delta x = dx = 0.1$.

   b. $x = 10, \quad \Delta x = dx = 1.$ d. $x = 10, \quad \Delta x = dx = -0.1.$

2. Use differentials to find approximations to

   a. $(1.01)^{100} - (1.01)^{25} - 5.$ c. $(1.99)^5.$ e. $\sqrt{9.2}.$

   b. $(0.99)^{10} - (0.99)^5 - 5.$ d. $(4.02)^{3/2}.$ f. $\sqrt[3]{65}.$

3. Find an approximation of $\frac{1}{9}$ by using differentials, $x = 10$, and $dx = -1$.

4. In measuring the acceleration of gravity g by means of a pendulum, the formula

$$g = \frac{4\pi^2 l}{T^2}$$

   is used, where l is the length of the pendulum in inches and T the period in seconds. Assuming error in measuring l is negligible, express the error dg in terms of the error dT. Also, express the relative error $(dg)/g$ in terms of the relative error $(dT)/T$.

5. A particle has $s(t) = t^3 + t^2 - 2t + 1$ as its law of linear motion. By using differentials find approximately:

   a. How far the particle moved between $t = 1$ and $t = 1.1$.

   b. How fast the particle is moving at $t = 1.9$ and at $t = 7.1$.

6. For $f(x) = x + \sqrt{x}$ find approximately the value of $f'(3.99)$.

7. Show that the Law of the Mean implies that:
   If f' exists throughout an open interval containing x and, with $\Delta x = dx$, if $x + \Delta x$ is in that interval, then

   a. There is a number c such that $0 < c < 1$ and

$$\Delta f(x) = f'(x + c\,\Delta x)\,\Delta x.$$

   b. There is a number ξ between x and $x + dx$ such that

$$f(x + dx) = f(x) + df(\xi).$$

CHAPTER 5

Elementary Transcendental Functions

DEFINITION. *A function is* **transcendental** *if it is not algebraic.*

This is a concise statement and is satisfactory if we know what an algebraic function is. The function f defined for $x > 3$ by

$$f(x) = \frac{x^3 - x + \sqrt{5 + x^2}}{(x - 3)^{5/4}}$$

should certainly be algebraic since it is built up from a (finite) number of the algebraic operations addition, subtraction, multiplication, division, and extraction of integer roots. But what does "built up" mean? The function $f(x) = \sqrt{x^2}$ is so built up, but is not algebraic since the alternative decision process in

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

is not one of the five algebraic operations.

Even under a precise definition of "algebraic function," (not to be given here) there are functions that, to date, have not been classified either as algebraic or as transcendental. It is, however, known that the functions sin, cos, tan, cot, sec, csc, and their inverses are transcendental as are $\log_b$ and the function E defined by $E(x) = b^x$ for $b > 0$ but $b \neq 1$. In fact, precisely these functions are classified as the **elementary transcendental functions.**

45. Trigonometric Functions

Each of the following formulas has been obtained in derivative notation (see page 80), but practice using only the first two has been carried on. After the present section familiarity with all six will be assumed.

1. $D_x \sin x = \cos x,$ $\qquad d \sin x = \cos x\, dx,$
2. $D_x \cos x = -\sin x,$ $\qquad d \cos x = -\sin x\, dx,$
3. $D_x \tan x = \sec^2 x,$ $\qquad d \tan x = \sec^2 x\, dx,$
4. $D_x \cot x = -\csc^2 x,$ $\qquad d \cot x = -\csc^2 x\, dx,$
5. $D_x \sec x = \sec x \tan x,$ $\qquad d \sec x = \sec x \tan x\, dx,$
6. $D_x \csc x = -\csc x \cot x,$ $\qquad d \csc x = -\csc x \cot x\, dx.$

Example 1. $D_x\sqrt{\tan x} = D_x(\tan x)^{1/2} = \frac{1}{2}(\tan x)^{-1/2}D_x \tan x$

$$= \frac{\sec^2 x}{2\sqrt{\tan x}}, \quad \tan x > 0.$$

Example 2. $d \sec^3 2t = 3 \sec^2 2t \, d \sec 2t$

$$= 3 \sec^2 2t(\sec 2t \tan 2t)\, d(2t)$$
$$= 6 \sec^3 2t \tan 2t \, dt.$$

Example 3. A straight, level highway passes $\frac{1}{2}$ mi from an airplane beacon making 2 rev/min. How fast is the lighted part of the highway traveling along the highway:

a. When it is at the nearest point to the beacon?

b. When it is $\frac{3}{4}$ mi from this nearest point?

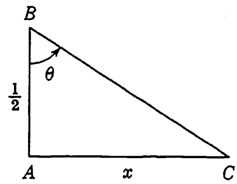


Figure 45

Solution. Let B be the beacon, A the nearest point of the highway to the beacon, C the point of the highway on which the light is shining at the instant the angle from AB to the light ray is θ (measured in radians). Letting $x = AC$ the whole problem is translated into:

Given: $x = (\frac{1}{2}) \tan \theta$ mi, $D_t\theta = 4\pi$ radians/min.

To find: $D_tx]_{x=0}$ and $D_tx]_{x=3/4}$.

Since $D_tx = (\frac{1}{2})D_t \tan \theta = (\frac{1}{2}) \sec^2 \theta D_t\theta = (\frac{1}{2})(\sec^2 \theta)4\pi = 2\pi \sec^2 \theta$,

(1) $$D_tx]_{t=0} = 2\pi \sec^2 0 = 2\pi \text{ mi/min}.$$

Also, when $x = \frac{3}{4}$, then $BC = \sqrt{13}/4$, $\sec \theta = \sqrt{13}/2$ and hence

(2) $$D_tx]_{x=3/4} = 2\pi(\sqrt{13}/2)^2 = 13\pi/2 \text{ mi/min}.$$

We could have used the identity $\sec^2 \theta \equiv 1 + \tan^2 \theta$ to obtain

$$D_tx = 2\pi \sec^2 \theta = 2\pi(1 + \tan^2 \theta) = 2\pi(1 + 4x^2)$$

and then we could substitute $x = 0$ and $x = \frac{3}{4}$ to obtain (1) and (2), respectively.

Example 4. Solve the differential equation $dy = x^2 \cot x^3 \csc^2 x^3 \, dx$.

Solution.

$$\begin{aligned} dy &= \tfrac{1}{3} \cot x^3 \csc^2 x^3 \, dx^3 && (\text{since } dx^3 = 3x^2\, dx) \\ &= -\tfrac{1}{3} \cot x^3 \, d \cot x^3 && (\text{since } d \cot u = -\csc^2 u \, du) \\ &= -\tfrac{1}{3}u \, du && (\text{where } u = \cot x^3) \\ &= -\tfrac{1}{6}\, du^2, && (\text{since } du^2 = 2u\, du) \\ y &= -\tfrac{1}{6}u^2 + c = -\tfrac{1}{6} \cot^2 x^3 + c. \end{aligned}$$

Example 5. Given $y = c \csc x - \cot x$ show that y' may be written as

$$y' = 1 - y \cot x.$$

Solution. Since $y' = D_x y$ we first obtain

$$y' = -c \csc x \cot x + \csc^2 x. \tag{3}$$

In the given equation, c is a constant, but the variables x and y are related in such a way that

$$y + \cot x = c \csc x.$$

Now in (3) replace $c \csc x$ by $y + \cot x$ to obtain

$$\begin{aligned} y' &= -(y + \cot x) \cot x + \csc^2 x \\ &= -y \cot x - \cot^2 x + \csc^2 x \\ &= -y \cot x + 1 \qquad (\text{since } 1 + \cot^2 x \equiv \csc^2 x). \end{aligned}$$

PROBLEMS

1. Find each of the following derivatives:

a. $D_x \sqrt{\cot x}$.

b. $D_x \sec^2 x$.

c. $D_x\, x \tan x$.

d. $\dfrac{d}{dx} \dfrac{\tan x}{x}$.

e. $D_x \sec x \tan x$.

f. $D_x(\sin x + \tan x)$.

g. $D_x(\sin x \tan x)$.

h. $\dfrac{d}{dx}(\tfrac{1}{3} \tan x^3)$.

i. $D_x \csc (2x^2 + 1)$.

j. $D_x \cot \sqrt{x^2 + 4}$.

k. $D_x(1 + \cot^2 x)$.

l. $\dfrac{d}{dx} \tan \left(\dfrac{x}{2}\right)$.

2. Establish each of the following:

a. $\dfrac{d}{dx} \dfrac{\tan^{p+1} x}{p + 1} = \tan^p x \sec^2 x$.

b. $\dfrac{d}{dx} \dfrac{1}{p} \sec^p x = \sec^p x \tan x$.

c. $\dfrac{d}{dx}\left(\dfrac{\cot ax}{a} + x\right) = -\cot^2 ax$.

d. $\dfrac{d}{dx} \cot \left(\dfrac{x}{2} - \dfrac{\pi}{4}\right) = \dfrac{-1}{1 - \sin x}$.

e. $\dfrac{d}{dx}(\cot^4 x - \csc^4 x) = 4 \csc^2 x \cot x$.

f. $\dfrac{d(\tan^2 x)}{dx} = \dfrac{d(\sec^2 x)}{dx}$.

3. The shorter of the two parallel sides of a right trapezoid is 4 in., and the oblique side is 8 in. Find the angle between the 8 in. side and the longer of the parallel sides if the area of the trapezoid is maximum.

4. Find the minimum of $f(x) = \tan x + \cot x$, $0 < x < \pi/2$. (Note: This function arises in connection with measuring electric current.)

5. For $f(x) = \pi x - \tan (\pi x/2)$ show that $f(\tfrac{1}{2} + n)$ is a relative minimum or maximum according to whether n is odd or even.

6. A light is to be hung above the center of a circular table so as to give maximum illumination at the edge of the table. Find the proper height, given that the illumination at a point in a plane varies directly as the sine of the angle between the plane and the light source and inversely as the square of the distance to the source.

7. A wall a ft high is b ft from a house. Find the length of the shortest ladder that will reach from the horizontal ground over the wall to the house.

8. OA is a crankshaft a in. long revolving about O at 300 rev/min. AB is a connecting rod b in. long ($b > a$) with B moving on a line through O. Find B's velocity t min after B is at its greatest distance from O.

9. A sphere rests upon a table. Find the volume of the right circular cone of minimum volume which will cover the sphere and rest with its circular base on the table. Use half vertex angle as independent variable.

10. Use the technique illustrated in Example 5 to show that y' can be written in the form shown.

a. $y = c \sec \theta - \tan \theta, \quad y' = y \tan \theta - 1.$

b. $y = c \sec \theta + \tan \theta, \quad y' = y \tan \theta + 1.$

c. $y = c(\sec \theta + \tan \theta), \quad y' = y \sec \theta.$

d. $y = c \tan x, \quad y' = 2y \csc 2x.$

46. Inverse Trigonometric Functions

Graphs of the trigonometric functions were drawn in the trigonometry course. We now know that at each point of

$$\{(x,y) \mid -\pi/2 < x < \pi/2 \quad \text{and} \quad y = \tan x\}$$

there is a tangent line with a positive slope (actually ≥ 1) since

$$D_x \tan x = \sec^2 x \quad \text{for} \quad x \neq \pi/2 + m\pi,$$

so the graph is, as drawn in Fig. 46.1, a smooth curve rising to the right with no horizontal or vertical tangent line. Also, for y any given number, there is one and only one number x such that both

$$-\pi/2 < x < \pi/2 \quad \text{and} \quad y = \tan x.$$

By interchanging the roles of x and y, it follows that for any given number x there is one and only one number y such that

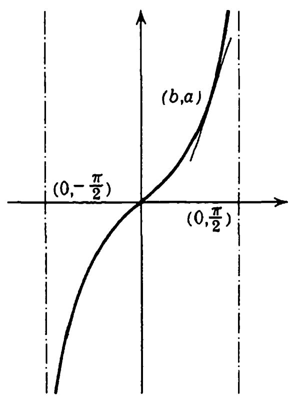


Figure 46.1

$$-\pi/2 < y < \pi/2 \quad \text{and} \quad x = \tan y,$$

and we write, to emphasize that x was chosen independently,

$$y = \tan^{-1} x, \; -\pi/2 < y < \pi/2$$

which is read "y is the inverse tangent at x." Hence,

$$\{(x,y) \mid -\pi/2 < y < \pi/2 \quad \text{and} \quad y = \tan^{-1} x\}$$

is a function (called the **inverse tangent** function) and the graph of this function is given in Fig. 46.2. Notice that for x any given number, then both

$$-\pi/2 < \tan^{-1}x < \pi/2$$

and $$\tan(\tan^{-1}x) = x.$$

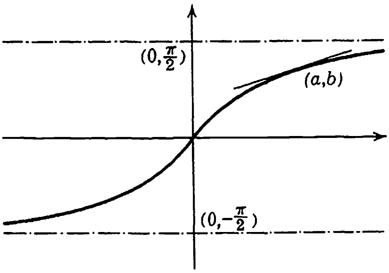


Figure 46.2

To obtain an auxiliary fact, let a_1, b_1, a_2, and b_2 be given numbers with $a_2 \neq a_1$. Then the line joining the points (a_1,b_1) and (a_2,b_2) has slope m_1 where

$$m_1 = \frac{b_2 - b_1}{a_2 - a_1}.$$

If in addition $b_2 \neq b_1$, then by using a previous ordinate for an abscissa and a previous abscissa for an ordinate, the line joining the points (b_1,a_1) and (b_2,a_2) has slope m_2 which is the reciprocal of m_1:

$$m_2 = \frac{a_2 - a_1}{b_2 - b_1} = \frac{1}{m_1}.$$

Now let a be any number and select the number b such that the point (a,b) is on the graph in Fig. 46.2. Hence, $-\pi/2 < b < \pi/2$ and $b = \tan^{-1} a$ or $a = \tan b$. Thus, the point (b,a) is on the graph in Fig. 46.1 and at this point the tangent line to this graph has slope

$$D_x \tan x\big]_{x=b} = \sec^2 b.$$

Returning now to the point (a,b) of Fig. 46.2, this graph has a tangent line at this point and the slope of this tangent line is the reciprocal of $\sec^2 b$; that is, $D_x \tan^{-1} x]_{x=a}$ exists and, moreover,

$$D_x \tan^{-1} x\big]_{x=a} = \frac{1}{\sec^2 b} = \frac{1}{\sec^2(\tan^{-1} a)}.$$

Since a was any number whatever, we therefore have

$$D_x \tan^{-1} x = \frac{1}{\sec^2(\tan^{-1} x)}.$$

But $\sec^2(\tan^{-1} x) = 1 + \tan^2(\tan^{-1} x) = 1 + x^2$ and we have the standard formula

(1) $$\boldsymbol{D_x \tan^{-1} x = \frac{1}{1 + x^2}}.$$

Hence, from the formula for the derivative of a composite function,

$$D_x \tan^{-1} u(x) = \frac{1}{1 + u^2(x)} D_x u(x),$$

for any function u whose derivative exists.

Should formula (1) be forgotten, it may be recovered by starting with $y = \tan^{-1} x$, so that $\tan y = x$, and, hence, writing

$$D_x \tan y = D_x x, \quad \sec^2 y\, D_x y = 1, \quad D_x y = \frac{1}{\sec^2 y}.$$

But $y = \tan^{-1} x$ so that

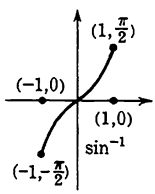
$$D_x \tan^{-1} x = \frac{1}{\sec^2(\tan^{-1} x)} = \frac{1}{1 + \tan^2(\tan^{-1} x)} = \frac{1}{1 + x^2}.$$

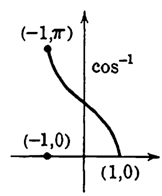
Each of the trigonometric functions has an inverse function with an appropriate domain. These are defined as follows:

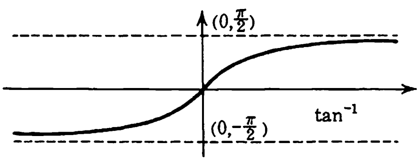
$$\begin{array}{llll}
\text{If } -1 \le x \le 1, & \text{then } -\pi/2 \le \sin^{-1} x \le \pi/2 & \text{and } \sin\ (\sin^{-1} x) = x. \\
\text{If } -1 \le x \le 1, & \text{then } 0 \le \cos^{-1} x \le \pi & \text{and } \cos\ (\cos^{-1} x) = x. \\
\text{If } -\infty < x < \infty, & \text{then } -\pi/2 < \tan^{-1} x < \pi/2 & \text{and } \tan\ (\tan^{-1} x) = x. \\
\text{If } -\infty < x < \infty, & \text{then } 0 < \cot^{-1} x < \pi & \text{and } \cot\ (\cot^{-1} x) = x. \\
\text{If } \left\{\begin{matrix} -\infty < x \le -1 \\ 1 \le x < \infty \end{matrix}\right\}, & \text{then } \left\{\begin{matrix} -\pi \le \sec^{-1} x < -\pi/2 \\ 0 \le \sec^{-1} x < \pi/2 \end{matrix}\right\} & \text{and } \sec\ (\sec^{-1} x) = x. \\
\text{If } \left\{\begin{matrix} -\infty < x \le -1 \\ 1 \le x < \infty \end{matrix}\right\}, & \text{then } \left\{\begin{matrix} -\pi < \csc^{-1} x \le -\pi/2 \\ 0 < \csc^{-1} x \le \pi/2 \end{matrix}\right\} & \text{and } \csc\ (\csc^{-1} x) = x.
\end{array}$$

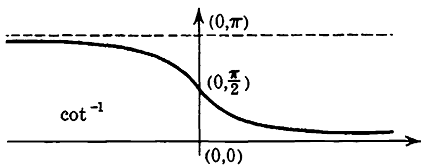
The graph of each of these inverse functions is given in an accompanying figure. It should be seen that in each case the domain and values of the inverse function are such that these values would be called **principal values** in a trigonometry course.

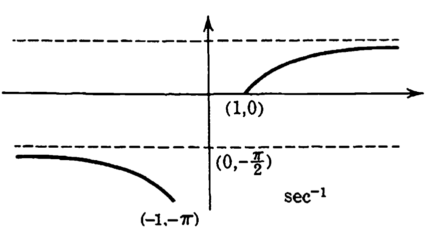
An argument similar to the one given for the inverse tangent function will show that each of the inverse trigonometric functions has

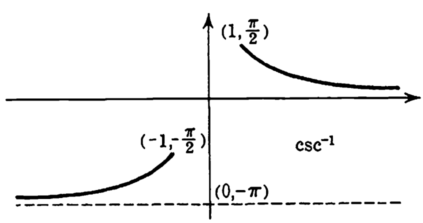












a derived function whose domain includes all except end points of the domain of the inverse function itself. For example, the function $\{(x,y) \mid y = \sin^{-1} x\}$ has domain $\{x \mid -1 \le x \le 1\}$, but the derived function $\{(x,y) \mid y = D_x \sin^{-1} x\}$ has domain $\{x \mid -1 < x < 1\}$. With the existence of a derived inverse trigonometric function established, a formula for the derivative may be obtained by the method suggested for recovering (1) should it be forgotten. Thus, to obtain a formula for $D_x \sin^{-1} x$ with $-1 < x < 1$ proceed as follows:

$$y = \sin^{-1} x \quad \text{so} \quad -\pi/2 < y < \pi/2,$$

$$\sin y = x, \quad D_x \sin y = D_x x = 1, \quad \cos y\, D_x y = 1,$$

$$D_x y = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} \quad \left(\begin{array}{l}\text{and not } -\sqrt{1 - \sin^2 y} \\ \quad \text{since } -\pi/2 < y < \pi/2 \text{ so } \cos y > 0\end{array}\right)$$

$$D_x \sin^{-1} x = \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1 - x^2}}.$$

Each of the following formulas should be obtained:

1. $$D_x \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1.$$

2. $$D_x \cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}}, \quad -1 < x < 1.$$

3. $$D_x \tan^{-1} x = \frac{1}{1 + x^2}.$$

4. $$D_x \cot^{-1} x = \frac{-1}{1 + x^2}.$$

5. $$D_x \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}, \quad x < -1 \quad \text{or} \quad x > 1.$$

6. $$D_x \csc^{-1} x = \frac{-1}{x\sqrt{x^2 - 1}}, \quad x < -1 \quad \text{or} \quad x > 1.$$

Example 1.

$$D_x \cot^{-1}(3x + 1) = \frac{-1}{1 + (3x + 1)^2} D_x(3x + 1) = \frac{-3}{9x^2 + 6x + 2}.$$

Example 2.

$$\frac{d \sec^{-1} x^2}{dx} = \frac{1}{x^2\sqrt{x^4 - 1}} \frac{dx^2}{dx} = \frac{2x}{x^2\sqrt{x^4 - 1}} = \frac{2}{x\sqrt{x^4 - 1}}.$$

Example 3. Find $\dfrac{d}{dx}\left(\dfrac{1}{a} \cos^{-1} \dfrac{a}{x}\right)$ where a is a constant.

Solution.

$$\frac{1}{a}\frac{d}{dx}\left(\cos^{-1}\frac{a}{x}\right)=\frac{1}{a}\frac{-1}{\sqrt{1-\left(\frac{a}{x}\right)^2}}\frac{d}{dx}\left(\frac{a}{x}\right)=\frac{-1}{\frac{a\sqrt{x^2-a^2}}{|x|}}\left(-\frac{a}{x^2}\right)$$
$$=\frac{1}{\sqrt{x^2-a^2}}\frac{|x|}{x^2}=\frac{1}{|x|\sqrt{x^2-a^2}}.$$

Example 4. A picture b ft from bottom to top hangs on a wall with lower edge a ft above the eye-level of an observer. How far from the wall should the observer stand in order that the picture subtends the largest angle at his eye?

Solution. When the observer is x ft from the wall let θ be the angle subtended by the picture, and let α and β be the angles of elevation of the lower and upper edges of the picture. Hence, (see Fig. 46.3), $\theta = \beta - \alpha$, $\beta = \cot^{-1}\dfrac{x}{a+b}$ and $\alpha = \cot^{-1}\dfrac{x}{a}$ and thus

$$\theta = \cot^{-1}\frac{x}{a+b} - \cot^{-1}\frac{x}{a}.$$

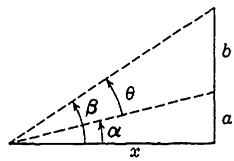


Figure 46.3

Therefore, $$\frac{d\theta}{dx}=\frac{-1/(a+b)}{1+[x/(a+b)]^2}+\frac{1/a}{1+(x/a)^2}=\frac{b(a^2+ab-x^2)}{[(a+b)^2+x^2][a^2+x^2]}.$$

Hence, $\left.\dfrac{d\theta}{dx}\right]_{x=\sqrt{a^2+ab}}=0$, $\left.\dfrac{d\theta}{dx}\right]_{x>\sqrt{a^2+ab}}<0$, $\left.\dfrac{d\theta}{dx}\right]_{x<\sqrt{a^2+ab}}>0$ and thus θ is largest when $x = \sqrt{a^2+ab}$.

PROBLEMS

1. Find $f'(x)$ for each of the following:

a. $f(x) = \sin^{-1}(5x)$. c. $f(x) = \tan^{-1}\sqrt{x}$. e. $f(x) = \sec^{-1}(\sin x)$.

b. $f(x) = \cos^{-1}\left(\dfrac{x}{4}\right)$. d. $f(x) = \cot^{-1}\left(\dfrac{x-1}{2}\right)$. *f.* $f(x) = \csc^{-1}\left(\dfrac{1}{x}\right)$.

2. With $a > 0$, b, and c constants, obtain each of the following.

a. $\dfrac{d}{dx}\sin^{-1}\dfrac{x}{a} = \dfrac{1}{\sqrt{a^2-x^2}}$.

c. $\dfrac{d}{dx}\dfrac{1}{a}\tan^{-1}\dfrac{x}{a} = \dfrac{1}{a^2+x^2}$.

b. $\dfrac{d}{dx}\cos^{-1}\dfrac{a-x}{a} = \dfrac{1}{\sqrt{2ax-x^2}}$.

d. $\dfrac{d}{dx}\dfrac{1}{a}\csc^{-1}\dfrac{x}{a} = \dfrac{-1}{x\sqrt{x^2-a^2}}$.

e. $\dfrac{d}{dx}\dfrac{1}{\sqrt{ab}}\tan^{-1}\left(x\sqrt{\dfrac{a}{b}}\right) = \dfrac{1}{ax^2+b}$, $b > 0$.

f. $\dfrac{d}{dx}\dfrac{1}{\sqrt{a}}\sin^{-1}\left(x\sqrt{\dfrac{a}{b}}\right) = \dfrac{1}{\sqrt{b - ax^2}}, \quad b > 0.$

g. $\dfrac{d}{dx}\dfrac{1}{\sqrt{b}}\sec^{-1}\left(x\sqrt{\dfrac{a}{b}}\right) = \dfrac{1}{x\sqrt{ax^2 - b}}, \quad b > 0.$

h. $\dfrac{d}{dx}\dfrac{2}{\sqrt{4ac - b^2}}\tan^{-1}\dfrac{2ax + b}{\sqrt{4ac - b^2}} = \dfrac{1}{ax^2 + bx + c}, \quad 4ac - b^2 > 0.$

i. $\dfrac{d}{dx}\dfrac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2\sin^{-1}\dfrac{x}{a}\right) = \sqrt{a^2 - x^2}.$

j. $\dfrac{d}{dx}\left(\dfrac{1}{a}\sqrt{1 - a^2x^2} + x\sin^{-1}ax)\right) = \sin^{-1}ax.$

3. Some result in Prob. 2 may be used in solving each of the following differential systems.

a. $dy = \dfrac{1}{4 + x^2}\,dx, \quad y = 5$ when $x = 2.$

b. $dy = \dfrac{1}{\sqrt{3 - 2x^2}}\,dx, \quad y = 1$ when $x = 0.$

c. $dy = \sqrt{3 - x^2}\,dx, \quad y = 0$ when $x = -\frac{3}{2}.$

d. $dy = \dfrac{1}{x^2 + x + 1}\,dx, \quad y = 0$ when $x = -1.$

4. a. Let t be such that $0 \leq t \leq 1$ and plot the points $(0,t)$ and $(\sin^{-1} t,0)$. Sketch $\{(x,y) \mid 0 \leq x \leq \sin^{-1} t,\ 0 \leq y \leq \sin x\}$ and show that this set may also be expressed as $\{(x,y) \mid 0 \leq y \leq t,\ \sin^{-1} y \leq x \leq \sin^{-1} t\}$.

b. Let t be a number such that $-1 \leq t \leq 1$. Show that

$$\left\{(x,y) \mid \sin^{-1} t \leq x \leq \frac{\pi}{2},\ t \leq y \leq \sin x\right\} = \left\{(x,y) \mid t \leq y \leq 1,\ \sin^{-1} y \leq x \leq \frac{\pi}{2}\right\}.$$

47. Exponents and Logarithms

Throughout the previous work in this book it was assumed that the laws of exponents were known at least for rational exponents; e.g., $a^{m/n} = (a^{1/n})^m$ for m and n integers and $a > 0$. So far we have had no reason to employ irrational exponents, but we must now do so.

Let $b > 0$ and p be given numbers. In a more advanced course it will be proved: There is a number L having the property that for each number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\{(x,y) \mid x \text{ rational}, |x - p| < \delta \text{ and } y = b^x\} \subset \{(x,y) \mid |x - p| < \delta \text{ and } |y - L| < \epsilon\}.$$

It will then follow in case p is rational that $b^p = L$. The advanced course will assign the value L to the symbolism b^p in case p is irrational and will proceed to prove that the laws of exponents continue to hold under this extended definition of b^p and that the functions

$$\{(x,y) \mid y = b^x\} \quad \text{and} \quad \{(x,y) \mid x > 0 \quad \text{and} \quad y = x^p\}$$

are continuous. We shall henceforth use these results as though we also had proved them. These results include facts about inequalities as well as equalities. For example:

$$\text{If } x_1 < x_2, \quad \text{then} \quad \begin{cases} b^{x_1} > b^{x_2} & \text{for} \quad 0 < b < 1 \\ b^{x_1} = b^{x_2} = 1 & \text{for} \quad b = 1 \\ b^{x_1} < b^{x_2} & \text{for} \quad b > 1. \end{cases}$$

Hence, in addition to being continuous, the function

(1) $$\{(x,y) \mid y = b^x\}$$

is decreasing if $0 < b < 1$, but is increasing if $b > 1$ (see Fig. 47.1), and in either case has domain the set of all numbers x and range $\{y \mid y > 0\}$. Thus, either for $0 < b < 1$ or for $b > 1$, the function (1) has an inverse function given by

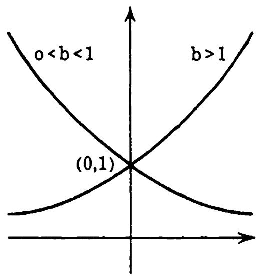


Figure 47.1

(2) $$\{(x,y) \mid x = b^y\}$$

whose domain is $\{x \mid x > 0\}$ and range the set of all numbers y. The graph of this function has the form of one or the other of the curves of Fig. 47.2.

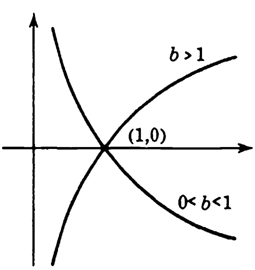


Figure 47.2

The function (2) is represented by $\log_b$ so that for x a positive number, then $\log_b x$ is a number (called the logarithm to the base b of x) and this number is such that

(3) $$b^{\log_b x} = x, \quad 0 < b < 1 \quad \text{or} \quad b > 1.$$

The ordinary laws of logarithms follow directly from the relation (3) as we now show. For example, with p any number, raise both sides of (3) to the power p:

$$(b^{\log_b x})^p = x^p \quad \text{so that} \quad b^{p \log_b x} = x^p.$$

Since (3) holds no matter what positive number is substituted for x, we replace x in (3) by x^p to obtain

$$b^{\log_b x^p} = x^p.$$

These expressions for x^p must be equal so that

$$b^{\log_b x^p} = b^{p \log_b x},$$

but now the exponents must be equal and this states that

(4) $$\log_b x^p = p \log_b x.$$

To obtain another law, let M and N be positive numbers so that also $MN > 0$. Now replace x in (3) first by M, then by N, and finally by MN:

$$b^{\log_b M} = M, \quad b^{\log_b N} = N, \quad \text{and} \quad b^{\log_b MN} = MN.$$

From the first two of these, and a law of exponents,

$$MN = b^{\log_b M} b^{\log_b N} = b^{(\log_b M + \log_b N)},$$

and this relation, with the third equation above, gives

$$b^{\log_b MN} = b^{(\log_b M + \log_b N)}$$

Again the exponents must be equal so that

(5) $$\log_b MN = \log_b M + \log_b N.$$

By a similar procedure we also obtain

(6) $$\log_b (M/N) = \log_b M - \log_b N.$$

Also, upon replacing x in (3) by b and then by 1 we have

$$b^{\log_b b} = b \quad \text{and} \quad b^{\log_b 1} = 1 \quad (=b^0),$$

which say, respectively, that $\log_b b = 1$ and $\log_b 1 = 0$.

THEOREM 47. *The function $\log_b$ is continuous; that is, if x_0 is any positive number, then*

$$\lim_{x \to x_0} \log_b x = \log_b x_0, \quad 0 < b < 1 \quad \textit{or} \quad b > 1.$$

PROOF. Consider first $b > 1$. Let ϵ be an arbitrary positive number. Then $1 < b^\epsilon$ and if x is such that $1 < x < b^\epsilon$, then

$$0 = \log_b 1 < \log_b x < \log_b b^\epsilon = \epsilon \log_b b = \epsilon.$$

Also, $b^{-\epsilon} < 1$ and for x such that $b^{-\epsilon} < x < 1$, then

$$-\epsilon = -\epsilon \log_b b = \log_b b^{-\epsilon} < \log_b x < \log 1 = 0.$$

Now let δ be the smaller of $b^\epsilon - 1$ and $1 - b^{-\epsilon}$ so that $\delta > 0$ and

$$\text{if} \quad 0 < |x - 1| < \delta, \quad \text{then} \quad |\log_b x| < \epsilon.$$

From the definition of a limit, this states that

$$\lim_{x \to 1} \log_b x = 0, \quad b > 1.$$

A similar procedure establishes this limit in case $0 < b < 1$.

Now let x_0 be any positive number. Then

$$\lim_{x \to x_0} \frac{x}{x_0} = 1 \quad \text{and hence} \quad \lim_{x \to x_0} \log_b \frac{x}{x_0} = 0.$$

Consequently, we may write the following, knowing the existence of each limit as we write it:

$$\log_b x_0 = 0 + \log_b x_0 = \left(\lim_{x \to x_0} \log_b \frac{x}{x_0}\right) + \log_b x_0 = \lim_{x \to x_0} \left(\log_b \frac{x}{x_0} + \log_b x_0\right)$$

$$= \lim_{x \to x_0} \log \left(\frac{x}{x_0} x_0\right) = \lim_{x \to x_0} \log_b x,$$

which states the relation we wished to establish.

48. Log Scales

Sections 48–50 are somewhat out of the main stream of the course, but they are included for engineering and scientific applications, and to show that in analytic geometry there need not always be equally spaced divisions on the axes.

We now consider common logarithms; that is, logarithms with base 10, and shall not write the base. Thus, log means $\log_{10}$ so that

$$\log 1 = 0, \ \log 10 = 1, \ \log 100 = 2, \ \log 0.1 = -1, \ \text{and} \ \log 0.01 = -2.$$

Draw a line and parallel to it draw a second line. On the first line select an ordinary linear coordinate system. Let M be a positive number. Now

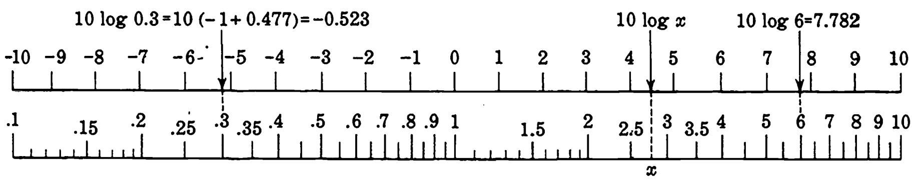


Figure 48

with x a positive number, locate the point on the first line with coordinate $M \log x$. Opposite this point of the first line is a point of the second line, and this second point we now label x. In particular the point opposite the origin 0 of the ordinary linear coordinate system is labeled 1 since $M \log 1 = M \cdot 0 = 0$. The second line so labeled is called a **logarithmic scale** (or log scale) with origin 1 and modulus M. It is usually convenient to select the modulus $M = 10$, and this we now do. Hence (see Fig. 48), the points 1, 10, and 0.1 of the log scale will be opposite the respective points

$$10 \log 1 = 0, \quad 10 \log 10 = 10, \quad \text{and} \quad 10 \log (0.1) = -10$$

of the ordinary linear coordinate system. Also, since

$$10 \log 2 = 10(0.3010^{+}) = 3.010^{+} \quad \text{and} \quad 10 \log(\tfrac{1}{2}) = -10 \log 2 = -(3.010^{+})$$

the points 2 and 0.5 are on opposite sides of the origin and each slightly more than 3 units from the origin.

Let $0 < x_1 < x_2$ be given. Then on the ordinary coordinate system the point $\log x_1$ precedes the point $\log x_2$, on the log scale the point x_1 precedes the point x_2, and the actual number of ordinary units between either pair of points is

$$10 \log x_2 - 10 \log x_1 = 10(\log x_2 - \log x_1) = 10 \log(x_2/x_1).$$

49. Semi-Log Coordinates

Draw two perpendicular lines. With the point of intersection as origin of both systems establish an ordinary linear coordinate system on the horizontal line, but on the vertical one put a log scale with 10 ordinary units as modulus. This is called a **semi-log** coordinate system,† and by means of it is established a one-to-one correspondence between points of the plane and ordered pairs (x,y) of numbers in which $y > 0$. In particular, the origin has coordinates $(0,1)$.

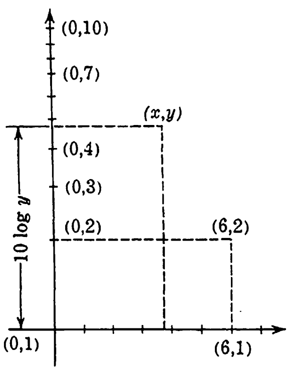


Figure 49.1

THEOREM 49. *For any line not parallel to the log scale there are numbers $a > 0$ and $b > 0$ such that the line has semi-log equation*

$$y = ba^x. \tag{1}$$

Conversely, with $a > 0$ and $b > 0$ the semi-log graph of the set $\{(x,y) \mid y = ba^x\}$ is a straight line.

PROOF. Consider a line not parallel to the log scale. This line then intersects the log scale say at $(0,b)$. Consequently $b > 0$. Let (x,y) be any point other than $(0,b)$ so that $x \neq 0$. The line has an actual slope m (which may be positive, negative, or zero) and note that

$$m = \frac{10 \log y - 10 \log b}{x - 0} = \frac{10}{x} \log \frac{y}{b}.$$

† If no commercially printed semi-log paper is available, a sheet may be made by marking equal divisions on the x-axis and then using a scale of the slide rule to make logarithmic divisions on the y-axis. The **B** scale is a convenient size.

Consequently, $\log (y/b) = mx/10$, $y/b = 10^{mx/10}$ and hence

$$y = b(10^{m/10})^x.$$

Upon setting $10^{m/10} = a$ we see that $a > 0$ and that (1) holds.

Conversely, let $a > 0$ and $b > 0$ be given and consider the semi-log graph of $\{(x,y) \mid y = ba^x\}$. Hence, the point $(0,b \cdot a^0)$, which is the point $(0,b)$, is on the graph. Now consider any point (x,y) of the graph other than $(0,b)$. Hence, $x \neq 0$ and the line joining (x,y) and $(0,b)$ has slope

$$\begin{aligned} m &= \frac{10 \log y - 10 \log b}{x - 0} = \frac{10}{x}[\log (ba^x) - \log b] \\ &= \frac{10}{x}(\log b + x \log a - \log b) = 10 \log a. \end{aligned}$$

Since the line joining $(0,b)$ and (x,y) of the graph has slope $m = 10 \log a$ (which contains neither x nor y) all points of the graph lie on this line. By the first part of the proof every point of the line satisfies the equation $y = ba^x$ so the line is the whole graph as we wished to prove.

Example. Given the line passing through the points whose semi-log coordinates are (2,15) and (6,9). Find the equation of this line in the form (1) with a and b accurate to two decimal places.

Solution. The problem is to determine constants a and b such that

$$15 = ba^2 \quad \text{and} \quad 9 = ba^6$$

or, by taking logs, such that

$$\begin{aligned} \log 15 &= \log b + 2 \log a \\ \log 9 &= \log b + 6 \log a. \end{aligned}$$

By subtraction, $\log 9 - \log 15 = 4 \log a$. From this equation we find $\log a$ which we substitute into one of the log equations (we choose the first) to obtain $\log b$:

$$\begin{aligned} \log a &= \tfrac{1}{4}(\log 9 - \log 15) \\ &= \tfrac{1}{4}(0.9542 - 1.1761) = -0.0555, \\ \log b &= \log 15 - 2(-0.0555) \\ &= 1.1761 + 0.1110 = 1.2871. \end{aligned}$$

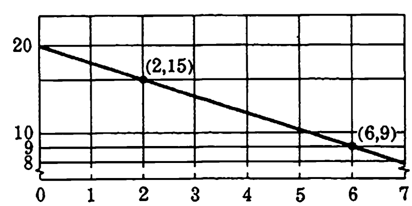


Figure 49.2

Thus, $\log a = 10 - 0.0555 - 10 = 9.9445 - 10$, $a = 0.88$, $b = 19.4$ and the desired equation is $y = 19.4(0.88)^x$.

The amount (principal plus interest) on a \$1 and on a \$5 investment at 6% compounded annually is given at 10 year intervals in the accompanying table. Both sets of data are plotted and the points joined by a smooth curve on an ordinary rectangular coordinate system in Fig. 49.3 and on a semi-log system in Fig. 49.4.

Year	0	10	20	30	40	50	60
\$1	1	1.79	3.20	5.74	10.29	18.42	32.98
\$5	5	8.95	16.03	28.70	51.43	92.10	164.83

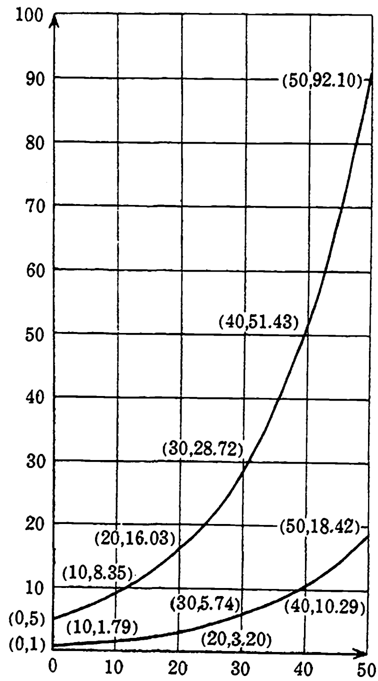


Figure 49.3

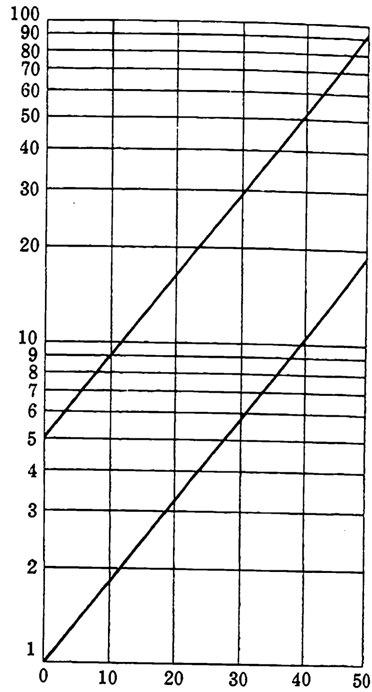


Figure 49.4

PROBLEMS

1. Sketch the graph of each of the following equations both on a rectangular and on a semi-log system.

   a. $y = 2x$. c. $y = -2x$. e. $y = x^2$.

   b. $y = 2^x$. d. $y = 2^{-x}$. f. $y = x^{-2}$.

2. Find equations in the form (1) of the line passing through the semi-log points:

   a. (0,2), (1,3). c. (2,3), (1,2). e. (1.86,3), (2.86,4).

   b. (0,2), (−1,3). d. $(3,\frac{8}{9})$, $(4,\frac{16}{27})$. f. (3,1.86), (4,2.86).

3. On a semi-log system (a) sketch the graphs of $y = 2^x$ and $y = 3^x$, (b) obtain a third graph by geometric addition of these two graphs, and (c) find an equation of this third graph.

4. Sketch the graphs of $y = 2^x$ and $y = 2^x + 3$ on a semi-log system. Show that the first is an asymptote of the second.

5. Represent the data of the table by points on a semi-log system, approximate these points by a straight line, select two points on this line, and using these points find an equation in the form (1) of this line.

a.	x	0.8	1.7	2.3	3.6	4.8
	y	2.9	4.1	5.4	8.8	10.5
b.	x	1.3	2.4	3.5	4.9	6.7
	y	9.5	7.6	6.3	5.2	3.7
c.	x	17	26	35	48	53
	y	3.4	1.6	0.95	0.38	0.24
d.	x	10.7	11.9	13.3	14.1	14.9
	y	48	105	230	370	630

6. Show that the pair of equations represent the same line on a semi-log system.
   a. $y = 2(3^{x+1}),\quad y = 6 \cdot 3^x.$
   b. $y = 0.831\ 10^{3-2x},\quad y = 831(0.01)^x.$
   c. $y = 9.86(2.37)^x,\quad y = 9.86(10^{0.3747})^x.$

7. Let t be a number such that $t \geq 1$. Show that

$$\{(x,y) \mid 1 \leq x \leq t, 0 \leq y \leq \log_2 x\} = \{(x,y) \mid 0 \leq y \leq \log_2 t, 2^y \leq x \leq t\}.$$

(Hint: Sketch the rectangular graph of $y = \log_2 x$ and note the region under the graph and above the interval joining the points (1,0) and $(t,0)$.)

50. Log-Log Coordinates

Draw two perpendicular lines. Use the same modulus and place a log scale on each of these lines with the origin of each at the intersection of the lines. The result is a **logarithmic** coordinate system (or log-log system). By means of this system a one-to-one correspondence is established between points of the plane and ordered pairs (x,y) of positive numbers. Notice that the origin has coordinates (1,1).

THEOREM 50. *For any non-vertical line there is a number $b > 0$ and a number m such that the line has log-log equation*

$$y = bx^m. \tag{1}$$

Conversely, with $b > 0$ and m any number, the log-log graph of $\{(x,y) \mid y = bx^m\}$ is a straight line.

The proof, similar to that of Theorem 49, is left as an exercise.

Example 1. Draw the log-log graph of $y = 8.6(x)^{-0.4}$.

Solution. Since the equation is in the form (1), the graph is a straight line so two points determine it. One point, obtained by setting $x = 1$, is (1,8.6). The easiest second point to compute is the one with $x = 10$:

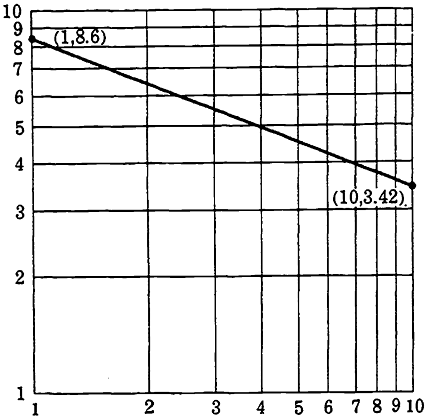


Figure 50

$$\begin{aligned}\log y &= \log 8.6 - 0.4 \log 10 \\ &= 0.9345 - 0.4 = 0.5345,\end{aligned}$$

so that $y = 3.42$ and (see Fig. 50) a second point is (10,3.42).

A method of geometrically constructing a second point depends upon the fact (known if Theorem 50 were proved) that the line whose equation is (1) has actual slope m. Thus, in this example the line has slope $m = -0.4 = -\frac{2}{5}$ and a second point may be obtained by using an ordinary ruler and measuring 5 inches (regardless of the modulus used) to the right from the point (1,8.6) and then down 2 inches.

Example 2. Find a log-log equation of the line through the points (2,85), (9,15).

Solution. We must have $85 = b \cdot 2^m$ and $15 = b \cdot 9^m$, or in log form

$$\log 85 = \log b + m \log 2 \quad \text{and} \quad \log 15 = \log b + m \log 9. \tag{2}$$

By subtraction and then division,

$$m = \frac{\log 85 - \log 15}{\log 2 - \log 9} = \frac{1.9294 - 1.1761}{0.3010 - 0.9542} = -\frac{0.7533}{0.6532}.$$

Thus, $\log |m| = \log 0.7533 - \log 0.6532 = 0.0620$. Therefore, $|m| = 1.154$ and $m = -1.154$. From the first of the equations (2)

$$\log b = \log 85 - m \log 2 = 1.9294 + (1.154)(0.3010) = 2.2768$$

and $b = 189.1$. The desired equation is therefore $y = 189.1(x^{-1.154})$.

Physical phenomena in which one quantity depends (or is assumed to depend) upon only one other quantity are often encountered. In any specific case a guess at a governing law may be made by making several observations in an experiment to obtain ordered pairs of numbers, plotting these ordered pairs, passing a smooth curve approximating these points, and then assuming that an equation of this curve represents the law to within an allowable error. Experience has shown that for non-periodic phenomena, the ordered pairs plotted on either a rectangular, or on a semi-log, or on a log-log coordinate system will appear to be linear in a sufficient number of cases to justify checking these three possibilities first.

51. The Number e

The binomial expansion of $(a + b)^n$ where n is a positive integer may be written as

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}a^0b^n,$$

where for n a positive integer and r an integer such that $0 < r \leq n$,

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

As specific examples note that

$$(1 + \tfrac{1}{2})^2 = 1 + 2(\tfrac{1}{2}) + (\tfrac{1}{2})^2 = 2 + \tfrac{1}{4},$$
$$(1 + \tfrac{1}{3})^3 = 1 + 3(\tfrac{1}{3}) + 3(\tfrac{1}{3})^2 + (\tfrac{1}{3})^3 = 2 + \tfrac{10}{27}, \quad \text{and}$$
$$(1 + \tfrac{1}{4})^4 = 1 + 4(\tfrac{1}{4}) + 6(\tfrac{1}{4})^2 + 4(\tfrac{1}{4})^3 + (\tfrac{1}{4})^4 = 2 + \tfrac{113}{256}.$$

THEOREM 51. *With n taking only integer values, the numbers*

$$u_n = (1 + 1/n)^n \tag{1}$$

are such that $2 < u_n < u_{n+1} < 3$ *for* $n = 2, 3, 4, \cdots$.

PROOF. First note, for $2 \leq n$ and $0 < r \leq n$ that

$$\binom{n}{r}\left(\frac{1}{n}\right)^r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\frac{1}{n^r}$$
$$= \frac{1}{r!}\frac{n}{n}\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-r+1}{n}\right)$$
$$= \frac{1}{r!}1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right) \tag{2}$$
$$< \frac{1}{r!} \quad \text{since} \quad 1-\frac{1}{n} < 1, \quad 1-\frac{2}{n} < 1, \cdots, 1-\frac{r-1}{n} < 1.$$

Since, however, $r! = 1 \cdot 2 \cdot 3 \cdots r \geq 1 \cdot 2 \cdot 2 \cdots 2 = 2^{r-1}$ we have

$$\binom{n}{r}\frac{1}{n^r} < \frac{1}{2^{r-1}} \quad \text{for} \quad 1 < r \leq n.$$

Now, by the binomial expansion with $a = 1$ and $b = 1/n$ we have for $n \geq 2$

$$(3) \quad u_n = 1 + 1 + \binom{n}{2}\frac{1}{n^2} + \binom{n}{3}\frac{1}{n^3} + \cdots + \binom{n}{r}\frac{1}{n^r} + \cdots + \binom{n}{n}\frac{1}{n^n}$$

$$< 2 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 2 + \left(1 - \frac{1}{2^{n-1}}\right) < 3$$

so that $2 < u_n < 3$ for any integer $n \geq 2$. Hence, for $n \geq 2$, then $n + 1 > 2$ and thus $2 < u_{n+1} < 3$. The rest of the proof is to show that $u_n < u_{n+1}$, and to do this we first use (2) with n replaced by $n + 1$ to see that

$$\binom{n+1}{r}\frac{1}{(n+1)^r} = \frac{1}{r!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{r-1}{n+1}\right)$$

$$> \frac{1}{r!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{r}{n}\right) = \binom{n}{r}\frac{1}{n^r}.$$

Now, from (3) with each term except the first two replaced by a larger value (and then extra terms added), we have

$$u_n < \left[1 + 1 + \binom{n+1}{2}\frac{1}{(n+1)^2} + \cdots + \binom{n+1}{r}\frac{1}{(n+1)^r} + \cdots \right.$$

$$\left. + \binom{n+1}{n}\frac{1}{(n+1)^n}\right] + \binom{n+1}{n+1}\frac{1}{(n+1)^{n+1}}.$$

But the expression on the right is (3) with n replaced by $n + 1$ and is thus equal to u_{n+1} so that $u_n < u_{n+1}$ for $n \geq 2$.

Now, let $S = \{x \mid x = u_n \text{ for some positive integer } n\}$. Hence, the set S is bounded above by 3 and thus (see the Axiom in Sec. 4) the set S has a smallest upper bound which we call e. Thus, e is a number, $2 < e \leq 3$, and $u_n \leq e$ for each positive integer, but no number smaller than e has this property. Now, let ϵ be an arbitrary positive number. Then, since $e - \epsilon < e$, let N be an integer such that $e - \epsilon < u_N$. Hence, from Theorem 51, if $n > N$ then $u_N < u_n$. Thus, if $n \geq N$, then $e - \epsilon < u_n \leq e$ which says that

$$\lim_{n\to\infty} u_n = e.$$

To return to the definition of u_n, as given by (1), we thus have

$$(4) \qquad \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e.$$

In (4), n takes only positive integer values. Later, we shall show how the number e may be approximated to any desired degree of accuracy. To 15 decimal places

$$e = 2.7\,1828\,1828\,45\,90\,45,$$

which may be used for numerical approximations. It should be noted as we proceed that no better approximation than $2 < e \leq 3$ need be known for all theoretical work. It is, however, necessary to know (as proved in Appendix A3), that $\lim_{h\to 0} (1 + h)^{1/h}$ exists and that also

$$\lim_{h\to 0} (1 + h)^{1/h} = e. \tag{5}$$

By changing the form we obtain that given a number x, then

$$\lim_{\Delta x\to 0} \left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x} = e \quad \text{for} \quad x \neq 0 \tag{6}$$

and it is this form we shall use in deriving a formula for $D_x \log_e |x|$.

In the next section it will be seen that the constant e appears naturally in the course of deriving formulas, and that some formulas are simplified by using logarithms to the base e. Although there are tables of logs to the base e, computation with them is more difficult (mainly because characteristics are harder to determine) than for logs to the base 10. When logs to the base e are involved in computation, it is generally better to "change the base" as given below.

We first derive a general formula. Recall that

$$b^{\log_b x} = x \quad \text{for} \quad 0 < b < 1 \quad \text{or} \quad b > 1. \tag{7}$$

Now let a be a number such that $0 < a < 1$ or $a > 1$. Hence, by taking the log to the base a of both sides of (7)

$$\log_a (b^{\log_b x}) = \log_a x$$

and then by using (4) of Sec. 47 with $p = \log_b x$ we have

$$\log_b x \log_a b = \log_a x. \tag{8}$$

The approximations $\log_{10} e = 0.4343$ and $1/0.4343 = 2.3026$ give

$$0.4343 \log_e x = \log_{10} x \quad \text{and} \quad \log_e x = 2.3026 \log_{10} x$$

for interchanging between logs of bases e and 10. Since it is almost universal to use $\ln x = \log_e x$ and $\log x = \log_{10} x$, the formulas for changing four place approximations of logarithms to the base 10 or e to the other base appear as

$$\log x = 0.4343 \ln x \quad \text{and} \quad \ln x = 2.3026 \log x. \tag{9}$$

PROBLEMS

1. For only positive values of the variables, sketch the log-log graphs of

a. $y = x$. c. $y = \sqrt{x}$. e. $\sqrt{xy} = 1$.

b. $y = x^2$. d. $xy = 1$. f. $x\sqrt{y} = 1$.

2. Find the equation in the form (1) of the log-log line through:

   a. (1,2), (2,3).
   b. (1,4.75), (10,0.96).
   c. (2.8,11.76), (9.1,124.2).
   d. (1.42,0.78), (8.75,21.4).

3. On a log-log system (a) sketch the graphs of $y = x^2$ and $y = x^3$; (b) obtain a third graph by geometric addition of these two; and (c) find an equation of the graph of part (b).

4. Work Prob. 3 using instead $y = x^{-2}$ and $y = x^3$.

5. Work Prob. 3 using instead $y = 1.5x^2$ and $y = 2.3x^3$.

6. Sketch the graphs of $y = x^2$ and $y = x^2 + 3$ and show that the first is an asymptote of the second.

7. Find a log-log equation of a line approximating the data:

a.

x	1.4	2.2	3.2	4.5	7.1
y	18	12	9.0	7.1	4.7

b.

x	4.7	7.1	9.0	12	18
y	7.1	4.5	3.2	2.2	1.3

c.

x	0.7	2	4.5	11	20
y	1.6	3	6	10	15

d.

x	2	4	10	20	40
y	5	4	3	2.5	2

8. The data of the table will appear nearly linear when plotted in one of the systems; rectangular, semi-log, or log-log. Find an approximating law.

a.

0.6	1.3	2.6	4.3	5.7
1.75	2.6	5.5	14.5	32

b.

0.35	1.25	3.0	4.5	6.5
1.89	1.65	1.15	0.75	0.15

c.

0.5	1.3	2.6	4.9	5.7
3	2	1	0.3	0.2

d.

0.1	0.3	0.5	0.8	1.5
0.62	1.2	1.6	2.1	3.0

9. Using laws of logarithms show that

   a. $\log(1/x^2) = -2\log|x|, \quad x \neq 0.$

   b. $\log\csc^2 x = -2\log|\sin x|, \quad x \neq m\pi.$

   c. $\log(1 - \cos x) + \log(1 + \cos x) = 2\log|\sin x|, \quad x \neq m\pi.$

   d. $\log(1 - \cos x) - \log(1 + \cos x) = 2\log|\tan(x/2)|, \quad x \neq m\pi.$

   e. If $a = b^c$, then $\log\log a = \log c + \log\log b$.

   f. If $a^x = b^y$, then $y = x\log_b a$.

   g. If $y = \ln(x + \sqrt{x^2 + 1})$, then $x = \frac{1}{2}(e^y - e^{-y})$.

   h. $\log_{10}(\log_2 5.26) = \log_{10}(\log_{10} 5.26) - \log_{10}(\log_{10} 2).$

52. Derivatives of Log Functions

We first prove the following theorem.

THEOREM 52. *The function* $\{(x,y) \mid y = \ln|x|\}$ *has derived function* $\left\{(x,y) \mid y = \dfrac{1}{x}\right\}$; that is,

$$D_x \ln|x| = \frac{1}{x}, \quad x \neq 0.$$

PROOF. Let x be a number different from 0. Choose Δx such that $0 < |\Delta x| < |x|$. Then $x + \Delta x$ and x are either both positive or both negative and

$$(1) \qquad 0 < \frac{x + \Delta x}{x} = 1 + \frac{\Delta x}{x}.$$

By using this relation and the laws of logarithms we have

$$\begin{aligned}
\frac{\ln|x+\Delta x| - \ln|x|}{\Delta x} &= \frac{1}{\Delta x}\ln\left(\frac{|x+\Delta x|}{|x|}\right) && \text{(by (6) of Sec. 47)}\\
&= \frac{1}{\Delta x}\ln\left(1 + \frac{\Delta x}{x}\right) && \left(\begin{array}{c}\text{absolute values}\\ \text{removed by (1)}\end{array}\right)\\
&= \frac{1}{x}\frac{x}{\Delta x}\ln\left(1 + \frac{\Delta x}{x}\right) && \left(\text{since } \frac{x}{x} = 1,\ x \neq 0\right)\\
&= \frac{1}{x}\ln\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x} && \text{(by (4) of Sec. 47).}
\end{aligned}$$

This gives us the clue for writing

$$\begin{aligned}
\frac{1}{x} = \frac{1}{x}\ln e &= \frac{1}{x}\ln\left[\lim_{\Delta x\to 0}\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right] && \text{(by (6) of Sec. 51)}\\
&= \frac{1}{x}\lim_{\Delta x\to 0}\left[\ln\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}\right] && \left(\begin{array}{c}\text{ln is a continuous}\\ \text{function, see Theorem 47}\end{array}\right)\\
&= \frac{1}{x}\lim_{\Delta x\to 0}\left[\frac{x}{\Delta x}\ln\left(1 + \frac{\Delta x}{x}\right)\right]\\
&= \frac{x}{x}\lim_{\Delta x\to 0}\left[\frac{1}{\Delta x}\ln\left(\frac{x+\Delta x}{x}\right)\right] && \left(\begin{array}{c}x\text{ is independent}\\ \text{of }\Delta x\end{array}\right)\\
&= \lim_{\Delta x\to 0}\frac{\ln|x+\Delta x| - \ln|x|}{\Delta x} && \text{(see above)}
\end{aligned}$$

which justifies each limit and proves the theorem.

In formalistic notation, the result of Theorem 52 is written as

(2) $$D_x \ln |x| = \frac{1}{x}, \quad x \neq 0 \quad \text{or}$$

$$D_u \ln |u| = \frac{1}{u}, \quad u \neq 0.$$

Consequently, assuming that $u'(x)$ exists,

(3) $$D_x \ln |u(x)| = \frac{1}{u(x)} D_x u(x), \quad u(x) \neq 0.$$

Example 1. $D_x \ln |\sin x| = \dfrac{1}{\sin x} D_x \sin x = \dfrac{\cos x}{\sin x} = \cot x.$

For $0 < b < 1$ or $b > 1$ the formula for changing from base b to base e is (see (8) of Sec. 51)

$$\log_b |x| = \log_b e \ln |x|.$$

Hence, $D_x \log_b |x| = D_x(\log_b e \ln |x|) = \log_b e\, D_x \ln |x|$ so that

(4) $$D_x \log_b |x| = (\log_b e) \frac{1}{x}, \quad x \neq 0 \quad \text{and}$$

(5) $$D_x \log_b |u(x)| = (\log_b e) \frac{1}{u(x)} D_x u(x), \quad u(x) \neq 0.$$

Example 2. $D_x \log |3x^2 + 1| = \dfrac{\log e}{3x^2 + 1} D_x(3x^2 + 1) = \dfrac{6(0.4343)x}{3x^2 + 1}.$

Example 3. Find $\dfrac{dy}{dx}$ if $y = \ln \left\{ |x^3 - 1| \sqrt{\dfrac{4x - 1}{x^2 + 1}} \right\}, \quad x \neq 1, \quad x > \frac{1}{4}.$

Solution. The problem is greatly simplified by first writing

$$y = \ln |x^3 - 1| + \tfrac{1}{2} \ln (4x - 1) - \tfrac{1}{2} \ln (x^2 + 1)$$

(which follows from the laws of logarithms) and then obtaining

$$\frac{dy}{dx} = \frac{1}{x^3 - 1} D_x(x^3 - 1) + \frac{1}{2} \frac{1}{4x - 1} D_x(4x - 1) - \frac{1}{2} \frac{1}{x^2 + 1} D_x(x^2 + 1)$$

$$= \frac{3x^2}{x^3 - 1} + \frac{2}{4x - 1} - \frac{x}{x^2 + 1}.$$

In finding anti-derivatives, it is suggestive to write (3) in differential notation as

(6) $$d \ln |u| = \frac{du}{u}.$$

Example 4. Find $f(x)$ given that $f'(x) = \tan x$.

Solution. Write $df(x) = \dfrac{\sin x}{\cos x}\,dx = -\dfrac{d\cos x}{\cos x}$ which (except for the minus sign) is the right hand side of (6) with $u = \cos x$, and hence, with c an arbitrary constant,

$$f(x) = -\ln|\cos x| + c.$$

Since

$$\frac{d}{dx}\frac{1}{p+1}x^{p+1} = x^p \quad \text{if} \quad p \neq -1, \quad \text{but} \quad \frac{d}{dx}\ln|x| = x^{-1}$$

we have the following result for finding anti-derivatives:

$$\textit{If} \quad f'(x) = x^p, \quad \textit{then} \quad f(x) = \begin{cases} \dfrac{1}{p+1}x^{p+1} + c & \textit{if} \quad p \neq -1 \\ \ln|x| + c & \textit{if} \quad p = -1. \end{cases}$$

PROBLEMS

1. Find dy/dx for each of the following:

a. $y = \ln|x^3|$.

b. $y = \log_b|x^3|$.

c. $y = x\ln|3x|$.

d. $y = \ln(x + \sqrt{1 + x^2})$.

e. $y = \dfrac{1}{x}\ln(x^2 + 1)$.

f. $y = \ln(3x + 5)^2$.

g. $y = \tan^{-1}(\ln|x|)$.

h. $y = \ln|\ln|x||$.

i. $y = \ln^2|x|$.

j. $y = \sin(\ln|x|)$.

2. Work out both sides of the following and explain why the equality holds:

a. $D_x x(x - 2b) = D_x(b - x)^2$.

b. $D_x \ln|x/a| = D_x \ln|x|$.

c. $D_x \ln\left(\dfrac{x + \sqrt{a^2 + x^2}}{a}\right) = D_x \ln(x + \sqrt{a^2 + x^2}), \quad a > 0.$

d. $D_x \ln|x - \sqrt{a^2 + x^2}| = -D_x \ln(x + \sqrt{a^2 + x^2}), \quad a \neq 0.$

3. Establish each of the following:

a. $D_x \ln|\tan x| = 2\csc 2x$.

b. $D_x \ln|\sec x + \tan x| = \sec x$.

c. $\dfrac{d}{dx}\dfrac{1}{2a}\ln\left|\dfrac{x - a}{x + a}\right| = \dfrac{1}{x^2 - a^2}$.

d. $D_x \ln|\cos x| = -\tan x$.

e. $D_x x(\ln|x| - 1) = \ln|x|$.

f. $\dfrac{d}{dx}\ln\left|\dfrac{x}{ax + b}\right| = \dfrac{b}{x(ax + b)}$.

g. $\dfrac{d}{dx}\{ax - b\ln|ax + b|\} = \dfrac{a^2x}{ax + b}$.

h. $\dfrac{d}{dx}\left\{-\dfrac{1}{bx} + \dfrac{a}{b^2}\ln\left|\dfrac{ax + b}{x}\right|\right\} = \dfrac{1}{x^2(ax + b)}$.

i. $\dfrac{d}{dx}\ln|x + \sqrt{x^2 \pm a^2}| = \dfrac{1}{\sqrt{x^2 \pm a^2}}$.

j. $D_x(x\tan^{-1}x - \ln\sqrt{1 + x^2}) = \tan^{-1}x$.

4. Some of the results in Prob. 3 will be useful in solving the following derivative or differential systems.

a. $f'(x) = \dfrac{1}{x^2 - 9}$, $f(2) = 1$.

b. $f'(x) = \dfrac{1}{x(2x + 3)}$, $f(-1) = 5$.

c. $dy = \dfrac{1}{\sqrt{x^2 - 9}}\,dx$, $y = 0$ when $x = 5$.

d. $dy = \dfrac{x}{2x + 3}\,dx$, $y = 5$ when $x = -1$.

e. A curve passes through the point $(-1,2)$ and at each point (x,y) on the curve the tangent to the curve has slope $m = \ln|x|$. Find the equation of the curve.

f. Work Part e, with $m = \tan^{-1} x$.

5. a. Show that the graph of $y = \ln|x|$ is concave downward with no relative maxima or minima. Sketch the graph.

b. For the graph of $y = x \ln|x|$, discuss the concavity properties and find the relative maximum and minimum points.

6. Find $D_x y$ given

a. $y = \ln\left[\dfrac{|x - 1|(\sqrt{x^3 + 3})}{\sqrt[4]{x^2 + 1}}\right]$, $x \neq 1$, $x > -\sqrt[3]{3}$.

b. $y = \ln\left[\dfrac{|\sin x|(x^2 + 9)}{1 + \cos x}\right]$, $x \neq m\pi$.

53. Exponential Functions

The function $\{(x,y) \mid y = e^x\}$ and the function $\{(x,y) \mid x > 0 \text{ and } y = \ln x\}$ are inverse of each other. Hence, for a any number and $b = e^a$, the point (a,b) is on the first graph and the point (b,a) is on the second graph. Moreover, the tangent to the second graph at the point (b,a) has slope $D_x \ln x]_{x=b} = 1/b$ and, hence, the tangent to the first graph exists at the point (a,b) and has slope the reciprocal of $1/b$; that is, $D_x e^x]_{x=a}$ exists and

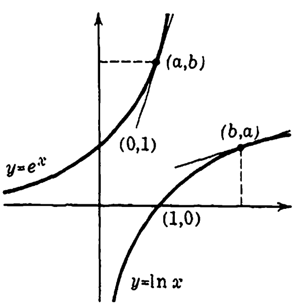


Figure 53

$$D_x e^x\big]_{x=a} = b = e^a.$$

Since a was any number we have that $D_x e^x$ exists and

(1) $$D_x e^x = e^x.$$

Hence, considering composition functions, if u is a function and x is a number such that $u'(x)$ exists, then

$$(2) \qquad D_x e^{u(x)} = e^{u(x)} D_x u(x).$$

Example 1. $D_x e^{\sin x} = e^{\sin x} D_x \sin x = (\cos x)e^{\sin x}$.

Since ln is merely another notation for $\log_e$, we have that for any positive number a, then $e^{\ln a} = a$. By raising both sides of this equation to the power x we have

$$(3) \qquad e^{x \ln a} = a^x, \quad a > 0.$$

This formula makes it possible to write any exponential expression as e to a power. Thus, by using (3) and (2):

$$(4) \qquad D_x a^x = D_x e^{x \ln a} = e^{x \ln a} D_x(x \ln a) = a^x \ln a, \quad a > 0.$$

Hence, for u a function and x a number such that $u'(x)$ exists, then

$$(5) \qquad D_x a^{u(x)} = a^{u(x)} \ln a \; D_x u(x), \quad a > 0.$$

Example 2. $D_x 10^{5x^3} = 10^{5x^3} \ln 10 \; D_x 5x^3$
$= 10^{5x^3}(2.3026)15x^2 = 34.5390x^2 10^{5x^3}$.

In differential notation (2) and (5) appear as

$$(6) \qquad d\, e^u = e^u\, du$$

$$(7) \qquad d\, a^u = a^u \ln a\, du, \quad \text{or}$$

$$(8) \qquad d\left(\frac{a^u}{\ln a}\right) = a^u\, du, \quad 0 < a < 1 \quad \text{or} \quad a > 1.$$

Example 3. Find $f(x)$ given $df(x) = \sin x\, 10^{\cos x}\, dx$, $f(\pi/2) = 1$.

Solution. Since $d \cos x = -\sin x\, dx$ we see that $df(x) = -10^{\cos x}\, d \cos x$ which (except for the minus sign) is (8) with $a = 10$ and $u = \cos x$, so that

$$f(x) = -\frac{10^{\cos x}}{\ln 10} + c = -(\log e)10^{\cos x} + c.$$

Hence, $1 = f(\pi/2) = -(\log e)10^0 + c = -\log e + c$ so that

$$f(x) = -(\log e)10^{\cos x} + 1 + \log e.$$

For computational purposes $f(x) = -(0.4343)10^{\cos x} + 1.4343$.

PROBLEMS

1. Find $D_x y$ and $D_x^2 y$ for each of the following:

a. $y = e^{-x}$.
b. $y = e^{-ax}$.
c. $y = e^{\tan x}$.
d. $y = e^{\tan^{-1} x}$.
e. $y = e^{1/x}$.
f. $y = xe^x$.
g. $y = x^2 e^x$.
h. $y = x^3 e^{x^3}$.

2. Find each of the following differentials:

a. $d(e^{-x}\sin 2x)$. c. $d(e^x - e^{-x})$. e. $d(10^{3x})$.

b. $de^{(2x^2-3x+1)}$. d. $d(x^2e^{-3/x})$. f. $d(x^2 2^x)$.

3. Establish each of the following:

a. If $y = e^x + e^{-x}$, then $D_x^2 y = y$.

b. If $y = \sin x + \cos x$, then $D_x^2 y = -y$.

c. If $y = ae^{cx} + be^{-cx}$, then $D_x^2 y = c^2 y$.

d. If $u = \frac{1}{2}(e^{cx} + e^{-cx})$ and $v = \frac{1}{2}(e^{cx} - e^{-cx})$, then $u^2 - v^2 = 1$, $D_x u = cv$, and $D_x v = cu$.

4. Find the relative maxima and minima of f if:

a. $f(x) = xe^{-x}$. c. $f(x) = xe^{-x^2}$.

b. $f(x) = x^2e^{-x}$. d. $f(x) = \sin x(e^{-\sin x})$.

5. For the graph of $y = e^{-x^2}$:

a. Find the maximum point.

b. Find the points of inflection and discuss the concavity properties of the graph.

c. Find the asymptotes of the graph. Sketch the graph.

d. Show that the rectangle of maximum area which has its base on the x-axis and two vertices on the curve has these vertices at the points of inflection.

6. For the graph of each of the following equations, find the equation of the tangent which passes through the origin.

a. $y = e^x$. b. $y = e^{-x}$. c. $y = e^{2x}$. d. $y = e^{-3x}$.

7. After examining concavity, maxima, minima, and asymptotes sketch the graph of

a. $y = e^{-x}\sin x$, $x \geq 0$. b. $y = e^{1/x}$.

8. Find $f(x)$ to within an additive arbitrary constant, given that:

a. $f'(x) = e^{2x}$. b. $f'(x) = xe^{x^2}$. c. $f'(x) = 10^x$. d. $f'(x) = (\sin x)e^{\cos x}$.

9. Establish each of the following:

a. $D_x(x - 1)e^x = xe^x$.

b. $\dfrac{d}{dx}(nx - \ln|a + be^{nx}|) = \dfrac{an}{a + be^{nx}}$.

c. $D_x e^{ax}(a \sin bx - b\cos bx) = (a^2 + b^2)e^{ax}\sin bx$.

d. $\dfrac{d}{dx}\tan^{-1}\left(\sqrt{\dfrac{a}{b}}\,e^{nx}\right) = \dfrac{n\sqrt{ab}}{ae^{nx} + be^{-nx}}$; $a > 0$, $b > 0$.

54. Variable Bases and Powers

For $a > 0$ a constant and u a function, all of the functions of Sec. 53 are defined by equations of the form

$$y = a^{u(x)}$$

and are described as "functions with constant base and variable power." Functions of the form

$$y = [u(x)]^a$$

have "variable base and constant power" whereas for v also a function

$$y = v(x)^{u(x)}$$

is a "function with variable base and variable power." Both the second and third types may be transformed to the first type as we now explain. Let s be a positive number and let t be any number. Then, since ln means $\log_e$, we have $s = e^{\ln s}$ and thus

$$s^t = e^{t \ln s}. \tag{1}$$

Hence, with a suitable substitution (and under proper conditions)

$$[u(x)]^a = e^{a \ln u(x)} \quad \text{and} \quad v(x)^{u(x)} = e^{u(x) \ln v(x)}$$

each with right hand side having constant base e and variable power.

Example 1. Considering $x > 0$, find $D_x x^{\sin x}$.

Solution. Thinking of (1) with $s = x$ and $t = \sin x$,

$$\begin{aligned} D_x x^{\sin x} &= D_x e^{(\sin x) \ln x} = e^{(\sin x) \ln x} D_x[(\sin x) \ln x] \\ &= x^{\sin x}[\sin x \, D_x \ln x + \ln x \, D_x \sin x] \\ &= x^{\sin x}\left[\frac{\sin x}{x} + \ln x \cos x\right]. \end{aligned}$$

The formula $D_x x^n = nx^{n-1}$ was derived for n a rational number. Now with $x > 0$ and p any number whatever we obtain the same formula by considering (1) with $s = x$ and $t = p$ so that

$$D_x x^p = D_x e^{p \ln x} = e^{p \ln x} D_x(p \ln x) = x^p \frac{p}{x} = px^{p-1}.$$

The relation (1) may also be used to avoid consideration of two cases when a function is defined in terms of absolute values.

Example 2. Find $D_x |\sin x|$, $x \neq m\pi$.

Solution. From (1) with $t = 1$ and $s = |\sin x|$ we have

$$\begin{aligned} D_x |\sin x| &= D_x e^{\ln |\sin x|} = e^{\ln |\sin x|} D_x \ln |\sin x| = |\sin x| \frac{D_x \sin x}{\sin x} \\ &= \frac{|\sin x|}{\sin x} \cos x = \begin{cases} \cos x & \text{if} \quad \sin x > 0 \\ -\cos x & \text{if} \quad \sin x < 0. \end{cases} \end{aligned}$$

A use of ln for finding derivatives of complicated expressions involving products and quotients is illustrated in the following example and is sometimes referred to as "logarithmic differentiation."

Example 3. Find $D_x y$ if $y = (x \sin x)/\sqrt{x^2 + 1}$.

Solution. From previous work it is known that $D_x y$ exists and hence that $D_x \ln |y|$ exists at least for $y \neq 0$. Since

$$|y| = \frac{|x|\,|\sin x|}{\sqrt{x^2 + 1}}, \quad \text{then} \quad \ln |y| = \ln |x| + \ln |\sin x| - \tfrac{1}{2} \ln (x^2 + 1),$$

$$D_x \ln |y| = D_x[\ln |x| + \ln |\sin x| - \tfrac{1}{2} \ln (x^2 + 1)]$$

$$\frac{1}{y} D_x y = \frac{1}{x} + \frac{1}{\sin x} D_x \sin x - \frac{1}{2}\frac{1}{x^2 + 1} D_x(x^2 + 1)$$

$$D_x y = y\left(\frac{1}{x} + \frac{\cos x}{\sin x} - \frac{1}{2}\frac{2x}{x^2 + 1}\right) = \frac{x \sin x}{\sqrt{x^2 + 1}}\left(\frac{1}{x} + \cot x - \frac{x}{x^2 + 1}\right).$$

PROBLEMS

1. Find $D_x y$ and $D_x^2 y$ for each of the following.

   a. $y = x^x$, $x > 0$. d. $y = x^{\ln x}$, $x > 0$. g. $y = |\sin x|^x$, $x \neq m\pi$.

   b. $y = |x|^x$, $x \neq 0$. e. $y = (\ln x)^x$, $x > 1$. h. $y = (\sqrt{x})^x$, $x > 0$.

   c. $y = (10x)^{3x}$, $x > 0$. f. $y = x^{\cos x}$, $x > 0$. i. $y = (x^2)^x$, $x \neq 0$.

2. Use logarithmic differentiation to find $D_x y$ for each of the following:

   a. $y = \dfrac{\sin x}{x^2 + 1}$. c. $y = \dfrac{\sin x \sqrt{1 + \cos^2 x}}{\tan^3 x}$.

   b. $y = \dfrac{(2x^2 + 3)\sqrt{x^2 + 1}}{\sqrt[3]{x^3 + 1}}$. d. $y = \dfrac{x^5(1 + x)^4}{(x - 1)^6}$.

3. Each of the following expressions for y involves two constants c_1 and c_2. Find $y' = D_x y$ and $y'' = D_x^2 y$ and then eliminate the constants c_1 and c_2 by using the three equations for y, y', and y'' to obtain the given second equation.

   a. $y = c_1 e^x + c_2 e^{-2x}$; $y'' + y' - 2y = 0$.

   b. $y = c_1 e^{ax} + c_2 e^{3ax}$; $y'' - 4ay' + 3a^2 y = 0$.

   c. $y = (c_1 + c_2 x)e^{-2x}$; $y'' + 4y' + 4y = 0$.

   d. $y = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x$; $y'' + 2y' + 5y = 0$.

   e. $y = c_1 + c_2 e^x + \tfrac{1}{2}(\cos x - \sin x)$; $y'' - y' = \sin x$.

   f. $y = c_1 e^{-x} + c_2 e^{2x} - 2x^2 + 2x - 3$; $y'' - y' - 2y = 4x^2$.

   g. $y = c_1 \cos x + c_2 \sin x + 6 - 2 \cos 2x$; $y'' + y = 12 \cos^2 x$.

   h. $y = c_1 e^x + c_2 e^{2x} + e^{2x} \ln \sec e^{-x}$; $y'' - 3y' + 2y = \sec^2 e^{-x}$.

55. Hyperbolic Functions

In some technical investigations, especially in connection with suspension cables and electrical transmission, the two expressions

$$\tfrac{1}{2}(e^x - e^{-x}) \quad \text{and} \quad \tfrac{1}{2}(e^x + e^{-x})$$

appear with sufficient frequency to justify special designations for them. There are relations between these expressions and a hyperbola similar to relations between trigonometric functions and a circle. The fact that there are such relations is unimportant for anything that follows and we mention it only to forestall the natural question of why these expressions are called the **hyperbolic sine** and **hyperbolic cosine** at x, respectively, and are represented by

(1) $$\mathbf{sinh}\, x = \tfrac{1}{2}(e^x - e^{-x}) \quad \text{and} \quad \mathbf{cosh}\, x = \tfrac{1}{2}(e^x + e^{-x}).$$

Further similarities with trigonometry are instigated by defining

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \quad x \neq 0.$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}, \quad x \neq 0.$$

Notice, furthermore, that the formula $D_x \sin x = \cos x$ is duplicated for hyperbolic functions:

(2) $$D_x\, \mathbf{sinh}\, x = \mathbf{cosh}\, x$$

since $D_x\tfrac{1}{2}(e^x - e^{-x}) = \tfrac{1}{2}(e^x + e^{-x})$. Also, $D_x\tfrac{1}{2}(e^x + e^{-x}) = \tfrac{1}{2}(e^x - e^{-x})$ so that

(3) $$D_x\, \mathbf{cosh}\, x = \mathbf{sinh}\, x.$$

which does not duplicate $D_x \cos x = -\sin x$. There are subtle similarities and differences between other pairs of trigonometric and hyperbolic functions. Examples are, $\sin^2 x + \cos^2 x = 1$, but

(4) $$\begin{aligned} \mathbf{sinh}^2\, x + \mathbf{cosh}^2\, x &= \tfrac{1}{4}(e^x - e^{-x})^2 + \tfrac{1}{4}(e^x + e^{-x})^2 \\ &= \tfrac{1}{4}(e^{2x} - 2 + e^{-2x} + e^{2x} + 2 + e^{-2x}) \\ &= \tfrac{1}{2}(e^{2x} + e^{-2x}) = \mathbf{cosh}\, \mathbf{2}x \end{aligned}$$

whereas $\cos^2 x - \sin^2 x = \cos 2x$, but

$$(5)\qquad \mathbf{\cosh^2 x - \sinh^2 x} = \tfrac{1}{4}(e^x + e^{-x})^2 - \tfrac{1}{4}(e^x - e^{-x})^2$$
$$= \tfrac{1}{4}[e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})] = \mathbf{1}.$$

A direct check shows that $\sinh x \cosh y + \cosh x \sinh y$

$$= \tfrac{1}{2}(e^x - e^{-x})\tfrac{1}{2}(e^y + e^{-y}) + \tfrac{1}{2}(e^x + e^{-x})\tfrac{1}{2}(e^y - e^{-y})$$
$$= \tfrac{1}{4}\{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-(x+y)} + e^{x+y} - e^{x-y} + e^{-x+y} - e^{-(x+y)}\}$$
$$= \tfrac{1}{2}\{e^{x+y} - e^{-(x+y)}\} = \sinh(x + y);\ \text{that is,}$$

$$(6)\qquad \mathbf{\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y.}$$

and in a similar way (although $\cos(x + y) = \cos x \cos y - \sin x \sin y$)

$$(7)\qquad \mathbf{\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y.}$$

Example. Discuss the graph of $y = \cosh x$.

Solution. Since $D_x \cosh x = \sinh x = \frac{1}{2}(e^x - e^{-x}) = 0$ if and only if $x = 0$, the graph has a horizontal tangent at the point (0,1) and at no other point. Since

$$D_x^2 \cosh x = D_x \sinh x = \cosh x = \tfrac{1}{2}(e^x + e^{-x}) > 0$$

the whole graph is concave upward and thus $\cosh x > \cosh 0 = 1$ for $x \neq 0$. The graph may be obtained by geometric addition of the graphs of $y = e^x$ and $y = e^{-x}$ and then taking half of each ordinate.

At each point of the graph of the function sinh, the tangent has positive slope (since $D_x \sinh x = \cosh x \geq 1$). Hence, sinh is an increasing function, and therefore has an inverse function. This inverse function is designated by $\sinh^{-1}$ so that for each number x

$$\sinh(\sinh^{-1} x) = x.$$

This inverse hyperbolic sine function is, however, not a new function, as we now show. Let x be any number and set

$$y = \sinh^{-1} x.$$

so that $\sinh y = \sinh(\sinh^{-1} x) = x$. Consequently,

$$x = \tfrac{1}{2}(e^y - e^{-y}) = \frac{e^{2y} - 1}{2e^y} \quad \text{and} \quad (e^y)^2 - 2xe^y - 1 = 0.$$

This last equation is quadratic in e^y and thus the formal solution is $e^y = x \pm \sqrt{x^2 + 1}$. But, $\sqrt{x^2 + 1} > x$ (whether x is positive, negative, or zero) and hence $x - \sqrt{x^2 + 1} < 0$ and must be discarded, since $e^y > 0$. Hence, the only solution for e^y is

$$e^y = x + \sqrt{x^2 + 1} > 0.$$

From this equation we have $y = \ln(x + \sqrt{x^2 + 1})$; that is,

$$\mathbf{\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}).}$$

PROBLEMS

1. Either by returning to the definitions or by using previously derived formulas, show that

   a. $\sinh(-x) = -\sinh x$.
   b. $\cosh(-x) = \cosh x$.
   c. $\tanh^2 x + \operatorname{sech}^2 x = 1$.
   d. $\coth^2 x - \operatorname{csch}^2 x = 1$.
   e. $\sinh 2x = 2 \sinh x \cosh x$.
   f. $\cosh 2x = 2 \cosh^2 x - 1$.
   g. $\cosh 2x = 2 \sinh^2 x + 1$.
   h. $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$.
   i. $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$.
   j. $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$.
   k. $\sinh u + \sinh v = 2 \sinh \dfrac{u + v}{2} \cosh \dfrac{u - v}{2}$.
   l. $\cosh u + \cosh v = 2 \cosh \dfrac{u + v}{2} \cosh \dfrac{u - v}{2}$.
   m. $(\cosh x + \sinh x)^2 = \cosh 2x + \sinh 2x$.
   n. $(\cosh x - \sinh x)^2 = \cosh 2x - \sinh 2x$.
   o. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$.

2. Establish each of the following formulas:

   a. $D_x \tanh x = \operatorname{sech}^2 x$.
   b. $D_x \coth x = -\operatorname{csch}^2 x$.
   c. $D_x \operatorname{sech} x = -\operatorname{sech} x \tanh x$.
   d. $D_x \operatorname{csch} x = -\operatorname{csch} x \coth x$.
   e. $D_x \ln(\cosh x) = \tanh x$.
   f. $D_x \ln|\sinh x| = \coth x$.
   g. $D_x \sinh^{-1} x = \dfrac{1}{\sqrt{1 + x^2}}$.

3. a. Discuss the graph of $\{(x,y) \mid y = \tanh x\}$, show that the function tanh has an inverse, show that this inverse has domain $\{x \mid -1 < x < 1\}$, and that

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}, \quad x^2 < 1.$$

   b. Obtain $D_x \tanh^{-1} x = \dfrac{1}{1 - x^2}$.

c. Establish the following pair of formulas:

$$\frac{d}{dx}\frac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\frac{ax+b}{-b}} = \frac{1}{x\sqrt{ax+b}} \quad \text{if} \quad b < 0, \quad \text{but}$$

$$\frac{d}{dx}\frac{-2}{\sqrt{b}}\tanh^{-1}\sqrt{\frac{ax+b}{b}} = \frac{1}{x\sqrt{ax+b}} \quad \text{if} \quad b > 0.$$

4. Establish each of the following.

a. $\dfrac{1}{\sqrt{x^2+a^2}} = \dfrac{d}{dx}\sinh^{-1}\dfrac{x}{a}, \quad a > 0.$

b. $\dfrac{1}{x\sqrt{x^2+a^2}} = \dfrac{d}{dx}\left\{-\dfrac{1}{a}\sinh^{-1}\dfrac{a}{x}\right\}, \quad x > 0.$

c. $\dfrac{1}{x^2\sqrt{x^2+a^2}} = \dfrac{d}{dx}\left\{-\dfrac{\sqrt{x^2+a^2}}{a^2x}\right\}.$

d. $\dfrac{x}{\sqrt{x^2+a^2}} = \dfrac{d}{dx}\sqrt{x^2+a^2}.$

e. $\dfrac{x^2}{\sqrt{x^2+a^2}} = \dfrac{d}{dx}\left\{\dfrac{x}{2}\sqrt{x^2+a^2} - \dfrac{a^2}{2}\sinh^{-1}\dfrac{x}{a}\right\}, \quad a > 0.$

5. Given the first equation, obtain the second:

a. $y = c_1\sinh(kx) + c_2\cosh(kx); \quad y'' - k^2y = 0.$

b. $y = c_1\sin(kx) + c_2\cos(kx); \quad y'' + k^2y = 0.$

c. $y = c_1 + c_2e^{-2x} + c_3\cosh^2 x; \quad y''' - 4y' = 0.$

6. Show that

a. $D_x(e^x\cosh x) = D_x(\frac{1}{2}e^{2x}).$ b. $D_x(e^{2x}\sinh x\cosh x) = D_x(\frac{1}{4}e^{4x}).$

7. Let $t > 0$ be given. Show that $\{(x,y) \mid 0 \le x \le t \text{ and } 0 \le y \le \sinh x\}$ may also be expressed as $\{(x,y) \mid 0 \le y \le \sinh t \text{ and } \ln(y + \sqrt{1+y^2}) \le x \le t\}$.

CHAPTER 6

Definite Integrals

As mentioned in the introduction to Chapter 2, calculus is traditionally divided into derivative (or differential) calculus and integral calculus. Do not, however, make the mistake of thinking, here at the beginning of a new subject, that a package labeled "Derivatives" can now be tied up and put on the shelf to be forgotten. In the fourth section of this chapter, an intimate relation between these two divisions is made (The Fundamental Theorem of Calculus), and thereafter derivatives and integrals will proceed hand in hand.

56. Sigma Notation

With $m \leq n$ integers and f a function whose domain contains all integers from m to n inclusive, the notation

$$\sum_{k=m}^{n} f(k) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

is used and is read "The sum of $f(k)$ from $k = m$ to $k = n$" or more briefly "Sigma $f(k)$ from m to n." For example, with $m = 1$, $n = 5$ and $f(x) = 2x^2 + 3$, then

$$\begin{aligned}\sum_{k=1}^{5} f(k) &= (2\cdot 1^2 + 3) + (2\cdot 2^2 + 3) + (2\cdot 3^2 + 3) + (2\cdot 4^2 + 3) + (2\cdot 5^2 + 3)\\ &= 2(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + 3 + 3 + 3 + 3 + 3\\ &= 2\sum_{k=1}^{5} k^2 + \sum_{k=1}^{5} 3.\end{aligned}$$

As indicated, sigma notation is used even for constant functions:

$$\sum_{k=m}^{n} c = c(n - m + 1).$$

for c any number. Also, for a and b numbers and g a function

$$\sum_{k=m}^{n} \{af(k) + bg(k)\} = a\sum_{k=m}^{n} f(k) + b\sum_{k=m}^{n} g(k).$$

Notice, also, that if p is an integer such that $m \le p < n$, then

$$\sum_{k=m}^{n} f(k) = \sum_{k=m}^{p} f(k) + \sum_{k=p+1}^{n} f(k).$$

"The sum of $f(k)$ from m to m" is interpreted to be $f(m)$:

$$\sum_{k=m}^{m} f(k) = f(m).$$

The letter k as used above is called a "dummy index," since the actual symbol does not appear in the final result; e.g.,

$$\sum_{k=1}^{4} k^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30 \quad \text{and} \quad \sum_{l=1}^{4} l^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30.$$

The same symbol may be used for dummy index with different ranges:

$$\sum_{k=1}^{n} (k-1)^2 = \sum_{k=0}^{n-1} k^2$$

since both sides represent $0^2 + 1^2 + 2^2 + \cdots + (n-1)^2$. Formally, we may think of substituting $k - 1 = l$ on the left (so $l = 0$ when $k = 1$ and $l = n - 1$ when $k = n$) and then replacing the dummy index l by k to obtain the right hand side. As an example, start with

$$2k - 1 = k^2 - (k-1)^2,$$

sum both sides from $k = 1$ to $k = n$, and obtain

$$\sum_{k=1}^{n} (2k-1) = \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} (k-1)^2,$$

$$2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = \sum_{k=1}^{n} k^2 - \sum_{k=0}^{n-1} k^2, \qquad \text{(see above)}$$

$$2\sum_{k=1}^{n} k - n = \left(\sum_{k=1}^{n-1} k^2 + n^2\right) - \left(0^2 + \sum_{k=1}^{n-1} k^2\right) = n^2,$$

(1) $$\sum_{k=1}^{n} k = \frac{n^2 + n}{2} = \tfrac{1}{2}n(n+1)$$

which is a formula for finding the sum of the first n positive integers. For example,

$$\sum_{k=1}^{100} k = \tfrac{1}{2}(100)(101) = 5050.$$

Now that formula (1) is established, it may be used together with

$$3k^2 - 3k + 1 = k^3 - (k-1)^3$$

to sum the squares of the first n positive integers. For now

$$3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} (k-1)^3,$$

$$3\sum_{k=1}^{n} k^2 - 3\,\frac{n^2+n}{2} + n = \left[\sum_{k=1}^{n-1} k^3 + n^3\right] - \sum_{k=0}^{n-1} k^3 = n^3,$$

$$\sum_{k=1}^{n} k^2 = \tfrac{1}{3}\{n^3 + \tfrac{3}{2}(n^2+n) - n\} = \tfrac{1}{6}(2n^3 + 3n^2 + n). \tag{2}$$

Example 1. Find $\sum_{k=1}^{100} (3k^2 - 2k + 4)$.

Solution. Considering how this sum would look written out, how terms could be grouped and common factors removed, and then how (1) and (2) may be used, we have

$$\begin{aligned}\sum_{k=1}^{100}(3k^2 - 2k + 4) &= 3\sum_{k=1}^{100} k^2 - 2\sum_{k=1}^{100} k + \sum_{k=1}^{100} 4\\ &= \tfrac{3}{6}(2\cdot 100^3 + 3\cdot 100^2 + 100) - 2\left(\frac{100^2 + 100}{2}\right) + 4\cdot 100\\ &= \tfrac{1}{2}(2{,}030{,}100) - 10{,}100 + 400 = 1{,}005{,}350.\end{aligned}$$

Example 2. Given $f(x) = 3x^2 + 2$, find a formula for $\sum_{k=1}^{n} f\left(1 + \frac{k}{n}\right)\frac{1}{n}$.

Solution. Since $f(1 + k/n) = 3(1 + k/n)^2 + 2 = 5 + 6k/n + 3k^2/n^2$ we have

$$\begin{aligned}\sum_{k=1}^{n} f\left(1 + \frac{k}{n}\right)\frac{1}{n} &= \sum_{k=1}^{n}\left(5 + \frac{6}{n}k + \frac{3}{n^2}k^2\right)\frac{1}{n}\\ &= \frac{1}{n}\left\{\sum_{k=1}^{n} 5 + \frac{6}{n}\sum_{k=1}^{n} k + \frac{3}{n^2}\sum_{k=1}^{n} k^2\right\}\\ &= \frac{1}{n}\left\{5n + \frac{6}{n}\cdot\frac{1}{2}(n^2+n) + \frac{3}{n^2}\cdot\frac{1}{6}(2n^3 + 3n^2 + n)\right\}\\ &= 5 + 3\left(1 + \frac{1}{n}\right) + \frac{1}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\\ &= 9 + \frac{9}{2}\frac{1}{n} + \frac{1}{2}\frac{1}{n^2}.\end{aligned}$$

Notice that we may choose n so large that $9 + \dfrac{9}{2n} + \dfrac{1}{2n^2}$ differs from 9 by as little as we please. According to the following definition, we write

$$\lim_{n\to\infty}\left(9 + \frac{9}{2n} + \frac{1}{2n^2}\right) = 9.$$

DEFINITION 56. *Let F be a function whose domain contains all positive integers, and let L be a number. If corresponding to each positive number ϵ there is an integer N such that, for n an integer,*

$$\text{whenever } n > N \text{ it follows that } |F(n) - L| < \epsilon,$$

then L is said to be the limit of $F(n)$ as n becomes infinite over integer values and we write

$$\lim_{n\to\infty} F(n) = L.$$

Theorems similar to those of Sec. 17 may be proved for limits as n becomes infinite over integer values. Thus, if f and g are functions and L_1 and L_2 are numbers such that

$$\lim_{n\to\infty} f(n) = L_1 \quad \text{and} \quad \lim_{n\to\infty} g(n) = L_2,$$

then

$$\lim_{n\to\infty}[f(n) + g(n)] = L_1 + L_2, \quad \lim_{n\to\infty} f(n)g(n) = L_1L_2, \quad \text{and}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{L_1}{L_2} \quad \text{provided} \quad L_2 \neq 0.$$

Example 3. Find $S = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \left\{\left(2 + k \cdot \frac{3}{n}\right)^2 - 5\right\} \frac{3}{n}$.

Solution. By only indicating the limit until we are sure of its value we write

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \left\{4 + \frac{12}{n}k + \frac{9}{n^2}k^2 - 5\right\} \frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \left\{\sum_{k=1}^{n} \left(-1 + \frac{12}{n}k + \frac{9}{n^2}k^2\right)\right\}$$

$$= 3 \lim_{n\to\infty} \frac{1}{n} \left\{\sum_{k=1}^{n}(-1) + \frac{12}{n}\sum_{k=1}^{n} k + \frac{9}{n^2}\sum_{k=1}^{n} k^2\right\}$$

$$= 3 \lim_{n\to\infty} \frac{1}{n} \left\{-n + \frac{12}{n}\frac{1}{2}(n^2 + n) + \frac{9}{n^2}\frac{1}{6}(2n^3 + 3n^2 + n)\right\}$$

$$= 3 \lim_{n\to\infty} \left\{-1 + 6\left(1 + \frac{1}{n}\right) + \frac{3}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right\}$$

$$= 3\left\{-1 + 6 + \frac{3}{2} \cdot 2\right\} = 24.$$

PROBLEMS

1. By using formulas (1) and/or (2) show that

a. $\sum_{k=1}^{10} k = 55.$

b. $\sum_{k=1}^{10} \frac{k}{10^2} = 0.55.$

c. $\sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$

d. $\sum_{k=1}^{n} \left(1 + k\frac{3}{n}\right)^2 \frac{3}{n} = 21 + \frac{45}{2n} + \frac{9}{2n^2}.$

2. a. Express $k^4 - (k-1)^4$ as a third degree polynomial in k, and then use (1) and (2) to show that

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}(n^4 + 2n^3 + n^2). \tag{3}$$

b. Derive the formula

$$\sum_{k=1}^{n} k^4 = \frac{1}{30}(6n^5 + 15n^4 + 10n^3 - n).$$

3. By using previously derived formulas show that:

a. $\sum_{k=1}^{n} (2k-1) = n^2.$

b. $\sum_{k=1}^{n} (3k^2 - 3k + 1) = n^3.$

c. $\sum_{k=1}^{n} (4k^3 - 2k + 1) = n^3(n+2).$

d. $\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n^3 + n^2 + \frac{2}{3}n.$

e. $\sum_{k=1}^{n} k(k-1) = \frac{1}{3}(n^3 - n).$

4. a. Check that $\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}$. Use this relation to prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}; \quad \text{that is,} \quad \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

b. Devise a proof of $\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{1}{4}\frac{3n^2 + 5n}{(n+1)(n+2)}.$

5. Establish each of the following:

a. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{2}.$

b. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{1}{3}.$

c. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^3}{n^4} = \frac{1}{4}.$

d. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(1 + \frac{k}{n}\right)\frac{1}{n} = \frac{3}{2}.$

e. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(a + \frac{k}{n}\right)^2 \frac{1}{n} = a^2 + a + \frac{1}{3}.$

f. $\lim_{n\to\infty} \sum_{k=1}^{n} \left[a + k\frac{b-a}{n}\right]^2 \frac{b-a}{n} = \frac{1}{3}(b^3 - a^3).$

57. Definite Integrals

With a and b numbers and n a positive integer let

(1) $$\Delta_n x = \frac{b-a}{n} \quad \text{and} \quad x_k = a + k\,\Delta_n x \quad \text{for} \quad k = 1, 2, 3, \cdots, n.$$

In Appendix A4 it is proved that if f is a function which is continuous on the closed interval with end points a and b, then both of the limits

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta_n x \quad \text{and} \quad \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\,\Delta_n x$$

exist and have the same value. The common value of these limits is called the **definite integral** of f from a to b and is represented by $\int_a^b f(x)\,dx$ so that

(2) $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_n x = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\,\Delta_n x.$$

In this setting $f(x)$ is referred to as the **integrand**. Also, a is called the **lower limit** of integration and b is called the **upper limit** of integration.

Example 1. Find $\int_1^3 x^2\,dx$.

Solution. Here $a = 1$ and $b = 3$ so that $b - a = 3 - 1 = 2$,

$$\Delta_n x = \frac{2}{n} \quad \text{and} \quad x_k = 1 + k\frac{2}{n} \quad \text{for} \quad k = 0, 1, 2, \cdots, n.$$

Since $f(x) = x^2$, then $f(x_k) = x_k^2 = \left(1 + k\frac{2}{n}\right)^2$. Hence, by using the first limit in (2),

$$\begin{aligned}
\int_1^3 x^2\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(1 + k\frac{2}{n}\right)^2 \cdot \frac{2}{n} = \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \left(1 + \frac{4}{n}k + \frac{4}{n^2}k^2\right) \\
&= 2 \lim_{n\to\infty} \frac{1}{n}\left\{\sum_{k=1}^{n} 1 + \frac{4}{n}\sum_{k=1}^{n} k + \frac{4}{n^2}\sum_{k=1}^{n} k^2\right\} \\
&= 2 \lim_{n\to\infty} \frac{1}{n}\left\{n + \frac{4}{n}\,\frac{n^2+n}{2} + \frac{4}{n^2}\,\frac{2n^3+3n^2+n}{6}\right\} \\
&= 2 \lim_{n\to\infty} \left\{1 + 2\left(1 + \frac{1}{n}\right) + \frac{2}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right\} \\
&= 2\{1 + 2 + \tfrac{2}{3}2\} = \tfrac{26}{3}.
\end{aligned}$$

An observation for later use is that the letter x does not appear on the right in the above computation. In

(3) $$\int_a^b f(x)\,dx$$

the letter x is called a **dummy variable** or **variable of integration.** As examples, the result of Example 1 above enables us to know that

$$\int_1^3 t^2\,dt = \frac{26}{3}, \quad \int_1^3 u^2\,du = \frac{26}{3}, \quad \int_1^3 v^2\,dv = \frac{26}{3}.$$

Also, the dx in (3) or dt, du, or dv in the line above merely indicates the variable of integration so that later on when

$$\int_1^3 (t + x^2)\,dx$$

is met it will be known that x (and not t) is the variable of integration.

The introduction of $\Delta_n x$ and x_k may be avoided by writing (2) as

(2′)
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}$$
$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}.$$

Also, the lower limit of integration a need not be less than the upper limit of integration b.

Example 2. $\displaystyle\int_{+1}^{-1}(x+4)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left\{\left[+1 + k\frac{-1-(+1)}{n}\right] + 4\right\}\frac{-1-(+1)}{n}$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\left\{\left[1 - \frac{2}{n}k\right] + 4\right\}\frac{(-2)}{n}$$
$$= (-2)\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{5 - \frac{2}{n}k\right\}$$
$$= (-2)\lim_{n\to\infty}\frac{1}{n}\left[5n - \frac{2}{n}\frac{n^2+n}{2}\right]$$
$$= (-2)\lim_{n\to\infty}\left[5 - \left(1 + \frac{1}{n}\right)\right] = (-2)[5-1] = -8.$$

It may now be seen that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

PROBLEMS

1. By making a separate calculation of each of the integrals involved, show that

a. $\displaystyle\int_{-1}^{1} x^2\,dx + \int_1^2 x^2\,dx = \int_{-1}^{2} x^2\,dx.$

b. $\displaystyle\int_2^4 (x-1)^2\,dx = \int_1^3 x^2\,dx = \int_0^2 (x+1)^2\,dx.$

c. $\int_2^4 (t-1)^2\,dt = \int_2^4 u^2\,du - 2\int_2^4 v\,dv + \int_2^4 1\cdot ds.$

d. $\int_1^4 3x^2\,dx + \int_1^4 x\,dx = \int_1^4 (3x^2 + x)\,dx.$

e. $\left(\int_1^4 3x^2\,dx\right)\left(\int_1^4 x\,dx\right) \neq \int_1^4 (3x^2)(x)\,dx.$

2. Compute the value of each of the integrals:

a $\int_{-1}^4 (x+1)^2\,dx.$

b. $\int_0^5 x^3\,dx.$

c. $\int_{-3/2}^{5/2} t(t-3)\,dt.$

d. $\int_{-3/2}^{3/2} x(x^2-3)\,dx.$

3. With $a \neq b$ prove that:

a. $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}.$

b. $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}.$

58. Area and Work

We now give two illustrations of the use of definite integrals in defining extensions of common notions.

DEFINITION 58.1. *If f is a continuous function which is never negative on the closed interval* I$[a,b]$, *then the region*

$$R = \{(x,y) \mid a \le x \le b \quad and \quad 0 \le y \le f(x)\} \tag{1}$$

is said to have

$$area = \int_a^b f(x)\,dx \ units^2 \tag{2}$$

The reasoning behind this definition of area is based upon consideration of graphs, such as Fig. 58.1, of a continuous curve rising to the right. With

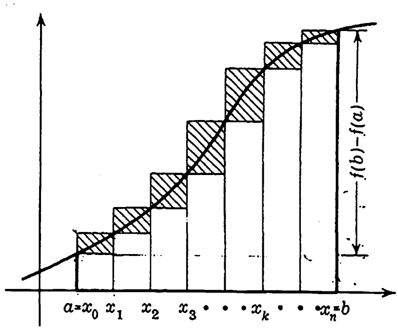


Figure 58.1

$$\Delta_n x = \frac{b-a}{n} \quad \text{and} \quad x_k = a + k\Delta_n x$$

$$\text{for} \quad k = 1, 2, 3, \cdots, n,$$

then $f(x_k)\,\Delta_n x$ units^2 is the area of a rectangle of altitude $f(x_k)$ units and base of $\Delta_n x$ units. If such a rectangle is drawn with right edge along the line perpendicular to the x-axis at the point $(x_k,0)$, then in Fig. 58.1 an upper corner of the rectangle is outside of R and

$$\sum_{k=1}^{n} f(x_k)\,\Delta_n x \tag{4}$$

appears too large for the number of square units in R. On the other hand, the same size rectangle on the other side of the perpendicular through $(x_k,0)$ lies in R so that

$$\sum_{k=0}^{n-1} f(x_k)\,\Delta_n x \tag{5}$$

appears too small for the number of square units in R. Since the sums in (4) and (5) both have the same limit as $n \to \infty$, it is natural to define (as in (2) above) this common limit to be the number of units^2 in R.

Notice also in Fig. 58.1 that all shaded portions may be fitted into the right most rectangle between the levels $y = f(a)$ and $y = f(b)$ so that

$$\sum_{k=1}^{n} f(x_k)\,\Delta_n x - \sum_{k=0}^{n-1} f(x_k)\,\Delta_n x = [f(b) - f(a)]\,\frac{b-a}{n}$$

and that the right side approaches 0 as $n \to \infty$.

Example 1. Find the area of the region

$$R = \{(x,y) \mid 1 \le x \le 3 \quad \text{and} \quad 0 \le y \le x^2\}.$$

Solution. According to Definition 58.1 this area is

$$\int_1^3 x^2\,dx \text{ units}^2.$$

Had we not already evaluated this integral (see Example 1 of Sec. 57) we would now do so, and find the area of R to be $\frac{26}{3}$ units^2.

Consider next the concept of work. If a 60 lb bucket is raised vertically 10 ft, then $60 \cdot 10 = 600$ ft $\cdot$ lb of work is done according to the definition:

If a constant force of f lb, acting in the direction of motion, moves an object h ft, then the work done is

$$W = f \cdot h \, ft \cdot lb.$$

Now let a force, which may not be constant, act in the direction of motion and move an object along a coordinate line (unit 1 ft) from a point with coordinate a to a point with coordinate b. Let the force be

$$f(x) \text{ lb}$$

when the object has coordinate x where $a \le x \le b$. With n a positive integer, let

$$\Delta_n x = \frac{b-a}{n} \quad \text{and} \quad x_k = a + k\,\Delta_n x \quad \text{for} \quad k = 1, 2, 3, \cdots, n.$$

If the force either steadily increases or steadily decreases as the object moves over the interval $I[x_{k-1},x_k]$, then the "work" in so moving the object is expected to be between

$$f(x_{k-1})\,\Delta_n x \text{ ft}\cdot\text{lb} \quad \text{and} \quad f(x_k)\,\Delta_n x \text{ ft}\cdot\text{lb}$$

and the "work" over the whole interval $I[a,b]$ should be between

$$\sum_{k=1}^{n} f(x_{k-1})\,\Delta_n x \text{ ft}\cdot\text{lb} \quad \text{and} \quad \sum_{k=1}^{n} f(x_k)\,\Delta_n x \text{ ft}\cdot\text{lb}.$$

The first sum may be written (by a change of dummy index) as

$$\sum_{k=0}^{n-1} f(x_k)\,\Delta_n x.$$

Again these sums have the same limit (as proved in Appendix A4), so it is natural to use the definite integral notation to give the following definition.

DEFINITION 58.2. *In moving an object along a coordinate line from a point with coordinate a to a point with coordinate b by a force f acting in direction of motion where*

$$f(x) \text{ lb}$$

is the force when the object has coordinate x, the work W done is

$$W = \int_a^b f(x)\,dx \text{ ft}\cdot\text{lb} \tag{3}$$

Example 2. A spring has natural length 12 in., and a force of $10x$ lb is required to hold this spring stretched x in. beyond its natural length. Find the work done in stretching the spring 6 in. beyond its natural length.

Solution. Establish a linear coordinate system (unit = 1 in.) with origin at the force end and -12 at the fixed end. Then the force function f is such that $f(x) = 10x$ lb for $0 \le x \le 6$. Hence, the required work is

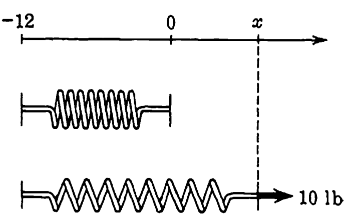


Figure 58.2

$$\begin{aligned} W &= \int_0^6 10x\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} 10\left(0 + k\frac{6}{n}\right)\frac{6}{n} \\ &= 10\cdot 6\cdot 6 \lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} k \\ &= 360 \lim_{n\to\infty} \frac{1}{n^2}\,\frac{n^2+n}{2} = \frac{360}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) \\ &= 180 \text{ in.}\cdot\text{lb}. \end{aligned}$$

A definite integral involves only a function and two constants and is an abstract notion free of geometric or physical terms. Nevertheless, as shown above, area and work are two illustrations of how definite integrals play a central role in different appearing subjects; later, other illustrations will be given.

Problems on area and work will not be given until after experience is gained in evaluating integrals by the method of the next section.

59. The Fundamental Theorem of Calculus

The concepts of the derivative of a function and the definite integral of a function are predominant notions of calculus. Differentiation grew out of early work on velocities, accelerations, and tangents to curves, whereas a direct path from Archimedes' (287?–212 B.C.) discussions of the circumference of a circle leads through considerations of area and work to definite integration. These notions were not suspected of being related until Leibnitz (1646–1716) in Germany and Newton (1642–1727) in England recognized and exploited the relation between derivatives and definite integrals as expressed in what is now called the fundamental theorem of calculus.

FUNDAMENTAL THEOREM OF CALCULUS. *If f is continuous on* $I[a,b]$, *then there is a function G such that $G'(x) = f(x)$ for $a \le x \le b$. Moreover, if F is any anti-derivative of f on* $I[a,b]$; *i.e.*

$$F'(x) = f(x) \quad \textit{for } a \le x \le b,$$

then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{1}$$

Leaving the proof of this theorem to the next section, we now illustrate the use of the theorem in evaluating definite integrals.

Example 2. Find $\int_0^{\pi/2} 3 \sin 2x\,dx$.

Solution. In this case $f(x) = 3 \sin 2x$, and thus to apply the fundamental theorem of calculus, we need a solution of the derivative equation $F'(x) = 3 \sin 2x$ and any solution will do. $F(x) = (-\frac{3}{2}) \cos 2x$ is such a solution (since $F'(x) = (-\frac{3}{2})(-\sin 2x)2 = 3 \sin 2x$), and therefore

$$\int_0^{\pi/2} 3 \sin 2x\,dx = F\left(\frac{\pi}{2}\right) - F(0) = -\frac{3}{2}\cos\left(2\frac{\pi}{2}\right) - \left(-\frac{3}{2}\cos 0\right)$$

$$= -\frac{3}{2}\cos\pi + \frac{3}{2}\cos 0 = -\frac{3}{2}(-1) + \frac{3}{2}(1) = 3.$$

This result may be interpreted either that the region

$$R = \left\{(x,y) \,\middle|\, 0 \le x \le \frac{\pi}{2} \quad \text{and} \quad 0 \le y \le 3 \sin 2x\right\}$$

has area 3 units^2, or else that the force function $f(x) = 3 \sin 2x$ lb does 3 ft · lb of work in moving an object along a coordinate line (unit = 1 ft) from the origin to the point $\pi/2$.

It is customary to use the notation $F(x)]_a^b = F(b) - F(a)$. In this notation the fundamental theorem of calculus is formalized as

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \quad \text{where} \quad F'(x) = f(x). \tag{2}$$

Example 3.

$$\int_3^1 \left(4x^3 - \sin\frac{\pi}{2}x\right) dx = \left[x^4 + \frac{2}{\pi}\cos\frac{\pi}{2}x\right]_3^1 = 1^4 + \frac{2}{\pi}\cos\frac{\pi}{2} - \left(3^4 + \frac{2}{\pi}\cos\frac{3\pi}{2}\right)$$

$$= 1 + \frac{2}{\pi}\cdot 0 - \left(81 + \frac{2}{\pi}\cdot 0\right) = -80.$$

Example 4.

$$\int_{-\pi/2}^{\pi/2} \sin^2 t\cos t\,dt = \frac{1}{3}\sin^3 t\Big]_{-\pi/2}^{\pi/2} = \frac{1}{3}\left[\sin^3\frac{\pi}{2} - \sin^3\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{3}[1 - (-1)] = \frac{2}{3}.$$

PROBLEMS

1. Find the value of each of the following definite integrals:

a. $\int_{-2}^{3}(3x^2 - 4x + 6)\,dx$.
e. $\int_0^{\pi}(u + \sin u)\,du$.
i. $\int_0^{\pi/4}(\cos x)^{-2}\sin x\,dx$.

b. $\int_0^{\pi/2} 5\sin 3x\,dx$.
f. $\int_0^4 \sqrt{x}(x+1)\,dx$.
j. $\int_0^{\pi/4}\tan^2 x\,dx$.

c. $\int_{-3}^{-4}(t+2)^{-1}\,dt$.
g. $\int_0^{\pi/2}\sqrt{\cos x}\sin x\,dx$.
k. $\int_{\pi/4}^{\pi/3}\tan x\,dx$.

d. $\int_0^{\pi/2}\cos^2 x\sin x\,dx$.
h. $\int_0^{\sqrt{3}}\sqrt{x^2+1}\,x\,dx$.
l. $\int_1^1 \log\tan x\,dx$.

2. Find an expression which does not involve derivatives or integrals for:

a. $\int_1^x t\,dt$.
d. $\int_{\pi/4}^{\pi/2}(D_x\sin x)\,dx$.
g. $\int_1^3\left\{\int_2^x 3t^2\,dt\right\}dx$.

b. $\int_1^4 t^2x\,dx$.
e. $D_x\left\{\int_1^x \sin t\,dt\right\}$.
h. $\int_1^{\pi/4}\left\{\int_2^x \sin t\,dt\right\}dx$.

c. $\int_1^4 t^2x\,dt$.
f. $D_t\left\{\int_1^t \sin x\,dx\right\}$.
i. $\left\{\int_2^3 (x+1)\,dx\right\}\left\{\int_2^3 x^2\,dx\right\}$.

3. Sketch the region and find its area:

a. $\{(x,y) \mid -2 \le x \le 2 \ \text{ and } \ 0 \le y \le x^2\}$.
b. $\{(x,y) \mid -2 \le x \le 2 \ \text{ and } \ 0 \le y \le 4 - x^2\}$.
c. $\{(x,y) \mid 0 \le x \le \pi \ \text{ and } \ 0 \le y \le x + \cos x\}$.
d. $\left\{(x,y) \mid 4 \le x \le 9 \text{ and } \ 0 \le y \le \dfrac{x + \sqrt{x}}{x}\right\}$.

4. Prove $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)=\ln 2$ by applying (2′) of Sec. 57 to $\int_1^2 \frac{dx}{x}$ and also evaluating this integral by the fundamental theorem of calculus.

5. Evaluate each of the following limits by first writing it as a definite integral and then using the fundamental theorem of calculus.

a. $\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{1+k\frac{1}{n}}\frac{1}{n}$.

b. $\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{2+k\frac{3}{n}}\frac{3}{n}$.

c. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+2k}$.

d. $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sin k\frac{\pi}{n}\right)\frac{\pi}{n}$.

e. $\lim_{n\to\infty}\sum_{k=1}^{n}\sin\left(\frac{\pi}{4}+k\frac{3}{4}\frac{\pi}{n}\right)\frac{3}{4}\frac{\pi}{n}$.

f. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{(n+k)^2}$.

60. Algebra of Integrals

Before proving the Fundamental Theorem of Calculus, we need some preliminary results, most of which are important for other reasons as well.

In Fig. 60 the region below the curve and above I[a,b] is divided into two sub-regions by a line segment. The relation between areas and definite integrals leads us to suspect that

1. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

holds, at least under some conditions. In Appendix A4 it is proved that **1.** holds if f is continuous on the closed interval from the smallest to the largest of a, b, and c. Thus a need not be the smallest, nor b the largest; nor need c lie between a and b.

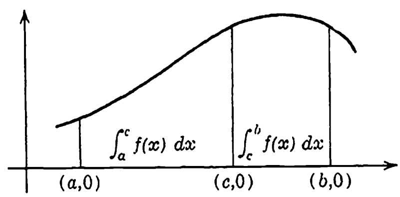


Figure 60

The way we shall use **1.** presently is first to switch to the dummy variable t. We then set $b = x + h$ and $c = x$ to have

$$\int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt.$$

The form of this equation which we shall use is

$$\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt. \tag{1}$$

The next result is a companion to the Law of the Mean for derivatives.

LAW OF THE MEAN FOR INTEGRALS. *If f is continuous on* I$[a,b]$, *then there is a number x^* such that $a < x^* < b$* and

$$(2) \qquad \int_a^b f(x)\,dx = f(x^*)(b-a).$$

PROOF. Since f is continuous on I$[a,b]$, it has a minimum and a maximum value, and we let $\underline{x}$ and $\overline{x}$ be such that $a \le \underline{x} \le b$, $a \le \overline{x} \le b$ and

$$f(\underline{x}) \le f(x) \le f(\overline{x}) \text{ for all } x \text{ of } \mathrm{I}[a,b].$$

Then for n a positive integer and $k = 1, 2, \cdots, n$ we have

$$(3) \qquad f(\underline{x}) \le f\left(a + k\,\frac{b-a}{n}\right) \le f(\overline{x}).$$

The right-hand inequality, and the definition of the definite integral, yields

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)\frac{b-a}{n} \le \lim_{n\to\infty} \sum_{k=1}^{n} f(\overline{x})\,\frac{b-a}{n}$$

$$= \lim_{n\to\infty} f(\overline{x})\,\frac{b-a}{n}\cdot n = \lim_{n\to\infty} f(\overline{x})(b-a) = f(\overline{x})(b-a).$$

This, and a similar use of the left-hand inequality of (3), shows first that $f(\underline{x})(b-a) \le \int_a^b f(x)\,dx \le f(\overline{x})(b-a)$ and therefore

$$(4) \qquad f(\underline{x}) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(\overline{x}).$$

We now call upon the intermediate value property of continuous functions (Corollary, page 86) to assert that there is a number x^* between $\underline{x}$ and $\overline{x}$ where f assumes the value of the middle term of (4); i.e.

$$f(x^*) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

The equivalence of this equation and (2) establishes the result.

These preliminary results make the following proof quite simple.

PROOF of the Fundamental Theorem of Calculus. Let G be the function defined on I$[a,b]$ by

$$(5) \qquad G(x) = \int_a^x f(t)\,dt.$$

Notice we have switched to t as the dummy variable of integration and that

$$(6) \qquad G(b) = \int_a^b f(t)\,dt \quad \text{whereas} \quad G(a) = \int_a^a f(t)\,dt = 0.$$

F(a) - F(a)

Now with $h \neq 0$, but such that $x + h$ is also on I[a,b], then

$$G(x + h) - G(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt \quad \text{by (1).}$$

We next apply the Law of the Mean for Integrals to the interval between x and $x + h$ and assert there is a number t_h^* between x and $x + h$ such that

$$\int_x^{x+h} f(t)\,dt = f(t_h^*)h.$$

Consequently

$$\frac{G(x + h) - G(h)}{h} = f(t_h^*).$$

Now let $h \to 0$. Then $t_h^* \to x$ and $f(t_h^*) \to f(x)$ from the continuity of f; i.e.

$$G'(x) = \lim_{h\to 0} \frac{G(x + h) - G(x)}{h} = \lim_{h\to 0} f(t_h^*) = f(x).$$

We have thus shown the existence of a function G such that $G'(x) = f(x)$ and, because of both parts of (6), $\int_a^b f(t)\,dt - \int_a^a f(t)\,dt$

$$\int_a^b f(t)\,dt = G(b) = G(b) - G(a).$$

The dummy variable t has now served its purpose and we return to using x. Thus if F is any anti-derivative of f, then both

$$F'(x) = f(x) \quad \text{and } G'(x) = f(x) \quad \text{for } x \text{ on I}[a,b].$$

But then $F(b) - F(a) = G(b) - G(a)$ by Theorem 39, page 119. Therefore

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

which concludes the proof of the Fundamental Theorem of Calculus.

If f and g are continuous functions on a closed interval I[a,b], then†

2. $$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

To see this let F, G, and H be functions such that $F'(x) = f(x)$, $G'(x) = g(x)$ and $H'(x) = f(x) + g(x)$. Hence, $H'(x) = F'(x) + G'(x)$ so

$$\begin{aligned} H(b) - H(a) &= [F(b) + G(b)] - [F(a) + G(a)] \\ &= [F(b) - F(a)] + [G(b) - G(a)] \end{aligned}$$

† This result and **3** are proved under more general conditions in Appendix A4.

which, by the Fundamental Theorem of Calculus, is 2 with the + sign. A similar argument may be made when + is replaced by −.

If f is continuous on the closed interval with end points a and b, and if c is a constant, then

3. $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

To see this let F be a function such that $F'(x) = f(x)$, and therefore such that $D_x cF(x) = cf(x)$. Then

$$c\int_a^b f(x)\,dx = c\Big[F(x)\Big]_a^b = \Big[cF(x)\Big]_a^b = \int_a^b cf(x)\,dx.$$

Example. Find $\int_0^3 x\sqrt{1+x}\,dx$.

Solution. It is not evident what function F is such that $F'(x) = x\sqrt{1+x}$. The following manipulation shows how this integral may be written as the sum of two integrals for each of which the related anti-derivative is easily found.

$$\begin{aligned}
\int_0^3 x\sqrt{1+x}\,dx &= \int_0^3(-1+1+x)\sqrt{1+x}\,dx && \text{(since } -1+1=0) \\
&= \int_0^3 -\sqrt{1+x}\,dx + \int_0^3 (1+x)\sqrt{1+x}\,dx && \text{(by 2)} \\
&= -\int_0^3 (1+x)^{1/2}\,dx + \int_0^3 (1+x)^{3/2}\,dx \\
&= -\Big[\tfrac{2}{3}(1+x)^{3/2}\Big]_0^3 + \Big[\tfrac{2}{5}(1+x)^{5/2}\Big]_0^3 = \tfrac{116}{15}
\end{aligned}$$

Hence if $F(x) = -\frac{2}{3}(1+x)^{3/2} + \frac{2}{5}(1+x)^{5/2}$, then $F'(x) = x\sqrt{1+x}$.

PROBLEMS

1. Find the value of each of the following integrals.

a. $\int_0^3 x\sqrt{4-x}\,dx$.
b. $\int_4^5 x\sqrt{x-4}\,dx$.
c. $\int_1^6 \frac{x}{\sqrt{3+x}}\,dx$.
d. $\int_0^2 x\sqrt{4-2x}\,dx$.
e. $\int_0^{7/3} x\sqrt[3]{1+3x}\,dx$.
f. $\int_0^3 \frac{x}{\sqrt{4-x}}\,dx$.
g. $\int_0^1 x\sqrt{1+3x^2}\,dx$.
h. $\int_2^3 x\sqrt{x^2-4}\,dx$.
i. $\int_0^2 x(4-2x)^{3/2}\,dx$.

2. Solve the given equation for c and give a geometric interpretation in terms of area.

a. $\int_0^c x^2\,dx = 9$.
b. $\int_0^c x^3\,dx = 4,\ c > 0$.
c. $\int_0^c x^2\,dx = \int_c^{10} x^2\,dx$.
d. $\int_0^c (x+1)^2\,dx = 21$.
e. $\int_0^1 (x+c)\,dx = 5$.
f. $\int_c^{c+1} x\,dx = 10$.

3. Solve each of the following equations for c.

a. $\int_1^3 (x - c)x^2\,dx = 0.$

b. $\int_0^4 (x - c)\sqrt{4 - x}\,dx = 0.$

c. $\int_0^{\pi/2} (c - \cos^2 x)\sin x\,dx = 0.$

d. $\int_1^6 (x - c)(3 + x)^{-1/2}\,dx = 0.$

4. Use the algebra of integrals to simplify before evaluating.

a. $\int_1^2 (x - \ln\sin x)\,dx + \int_1^2 \ln\sin x\,dx.$

b. $\int_1^2 x(1 + x\cos x)\,dx + \int_2^1 x^2(1 + \cos x)\,dx.$

c. $\int_1^2 x\sin x\,dx + \int_1^3 x(1 - \sin x)\,dx + \int_2^3 x(1 + \sin x)\,dx.$

d. $\int_1^2 \ln(x + \sqrt{1 + x^2})\,dx + \int_1^2 \ln(\sqrt{1 + x^2} - x)\,dx.$

5. Let a and b be numbers such that $a < b$ and let f be a function such that f is positive and f' exists on $\mathrm{I}[a,b]$.

a. Show that for $a \le x \le b$, then $\int_a^x f(t)\,dt \ge 0.$

b. Does it follow that $f'(x) \ge 0$ for $a \le x \le b$?

61. Area Between Curves

As a supplement to the algebra of integrals, the following inequality property is given:

If $a < b$, if f and g are continuous functions on $\mathrm{I}[a,b]$ *and if $g(x) \le f(x)$ for $a \le x \le b$, then*

$$\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx. \tag{1}$$

For, in the usual notation $g(x_k) \le f(x_k)$ and $\Delta_n x = (b - a)/n$ is positive, so

$$g(x_k)\,\Delta_n x \le f(x_k)\,\Delta_n x.$$

By summing both sides from $k = 1$ to n and then letting n increase without bound, we obtain (1).

Example 1. First recall that $D_x \tan^{-1} x = 1/(1 + x^2)$ so that

$$\int_0^1 \frac{1}{1 + x^2}\,dx = \tan^{-1} x\Big]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}.$$

The ordinary long division of 1 by $1 + x^2$ to four and five steps gives

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \frac{x^8}{1 + x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \frac{x^{10}}{1 + x^2}.$$

Since $x^8/(1 + x^2) \geq 0$ and $-x^{10}/(1 + x^2) \leq 0$, we have

$$1 - x^2 + x^4 - x^6 \leq \frac{1}{1 + x^2} \leq 1 - x^2 + x^4 - x^6 + x^8, \qquad \text{then by (1),}$$

$$\int_0^1 (1 - x^2 + x^4 - x^6)\, dx \leq \int_0^1 \frac{1}{1 + x^2}\, dx \leq \int_0^1 (1 - x^2 + x^4 - x^6 + x^8)\, dx,$$

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\Big]_0^1 \leq \tan^{-1} x\Big]_0^1 \leq x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}\Big]_0^1$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \leq \frac{\pi}{4} \leq 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}.$$

Therefore $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ approximates $\pi/4$ to within $\frac{1}{9}$, but the method illustrates how $\pi/4$ (and thus π) may be approximated to any desired degree of accuracy.

Note: It was proved by the early Greeks that, using any unit of length, the ratio of the circumference to the diameter of any circle is the same as for any other circle and this number they represented by π. By actual measurements the approximations 3.1 and $\frac{22}{7}$ were obtained, but it is now known that π is an irrational number. The procedure described above, permitting an evaluation of π to any desired degree of accuracy, was one of the earliest and most striking triumphs of calculus. The discovery of the connection between π and the series of reciprocals of odd integers with alternating signs is commonly attributed to Leibnitz (1646–1716); actually, it was known earlier to a Scotch mathematician, James Gregory (1638–1675). Since π was defined in terms of lengths, it seemed incredible that π could be approximated by any means other than actual measurements, but we now see that no instruments whatever are necessary to obtain any desired degree of approximation.

At present there is no reason for wanting π to 150 decimal places (although even more have been computed). There are, however, good reasons for being sure of the approximation of π to one more place than an instrument can measure, for then this value can be used to check the accuracy of the instrument.

The method approximating π is an illustration of how mathematics may be used in practical applications. For if a physical constant is sufficiently well defined, then by using no measurements whatever, but only mathematical methods, the constant may be determined or approximated and the result used to check the accuracy of subsequent physical observations.

Under the conditions which yield (1), then

$$0 \leq \int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b [f(x) - g(x)]\, dx$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left\{ f\left(a + k\frac{b-a}{n}\right) - g\left(a + k\frac{b-a}{n}\right)\right\} \frac{b-a}{n}.$$

Upon noting how each term of each sum is related to the area of a rectangle and how the union of the rectangles is related to the region of the following definition, we see why the following definition is a natural one to make.

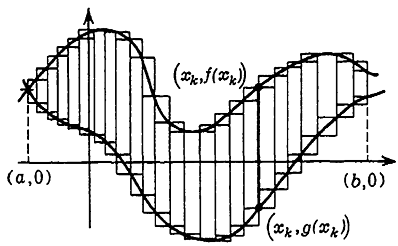


Figure 61.1

DEFINITION 61. *If $a < b$, if f and g are continuous functions on $I[a,b]$, and if $g(x) \le f(x)$ for $a \le x \le b$, then the region*

$$\{(x,y) \mid a \le x \le b,\ g(x) \le y \le f(x)\}$$

is said to have

$$area = \int_a^b [f(x) - g(x)]\, dx \text{ units.}^2$$

Example 2. Find the area of the region of the plane bounded by the graphs of

$$2x^2 + 9y = 36 \quad \text{and} \quad 2x + 3y = 0.$$

Solution. The first graph is a parabola, the second is a straight line, and (by simultaneous solution of the equations) they intersect at the points $(-3,2)$ and

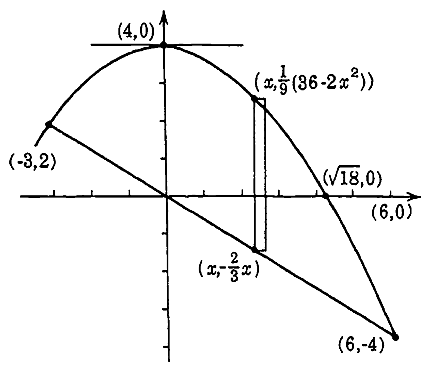


Figure 61.2

$(6,-4)$ between which the parabola is above the line. Upon solving each equation for y, the region is seen to be

$$\{(x,y) \mid -3 \le x \le 6,\quad -\tfrac{2}{3}x \le y \le \tfrac{1}{9}(36 - 2x^2)\}$$

and, according to the above definition, has area

$$\int_{-3}^{6} [\tfrac{1}{9}(36 - 2x^2) - (-\tfrac{2}{3}x)]\, dx = \int_{-3}^{6} (4 - \tfrac{2}{9}x^2 + \tfrac{2}{3}x)\, dx = 27 \text{ units.}^2$$

Example 3. Find the area A of the region bounded by the graphs of

Solution. The points of intersection are $(-3,-1)$ and $(5,3)$ and the vertex of the parabola is at $(-4,0)$. Notice how some vertical segments across the region have both ends on the parabola, whereas others have upper end on the parabola but lower end on the line; the division between the two classes occurring at the abscissa $x = -3$. Thus, the given region is the union of

$$\{(x,y) \mid -4 \le x \le -3, \quad -\sqrt{x+4} \le y \le \sqrt{x+4}\} \qquad \text{and}$$

$$\{(x,y) \mid -3 \le x \le 5, \quad \tfrac{1}{2}(x+1) \le y \le \sqrt{x+4}\}.$$

The area is therefore the sum of two integrals:

$$\begin{aligned} A &= \int_{-4}^{-3} [\sqrt{x+4} - (-\sqrt{x+4})]\,dx + \int_{-3}^{5} [\sqrt{x+4} - \tfrac{1}{2}(x+1)]\,dx \\ &= 2\int_{-4}^{-3} \sqrt{x+4}\,dx + \int_{-3}^{5} (\sqrt{x+4} - \tfrac{1}{2}x - \tfrac{1}{2})\,dx \\ &= 2\Big[\tfrac{2}{3}(x+4)^{3/2}\Big]_{-4}^{-3} + \Big[\tfrac{2}{3}(x+4)^{3/2} - \tfrac{1}{4}x^2 - \tfrac{1}{2}x\Big]_{-3}^{5} \\ &= \tfrac{4}{3} + [\tfrac{2}{3}(27-1) - \tfrac{1}{4}(25-9) - \tfrac{1}{2}(5+3)] = \tfrac{32}{3} \text{ units}^2. \end{aligned}$$

As another attack, notice that every horizontal segment across the region has left end on the parabola and right end on the line, so the region is

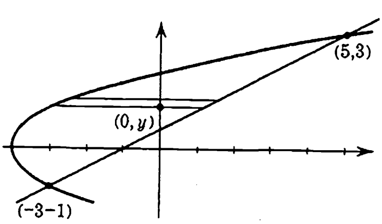


Figure 61.3

$$\{(x,y) \mid -1 \le y \le 3, y^2 - 4 \le x \le 2y - 1\}.$$

Hence, by switching to the variable y,

$$\begin{aligned} A &= \int_{-1}^{3} [(2y-1) - (y^2 - 4)]\,dy \\ &= \int_{-1}^{3} (3 + 2y - y^2)\,dy \\ &= 3y + y^2 - \tfrac{1}{3}y^3\Big]_{-1}^{3} = \tfrac{32}{3} \text{ units}^2. \end{aligned}$$

This second method is the so-called "Method of Horizontal Strips."

PROBLEMS

1. Find the area of region bounded by the graphs of:

a. $y = x^2, \quad y = x.$

b. $y = 9 - x^2, \quad y = x + 3.$

c. $y = \sqrt{x}, \quad y = x^2.$

d. $2y^2 + 9x = 36, \quad 14y = 9x.$

e. $x + y^2 = 4, \quad y = -(x+2).$

f. $y = x + \sin x, \quad y = x; \quad 0 \le x \le \pi.$

g. $y = \sin x, \quad y = \cos x; \quad 0 \le x \le \pi/4.$

h. $y(1 + x^2) = 1, \quad 15y = x^2 - 1.$

i. $xy = 8, \quad x + y = 6.$

j. $y^2 = 4 - 4x, \quad y^2 = 4 - 2x.$

2. Derive a formula for the area of each of the following, but first check that

$$\frac{d}{dx}\frac{1}{2}\left\{x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right\} = \sqrt{a^2 - x^2}, \quad \text{where} \quad a > 0.$$

   a. A circle of radius r.
   b. The portion of a circle of radius r cut off by a line h units from the center where $0 < h < r$.
   c. A sector of a circle with central angle α radians.
   d. An ellipse with major axis $2a$ and minor axis $2b$.
   e. The region bounded by a parabola and its right focal chord.
   f. The region bounded by the graphs of $y = e^{ax}$, $y = e^{-bx}$, and $y = c$ where a and b are positive constants and c is a constant greater than 1.

3. In Fig. Prob. 3 the curve is the graph of $xy = 1$, $x > 0$. Use the fact that the shaded region has area less than that of the trapezoid and show that

$$\ln\frac{b}{a} < \frac{b - a}{2}\left(\frac{1}{a} + \frac{1}{b}\right), \qquad 0 < a < b.$$

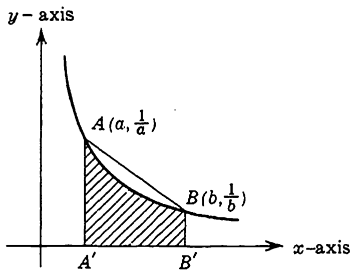


Figure Prob. 3

4. a. Notice that for $k < x < k + 1$, the graph of $xy = 1$ lies between the graphs of $ky = 1$ and $(k + 1)y = 1$. Use this fact to show that

$$\frac{1}{k+1} < \int_k^{k+1} \frac{dx}{x} < \frac{1}{k}.$$

   b. Use the result of 4a to show that for each integer $n > 1$

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}.$$

   c. Use the result of 4b to show that for each integer $n > 1$

$$\frac{1}{n} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n < 1.$$

5. Prove the formula $\pi/4 = 4\tan^{-1}(\frac{1}{5}) - \tan^{-1}(\frac{1}{239})$ and thus see that

$$\frac{\pi}{4} = 4\int_0^{1/5} \frac{1}{1 + x^2}\,dx - \int_0^{1/239} \frac{1}{1 + x^2}\,dx.$$

Use the method of Example 1 to approximate each of these integrals to obtain π accurate to 10^{-6}.†

6. a. Let t be a number such that $0 \le t \le 1$ and sketch the region

$$\{(x,y) \mid 0 \le x \le \sin^{-1} t, \quad 0 \le y \le \sin x\}.$$

† For similar formulas where the same accuracy may be obtained with even fewer terms, see D. H. Lehmer, "On Arcotangent Relations for π," *American Mathematical Monthly*, Vol. 45 (1938), pp. 657–667.

In terms of integrals express the area of this region both by the method of vertical strips and by horizontal strips to obtain

$$\int_0^{\sin^{-1}t} \sin x\, dx = \int_0^t (\sin^{-1} t - \sin^{-1} y)\, dy.$$

From this equation obtain that

$$\int_0^t \sin^{-1} y\, dy = t \sin^{-1} t + \sqrt{1 - t^2} - 1, \quad 0 \le t \le 1.$$

b. Let $t > 1$ be given and sketch $\{(x,y) \mid 0 \le x \le \ln t,\ e^x \le y \le t\}$. By the method of 6a obtain

$$\int_1^t \ln y\, dy = t \ln t - t + 1, \quad t > 1.$$

7. For f a continuous function on $I[a,b]$, let M_n be the mean value of the n values $f(x_1), f(x_2), \cdots, f(x_n)$ where $x_k = a + k\dfrac{b-a}{n}$ for $k = 1, 2, \cdots, n$. Show that

$$\lim_{n\to\infty} M_n = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

62. Pump Problems

The following example illustrates another application of definite integration to the concept of work.

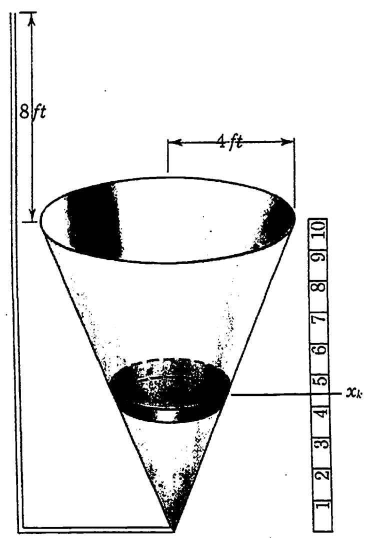


Figure 62

Example. A tank is in the form of a right circular cone with vertex down, axis vertical, altitude 10 ft, and radius of base 4 ft. The tank is full of water, weighing 62.5 lb/ft³. Find the work done in pumping all of the water to a level 8 ft above the top of the tank.

Solution. Place a vertical scale (unit = 1 ft) beside the tank as shown. Divide the scale into n equal parts by using the numbers $0 = x_0 < x_1 < x_2 < \cdots < x_n = 10$ where $x_k = k10/n$, $k = 0, 1, 2, \cdots, n$. Consider water pumped out until the surface stands level with x_k. The surface is a circle of radius r_k where, by similar triangles,

$$\frac{r_k}{x_k} = \frac{4}{10} \quad \text{so that} \quad r_k = 0.4x_k.$$

Visualize the top layer of water $10/n$ ft thick. If this layer were cylindrical it would have volume and weight

$$\pi r_k^2 \frac{10}{n} = \pi(0.4x_k)^2 \frac{10}{n}\ \text{ft}^3 \quad \text{and} \quad 62.5\pi(0.4)^2 x_k^2 \frac{10}{n}\ \text{lb},$$

respectively; the second being a downward force. Each particle of water in this layer is to be raised approximately $(18 - x_k)$ ft against this force so that the work to raise this one layer should be approximately

$$62.5\pi(0.4)^2\, x_k^2(18 - x_k)\frac{10}{n}\text{ ft}\cdot\text{lb}$$

and the sum from $k = 1$ to n of such terms should be (if n is large) about the work expected to accomplish the whole task. Reminiscence prompts us to define the work to be

(1) $$W = \int_0^{10} 62.5\pi(0.4)^2x^2(18 - x)\, dx\text{ ft}\cdot\text{lb}.$$

Hence, by the Fundamental Theorem of Calculus,

$$W = 62.5\pi(0.4)^2\int_0^{10}(18x^2 - x^3)\, dx = 62.5\pi(0.4)^2\Big[6x^3 - \tfrac{1}{4}x^4\Big]_0^{10}$$

$$= 62.5(0.4)^2 10^3(6 - \tfrac{5}{2})\pi = 35 \times 10^3\pi\text{ ft}\cdot\text{lb}.$$

After becoming thoroughly familiar with defining work in terms of a definite integral (i.e., as the limit of sums), it is only necessary to obtain the form of the integrand over the range of integration. A possible synthetic thought process for the above example is:

For $0 \leq x \leq 10$, when water stands x ft deep, the surface has radius r where $r/x = \frac{4}{10}$ so $r = 0.4x$ ft, the surface area is $\pi(0.4x)^2$ ft^2, a layer of thickness Δx exerts a downward force approximately $62.5\pi(0.4x)^2\,\Delta x$ lb which must be acted against for $(18 - x)$ ft and requires about

(2) $$62.5\pi(0.4x)^2(18 - x)\,\Delta x\text{ ft}\cdot\text{lb}$$

of work. This expression furnishes the pattern for the integrand in (1) and is sometimes referred to as "The element of work."

Note: Actually the symbol $\int$ is an elongated S standing for "sum" and the dx appearing in

(3) $$\int_a^b f(x)\, dx$$

is a rudiment of an early vague notion that it was "something" approached by Δx as Δx tends toward zero but in some unexplainable way was "not quite zero." After the results obtained by some very clever men revealed the power of calculus, other men as clever analyzed the processes and devised an unambiguous definition of a limit which removed the uncertainties that plagued earlier exposition and communication. Now (as in Sec. 57) the total symbolism (3) is defined with no specific meaning attached either to $\int$ or to dx, although the dx appearing here looks the same as the dx used in connection with differentiation.

Notice that in the above example no coordinate system was given, and it was up to us to pick one; we chose the origin at the bottom and 10 at the top. As an alternative method, consider the scale in the figure turned over with the origin at the top and 10 at the bottom. In order not to confuse notation, let y be the dummy variable. The synthetic analysis of the problem may now appear as:

For $0 \leq y \leq 10$, a section y feet from the top is $(10 - y)$ ft from the vertex, and hence this section has radius r where

$$\frac{r}{10 - y} = \frac{4}{10} \quad \text{so that} \quad r = 0.4(10 - y)$$

and the area of the section is $\pi(0.4)^2(10 - y)^2$. Hence, a layer Δy ft thick exerts a downward force of $62.5\pi(0.4)^2(10 - y)^2 \,\Delta y$ lb, and this must be acted against for $(8 + y)$ ft, so the element of work is $62.5\pi(0.4)^2(10 - y)^2(8 + y)\,\Delta y$ ft · lb. The definition of the total work is now

$$W = \int_0^{10} 62.5\pi(0.4)^2(10 - y)^2(8 + y)\, dy \text{ ft} \cdot \text{lb},$$

which, as a check will show, yields the same value as before.

PROBLEMS

1. For the problem of the above example place a vertical scale with $s = 0$ at the level to which water is to be raised and $s = 18$ at the level of the vertex. For this choice of scale see that W would be defined as

$$W = \int_8^{18} 62.5\pi(0.4)^2(18 - s)^2 s\, ds.$$

2. A tank is full of water. Find the work done in pumping all the water to a level H ft above the top of the tank if the tank is in the form of:
   a. A right circular cone, vertex down, axis vertical, radius of base 5 ft, altitude 8 ft, and $H = 10$.
   b. A cone as in a, with radius r ft, altitude h ft, and $H \geq 0$.
   c. The cone of b, but with vertex up.
   d. A hemispherical bowl with radius 5 ft and $H = 23$.
   e. A right circular cylinder, axis vertical, $r = 5$ ft, $h = 8$ ft, and $H = 15$.
   f. The bowl of d surmounted by the cylinder of e; $H = 15$.
   g. A sphere of radius 5 ft, and $H = 10$.
   h. A trough of length 6 ft, whose vertical cross section is an isosceles triangle with base 3 ft and altitude 2 ft, and $H = 10$.
   i. A trough of length 6 ft whose vertical cross section is a semicircle of radius 2 ft, and $H = 0$.
   j. The trough of i but with $H = 5$. (Hint: The derivative in Prob. 2, p. 187 will be useful.)
   k. A trough of length 6 ft, whose vertical cross section is an isosceles trapezoid of altitude 4 ft, upper base 3 ft, lower base 2 ft, and $H = 20$.

3. An upright cylindrical storage tank has radius 6 ft and altitude 5 ft. Next to the storage tank is a 20 ft tower on which stands an upright cylindrical utility tank with radius 2 ft and altitude 9 ft. If the storage tank is full and the utility tank is empty, what is the work required to fill the utility tank by pumping water through a pipe from the lower base of the storage tank to the lower base of the utility tank?

63. Hydrostatic Force

An incompressible substance is one which will neither contract nor expand merely because force is applied or released. Air is compressible and a cubic foot of air at the earth's surface weighs considerably more than a cubic foot of air in the stratosphere. Water is essentially incompressible, and a cubic foot of water weighs about 62.5 lb whether at the surface or 100 ft below the surface. Thus, the number of pounds of water in a container is the number of cubic feet of water times 62.5. A law of hydrostatics is that an incompressible fluid exerts force equally in all directions. For example, a flask and a bowl of the same altitude h ft when full of water will sustain due to the water the same force per square unit on their bases, and if each base is A ft^2 the force on each base due to the water is

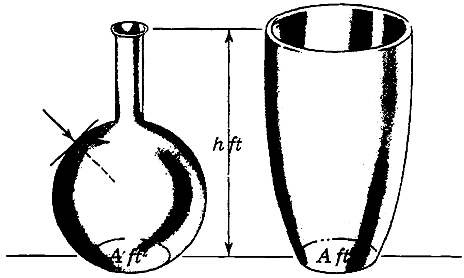


Figure 63.1

$$62.5\,Ah\text{ lb.}$$

Hydrostatic force on a vertical plane section requires an integral for its definition as illustrated in the following examples.

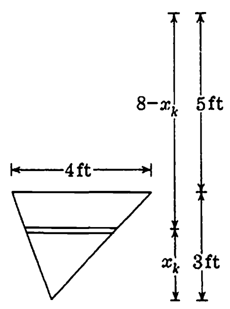


Figure 63.2

Example 1. On a vertical dam a triangle is marked with horizontal base 4 ft, vertex down, altitude 3 ft, and the base is 5 ft below the surface. Define and then find the force on this triangle.

Solution. Let x_k be a number such that $0 \leq x_k \leq 3$. Across the triangle consider a line segment x_k ft from the vertex. By similar triangles the length of this segment is $(\frac{4}{3})x_k$ ft, and a strip of width Δx ft at this level has area about $(\frac{4}{3})x_k\,\Delta x$ ft^2. The segment is x_k ft from the vertex, and thus $5 + 3 - x_k = 8 - x_k$ ft from the surface. If, then, the strip were turned edgewise to the dam it would sustain a force of about $62.5(\frac{4}{3})x_k(8 - x_k)\Delta x$ lb and, since hydrostatic force of an incompressible fluid is exerted equally in all directions, this strip would sustain about this same force in its actual position. These considerations lead to the definition

$$F = \int_0^3 62.5(\tfrac{4}{3})x(8 - x)\,dx = (62.5)\tfrac{4}{3}\int_0^3 (8x - x^2)\,dx = (62.5)(36)\text{ lb.}$$

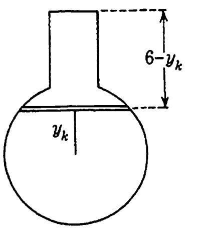


Figure 63.3

Example 2. A horizontal supply conduit is 3 ft in diameter and is blocked by a vertical valve head, directly in front of which is a lead-off pipe 1 ft in diameter. Find the force on the valve head when the water level in the lead-off pipe is 6 ft above the center line of the conduit.

Solution. First notice that the diameter of the lead-off pipe is superfluous information, but the 6 ft rise is all important. Establish a coordinate system (unit = 1 ft) with the origin at the center of the conduit and the point (0,6) at the surface of the water. With $-\frac{3}{2} \le y_k \le \frac{3}{2}$, notice that at the point with coordinate y_k the horizontal line segment across the conduit has length $2\sqrt{(\frac{3}{2})^2 - y_k^2}$ ft. At this level a thin strip of width Δy ft has area approximately $2\sqrt{\frac{9}{4} - y_k^2}\,\Delta y$ ft^2 and each of its points is about $6 - y_k$ ft below the surface, regardless of whether y_k is positive or negative (but we must have $-\frac{3}{2} \le y_k \le \frac{3}{2}$). Thus, this strip must sustain a force near to

$$(62.5)2\sqrt{\tfrac{9}{4} - y_k^2}(6 - y_k)\,\Delta y\text{ lb.}$$

Consequently, we define and compute

$$F = (62.5)2\int_{-3/2}^{3/2} \sqrt{\tfrac{9}{4} - y^2}(6 - y)\,dy$$

$$= 125\left\{6\int_{-3/2}^{3/2} \sqrt{\tfrac{9}{4} - y^2}\,dy - \int_{-3/2}^{3/2} \sqrt{\tfrac{9}{4} - y^2}\,y\,dy\right\}$$

$$= 125\left\{\frac{6}{2}\left[y\sqrt{\tfrac{9}{4} - y^2} + \frac{9}{4}\sin^{-1}\frac{2y}{3}\right]_{-3/2}^{3/2} + \left[\frac{1}{3}(\tfrac{9}{4} - y^2)^{3/2}\right]_{-3/2}^{3/2}\right\}$$

(check each of these by taking derivatives)

$$= 125\left\{3\left[0 - 0 + \frac{9}{4}\sin^{-1}1 - \frac{9}{4}\sin^{-1}(-1)\right] + \frac{1}{3}\left[3\cdot 0 - 3\cdot 0\right]\right\}$$

$$= 125\left\{3\cdot\frac{9}{4}\pi\right\} = 2.65 \times 10^3\text{ lb.}$$

64. Integration By Parts

The Fundamental Theorem of Calculus states for f a continuous function, that

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \quad\text{where}\quad F'(x) = f(x).$$

This formula may be written in either of the forms

$$\int_a^b F'(x)\,dx = F(x)\Big]_a^b \quad \text{or} \quad \int_a^b D_xF(x)\,dx = F(x)\Big]_a^b.$$

Hence, for u and v functions having continuous derivatives,

$$\int_a^b D_x\{u(x)v(x)\}\,dx = u(x)v(x)\Big]_a^b.$$

But $D_x\{u(x)v(x)\} = u(x)v'(x) + v(x)u'(x)$, and thus we have

$$\int_a^b \{u(x)v'(x) + v(x)u'(x)\}\,dx = u(x)v(x)\Big]_a^b.$$

The left side written as the sum of two integrals, and then the second one transposed, yields

$$(1) \qquad \int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\,dx$$

which is known as the formula for **Integration by Parts.**

Example 1. Find $\int_0^{\pi/2} x \sin x\,dx$.

Solution. Upon setting $u(x) = x$ and $v'(x) = \sin x$, so that $u'(x) = 1$ and (by finding an antiderivative) $v(x) = -\cos x$, we obtain from (1)

$$\begin{aligned}\int_0^{\pi/2} x \sin x\,dx &= x(-\cos x)\Big]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot (-\cos x)\,dx \\ &= -\frac{\pi}{2}\cos\frac{\pi}{2} + 0 \cdot \cos 0 + \int_0^{\pi/2} \cos x\,dx \\ &= -\frac{\pi}{2} \cdot 0 + 0 + \Big[\sin x\Big]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1.\end{aligned}$$

Integration by parts is a means by which one integral may be expressed in terms of another integral. Unless a suitable substitution is made, the second integral may turn out to be more complicated than the original one. As an illustration, in Example 1 the substitutions

$$u(x) = \sin x, \quad v'(x) = x$$

could have been made, so that $u'(x) = \cos x$, $v(x) = x^2/2$, and

$$\int_0^{\pi/2} x \sin x\,dx = (\sin x)\frac{x^2}{2}\Big]_0^{\pi/2} - \int_0^{\pi/2} \frac{x^2}{2} \cos x\,dx.$$

This equation is valid, but the substitution is not advisable since the second integral is more complicated than the first.

A fairly reliable rule is:

Set $u(x)$ a part of the integrand such that $u'(x)$ is simpler than $u(x)$.

Having selected a part of the integrand to set equal to $u(x)$, it is necessary that the remaining part of the integrand (to be set equal to $v'(x)$) be recognizable as a derivative of some function. For example, if

$$\int_1^e x \ln x\, dx$$

is sought then it is inadvisable to set $u(x) = x$, for (although $u'(x) = 1$ is simpler than $u(x) = x$) then $v'(x) = \ln x$ cannot easily be solved for $v(x)$. Hence, try $u(x) = \ln x$.

Sometimes a double exchange is necessary.

Example 2. Find the value of $I = \int_0^4 x^2\sqrt{4 - x}\, dx$.

Solution. Set $u(x) = x^2, v'(x) = \sqrt{4 - x}$. Then $u'(x) = 2x, v(x) = -\frac{2}{3}(4 - x)^{3/2}$,

$$I = x^2(-\tfrac{2}{3})(4 - x)^{3/2}\Big]_0^4 - \int_0^4(-\tfrac{2}{3})(4 - x)^{3/2}2x\, dx$$

$$= 0 + \tfrac{4}{3}\int_0^4 x(4 - x)^{3/2}\, dx.$$

Having finished with the above substitutions for u and v, we are now free to make new substitutions. In this new integral we set:

$u(x) = x$, $v'(x) = (4 - x)^{3/2}$ so that $u'(x) = 1$, $v(x) = -\frac{2}{5}(4 - x)^{5/2}$, and

$$I = \frac{4}{3}\left\{x\left(-\frac{2}{5}\right)(4 - x)^{5/2}\Big]_0^4 - \int_0^4\left(-\frac{2}{5}\right)(4 - x)^{5/2}(1)\, dx\right\}$$

$$= \frac{4}{3}\left\{0 + \frac{2}{5}\int_0^4(4 - x)^{5/2}\, dx\right\} = \frac{8}{15}\int_0^4(4 - x)^{5/2}\, dx$$

$$= \frac{8}{15}\left(-\frac{2}{7}\right)(4 - x)^{7/2}\Big]_0^4 = \frac{-16}{105}[0 - 4^{7/2}] = \frac{2048}{105}.$$

Example 3. Find $I = \int_0^{\pi/2} x^2 \cos x\, dx$.

Solution. Set $u(x) = x^2$, $v'(x) = \cos x$ so that $u'(x) = 2x$, $v(x) = \sin x$ and

$$I = x^2 \sin x\Big]_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x\, dx = \left(\frac{\pi}{2}\right)^2 - 2\underbrace{\int_0^{\pi/2} x \sin x\, dx}_{=1}.$$

Hence, by using Example 1, $I = (\pi/2)^2 - 2$.

PROBLEMS

1. A plane triangle with horizontal base 8 ft and altitude 16 ft is held vertically in water. Find the force on one face if the triangle has:

   a. Vertex 10 ft below the surface and above the base.

b. Base 10 ft below the surface and above the vertex.

c. Vertex up and at the surface.

d. Base up and at the surface.

e. In each case check that the answer is 62.5 times the area of the triangle times the depth of the intersection of the medians of the triangle.

2. A trough is full of water. Find the force on the end if the end is:

a. An isosceles triangle of base 4 ft and altitude 3 ft.

b. A semicircle of radius 4 ft.

c. A semiellipse whose semimajor axis is horizontal and 4 ft long and whose semiminor axis is 3 ft long.

d. A trapezoid with upper base 4 ft, lower base 2 ft, and altitude 3 ft.

3. A plane disk is submerged vertically in water. Find the force on one face if the disk is:

a. A circle of radius 3 ft with center 3 ft below the surface.

b. The circle of Part a, with the center 10 ft below the surface.

c. An ellipse with major axis horizontal, major axis $2a$ ft long, minor axis $2b$ ft long, and center c ft below the surface where $c \geq b$.

d. The ellipse of Part c with major axis vertical and $c \geq a$.

e. In each case check that the answer is 62.5 times the area of the disk times the depth of the center.

4. The vertical end of a tank is an isosceles trapezoid with lower base 10 ft, upper base 16 ft, and altitude 12 ft. Water stands 4 ft deep and then there is a 5 ft layer of oil weighing 50 lb/ft^3. Find the force on the end of the tank in terms of integrals.

5. Evaluate each of the definite integrals:

a. $\int_0^\pi x \sin 2x \, dx.$ d. $\int_2^5 x^2\sqrt{x-1} \, dx.$ g. $\int_0^3 \sqrt{25 - x^2}\, x \, dx.$

b. $\int_{-\pi/2}^{\pi/2} x \cos x \, dx.$ e. $\int_2^5 \frac{x}{\sqrt{x-1}} \, dx.$ h. $\int_{-\pi}^{\pi} x^2 \cos \frac{x}{2} \, dx.$

c. $\int_1^2 x \ln x \, dx.$ f. $\int_0^3 \sqrt{25 - 3x}\, x \, dx.$ i. $\int_0^1 x^2 e^x \, dx.$

6. a. A rectangle with horizontal base 5 ft and altitude 8 ft is marked on the face of a vertical dam with the upper base at the surface. Find a depth such that the force on the rectangle above this depth is equal to the force in the portion below this depth.

b. Replace the rectangle of Part a by a triangle with base of 5 ft in the surface and altitude 8 ft.

c. Turn the triangle of Part b over so the vertex is in the surface.

d. Replace the triangle of either Part b or Part c by a triangle with base b ft and altitude h ft. Show that the desired depth is $h/2$ ft when the base is up, but is $h/\sqrt[3]{2}$ when the vertex is up.

7. Start with $D_x\{u(x)v'(x) - u'(x)v(x)\}$ and obtain the formula

$$\int_a^b u(x)v''(x)\,dx = \Big[u(x)v'(x) - u'(x)v(x)\Big]_a^b + \int_a^b u''(x)v(x)\,dx.$$

Use this formula to obtain the integrals in Examples 2 and 3. Also, use this formula to find

a. $\int_0^{\pi/2} x\cos x\,dx.$ b. $\int_0^{\pi} e^x \sin x\,dx.$

65. First Moments and Centroids

Consider a strong light rod with a 10-lb weight at one end and a 20-lb weight at the other. In carrying this contraption with one hand it would be natural to grasp the rod twice as far from the 10-lb weight as from the 20-lb weight. A rod d ft long with weights of W_1 lb and W_2 lb at the ends has balance point (neglecting the weight of the rod) x ft from W_1-lb weight if

$$(1) \qquad xW_1 = (d - x)W_2.$$

By using a coordinate system, a generalization to more than two weights may be made. Let $x_1 < x_2$ be given and let weights of W_1 lb and W_2 lb be at the points $(x_1,0)$ and $(x_2,0)$, and let $(\bar{x},0)$ be the balance point. Thus $\bar{x} - x_1 > 0$, $x_2 - \bar{x} > 0$, and from (1)

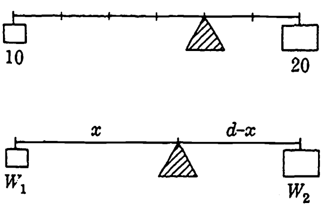


Figure 65.1

$$(\bar{x} - x_1)W_1 = (x_2 - \bar{x})W_2.$$

Since, however, $\bar{x} - x_1 = -(x_1 - \bar{x})$, this equation may be written as

$$(2) \quad (x_1 - \bar{x})W_1 + (x_2 - \bar{x})W_2 = 0, \quad \text{or as}$$

$$(3) \qquad \bar{x} = \frac{x_1W_1 + x_2W_2}{W_1 + W_2}.$$

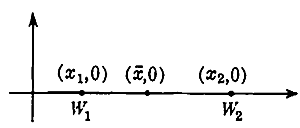


Figure 65.2

Now let particles of masses $m_1, m_2, \cdots, m_n$ be at points (x_1,y_1), (x_2,y_2), $\cdots$, (x_n,y_n) in a plane considered as lying horizontal. Let c be a number and let L be the line through the point $(c,0)$ perpendicular to the x-axis. By definition, the numbers

$$(x_1 - c)m_1, (x_2 - c)m_2, \cdots, (x_n - c)m_n$$

are the **first moments** relative to L of the individual masses,

$$M_L = (x_1 - c)m_1 + (x_2 - c)m_2 + \cdots + (x_n - c)m_n = \sum_{k=1}^{n} x_k m_k - c\sum_{k=1}^{n} m_k$$

is the **first moment** relative to L of the system, and the solution

$$(4) \qquad \bar{x} = \left(\sum_{k=1}^{n} x_k m_k\right) \Big/ \sum_{k=1}^{n} m_k$$

of the equation $M_L = 0$ for c is called the **abscissa of the centroid** of the system. In the same way, first moments relative to a line perpendicular to the y-axis are defined and the point $(\bar{x},\bar{y})$ where

$$(5) \qquad \bar{y} = \left(\sum_{k=1}^{n} y_k m_k\right) \Big/ \sum_{k=1}^{n} m_k$$

is called the **centroid** (or **center of mass**) of the system. Notice that in (4) and (5) the denominator is the mass of the system, while the numerators are, respectively, the first moment relative to the y-axis and the first moment relative to the x-axis.

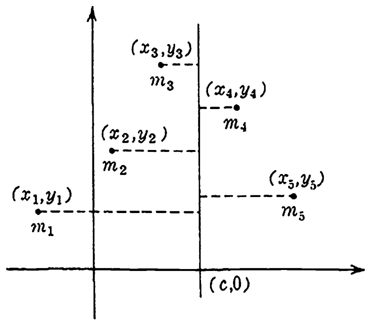


Figure 65.3

Consider a homogeneous sheet of metal of uniform thickness τ and density ρ. Thus, if A units^2 is the area of one face of the sheet, then the volume is $A\tau$ units^3 and the weight is $A\tau\rho$ weight units. Let the sheet be placed on a horizontal plane in which a coordinate system has been established. With a the smallest and b the largest abscissa of the region covered by the sheet and with n a positive integer, let

$$\Delta x = (b - a)/n \quad \text{and} \quad x_k = a + k \cdot \Delta x, \qquad k = 0, 1, 2, \cdots, n.$$

The vertical line through the point $(x_k,0)$ cuts the region in a segment whose length, using functional notation, we denote by $s(x_k)$. The strip of length $s(x_k)$ and width Δx has approximate weight $\rho\tau s(x_k)\,\Delta x$. With c a number, each point of the strip is approximately $|x_k - c|$ units from the line L perpendicular to the x-axis through the point $(c,0)$ and with respect to L, the first moment of the strip should be approximately $(x_k - c)\rho\tau s(x_k)\,\Delta x$. Considering all such strips, the moment of the sheet relative to L should be about

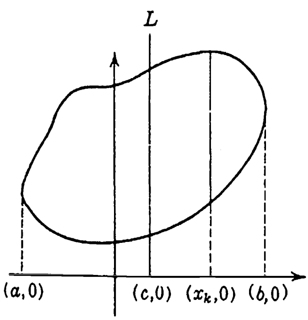


Figure 65.4

$$\sum_{k=1}^{n} (x_k - c)\rho\tau s(x_k)\,\Delta x.$$

We then define the limit as $n \to \infty$ of such sums as the **first moment** M_L of the sheet with respect to L, and thus have

$$M_L = \int_a^b \rho\tau(x - c)s(x)\,dx.$$

Upon letting $\bar{x}$ be the value of c for which $M_L = 0$, we call $\bar{x}$ the abscissa of the **centroid** of the sheet. Thus, with M_y the first moment of the sheet relative to the y-axis,

$$\bar{x} = \frac{\int_a^b \rho\tau x s(x)\,dx}{\int_a^b \rho\tau s(x)\,dx} = \frac{M_y}{\text{Total mass}}. \tag{6}$$

We are considering ρ and τ as constants. Thus, we shall have all essentials for finding first moments and centroids of sheets if for regions (area A units²) we define the first moment M_L and the abscissa $\bar{x}$ of the centroid as

$$M_L = \int_a^b (x - c)s(x)\,dx \quad \text{and} \quad \bar{x} = \frac{\int_a^b xs(x)\,dx}{\int_a^b s(x)\,dx} = \frac{M_y}{A}. \tag{7}$$

Example 1. Find the moment M_y (the moment with respect to the y-axis) of the region $\{(x,y) \mid 0 \le x \le \pi/2, 0 \le y \le \cos x\}$.

Solution. Upon setting $s(x) = \cos x$ and $c = 0$ in (7), we have

$$M_y = \int_0^{\pi/2} x\cos x\,dx = x\sin x\Big]_0^{\pi/2} - \int_0^{\pi/2} \sin x\,dx = \frac{\pi}{2} - 0 - \Big[-\cos x\Big]_0^{\pi/2} = \frac{\pi}{2} - 1.$$

In a similar way, first moments with respect to lines perpendicular to the y-axis and the ordinate of the centroid are defined.

Example 2. Find the centroid of the region bounded by the graphs of $y = x^2$ and $y^2 = 8x$.

Solution. The curves intersect at the origin and at the point (2,4) and the region is

$\{(x,y) \mid 0 \le x \le 2, x^2 \le y \le \sqrt{8x}\}$. Thus,

$$\bar{x} = \frac{\int_0^2 x(\sqrt{8x} - x^2)\,dx}{\int_0^2 (\sqrt{8x} - x^2)\,dx} = \frac{\int_0^2 (2\sqrt{2}x^{3/2} - x^3)\,dx}{\int_0^2 (2\sqrt{2}x^{1/2} - x^2)\,dx} = \frac{\frac{12}{5}}{\frac{8}{3}}$$

The same region is $\{(x,y) \mid 0 \le y \le 4, y^2/8 \le x \le \sqrt{y}\}$. Since the area of the region is $\frac{8}{3}$ unit², it need not be recomputed and

$$\bar{y} = \frac{\int_0^4 y(\sqrt{y} - y^2/8)\,dy}{\frac{8}{3}} = \frac{\frac{24}{5}}{\frac{8}{3}}.$$

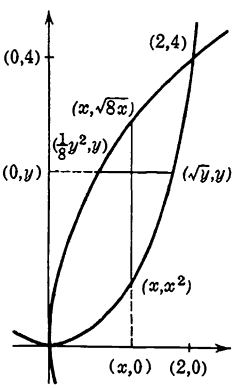


Figure 65.5

Thus, the centroid is the point $(\frac{9}{10},\frac{9}{5})$.

Example 3. For the region $\{(x,y) \mid 0 \leq x \leq \pi/2,\ 0 \leq y \leq \sin x\}$ find $\bar{x}$ and express $\bar{y}$ in terms of integrals.

Solution. In this case $s(x)$ of the general discussion is $\sin x$ so that

$$\bar{x} = \frac{\int_0^{\pi/2} x \sin x\, dx}{\int_0^{\pi/2} \sin x\, dx} = \frac{x(-\cos x)\Big]_0^{\pi/2} - \int_0^{\pi/2} (-\cos x)\, dx}{-\cos x\Big]_0^{\pi/2}} = \frac{0 + \sin x\Big]_0^{\pi/2}}{-0 + 1} = 1.$$

The region may also be expressed as $\{(x,y) \mid 0 \leq y \leq 1,\ \sin^{-1} y \leq x \leq \pi/2\}$. Thus, since the area of the region is 1 unit^2, we have

$$\bar{y} = \int_0^1 y\left(\frac{\pi}{2} - \sin^{-1} y\right) dy$$

$$= \frac{\pi}{2}\int_0^1 y\, dy - \int_0^1 y \sin^{-1} y\, dy$$

$$= \frac{\pi}{4} - \int_0^1 y \sin^{-1} y\, dy.$$

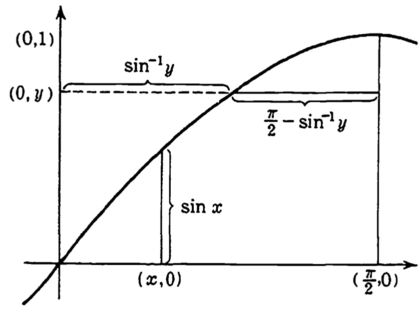


Figure 65.6

(Note: Later, methods for evaluating this integral will be given.)

Example 4. Find the centroid of the triangle with vertices (0,0), (1,4), and (5,0).

Solution. Equations of the sides through (1,4) are $y = 4x$ and $y = -x + 5$. Thus, the region may be written either as the union of

$$\{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 4x\} \quad \text{and} \quad \{(x,y) \mid 1 \leq x \leq 5,\ 0 \leq y \leq -x + 5\}$$
$$\text{or as} \quad \{(x,y) \mid 0 \leq y \leq 4,\ y/4 \leq x \leq 5 - y\}.$$

Since the area of the triangle is 10 units, we have

$$\bar{x} = \frac{1}{10}\left\{\int_0^1 x(4x)\, dx + \int_1^5 x(-x + 5)\, dx\right\} = 2,$$

$$\bar{y} = \frac{1}{10}\int_0^4 y\left(5 - y - \frac{y}{4}\right) dy = \frac{4}{3}.$$

PROBLEMS

1. Find the centroid of the region bounded by the curves whose equations are given.

a. $y = x^2,\quad y = 2x.$

b. $y = x^2,\quad y = mx;\quad m > 0.$

c. $y = x^3,\quad x = 0,\quad y = b;\quad b > 0.$

d. $y = 2x,\quad x + 2y = 5,\quad y = 0.$

e. $y^2 = x,\quad x - 2y = 3.$

f. $y^2 = 4 - 4x,\quad y^2 = 4 - 2x.$

g. $y = \ln x,\quad y = 0,\quad x = 2.$

h. $y = e^x,\quad x = 0,\quad y = 0,\quad x = 2.$

2. Prove that the centroid of a triangular region is at the intersection of the medians of the triangle.

3. First check the derivatives in a' and b' and then find the centroids of the regions in a to c.

a′. $D_x \frac{1}{2}\left[x\sqrt{a^2 - x^2} - a^2 \cos^{-1}\frac{x}{a}\right] = \sqrt{a^2 - x^2}, \quad a > 0.$

b′. $D_x\left\{\left(\frac{x^2}{2} - \frac{1}{4}\right)\cos^{-1} x - \frac{1}{4}x\sqrt{1 - x^2}\right\} = x\cos^{-1} x.$

a. A semicircle.

b. A semiellipse with semiaxes a and b.

c. $\{(x,y) \mid -\pi/2 \le x \le \pi/2, \quad 0 \le y \le \cos x\}.$

4. A vertical rectangular flood gate 4 ft broad and 6 ft high is swung on a horizontal bolt. Find where the bolt should be located so there will be no strain on the fastening device when the water surface is 1 ft above the top of the gate.

5. Prove that the force on a submerged portion of a vertical dam is 62.5 times the area of the portion times the depth of the centroid of the portion.

6. Two regions R_1 and R_2 have at most boundary points in common. If the areas of the regions are A_1 and A_2 and have centroids $(\bar{x}_1,\bar{y}_1)$ and $(\bar{x}_2,\bar{y}_2)$ show that the union of the two regions has centroid $(\bar{x},\bar{y})$ where

$$\bar{x} = \frac{\bar{x}_1A_1 + \bar{x}_2A_2}{A_1 + A_2} \quad \text{and} \quad \bar{y} = \frac{\bar{y}_1A_1 + \bar{y}_2A_2}{A_1 + A_2}.$$

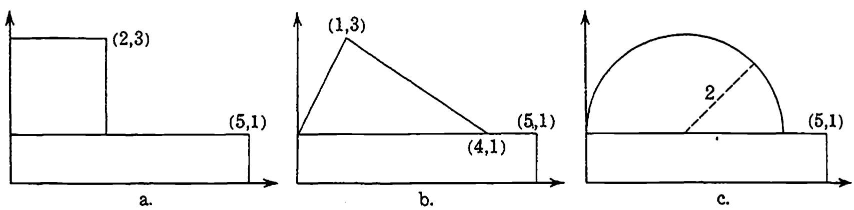


Figure Prob. 6

This result is known as a theorem of Pappus. Use the theorem and known centroids to obtain the centroids of the figures a, b, and c. Find the centroid of the region obtained by removing the circle of radius r and center $(r,0)$ from the:

d. Circle with radius $3r$ and center at the origin.

e. Rectangle with corners $(0,-3r)$, $(3r,-3r)$, $(3r,3r)$, $(0,3r)$.

66. Second Moments and Kinetic Energy

A particle of mass m moving with velocity v is said, in a branch of mechanics called dynamics, to have **kinetic energy** E where, in terms of the units used,

$$E = \tfrac{1}{2}mv^2. \tag{1}$$

Let a particle of mass m move around a circle with center at the origin and radius r and at time t (say measured in seconds) let the angle measured in radians made by the radius vector to the particle be $\theta(t)$. The particle is said to have angular velocity $\omega(t)$ where†

$$\omega(t) = \theta'(t) = \frac{d\theta(t)}{dt} = \lim_{h\to 0} \frac{\theta(t+h) - \theta(t)}{h} \text{ radians/sec.}$$

The length of an arc of a circle is the radius times the number of radians in the subtended angle and, therefore, this particle has velocity $v(t)$ and kinetic energy $E(t)$ where

$$v(t) = r\omega(t) \quad \text{and} \quad E(t) = \tfrac{1}{2}m\{r\omega(t)\}^2 = mr^2\frac{\omega^2(t)}{2}.$$

This particle is said to have **second moment** (or **moment of inertia**) I with respect to the origin, where

$$I = mr^2. \tag{2}$$

Notice that the second moment is independent of time and

$$E(t) = I\frac{\omega^2(t)}{2}. \tag{3}$$

The definition of second moment of a particle with respect to a point is extended, by means of the limit of sums, to the definition of the second moment of a sheet with respect to a line. On a coordinate system represent a uniform sheet of thickness τ and density ρ and let L be a line perpendicular to the x-axis through a point $(c,0)$ (although c need not be between a and b as illustrated). Since each particle having abscissa x where $a \le x \le b$ is $|x - c|$ units from L, the **second moment** I_L with respect to L of the whole sheet is defined to be

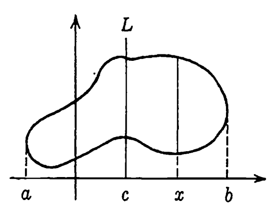


Figure 66

$$\begin{aligned} I_L &= \int_a^b \rho\tau\,|x - c|^2 s(x)\,dx \\ &= \int_a^b \rho\tau(x - c)^2 s(x)\,dx \end{aligned} \tag{4}$$

where $s(x)$ is the length of the segment across the sheet at the abscissa x. Whenever ρ and τ are constants, the region covered by the sheet is said to have **second moment** with respect to L where

$$I_L = \int_a^b (x - c)^2 s(x)\,dx. \tag{5}$$

† Angular velocity for a particle moving in a curve other than a circle will be given later.

Example 1. Find the second moment with respect to the y-axis of the region $\{(x,y) \mid 0 \le x \le \pi/2, 0 \le y \le \cos x\}$.

Solution. Letting I_y be this second moment (so $c = 0$), we have

$$I_y = \int_0^{\pi/2} x^2 \cos x \, dx.$$

The use of integration by parts twice (see Example 3, Sec. 64) yields $I_y = (\pi/2)^2 - 2$.

Kinetic energy of a body is of great importance in practical problems and in such problems the angular velocity is generally known. Thus, to find the kinetic energy of a body it is only necessary to compute the second moment by means of (5), multiply by $\rho\tau$, and then to use (3).

From (5) it is seen that

$$\begin{aligned} I_L &= \int_a^b (x^2 - 2cx + c^2)s(x)\,dx \\ &= \int_a^b x^2 s(x)\,dx - 2c\int_a^b xs(x)\,dx + c^2\int_a^b s(x)\,dx. \end{aligned}$$

The first term is the second moment I_y with respect to the y-axis, the middle term is $-2c$ times the first moment M_y with respect to the y-axis, and the last term is c^2 times the area A; hence, the formula

$$I_L = I_y - 2cM_y + c^2 A. \tag{6}$$

The statement (6) is referred to as "The Parallel Axis Theorem."

Example 2. Find the second moment with respect to the vertical line L through (3,0) of the region of Example 1.

Solution. $I_y = (\pi/2)^2 - 2$ from Example 1, $M_y = (\pi/2) - 1$ from Example 1 Sec. 65, and $A = 1$. Thus, from (6) with $c = 3$

$$I_L = \left(\frac{\pi^2}{4} - 2\right) - 2 \cdot 3\left(\frac{\pi}{2} - 1\right) + 3^2 \cdot 1 = \frac{\pi^2}{4} - 3\pi + 13.$$

The pitch and roll of a ship at sea and the stability of an airplane in flight are determined quite largely by moments of inertia and it is desirable to have these moments as small as possible. To find where L should be located to make I_L the minimum, note that I_y, M_y, and A are independent of c and that

$$D_c I_L = D_c(I_y - 2cM_y + c^2 A) = -2M_y + 2cA \quad \text{and} \quad D_c^2 I_L = 2A > 0.$$

Thus, the solution of $D_c(I_L) = 0$ for c furnishes the minimum. Hence, $c = M_y/A = \bar{x}$ shows that by selecting L through the centroid, the smallest second moment will be obtained.

PROBLEMS

1. With a and b positive numbers, find I_y for the rectangular region with vertices $(0,0)$, $(a,0)$, (a,b), $(0,b)$.

2. Replace the rectangle of Prob. 1 by the triangle with vertices $(0,0)$, $(a,0)$, $(0,b)$.

3. Replace the third vertex in Prob. 2 by (c,b) with $0 < c \neq a$.

4. Replace the rectangle of Prob. 1 by the quadrant of a circle of radius a and bounding radii along the axes, but first check that

$$D_x\left\{-\frac{x}{4}(a^2 - x^2)^{3/2} + \frac{a^2}{8}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right]\right\} = x^2\sqrt{a^2 - x^2}.$$

5. Substitute "ellipse" for "circle" in Prob. 4.

6. Replace the rectangle of Prob. 1 by $\{(x,y) \mid 0 \le x \le p, 0 \le y \le 2\sqrt{px}\}$.

7. Replace the region of Prob. 6 by $\{(x,y) \mid 0 \le y \le p, 0 \le x \le 2\sqrt{py}\}$.

67. Solids of Revolution

Let $a < b$ be numbers, let f be a continuous function, and consider the region

(1) $$\{(x,y) \mid a \le x \le b, 0 \le y \le |f(x)|\}.$$

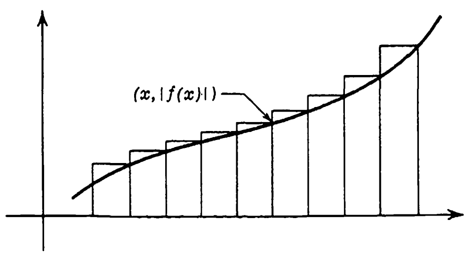


Figure 67.1

By revolving this region about the x-axis a solid of revolution is obtained whose **volume** is defined to be

(2) $$V = \int_a^b \pi[f(x)]^2\, dx.$$

The reasoning behind this definition is based upon first approximating the region (1) by rectangles. For $x_k = a + k(b - a)/n$, the rectangle of altitude $|f(x_k)|$ units and base $\Delta x = (b - a)/n$ units revolves into a solid disk which is a right circular cylinder having base radius $|f(x_k)|$ units, altitude Δx units, and volume

$$\pi|f(x_k)|^2\,\Delta x = \pi[f(x_k)]^2\,\Delta x \text{ units}^3.$$

The sum from $k = 1$ to n of such volumes and then the limit as $n \to \infty$ leads to the definition (2).

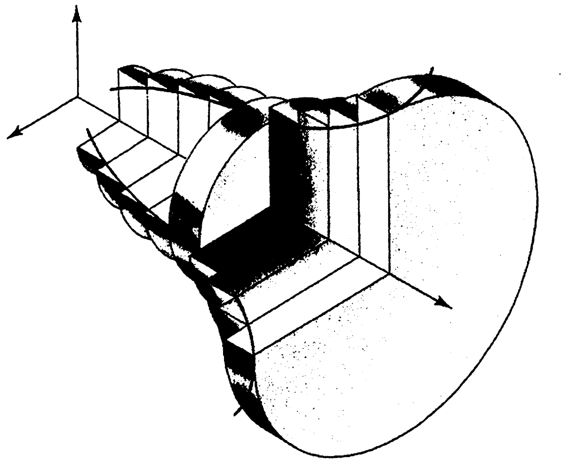


Figure 67.2

Example 1. The region $\{(x,y) \mid 1 \le x \le 3,\ 0 \le y \le x^{3/2}\}$ revolved about the x-axis generates a solid of volume

$$V = \int_1^3 \pi(x^{3/2})^2\, dx = \pi \int_1^3 x^3\, dx = 20\pi.$$

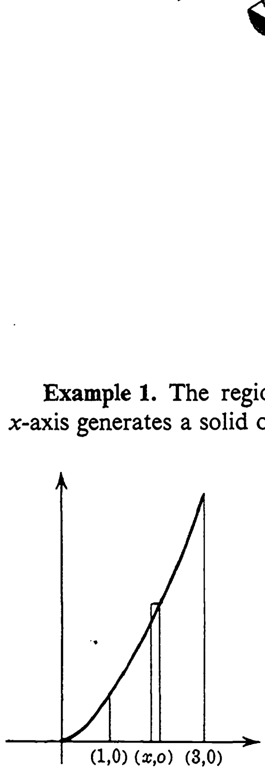


Figure 67.3

Example 2. Revolve the region of Example 1 about the y-axis and find the volume of the resulting solid.

First Solution. Notice that the region does not abut on the y-axis and that the curved portion of the boundary has end points $(1,1)$ and $(3,3^{3/2})$. Upon revolving about the y-axis it is natural to switch the dummy variable to y, and to express a point on the curve, not as $(x,x^{3/2})$ but, as $(y^{2/3},y)$. Now the given region is best expressed as the union of two regions:

$$\{(x,y) \mid 0 \le y \le 1,\ 1 \le x \le 3\}$$
$$\text{and} \quad \{(x,y) \mid 1 \le y \le 3^{3/2},\ y^{2/3} \le x \le 3\}$$

and the volume will be defined as the sum of two integrals. Each horizontal approximating rectangle revolves into a washer-shaped solid. Such a solid of inner radius r_1, outer radius r_2, and thickness Δy has volume $(\pi r_1^2 - \pi r_2^2)\,\Delta y$. Hence, the volume turns out as

$$V = \int_0^1 \{\pi \cdot 3^2 - \pi \cdot 1^2\}\, dy + \int_1^{3\sqrt{3}} \{\pi \cdot 3^2 - \pi(y^{2/3})^2\}\, dy$$
$$= \pi \int_0^1 8\, dy + \pi \int_1^{3\sqrt{3}} (9 - y^{4/3})\, dy = \tfrac{4}{7}\pi(27\sqrt{3} - 1).$$

Second Solution. To illustrate another approach we return to the dummy variable x. For $1 \le x \le 3$ the vertical segment from the point $(x,0)$ up to the curve has length $x^{3/2}$ units. This segment resolved about the y-axis generates a cylindrical surface of radius x units, altitude $x^{3/2}$ units, and thus area $2\pi x(x^{3/2})$ units^2. If instead of a segment we revolve a thin rectangle of width Δx units, we obtain a cylindrical shell whose volume is approximately

$$2\pi x(x^{3/2})\,\Delta x = 2\pi x^{5/2}\,\Delta x \text{ units}^3.$$

By the usual reasoning we define

$$V = \int_1^3 2\pi x^{5/2}\,dx = 2\pi\left[\tfrac{2}{7}x^{7/2}\right]_1^3$$
$$= \tfrac{4}{7}[27\sqrt{3} - 1]\pi \text{ units}^3.$$

The scheme used in the second solution is referred to as "The Method of Cylindrical Shells."

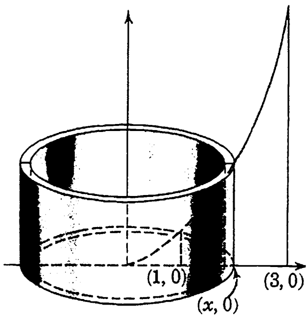


Figure 67.4

Example 3. Given the region $\{(x,y) \mid 1 \le x \le 3, 1 \le y \le x^{3/2}\}$ find the volume of the solid obtained by revolving this region about:

a. The x-axis. b. The line $y = 1$.

Solution. By visualizing "washers" in the first case and "disks" in the second case, we obtain:

$$V_a = \pi\int_1^3 [(x^{3/2})^2 - 1^2]\,dx$$
$$= \pi\int_1^3 (x^3 - 1)\,dx = 18\pi \text{ units}^3.$$
$$V_b = \pi\int_1^3 [x^{3/2} - 1]^2\,dx$$
$$= \pi\int_1^3 (x^3 - 2x^{3/2} + 1)\,dx$$
$$= \frac{\pi}{5}(114 - 36\sqrt{3}) \text{ units}^3.$$

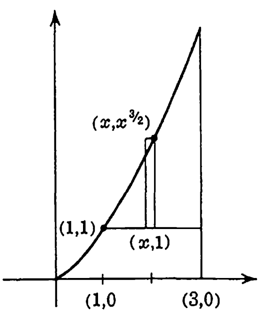


Figure 67.5

Consider a circular disk of homogeneous material perpendicular to the x-axis with a point $(x,0)$ the center of one face and let P_c be the plane perpendicular to the x-axis at a point $(c,0)$. If the disk has radius r, thickness Δx, and density ρ, then (by analogy with the plane "sheet" case) the first and second moments of the disk with respect to P_c are approximately

(3) $(x - c)\rho\pi r^2\,\Delta x$ and $(x - c)^2\rho\pi r^2\,\Delta x$.

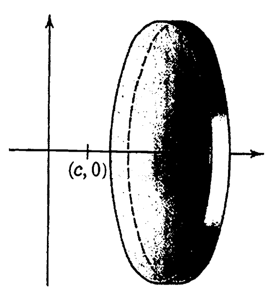


Figure 67.6

For solids of revolution the first moment, the centroid, and the second moment are defined in terms of integrals with integrands patterned upon the expressions (3). If the material is homogeneous, then by symmetry the centroid is on the axis of revolution.

Example 4. Find the centroid of a homogeneous hemispherical solid. Also, find the moment of inertia with respect to the plane of the base.

Solution. Consider the hemisphere as having been generated by revolving the region $\{(x,y) \mid 0 \le x \le r,\ 0 \le y \le \sqrt{r^2 - x^2}\}$ about the x-axis. Then $\bar{x}$ is the abscissa of the centroid if the first moment about $P_{\bar{x}}$ is zero:

$$\int_0^r (x - \bar{x})\rho\pi(\sqrt{r^2 - x^2})^2\,dx = 0.$$

Since ρ, π, and $\bar{x}$ are constants while x is the dummy variable

$$\bar{x} = \frac{\int_0^r x(r^2 - x^2)\,dx}{\int_0^r (r^2 - x^2)\,dx} = \frac{\left[\frac{1}{2}r^2x^2 - \frac{1}{4}x^4\right]_0^r}{\left[r^2x - \frac{1}{3}x^3\right]_0^r} = \tfrac{3}{8}r.$$

The centroid is on the axis of rotation three-eighths of the radius from the base.

The desired moment of inertia is

$$\int_0^r (x - 0)^2\rho\pi(\sqrt{r^2 - x^2})^2\,dx = \rho\pi\int_0^r (r^2x^2 - x^4)\,dx = \tfrac{2}{15}\rho\pi r^5.$$

PROBLEMS

1. Revolve the region $\{(x,y) \mid 0 \le x \le 4,\ 0 \le y \le x^2\}$ about the lines indicated. In each case find the volume and centroid of the resulting solid.
   a. x-axis
   b. y-axis
   c. Line $x = 4$.
   d. Line $y = 16$.
   e. Line $y = -4$.
   f. Line $x = -4$.
2. Find the volume and centroid of each of the solids described.
   a. A sphere of radius r.
   b. A right circular cone with base radius r and altitude h.
   c. A frustum of a cone with base radii r and R and altitude h.
   d. The solid obtained by revolving the region bounded by an ellipse about the major axis.
   e. A slice of a sphere of radius r with center at the origin made by planes perpendicular to the x-axis at $(a,0)$ and $(a + h,0)$ where $-r \le a < a + h \le r$.
   f. A right pyramid of altitude h and square base of side b.
   g. Obtained by revolving the region $\{(x,y) \mid 0 \le x \le \pi/2,\ 0 \le y \le \sin x\}$ about the x-axis. (Hint: $\sin^2 x \equiv \frac{1}{2}(1 - \cos 2x)$.)
3. Find the moment of inertia with respect to the plane of the base and for c, d, and e, with respect to the parallel plane through the vertex.
   a. A cylindrical column of altitude H and base radius r.
   b. A column of altitude H and square cross section of side a.
   c. A right circular cone of altitude H and base radius r.

d. A right pyramid of altitude H and square base of side a.

e. A right pyramid of altitude H whose base is a triangle having base b and altitude h.

4. A wedge is 8 in. long and each cross section is an isosceles triangle of altitude 6 in. and base 1 in. Find the centroid and find the moment of inertia both with respect to the base and to the parallel plane through the edge.

68. Improper Integrals

With $b > 1$, let $A_1(b)$ and $A_2(b)$ be given by

$$A_1(b) = \int_1^b \frac{1}{x}\,dx = \ln b \quad \text{and} \quad A_2(b) = \int_1^b \frac{1}{x^2}\,dx = 1 - \frac{1}{b}.$$

Hence, $A_1(b)$ units^2 and $A_2(b)$ units^2 are the areas of the regions

$$\{(x,y) \mid 1 \le x \le b, 0 \le y \le 1/x\} \quad \text{and} \quad \{(x,y) \mid 1 \le x \le b, 0 \le y \le 1/x^2\},$$

respectively. Given any large positive number B, then b may be chosen sufficiently large so that $A_1(b) > B$; but $A_2(b) < 1$ no matter now large b is taken, although $\lim_{b\to\infty} A_2(b) = 1$. In the following terminology we shall say

$$\int_1^\infty \frac{1}{x}\,dx \text{ does not exist, but } \int_1^\infty \frac{1}{x^2}\,dx \text{ exists and } = 1.$$

Recall that in the definition of the definite integral

$$\int_a^b f(x)\,dx,$$

a and b are numbers and f is a continuous function on the interval with end points a and b. Hence, f is bounded† on the interval. (See Appendix A4.)

The use of the definite integral symbolism is extended by defining **improper integrals**

(1) $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx \qquad \text{if the limit exists,}$$

(2) $$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx \qquad \text{if the limit exists, and}$$

(3) $$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^0 f(x)\,dx + \lim_{b\to\infty} \int_0^b f(x)\,dx \quad \text{if both limits exist.}$$

Another type of improper integral is defined if the integrand is unbounded.

† In the definition of a definite integral in Appendix A4, continuity of f is relinquished but boundedness of f is required.

If, for example, $a < c < b$ and $\lim_{x\to c} |f(x)| = \infty$, but $f(x)$ is defined for $a \leq x < c$ and for $c < x \leq b$, then:

~~A. Let~~ ϵ and η be positive numbers and compute both

$$\int_a^{c-\epsilon} f(x)\,dx \quad\text{and}\quad \int_{c+\eta}^{b} f(x)\,dx.$$

B. See if both $\lim_{\epsilon\to 0} \int_a^{c-\epsilon} f(x)\,dx$ and $\lim_{\eta\to 0} \int_{c+\eta}^{b} f(x)\,dx$ exist.

C. If both these limits exist, then by definition

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\eta\to 0} \int_{c+\eta}^{b} f(x)\,dx. \tag{4}$$

Example 1. Show that $\int_{-1}^{1} \frac{1}{x^2}\,dx$ does not exist.

Solution. One might set $\int_{-1}^{1} \frac{1}{x^2}\,dx$ equal to $-\frac{1}{x}\Big]_{-1}^{1} = -\frac{1}{1} + \frac{1}{-1} = -2$, but this should not be done since $\lim_{x\to 0} (1/x^2) = \infty$, i.e., the function f defined by $f(x) = 1/x^2$, $x \neq 0$ is unbounded on the interval I$[-1,1]$.

The proper procedure is to first compute

$$\int_{-1}^{0-\epsilon} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_{-1}^{-\epsilon} = -\frac{1}{-\epsilon} + \frac{1}{-1} = \frac{1}{\epsilon} - 1 \quad \text{for} \quad \epsilon > 0$$

and then to see that $\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon} - 1\right)$ does not exist. Hence, it is not even necessary to compute $\int_{\eta}^{1} \frac{1}{x^2}\,dx$ to know that $\int_{-1}^{1} \frac{1}{x^2}\,dx$ does not exist.

Example 2. Show that the improper integral $\int_0^3 \frac{2x}{(x^2 - 1)^{2/3}}\,dx$ exists.

Solution. The integrand is defined on I$[0,1)$ and† on I$(1,3]$ but

$$\lim_{x\to 1} \frac{2x}{(x^2 - 1)^{2/3}} = \infty$$

so we must use (4). First

$$\begin{aligned}
\lim_{\epsilon\to 0+} \int_0^{1-\epsilon} \frac{2x}{(x^2 - 1)^{2/3}}\,dx &= \lim_{\epsilon\to 0+} \Big[3(x^2 - 1)^{1/3}\Big]_0^{1-\epsilon} \\
&= \lim_{\epsilon\to 0+} 3[(-2\epsilon + \epsilon^2)^{1/3} - (-1)^{1/3}] = 3, \text{ and}
\end{aligned}$$

$$\lim_{\eta\to 0} \int_{1+\eta}^{3} \frac{2x}{(x^2 - 1)^{2/3}}\,dx = \text{etc.} = 6.$$

† Recall the definition of the half open-half closed intervals I$[a,b) = \{x \mid a \leq x < b\}$ and I$(a,b] = \{x \mid a < x \leq b\}$.

Since both limits exist, the improper integral exists and

$$\int_0^3 \frac{2x}{(x^2-1)^{2/3}}\,dx = 3 + 6 = 9.$$

Also, if f is continuous on $I(a,b]$ but $\lim\limits_{x\to a+} |f(x)| = \infty$, then by definition

$$\int_a^b f(x)\,dx \quad \text{exists and equals} \quad \lim_{\epsilon\to 0+} \int_{a+\epsilon}^b f(x)\,dx$$

if and only if this limit exists. A similar definition is given if f is continuous on $I[a,b)$ and $\lim\limits_{x\to b-} |f(x)| = \infty$.

Several kinds of "improperness" may be combined at once. For example, if f is defined and continuous at all x except possibly at c where $\lim\limits_{x\to c} |f(x)| = \infty$, then we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\left\{\lim_{\epsilon\to 0+}\int_a^{c-\epsilon} f(x)\,dx\right\} + \lim_{b\to\infty}\left\{\lim_{\eta\to 0+}\int_{c+\eta}^{b} f(x)\,dx\right\}$$

and say that $\int_{-\infty}^{\infty} f(x)\,dx$ exists if and only if all limits involved exist.

PROBLEMS

1. Examine each of the following improper integrals to see if it exists, and if it exists, find its value.

a. $\int_1^\infty \frac{dx}{x^{3/2}}$.

b. $\int_1^\infty \frac{dx}{x^{2/3}}$.

c. $\int_{-1}^1 \frac{dx}{x^{2/3}}$.

d. $\int_{-1}^1 \frac{dx}{x^{4/3}}$.

e. $\int_{-\infty}^{-1} \frac{dx}{x^{4/3}}$.

f. $\int_0^1 \frac{dx}{\sqrt{x}}$.

g. $\int_3^6 \frac{dx}{5-x}$.

h. $\int_3^6 \frac{dx}{\sqrt[3]{5-x}}$.

i. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

2. For $a > 0$, show that

a. $\int_0^a \frac{dx}{(a-x)^p}$ $\begin{cases}\text{exists if } p < 1\\ \text{does not exist if } p \geq 1.\end{cases}$

b. $\int_0^\infty \frac{dx}{(a+x)^p}$ $\begin{cases}\text{exists if } p > 1\\ \text{does not exist if } p \leq 1.\end{cases}$

3. Show that

a. $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+1}$ does not exist, but $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$.

b. $\int_{-\infty}^{\infty} \frac{dx}{4-x^2}$ does not exist, but $\int_{-\infty}^{\infty} \frac{dx}{4+x^2} = \frac{\pi}{2}$.

c. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx$ does not exist, but $\displaystyle\lim_{t\to\infty} \int_{-t}^{t} \frac{x}{x^2+1}\,dx = 0.$

d. $\displaystyle\int_{-\infty}^{\infty} \sin x\,dx$ does not exist, but $\displaystyle\lim_{t\to\infty} \int_{-t}^{t} \sin x\,dx = 0.$

e. $\displaystyle\int_{-1}^{1} \frac{dx}{x}$ does not exist, but $\displaystyle\lim_{\epsilon\to 0+} \left\{\int_{-1}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{1} \frac{dx}{x}\right\} = 0.$

f. $\displaystyle\int_{1}^{\infty} \frac{dx}{2x}$ and $\displaystyle\int_{1}^{\infty} \frac{dx}{3+2x}$ do not exist, but

$$\int_{1}^{\infty} \left(\frac{1}{2x} - \frac{1}{3+2x}\right) dx = \frac{1}{2} \ln \frac{5}{2}.$$

4. Show there is no value of t for which $\displaystyle\int_{1}^{t} \frac{1}{x^2}\,dx = 4.$

5. Show that the area of the region $\{(x,y) \mid 1 \le x \text{ and } 0 \le y \le 1/x\}$ is infinite, but upon revolving this region about the x-axis a solid of finite volume is obtained.

6. Let f be a function such that $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ exists and such that $f'(x)$ exists for $-\infty < x < \infty$. Prove the existence of the integral and the equality

$$\int_{-\infty}^{\infty} f(x) f'(x)\,dx = 0.$$

(Hint: $\displaystyle\lim_{x\to\pm\infty} f(x) = 0$ and $\displaystyle\frac{1}{2}\frac{d}{dx} f^2(x) = ?$)

CHAPTER 7

Indefinite Integration

If f is a continuous function on an interval with end points a and b, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n} \tag{1}$$

and also (by the Fundamental Theorem of Calculus)

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \quad \text{where} \quad F'(x) = f(x). \tag{2}$$

The relation (1) furnishes a means for making an analysis to discover the appropriate integral in defining extensions of some common notions, such as area, work, centroid, etc., but once the form of the integral is obtained the value of the integral is seldom determined by actual evaluation of the limit of sums in (1); instead the property (2) is used. Thus, after the analysis by means of (1) is made, the finding of a function F whose derived function F' is known is tantamount to solving many physical problems. It is this process of finding a function whose derived function is given which is systematized and formalized in this chapter under the name of "indefinite integration."

DEFINITION. *If two functions F and f are related by the equation*

$$F'(x) = f(x),$$

*then F is said to be an **indefinite integral** of f and the notation*

$$\int f(x)\,dx = F(x) + c$$

*is used in which $f(x)\,dx$ is called the **integrand** and c is any constant.*

Hence, the three statements all pose the same problem:
First: Find $F(x)$ if $F'(x) = 3\sin^2 x \cos x$.
Second: Find $F(x)$ if $dF(x) = 3\sin^2 x \cos x\,dx$.
Third: Find $\int 3\sin^2 x \cos x\,dx$.
Since $D_x \sin^3 x = 3\sin^2 x \cos x$, a solution of the problem is: "Any function F defined by $F(x) = \sin^3 x + c$" and now we write

$$\int 3\sin^2 x \cos x\,dx = \sin^3 x + c.$$

It is more realistic to think of the solution of the problem as a whole class of functions where the function F defined by $F(x) = \sin^3 x$ is in the class and any two members of the class differ at most by an additive constant. (See Theorem 39 for a proof of this "class" property.)

69. Four Basic Formulas

The derivative of a constant k times a function is k times the derivative of the function, and thus for the inverse process of indefinite integration

1. $$\int ku(x)\,dx = k\int u(x)\,dx.$$

The sum or difference of the derivatives of two functions is the sum or difference of the derivatives of the functions and hence

2. $$\int [u(x) \pm v(x)]\,dx = \int u(x)\,dx \pm \int v(x)\,dx.$$

The following formula is necessarily given in two parts:

3. $$\int u^p\,du = \begin{cases} \dfrac{u^{p+1}}{p+1} + c & \text{if } p \neq -1 \\ \ln|u| + c & \text{if } p = -1 \end{cases}$$

which are inverses of the two differential formulas

$$d\left(\frac{u^{p+1}}{p+1}\right) = u^p\,du \quad \text{if} \quad p+1 \neq 0, \quad \text{but} \quad d\ln|u| = \frac{1}{u}\,du = u^{-1}\,du.$$

Example 1. $$\int\left(5x^3 + 3x - \frac{2}{x}\right)dx = \int 5x^3\,dx + \int 3x\,dx - \int \frac{2}{x}\,dx \qquad \text{(by 2)}$$

$$= 5\int x^3\,dx + 3\int x\,dx - 2\int \frac{dx}{x} \qquad \text{(by 1)}$$

$$= 5\left(\frac{x^4}{4} + c_1\right) + 3\left(\frac{x^2}{2} + c_2\right) - 2(\ln|x| + c_3). \qquad \text{(by 3)}$$

Since c_1, c_2, and c_3 are arbitrary constants it follows that $5c_1 + 3c_2 - 2c_3$ is an arbitrary constant, and thus we write

$$\int\left(5x^3 + 3x - \frac{2}{x}\right)dx = \frac{5}{4}x^4 + \frac{3}{2}x^2 - 2\ln|x| + c.$$

Given any differentiable function F with independent variable denoted by x, then

(1) $$\int F'(x)\,dx = F(x) + c.$$

As examples, $\int D_x \sin x\,dx = \sin x + c$, $\int D_x \ln|x|\,dx = \ln|x| + c$, and if f and u are differentiable functions for which $D_x f[u(x)]$ exists, then

$$\int D_x f(u(x))\,dx = f(u(x)) + c.$$

By the chain rule for the derivative of a composition function $D_x f(u(x)) = f'(u(x))u'(x)$ and hence

$$\int f'(u(x))u'(x)\,dx = f(u(x)) + c.$$

But by the definition of differentials $du(x) = u'(x)\,dx$, and hence

$$\int f'(u(x))\,du(x) = f(u(x)) + c.$$

Now by suppressing any evidence of the independent variable x,

4. $$\int f'(u)\,du = f(u) + c,$$

which has the same form as (1) even though u is now a function instead of the independent variable. This formula is useful in a purely operational way to change an integrand into a form more easily recognizable as a derivative.

Example 2. Find $\int \sin^6 x \cos x\,dx$.

Solution. Substitute $u = \sin x$ so $du = \cos x\,dx$, and

$$\int \sin^6 x \cos x\,dx = \int u^6\,du = \tfrac{1}{7}u^7 + c = \tfrac{1}{7}\sin^7 x + c.$$

In the previous example, the substitution $u = \sin x$ transformed the integral into a form to which Formula **3** was applicable. Much time and trouble may be saved, however, by avoiding the actual introduction of $u = \sin x$ and merely writing

$$\int \sin^6 x \cos x\,dx = \int \sin^6 x\,d\sin x = \tfrac{1}{7}\sin^7 x + c.$$

Example 3. $$\int \sec^3 x \tan x\,dx = \int \sec^2 x(\sec x \tan x)\,dx$$
$$= \int \sec^2 x\,d\sec x = \tfrac{1}{3}\sec^3 x + c.$$

Example 4. Find $\int \sqrt{5x^3 + 3x}(5x^2 + 1)\,dx$.

Solution. Notice that $d(5x^3 + 3x) = (15x^2 + 3)\,dx = 3(5x^2 + 1)\,dx$, which differs from $(5x^2 + 1)\,dx$ only by the multiplicative constant 3, and thus

$$\begin{aligned}\int \sqrt{5x^3 + 3x}(5x^2 + 1)\,dx &= \int (5x^3 + 3x)^{1/2}\,\tfrac{1}{3}d(5x^3 + 3x)\\ &= \frac{1}{3}\int (5x^3 + 3x)^{1/2}\,d(5x^3 + 3x)\\ &= \frac{1}{3}\frac{(5x^3 + 3x)^{3/2}}{3/2} + c = \frac{2}{9}(5x^3 + 3x)^{3/2} + c.\end{aligned}$$

The fact that the differential of a constant is zero may be used in forcing an integrand into the exact pattern of Formula **4**.

Example 5. $\int x \sin\left(x^2 + \frac{\pi}{3}\right) dx = \int \sin\left(x^2 + \frac{\pi}{3}\right)(x\,dx)$

$$= \int \sin\left(x^2 + \frac{\pi}{3}\right)\left(\frac{1}{2}\,dx^2\right) = \frac{1}{2}\int \sin\left(x^2 + \frac{\pi}{3}\right) dx^2$$

$$= \frac{1}{2}\int \sin\left(x^2 + \frac{\pi}{3}\right) d\left(x^2 + \frac{\pi}{3}\right)$$

$$\left(\text{since } d\left(x^2 + \frac{\pi}{3}\right) = dx^2 + d\frac{\pi}{3} = dx^2\right)$$

$$= -\frac{1}{2}\cos\left(x^2 + \frac{\pi}{3}\right) + c.$$

PROBLEMS

1. Find each of the following indefinite integrals:

a. $\int (x^3 - x + 5)\,dx.$

b. $\int \left(\frac{1}{x^3} - \frac{1}{x} + \frac{1}{5}\right) dx.$

c. $\int (x^{3/2} - 3\sqrt{x})\,dx.$

d. $\int \cos^3 x \sin x\,dx.$

e. $\int x\sqrt{x^2 + 1}\,dx.$

f. $\int \frac{x + 1}{x}\,dx.$

g. $\int \frac{x}{x + 1}\,dx.$

h. $\int \frac{x}{\sqrt{x^2 + 1}}\,dx.$

i. $\int \frac{\sin x}{\cos x}\,dx.$

j. $\int \frac{\sinh x}{\cosh x}\,dx.$

k. $\int \sqrt{a + bx}\,dx.$

l. $\int x\sqrt{a + bx^2}\,dx.$

m. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx.$

n. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx.$

o. $\int \sin^2 \frac{x}{2} \cos \frac{x}{2}\,dx.$

p. $\int \frac{x^4 + 3x^3 + x^2 - 5}{x^4}\,dx.$

q. $\int \cos(\pi x + 2)\,dx.$

r. $\int \frac{x}{3x^2 + 5} \ln(3x^2 + 5)\,dx.$

s. $\int \frac{x^3}{\sqrt{a^2 + x^4}}\,dx.$

2. The independent variable need not be denoted by x. Find:

a. $\int \frac{a + b\sqrt{t}}{\sqrt{t}}\,dt.$

b. $\int \sin^4 t \cos t\,dt.$

c. $\int \sin 2t \cos t\,dt.$

d. $\int \cos 2t \sin t\,dt.$

e. $\int z\sqrt{2z}\,dz.$

f. $\int \frac{\ln |u|}{u}\,du.$

g. $\int \frac{e^{at} - e^{-at}}{e^{at} + e^{-at}}\,dt.$

h. $\int \sin\left(2s + \frac{\pi}{4}\right) \sin s\,ds.$

70. Trigonometric Integrals

Some of the following formulas are already known, but are included for ready reference.

5. $\displaystyle\int \sin u\,du = -\cos u + c.$ (since $d(-\cos u) = \sin u\,du$)

6. $\displaystyle\int \cos u\,du = \sin u + c.$

7. $\displaystyle\int \tan u\,du = \int \frac{\sin u}{\cos u}\,du = -\int \frac{d\cos u}{\cos u} = -\ln|\cos u| + c.$

8. $\displaystyle\int \cot u\,du = \ln|\sin u| + c.$

9. $\displaystyle\int \sec u\,du = \int \sec u\left(\frac{\sec u + \tan u}{\sec u + \tan u}\right)du = \int \frac{\sec^2 u + \sec u\tan u}{\sec u + \tan u}\,du$

$$= \int \frac{d(\tan u + \sec u)}{\tan u + \sec u} = \ln|\sec u + \tan u| + c.$$

10. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + c.$

Inverse to $d\sec u = \sec u\tan u\,du$ and $d\csc u = -\csc u\cot u\,du$ are

11. $$\int \sec u\tan u\,du = \sec u + c.$$

12. $$\int \csc u\cot u\,du = -\csc u + c.$$

From the trigonometric identities $\cos 2u \equiv 2\cos^2 u - 1 \equiv 1 - 2\sin^2 u$ and $\sin 2u = 2\sin u\cos u$ we have

13. $$\int \sin^2 u\,du = \int \tfrac{1}{2}(1 - \cos 2u)\,du = \tfrac{1}{2}\left\{\int du - \int \cos 2u\,du\right\}$$
$$= \tfrac{1}{2}(u - \tfrac{1}{2}\sin 2u) + c$$
$$= \tfrac{1}{2}(u - \sin u\cos u) + c.$$

14. $\displaystyle\int \cos^2 u\,du = \tfrac{1}{2}(u + \tfrac{1}{2}\sin 2u) + c = \tfrac{1}{2}(u + \sin u\cos u) + c.$

Since $d\tan u = \sec^2 u\,du$ and $d\cot u = -\csc^2 u$ we first obtain **17** and **18**, and then **15** and **16** from the identities $\tan^2 u \equiv \sec^2 u - 1$ and $\cot^2 u \equiv \csc^2 u - 1$.

15. $$\int \tan^2 u\,du = \int (\sec^2 u - 1)\,du = \tan u - u + c.$$

16. $$\int \cot^2 u\,du = -\cot u - u + c.$$

17. $$\int \sec^2 u\,du = \tan u + c.$$

18. $$\int \csc^2 u\,du = -\cot u + c.$$

PROBLEMS

1. Find each of the indefinite integrals:

a. $\int x \sin 3x^2\, dx.$

b. $\int x^2 \tan (5x^3 + 1)\, dx.$

c. $\int e^x \sec e^x\, dx.$

d. $\int e^{3x} \csc e^{3x}\, dx.$

e. $\int \frac{\sec \sqrt{x} \tan \sqrt{x}}{\sqrt{x}}\, dx.$

f. $\int (\sec x - \tan x)^2\, dx.$

g. $\int \frac{1 - \cos x}{\sin^2 x}\, dx.$

h. $\int \frac{1 + \tan x}{\cot x}\, dx.$

2. Notice that

$$\frac{1}{1 + \cos x} \equiv \frac{1}{1 + \cos x}\,\frac{1 - \cos x}{1 - \cos x} \equiv \frac{1 - \cos x}{\sin^2 x} \equiv \csc^2 x - \csc x \cot x.$$

Use this, and similar identities, to obtain

a. $\int \frac{1}{1 + \cos x}\, dx.$

b. $\int \frac{1}{1 - \cos x}\, dx.$

c. $\int \frac{\cos x}{1 - \cos x}\, dx.$

d. $\int \frac{1}{1 + \sin x}\, dx.$

e. $\int \frac{1}{1 + \csc x}\, dx.$

f. $\int \frac{\tan x}{1 + \tan x}\, dx.$

3. Use $\sin (A + B) + \sin (A - B) \equiv 2 \sin A \cos B$ and similar identities for:

a. $\int \sin 2x \cos x\, dx.$

b. $\int \cos 3x \sin 2x\, dx.$

c. $\int \cos 3x \cos 5x\, dx.$

d. $\int \sin 3x \sin 3x\, dx.$

Also, with m and n positive integers, show that:

e. $$\int_0^{2\pi} \sin mx \sin nx\, dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ \pi & \text{if} \quad m = n. \end{cases}$$

f. $$\int_0^{2\pi} \cos mx \cos nx\, dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ \pi & \text{if} \quad m = n. \end{cases}$$

4. Notice that $\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2 \cos x\, dx = \int (1 - 2 \sin^2 x + \sin^4 x)\, d \sin x = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + c$, and $\int \sec^6 x\, dx = \int \sec^4 x \sec^2 x\, dx = \int (1 + \tan^2 x)^2\, d \tan x =$ etc.
Each of the following integrals may be found by similar methods.

a. $\int \sin^3 2x\, dx.$

b. $\int \sec^4 x\, dx.$

c. $\int \sin^2 x \cos^5 x\, dx.$

d. $\int \tan^5 x \sec^3 x\, dx.$

e. $\int \frac{\cos^3 x}{\sqrt{\sin x}}\, dx.$

f. $\int \cot^3 \frac{1}{2}x\, dx.$

5. a. Check each of the following steps:

$$\int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx = \int \tan^{n-2} x(\sec^2 x - 1)\,dx,$$
$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$
$$= \int \tan^{n-2} x\,d\tan x - \int \tan^{n-2} x\,dx$$
$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx, \quad n \neq 1.$$

This is an example of a "Reduction Formula." Use the result to find

$$\int \tan^2 x\,dx, \quad \int \tan^3 x\,dx, \quad \text{and} \quad \int \tan^4 x\,dx.$$

b. Obtain a similar reduction formula for $\int \cot^n x\,dx$.

6. Derive formulas analogous to Formulas **5–18** for hyperbolic functions.

71. Algebraic Transcendental Integrals

The formulas of this section contain algebraic integrands, but transcendental results.

From the differential relation (see Sec. 46)

$$d \sin^{-1}\frac{u}{a} = \frac{1}{\sqrt{1-(u/a)^2}}\,d\left(\frac{u}{a}\right) = \frac{\sqrt{a^2}}{\sqrt{a^2-u^2}}\frac{du}{a} = \frac{du}{\sqrt{a^2-u^2}}, \quad a > 0$$

and a similar relation for $d\cos^{-1}(u/a)$ we have

19. $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c = -\cos^{-1}\frac{u}{a} + c_1, \quad a > 0.$$

Also $d\tan^{-1}\frac{u}{a} = \frac{1}{1+(u/a)^2}\,d\frac{u}{a} = \frac{a}{a^2+u^2}\,du$, and thus

20. $$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c_1.$$

By finding $d\sec^{-1}(u/a)$, $a > 0$, it will be seen that

21. $$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + c = -\frac{1}{a}\csc^{-1}\frac{u}{a} + c_1.$$

The following integral should be checked by taking a derivative

22. $$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln|u + \sqrt{u^2 \pm a^2}| + c.$$

Example 1. Find $\int \frac{x^4 - 4x^3 + 2x^2 - 5x - 1}{x^2+2}\,dx.$

Solution. By the algebraic process of dividing the numerator by the denominator:

$$\frac{x^4 - 4x^3 + 2x^2 - 5x - 1}{x^2 + 2} = x^2 - 4x + \frac{3x - 1}{x^2 + 2},$$

where the remainder has numerator $3x - 1$ of lower degree than denominator $x^2 + 2$. Thus, the desired integral is equal to

$$\begin{aligned}\int x^2\,dx - 4\int x\,dx + \int \frac{3x - 1}{x^2 + 2}\,dx &= \frac{1}{3}x^3 - 2x^2 + 3\int \frac{x}{x^2 + 2}\,dx - \int \frac{1}{x^2 + 2}\,dx \\ &= \frac{1}{3}x^3 - 2x^2 + \frac{3}{2}\int \frac{d(x^2 + 2)}{x^2 + 2} - \int \frac{dx}{x^2 + (\sqrt{2})^2} \\ &= \frac{1}{3}x^3 - 2x^2 + \frac{3}{2}\ln(x^2 + 2) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c.\end{aligned}$$

Among all the indefinite procedures of indefinite integration, the following hard and fast rule (illustrated by the above example) must unequivocally be followed whenever the situation arises.

RULE. *If an integrand is the ratio of two polynomials with numerator of equal or higher degree than the denominator, then divide the numerator by the denominator until a remainder of lower degree than the denominator is obtained.*

Notice the similarity of the integrands in **20** and **23**:

23. $$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u - a}{u + a}\right| + c,$$

which follows from the identity $\dfrac{1}{u^2 - a^2} \equiv \dfrac{1}{2a}\left[\dfrac{1}{u - a} - \dfrac{1}{u + a}\right]$ and **3**.

Example 2. Find $\displaystyle\int \frac{dx}{x^2 + 4x + 1}$, $\displaystyle\int \frac{dx}{x^2 + 4x + 4}$, and $\displaystyle\int \frac{dx}{x^2 + 4x + 6}$.

Solution. "Completing the square" should be tried on quadratic expressions:

$$\begin{aligned}\int \frac{dx}{x^2 + 4x + 1} &= \int \frac{dx}{(x^2 + 4x + 4) + 1 - 4} = \int \frac{dx}{(x + 2)^2 - 3} = \int \frac{d(x + 2)}{(x + 2)^2 - (\sqrt{3})^2} \\ &= \frac{1}{2\sqrt{3}}\ln\left|\frac{x + 2 - \sqrt{3}}{x + 2 + \sqrt{3}}\right| + c \qquad \text{by } \mathbf{23}.\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{x^2 + 4x + 4} &= \int \frac{dx}{(x + 2)^2} = \int (x + 2)^{-2}\,d(x + 2) = \frac{(x + 2)^{-1}}{-1} + c \\ &= -\frac{1}{x + 2} + c. \qquad (\text{by } \mathbf{3})\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{x^2 + 4x + 6} &= \int \frac{dx}{(x^2 + 4x + 4) + 2} = \int \frac{dx}{(x + 2)^2 + 2} = \int \frac{d(x + 2)}{(x + 2)^2 + (\sqrt{2})^2} \\ &= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x + 2}{\sqrt{2}} + c \qquad (\text{by } \mathbf{20}).\end{aligned}$$

Example 3. Find $\displaystyle\int \frac{x\,dx}{2x^2 + 3x - 4}$.

Solution. The differential of the denominator is $(4x + 3)\,dx$ which differs from the numerator only by a multiplicative and an additive constant. Hence

$$\int \frac{x\,dx}{2x^2 + 3x - 4} = \frac{1}{4}\int \frac{4x\,dx}{2x^2 + 3x - 4} = \frac{1}{4}\int \frac{(4x + 3 - 3)\,dx}{2x^2 + 3x - 4}$$

$$= \frac{1}{4}\int \frac{(4x + 3)\,dx}{2x^2 + 3x - 4} - \frac{3}{4}\int \frac{dx}{2x^2 + 3x - 4}$$

$$= \frac{1}{4}\int \frac{d(2x^2 + 3x - 4)}{2x^2 + 3x - 4} - \frac{3}{4}\cdot\frac{1}{2}\int \frac{dx}{x^2 + \frac{3}{2}x - 2}$$

$$= \frac{1}{4}\ln|2x^2 + 3x - 4| - \frac{3}{8}\int \frac{dx}{x^2 + \frac{3}{2}x + (\frac{3}{4})^2 - 2 - \frac{9}{16}}$$

$$= \frac{1}{4}\ln|2x^2 + 3x - 4| - \frac{3}{8}\int \frac{dx}{(x + \frac{3}{4})^2 - (\sqrt{41}/4)^2}$$

$$= \frac{1}{4}\ln|2x^2 + 3x - 4| - \frac{3}{8}\int \frac{d(x + \frac{3}{4})}{(x + \frac{3}{4})^2 - (\sqrt{41}/4)^2}$$

$$= \frac{1}{4}\ln|2x^2 + 3x - 4| - \frac{3}{8}\,\frac{1}{2\sqrt{41}/4}\ln\left|\frac{x + \frac{3}{4} - \sqrt{41}/4}{x + \frac{3}{4} + \sqrt{41}/4}\right| + c \qquad \text{(by 23)}$$

$$= \frac{1}{4}\ln|2x^2 + 3x - 4| - \frac{3}{4\sqrt{41}}\ln\left|\frac{4x + 3 - \sqrt{41}}{4x + 3 + \sqrt{41}}\right| + c.$$

PROBLEMS

1. Each of the formulas **19–23** is used for some integral in this group.

a. $\displaystyle\int \frac{dx}{\sqrt{4 - x^2}}$. c. $\displaystyle\int \frac{dx}{(x + 3)\sqrt{x^2 + 6x}}$. e. $\displaystyle\int \frac{x^2 + 1}{4x^2 + 4x + 2}\,dx$.

b. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 6x}}$. d. $\displaystyle\int \frac{dx}{\sqrt{4x^2 + 4x + 2}}$. f. $\displaystyle\int \frac{(x + 1)^2}{(x + 2)^2 - 9}\,dx$.

2. Further practice may be obtained by finding:

a. $\displaystyle\int \frac{(4x + 3)}{x^2 + 4x - 3}\,dx$. e. $\displaystyle\int \frac{x^2\,dx}{x^2 + 4x - 12}$.

b. $\displaystyle\int \frac{dx}{\sqrt{x^2 + 4x - 3}}$. f. $\displaystyle\int \frac{x^4 + 3x^3 + x^2 + 3x + 1}{x^2 + 2x}\,dx$.

c. $\displaystyle\int \frac{dx}{\sqrt{3 - 4x - x^2}}$. g. $\displaystyle\int \frac{x\,dx}{-3x^2 + 2x - 1}$.

d. $\displaystyle\int \frac{dx}{4x^2 + 4x + 1}$. h. $\displaystyle\int \frac{2x^2 + 3x - 4}{2x^2 + 3x - 5}\,dx$.

3. With $a \neq 0$, $X = ax^2 + bx + c$, and $q = b^2 - 4ac$ derive the formulas:

a. $$\int \frac{dx}{X} = \begin{cases} \dfrac{1}{\sqrt{q}} \ln \left| \dfrac{2ax + b - \sqrt{q}}{2ax + b + \sqrt{q}} \right| + k & \text{if } q > 0 \\[2ex] \dfrac{2}{\sqrt{-q}} \tan^{-1} \dfrac{2ax + b}{\sqrt{-q}} + k & \text{if } q < 0 \\[2ex] -\dfrac{2}{2ax + b} + k & \text{if } q = 0 \end{cases}$$

b. $$\int \frac{dx}{\sqrt{X}} = \begin{cases} \dfrac{1}{\sqrt{a}} \ln \left| \sqrt{X} + \dfrac{2ax + b}{2\sqrt{a}} \right| + k & \text{if } a > 0 \\[2ex] \dfrac{1}{\sqrt{-a}} \sin^{-1} \dfrac{-2ax - b}{\sqrt{g}} + k & \text{if } a < 0. \end{cases}$$

72. Exponential Integrals

Since $de^u = e^u\,du$ (see Sec. 53) we have

24. $$\int e^u\,du = e^u + c.$$

Because of the special character of the constant e, this formula may seem to have limited use, but **24** together with

(1) $$b^t = e^{t \ln b}, \quad 0 < b$$

permits the indefinite integration of apparently more general exponential functions.

Example. $\displaystyle\int 2^{\sin x} \cos x\,dx = \int e^{(\sin x)\ln 2}\,d \sin x$ (by (1))

$\displaystyle = \frac{1}{\ln 2} \int e^{(\sin x)\ln 2}\,d(\sin x \ln 2) = \frac{1}{\ln 2} e^{\sin x \ln 2} + c$ (by **24**)

$\displaystyle = \frac{1}{\ln 2} 2^{\sin x} + c.$

A formula often listed for $0 < b$ but $b \neq 1$ is:

$$\int \boldsymbol{b^u\,du} = \int e^{u \ln b}\,du = \frac{1}{\ln b} \int e^{u \ln b}\,d(u \ln b) = \frac{1}{\ln b} e^{u \ln b} + c$$

$$= \frac{\mathbf{1}}{\mathbf{\ln b}} \boldsymbol{b^u} + c.$$

73. Trigonometric Substitutions

Of the three trigonometric identities

$$\sin 2A \equiv 2 \sin A \cos A \tag{1}$$

$$\cos 2A \equiv 2 \cos^2 A - 1 \tag{2}$$

$$\equiv 1 - 2 \sin^2 A, \tag{3}$$

(2) and (3) relate the square of one function with the first power of another and were used to "reduce the power" in deriving **13** and **14**. The next example illustrates how these formulas may be used to "rationalize" an integrand.

Example 1. Find $\int \cos x \sqrt{1 + \cos x}\, dx, \; -\pi < x \leq \pi.$

Solution. $1 + \cos x = 1 + \cos 2\left(\frac{x}{2}\right) = 1 + \left(2 \cos^2 \frac{x}{2} - 1\right) = 2 \cos^2 \frac{x}{2}$ and

$$\sqrt{1 + \cos x} = \sqrt{2}\left|\cos \frac{x}{2}\right| = \sqrt{2} \cos \frac{x}{2}$$

(and not $-\sqrt{2} \cos (x/2)$ since whatever the quadrant of x, then $x/2$ is in the first or fourth quadrant where the cosine function is positive). Consequently, the given integral is equal to

$$\int \cos x\left(\sqrt{2} \cos \frac{x}{2}\right) dx = \sqrt{2} \int \left(1 - 2 \sin^2 \frac{x}{2}\right) \cos \frac{x}{2}\, dx \qquad \text{(from (3))}$$

$$= \sqrt{2}\left\{\int \cos \frac{x}{2}\, dx - 2 \int \sin^2 \frac{x}{2} \cos \frac{x}{2}\, dx\right\}$$

$$= \sqrt{2}\left\{2 \int \cos \frac{x}{2}\, d\left(\frac{x}{2}\right) - 2 \cdot 2 \int \sin^2 \frac{x}{2}\, d \sin \frac{x}{2}\right\}$$

$$= \sqrt{2}\left\{2 \sin \frac{x}{2} - \frac{4}{3} \sin^3 \frac{x}{2}\right\} + c.$$

Given a number $a > 0$ and a number u, then from trigonometry there is an angle of t radians such that

(i) $\quad -\pi/2 \leq t \leq \pi/2 \quad$ and $\quad u = a \sin t \quad$ if $\quad |u| \leq a,$

(ii) $\quad -\pi/2 < t < \pi/2 \quad$ and $\quad u = a \tan t,$

(iii) $\quad -\pi \leq t < -\pi/2 \quad$ or $\quad 0 \leq t < \pi/2 \quad$ and $\quad u = a \sec t \quad$ if $\quad |u| \geq a.$

These facts together with the trigonometric identities

(iv) $\quad \sin^2 t + \cos^2 t \equiv 1, \quad 1 + \tan^2 t \equiv \sec^2 t, \quad$ and $\quad \sec^2 t - 1 = \tan^2 t$

are exploited to find indefinite integrals whose integrands contain expressions which can be put into one of the forms

$$\sqrt{a^2 - u^2}, \quad \sqrt{a^2 + u^2}, \quad \text{or} \quad \sqrt{u^2 - a^2}.$$

Example 2. Derive the formula

25. $$\int \sqrt{a^2 - u^2}\, du = \frac{1}{2}\left(u\sqrt{a^2 - u^2} + a^2 \sin^{-1}\frac{u}{a}\right) + c.$$

Solution. It is, of course, understood that $a^2 - u^2 \geq 0$ so that $|u| \leq a$ which appears in (i). With t limited as in (i) we set

(4) $$\sqrt{a^2 - u^2} = \sqrt{a^2 - (a \sin t)^2} = \sqrt{a^2(1 - \sin^2 t)} = \sqrt{a^2 \cos^2 t}$$
$$= |a \cos t| = a \cos t$$

since $a > 0$ and $-\pi/2 \leq t \leq \pi/2$. Also, from $u = a \sin t$ we have $du = a \cos t\, dt$

and $$\int \sqrt{a^2 - u^2}\, du = \int a \cos t(a \cos t\, dt) = a^2 \int \cos^2 t\, dt$$
$$= \frac{a^2}{2}(t + \sin t \cos t) + c \qquad \text{(from **14**)}.$$

But the range for t in (i) is exactly the range for the inverse sine function so that $t = \sin^{-1}(u/a)$. Also, from (4), $\cos t = \sqrt{a^2 - u^2}/a$ so that

$$\int \sqrt{a^2 - u^2}\, du = \frac{a^2}{2}\left(\sin^{-1}\frac{u}{a} + \frac{u}{a}\frac{\sqrt{a^2 - u^2}}{a}\right) + c$$
$$= \frac{1}{2}\left(u\sqrt{a^2 - u^2} + a^2 \sin^{-1}\frac{u}{a}\right) + c.$$

By using (ii) and (iii) we obtain, respectively

26. $$\int \sqrt{u^2 \pm a^2}\, du = \tfrac{1}{2}(u\sqrt{u^2 \pm a^2} \pm a^2 \ln |u + \sqrt{u^2 \pm a^2}|) + c.$$

Example 3. Find $\displaystyle\int \frac{dx}{(x + 1)^2\sqrt{x^2 + 2x + 2}}$.

Solution. Since $x^2 + 2x + 2 = (x + 1)^2 + 1$, set $u = x + 1$, $du = dx$, and obtain

$$\int \frac{du}{u^2\sqrt{u^2 + 1}}.$$

Now set $u = \tan t$ with $-\pi/2 < t < \pi/2$ (see (ii)) so that $du = \sec^2 t\, dt$, $\sqrt{u^2 + 1} = \sqrt{\tan^2 t + 1} = \sqrt{\sec^2 t} = |\sec t| = \sec t$ and the integral is

$$\int \frac{\sec^2 t\, dt}{\tan^2 t \sec t} = \int \frac{\sec t}{\tan^2 t}\, dt = \int \frac{\cos t}{\sin^2 t}\, dt = \int (\sin t)^{-2}\, d \sin t$$

$$= \frac{-1}{\sin t} + c = -\frac{\cos t \sec t}{\sin t} + c = -\frac{\sec t}{\tan t} + c = -\frac{\sqrt{1 + \tan^2 t}}{\tan t} + c$$

$$= -\frac{\sqrt{1 + u^2}}{u} + c = -\frac{\sqrt{1 + (x + 1)^2}}{x + 1} + c = -\frac{\sqrt{x^2 + 2x + 2}}{x + 1} + c.$$

PROBLEMS

1. Find:

a. $\int e^{3x}\,dx.$ d. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$ g. $\int \frac{1+e^{2x}}{e^{x}}\,dx.$

b. $\int \frac{1}{e^{3x}}\,dx.$ e. $\int (2e)^{x}\,dx.$ h. $\int \sqrt{1+\sinh^2 x}\,dx.$

c. $\int 10^{x}\,dx.$ f. $\int \frac{e^{x}}{1+e^{2x}}\,dx.$ i. $\int e^{3x}e^{2x}\,dx.$

2. Whenever it occurs in the following integrals, a is a positive constant.

a. $\int \frac{dx}{(25+x^2)^{3/2}}.$ c. $\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx.$ e. $\int x^3\sqrt{1+x^2}\,dx.$

b. $\int \frac{x^2\,dx}{(25-x^2)^{3/2}}.$ d. $\int \frac{dx}{(a^2-x^2)^{3/2}}.$ f. $\int \frac{dx}{x\sqrt{x^2-9}}.$

3. For the smaller region into which the line having equation $x = 2$ divides the ellipse having equation $9x^2 + 16y^2 = 144$, find:

a. The area. b. The centroid.

c. The second moment with respect to the y-axis.

74. Integral of a Product

The formula for integration by parts

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\,dx$$

was derived in Sec. 64. Upon setting $v'(x)\,dx = dv(x)$ and $u'(x)\,dx = du(x)$ and then stripping the formula of the dummy variable x and limits of integration we obtain

27. $$\int u\,dv = uv - \int v\,du.$$

Example 1. $$\int \underbrace{x}_{u}\,\underbrace{\sin x\,dx}_{dv} = \underbrace{x}_{u}\,\underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v}\,\underbrace{dx}_{du}$$
$$= -x\cos x + \sin x + c.$$

In **27**, upon setting $u = f(x)$ and $dv = g(x)\,dx$ so that

$$du = f'(x)\,dx \quad \text{and} \quad v = \int g(x)\,dx \quad \text{we have}$$

28. $$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int \left(\int g(x)\,dx\right) f'(x)\,dx.$$

Example 2. $\int x \sin x\,dx = x\int \sin x\,dx - \int(\int \sin x\,dx)\,dx$

$$= x(-\cos x) - \int(-\cos x)\,dx = -x\cos x + \sin x + c.$$

The derivative of a sum is the sum of the derivatives, but the derivative of a product is **not** the product of the derivatives. Also, the integral of a sum is the sum of the integrals, but the integral of a product is **not** the product of the integrals as **28** vividly displays.

Sometimes more than one application of **28** is necessary.

Example 3. $\int x^2e^{2x}\,dx = x^2\int e^{2x}\,dx - \int(\int e^{2x}\,dx)D_x x^2\,dx$

$$= x^2(\tfrac{1}{2}e^{2x}) - \int \tfrac{1}{2}e^{2x}2x\,dx = \tfrac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx \quad \text{(use } \mathbf{28} \text{ again)}$$

$$= \tfrac{1}{2}x^2e^{2x} - \left\{x\int e^{2x}\,dx - \int(\int e^{2x}\,dx)D_x x\,dx\right\}$$

$$= \tfrac{1}{2}x^2e^{2x} - \left\{x\tfrac{1}{2}e^{2x} - \int \tfrac{1}{2}e^{2x}\,dx\right\}$$

$$= \tfrac{1}{2}x^2e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x} + c.$$

It may be that the integrand does not appear to be a product of two functions, but **28** may be used by considering the integrand multiplied by $g(x) \equiv 1$.

Example 4. Derive the formula

29. $$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + c.$$

Solution. $\int(\sin^{-1} x)\cdot 1\cdot dx = \sin^{-1} x\int 1\cdot dx - \int(\int dx)D_x(\sin^{-1} x)\,dx$

$$= (\sin^{-1} x)x - \int x\frac{1}{\sqrt{1-x^2}}\,dx = x\sin^{-1} x + \frac{1}{2}\int\frac{-2x}{\sqrt{1-x^2}}\,dx$$

$$= x\sin^{-1} x + \frac{1}{2}\int(1-x^2)^{-1/2}\,d(1-x^2) = x\sin^{-1} x + \frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + c.$$

After a little practice the "little integrals" of **28** may be done in the head; for example

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx \quad (\text{since } \int \sec^2 x\,dx = \tan x \text{ and } D_x x = 1)$$

$$= x\tan x + \ln|\cos x| + c.$$

It is sometimes possible to use **28** twice and have the original integral reappear as illustrated in the next example.

Example 5. Again we do not write down the little integrals to obtain:

$$\begin{aligned}\int e^{2x}\sin x\,dx &= e^{2x}(-\cos x) - \int(-\cos x)2e^{2x}\,dx\\ &= -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx \qquad \text{(now use \textbf{28} again)}\\ &= -e^{2x}\cos x + 2\left\{e^{2x}\sin x - \int(\sin x)2e^{2x}\,dx\right\}\\ &= e^{2x}(-\cos x + 2\sin x) - 4\int e^{2x}\sin x\,dx.\end{aligned}$$

The original integral reappeared on the right; transpose it, solve for it, and then add an arbitrary constant to obtain

$$\int e^{2x}\sin x\,dx = \tfrac{1}{5}e^{2x}(-\cos x + 2\sin x) + c.$$

In working out the little integrals, an arbitrary constant may be added and later specialized to zero or any other value chosen to simplify the big integral.

Example 6.

$$\begin{aligned}\int x\ln|x+3|\,dx &= \ln|x+3|\textstyle\int x\,dx - \displaystyle\int(\textstyle\int x\,dx)D_x\ln|x+3|\,dx\\ &= \ln|x+3|\left[\frac{x^2}{2}+c\right] - \int\left[\frac{x^2}{2}+c\right]\frac{1}{x+3}\,dx\end{aligned}$$

which is true regardless of the value of c. Upon choosing $c = -\frac{9}{2}$ we have

$$\begin{aligned}\int x\ln|x+3|\,dx &= \left(\frac{x^2}{2}-\frac{9}{2}\right)\ln|x+3| - \frac{1}{2}\int(x^2-9)\frac{1}{x+3}\,dx\\ &= \frac{1}{2}(x^2-9)\ln|x+3| - \frac{1}{2}\int(x-3)\,dx\\ &= \frac{1}{2}(x^2-9)\ln|x+3| - \frac{1}{2}\left(\frac{x^2}{2}-3x\right) + c.\end{aligned}$$

PROBLEMS

1. Use **28** (with $g(x) \equiv 1$) to obtain a formula for each of the inverse trigonometric functions.

2. Find each of the integrals:

$\int x\cos x\,dx.$	e. $\int x\sinh x\,dx.$	i. $\int x\{\ln|x| + \sin x\}\,dx.$
b. $\int x^2\cos x\,dx.$	f. $\int x\ln|x|\,dx.$	j. $\int x\tan^{-1}x^2\,dx.$
c. $\int x\cos x^2\,dx.$	g. $\int e^{2x}x^3\,dx.$	k. $\int \ln^2|x|\,dx.$
d. $\int x\sqrt{x-1}\,dx.$	h. $\int\sqrt{x+1}\,x^2\,dx.$	l. $\int\sqrt{x^2+1}\,x\,dx.$

3. By first using **28** and then changing the resulting integral on the right (possibly using **28** again) make the given integral reappear on the right and thus evaluate the given integral.

a. $\int e^{3x} \cos 2x \, dx.$

b. $\int \sin x \sin x \, dx.$

c. $\int \sin x \sin 3x \, dx.$

d. $\int \sec^3 x \, dx.$

4. Let $A = \int e^{ax} \cos px \, dx$ and $B = \int e^{ax} \sin px \, dx$. Use **28** to obtain

$$A = \frac{1}{p}(e^{ax} \sin px - aB) \quad \text{and} \quad B = \frac{1}{p}(-e^{ax} \cos px + aA).$$

Now solve these equations for A and B, thus evaluating the given integrals.

5. Use **28** to obtain each of the reduction formulas:

a. $\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x \, dx.$

b. $\int x^n b^{ax} \, dx = \frac{x^n b^{ax}}{a \ln b} - \frac{n}{a \ln b}\int x^{n-1} b^{ax} \, dx.$

c. $\int (\ln |ax|)^n \, dx = x(\ln |ax|)^n - n\int (\ln |ax|)^{n-1} \, dx.$

75. Integral Tables

In the preceding sections on indefinite integration, twenty-nine formulas were listed and several methods were illustrated. There are a great many more formulas and methods which could be learned, but the usual procedure is to rely quite heavily upon tables of integrals. A short table is included at the end of this book.

Before starting to use a table of integrals, the arrangement should be carefully studied. Most tables begin with a few fundamental forms and then group formulas according to different types of integrands.

If one wished to use a table at the end of the book to find

$$\int \sin^5 x \cos^5 x \, dx,$$

he could settle on Formula 108_2, which is

$$\int \sin^r u \cos^s u \, du = \frac{-\sin^{r-1} u \cos^{s+1} u}{r+s} + \frac{r-1}{r+s}\int \sin^{r-2} u \cos^s u \, du,$$

and "reduces the power of the sine." First with $r = s = 5$ (and $u = x$),

$$\int \sin^5 x \cos^5 x = -\frac{\sin^4 x \cos^6 x}{10} + \frac{4}{10}\int \sin^3 x \cos^5 x \, dx,$$

next from the same formula with $r = 3$ and $s = 5$

$$\int \sin^3 x \cos^5 x = -\frac{\sin^2 x \cos^6 x}{8} + \frac{2}{8}\int \sin x \cos^5 x \, dx.$$

The solver might then look up Formula 105, which is

$$\int \cos^p u \sin u \, du = -\frac{\cos^{p+1} u}{p+1}$$

(although he should know this formula) and obtain

$$\int \sin x \cos^5 x \, dx = -\tfrac{1}{6}\cos^6 x.$$

The additive constant is usually omitted in tables. By putting all this back together it then follows that

$$\int \sin^5 x \cos^5 x \, dx = -\frac{\sin^4 x \cos^6 x}{10} - \frac{\sin^2 x \cos^6 x}{20} - \frac{\cos^6 x}{60} + c.$$

Without the use of tables we would have

$$\begin{aligned}\int \sin^5 x \cos^5 x \, dx &= \int \sin^5 x (1 - \sin^2 x)^2 \cos x \, dx \\ &= \int (\sin^5 x - 2\sin^7 x + \sin^9 x) \, d \sin x \\ &= \tfrac{1}{6}\sin^6 x - \tfrac{1}{4}\sin^8 x + \tfrac{1}{10}\sin^{10} x + c\end{aligned}$$

and, although the answers look quite different, they can differ at most by an additive constant.

Tables of indefinite integrals are, however, not foolproof and cannot be used without considerable thought. For example, few will include formulas for finding

(1) $$\int \frac{1 + x^{1/2}}{x^{2/3}} dx \quad \text{or} \quad \int \frac{x^{2/3}}{1 + x^{1/2}} dx$$

directly. Of course the first integral is obtained as

$$\int \frac{1 + x^{1/2}}{x^{2/3}} dx = \int x^{-2/3} \, dx + \int x^{-1/6} \, dx = 3x^{1/3} + \frac{6}{5} x^{5/6} + c$$

and the second may be transformed by using the following:

(2) *If an integral involves $x^{p/q}$ and $x^{r/s}$, then substitute $x = z^c$, c the common denominator of p/q and r/s.*

Example 1. For the second integral in (1), $c = 6$ so we set $x = z^6$ with $z > 0$. Now $x^{1/2} = z^3$, $x^{2/3} = z^4$ and (it is easy to forget to substitute for dx) $dx = 6z^5\,dz$. Thus

$$\begin{aligned}\int \frac{x^{2/3}}{1 + x^{1/2}}\,dx &= \int \frac{z^4}{1 + z^3}\,6z^5\,dz = 6\int \frac{z^9}{z^3 + 1}\,dz \\ &= 6\int \left\{z^6 - z^3 + 1 - \frac{1}{z^3 + 1}\right\} dz \\ &= 6\left\{\frac{1}{7}z^7 - \frac{1}{4}z^4 + z - \int \frac{1}{z^3 + 1}\,dz\right\}\end{aligned}$$

and to obtain the last integral we may use Formula 27 of the table of integrals. Upon doing so, and then returning to x via $z = x^{1/6}$, we obtain the answer

$$6\left\{\frac{1}{7}x^{7/6} - \frac{1}{4}x^{2/3} - x^{1/6}\right\} - 2\left\{\sqrt{3}\tan^{-1}\frac{2x - 1}{\sqrt{3}} + \ln\left|\frac{1 + x}{\sqrt{1 - x + x^2}}\right|\right\} + c.$$

Example 2. Find $\displaystyle\int \frac{x}{2 + \sqrt{x + 1}}\,dx.$

Solution. Let $x + 1 = z^2$, $z > 0$ so that $x = z^2 - 1$, $dx = 2z\,dz$ and the given integral is transformed into

$$\begin{aligned}\int \frac{z^2 - 1}{2 + z}\,2z\,dz &= 2\int \frac{z^3 - z}{z + 2}\,dz = 2\int \left(z^2 - 2z + 3 - \frac{6}{z + 2}\right) dz \\ &= 2\{\tfrac{1}{3}z^3 - z^2 + 3z - 6\ln(z + 2)\} + c \\ &= 2\{\tfrac{1}{3}(x + 1)^{3/2} - (x + 1) + 3\sqrt{x + 1} - 6\ln(2 + \sqrt{x + 1})\} + c.\end{aligned}$$

PROBLEMS

1. By looking for the appropriate formula in the tables find:

a. $\displaystyle\int \frac{\sqrt{3x + 2}}{x}\,dx.$ c. $\displaystyle\int \frac{dx}{3x^2 + 2x + 1}.$ e. $\displaystyle\int \sin(\ln|x|)\,dx.$

b. $\displaystyle\int \frac{\sqrt{3x - 2}}{x}\,dx.$ d. $\displaystyle\int \frac{dx}{3x^2 + 2x - 1}.$ f. $\displaystyle\int \frac{dx}{2 + 3e^{4x}}.$

2. Use a reduction formula (if necessary) from the tables to find:

a. $\displaystyle\int \sin^4 x\,dx.$ c. $\displaystyle\int x^3\sqrt{x + 2}\,dx.$ e. $\displaystyle\int \sqrt{-3x^2 + 2x + 1}\,dx.$

b. $\displaystyle\int \cos^5 x\,dx.$ d. $\displaystyle\int \frac{\sqrt{x - x^2}}{x}\,dx.$ f. $\displaystyle\int x^2(2x^3 + 5)^{1/2}\,dx.$

3. Find:

a. $\displaystyle\int \frac{dx}{x + \sqrt{x}}.$ c. $\displaystyle\int \frac{dx}{x + 3\sqrt{x} - 2}.$ e. $\displaystyle\int \frac{x^{3/2}}{x^{1/3} - x^{1/4}}\,dx.$

b. $\displaystyle\int \frac{dx}{x + \sqrt[3]{x}}.$ d. $\displaystyle\int \frac{x^{1/3} - x^{1/4}}{x^{3/2}}\,dx.$ f. $\displaystyle\int \frac{x - 3}{x\sqrt{x + 2}}\,dx.$

76. Partial Fractions

A standard type of problem of algebra is: "Put

$$\frac{3}{x-1} - \frac{4x+5}{x^2+2x+3} \tag{1}$$

over a common denominator." The answer is

$$\frac{-x^2+5x+14}{x^3+x^2+x-3}. \tag{2}$$

In some algebra courses an expression such as (2) is given (not knowing that it came from (1)) and the problem is to show that (2) can be split into the simpler fractions of (1). A reason for this type of problem is that if we want to find

$$\int \frac{-x^2+5x+14}{x^3+x^2+x-3}\,dx$$

and can find that (2) is identical with (1), then this integral is equal to the difference

$$\int \frac{3}{x-1}\,dx - \int \frac{4x+5}{x^2+2x+3}\,dx$$

and each of these simpler integrals may be found by previously developed methods or by using a table of integrals.

There are standard methods for expressing the ratio of two polynomials as the sum of simpler fractions called **partial fractions**. The methods fall into four cases, depending upon the types of factors in the denominator. Before considering these cases we state:

The numerator **must** *be of lower degree than the denominator.* Remember the hard and fast rule stated in Sec. 71; it still applies here.

CASE I. The denominator has only first degree factors, none of which is repeated. *The given expression is equal to the sum of partial fractions each consisting of a constant divided by a factor of the denominator.*†

A method of determining the constants is illustrated below.

Example 1. Split $\dfrac{5x^2+16x-12}{x^3+x^2-6x}$ into partial fractions.

† No reason will be given here for this statement or similar ones in the other cases. Such facts are proved in an advanced algebra course.

Solution. Factor the denominator: $x(x^2 + x - 6) = x(x - 2)(x + 3)$, see that all factors are linear and none is repeated, and set

$$\frac{5x^2 + 16x - 12}{x^3 + x^2 - 6x} \equiv \frac{A}{x} + \frac{B}{x - 2} + \frac{C}{x + 3}; \quad A, B, \text{ and } C \text{ constants}$$

$$\equiv \frac{A(x - 2)(x + 3) + Bx(x + 3) + Cx(x - 2)}{x(x - 2)(x + 3)}$$

$$\equiv \frac{(A + B + C)x^2 + (A + 3B - 2C)x - 6A}{x^3 + x^2 - 6x}$$

This, being an identity, like powers of x in both numerators must have equal coefficients; that is,

$$\begin{aligned} A + B + C &= 5 \\ A + 3B - 2C &= 16 \\ -6A \qquad\qquad &= -12, \end{aligned}$$

and, by simultaneous solution, $A = 2$, $B = 4$ and $C = -1$. Hence

$$\frac{5x^2 + 16x - 12}{x^3 + x^2 - 6x} \equiv \frac{2}{x} + \frac{4}{x - 2} - \frac{1}{x + 3}.$$

Had the original request been for the integral of the expression in Example 1, we would perform the above algebra and then write

$$\int \frac{5x^2 + 16x - 12}{x^3 + x^2 - 6x}\, dx = 2\int \frac{dx}{x} + 4\int \frac{dx}{x - 2} - \int \frac{dx}{x + 3}$$

$$= 2 \ln |x| + 4 \ln |x - 2| - \ln |x + 3| + c.$$

CASE II. The denominator has only first degree factors, but some are repeated. *Corresponding to a first degree factor to the power k there are k partial fractions each a constant divided by the factor raised to one of the powers* 1, 2, 3, $\cdots$, k.

Example 2. Find $\displaystyle\int \frac{8x^5 + 20x^4 + 12x^3 + 11x^2 + 13x + 6}{x^3(x + 1)^2(x + 2)}\, dx.$

Solution. We first note by inspection that in the denominator x^6 is the highest power of x, and thus the numerator is of lower degree than the denominator. Noting the repetition of factors in the denominator, we set

$$\frac{8x^5 + 20x^4 + 12x^3 + 11x^2 + 13x + 6}{x^3(x + 1)^2(x + 2)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x + 1} + \frac{E}{(x + 1)^2} + \frac{F}{x + 2}.$$

Upon putting the right side over a common denominator, collecting powers of x in the resulting numerator, and then equating like powers of x in the two numerators (a lot of algebraic manipulation is involved) we obtain the six equations

$$\begin{aligned} A \qquad\quad + D \qquad + F &= 8 & 2A + 5B + 4C &= 11 \\ 4A + B \qquad + 3D + E + 2F &= 20 & 2B + 5C &= 13 \\ 5A + 4B + C + 2D + 2E + F &= 12 & 2C &= 6 \end{aligned}$$

and the simultaneous solution $A = 2$, $B = -1$, $C = 3$, $D = 5$, $E = -4$, $F = 1$. Having these values we can mentally substitute them, perform the simple integrations, and obtain the answer

$$2 \ln |x| + \frac{1}{x} - \frac{3}{2}\frac{1}{x^2} + 5 \ln |x + 1| + \frac{4}{x + 1} + \ln |x + 2| + c$$

CASE III. The denominator has one or more factors of second degree, none of which is repeated. *To each factor of the form $ax^2 + bx + c$ (which has no real linear factors) corresponds a partial fraction*

$$\frac{Ax + B}{ax^2 + bx + c}$$

and any linear factors are handled as in Case I *or* II.

Example 3. $\displaystyle\int \frac{2x^4 + 3x^3 + 2x^2 + 11}{(x - 1)(x^2 + 2x + 3)}\,dx.$

Solution. The numerator is not of lower degree than the denominator so we first multiply out the denominator, divide, and see that the integrand is equal to

$$2x + 1 + \frac{-x^2 + 5x + 14}{(x - 1)(x^2 + 2x + 3)}. \tag{3}$$

The first two terms are easily integrated, but for the remainder we set

$$\frac{-x^2 + 5x + 14}{(x - 1)(x^2 + 2x + 3)} \equiv \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 2x + 3}$$
$$\equiv \frac{x^2(A + B) + x(2A - B + C) + (3A - C)}{(x - 1)(x^2 + 2x + 3)}.$$

Thus $\left\{\begin{array}{l} A + B = -1 \\ 2A - B + C = 5 \\ 3A - C = 14 \end{array}\right\}$ from which $A = 3$, $B = -4$, $C = -5$ and therefore the given integral is equal to (remember the first two terms in (3))

$$x^2 + x + \int \frac{3}{x - 1}\,dx - \int \frac{4x + 5}{x^2 + 2x + 3}\,dx.$$

The first integral $= 3 \ln |x - 1|$ whereas for the second we use 74 of the tables and finally obtain the answer

$$x^2 + x + 3 \ln |x - 1| - 2 \ln (x^2 + 2x + 3) - \frac{1}{\sqrt{2}} \tan^{-1} \frac{x + 1}{\sqrt{2}} + c.$$

CASE IV. The denominator has repeated quadratic factors. *To each term of the denominator of the form $(ax^2 + bx + c)^k$ where $ax^2 + bx + c$ is a prime factor, corresponds a sum of partial fractions*

$$\frac{A_1x + B_1}{ax^2 + bx + c}, \frac{A_2x + B_2}{(ax^2 + bx + c)^2}, \ldots, \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

and any linear factors or non-repeated quadratic factors are handled as in Case I, II, *or* III.

With this information, the technique for determining the constants A_1, B_1, A_2, B_2, etc., is the same as in the previous illustrations. The only additional feature about integrating such expressions is that the integrand will have a linear numerator, but a quadratic denominator raised to some integer power p. If nothing better can be thought of, the following formulas may be used to reduce this power to 1 and then the table Formula 73 used:

$$\int \frac{dx}{X^p} = -\frac{1}{(p-1)q}\left\{\frac{2ax+b}{X^{p-1}} + 2a(2p-3)\int \frac{dx}{X^{p-1}}\right\}$$

$$\int \frac{x\,dx}{X^p} = -\frac{1}{2(p-1)}\frac{1}{X^{p-1}} - \frac{b}{2a}\int \frac{dx}{X^{p-1}}$$

where $X = ax^2 + bx + c$ and $q = b^2 - 4ac$. Table Formulas 20 and 22 are equivalent to these, but are a little harder to use for a general quadratic.

The reason we need consider no more cases (i.e., denominators with factors of third, fourth, or higher degree) is because of the following theorem of algebra:

Any polynomial with real coefficients may be factored into prime factors of first and second degree.

PROBLEMS

1. The four cases are illustrated by the following four integrals:

a. $\displaystyle\int \frac{x}{2x^2 - x - 1}\,dx.$ c. $\displaystyle\int \frac{x^2 - 6}{2x^3 + 3x^2 + 3x}\,dx.$

b. $\displaystyle\int \frac{-x^2 + 3x + 7}{(2x+3)(x+1)^2}\,dx.$ d. $\displaystyle\int \frac{9}{x(2x^2 + 3x + 3)^2}\,dx.$

2. Further practice may be gained by finding the following integrals:

a. $\displaystyle\int \frac{dx}{x^2 + 3x + 2}.$ c. $\displaystyle\int \frac{dx}{x^3 + x}.$ e. $\displaystyle\int \frac{3x^2 + 2x + 1}{x(x^2 - 1)^2}\,dx.$

b. $\displaystyle\int \frac{x^2\,dx}{x^2 + 3x + 2}.$ d. $\displaystyle\int \frac{x^4\,dx}{x^3 + 1}.$ f. $\displaystyle\int \frac{x(7x^2 + 5)}{2(x^2 - 1)(x^2 + 2)}\,dx.$

3. Find the value of the definite integrals

a. $\displaystyle\int_1^2 \frac{dx}{(x-3)(x+2)}.$ b. $\displaystyle\int_0^1 \frac{(x+1)(x-2)}{(x+3)(x+2)}\,dx.$

4. With a, b, c, and d constants with $bc \neq ad$, use partial fractions to derive the formulas

a. $\displaystyle\int \frac{dx}{(ax+b)(cx+d)} = \frac{1}{bc - ad}\ln\left|\frac{cx+d}{ax+b}\right| + k.$

b. $\displaystyle\int \frac{x\,dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad}\left\{\frac{b}{a}\ln|ax+b| - \frac{d}{c}\ln|cx+d|\right\} + k.$

c. $\displaystyle\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + k.$

5. Determine A, B, etc., so each of the following will be an identity in k. Use the resulting identity to find $\sum_{k=1}^{n} k^p$, for $p = 2, 3, 4$, and 5.

a. $k^2 \equiv Ak\{(k+1)(2k+1) - (k-1)(2k-1)\}.$

b. $k^3 \equiv Ak^2\{(k+1)^2 - (k-1)^2\}.$

c. $k^4 \equiv Ak^2\{(k+1)^2(2k+1) - (k-1)^2(2k-1)\}$
$\qquad + Bk\{(k+1)(2k+1) - (k-1)(2k-1)\}.$

d. $k^5 \equiv Ak^3\{(k+1)^3 - (k-1)^3\} + Bk^2\{(k+1)^2 - (k-1)^2\}.$

77. Resubstitution Avoided

Recall that for definite integrals

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b \quad \text{where} \quad F'(x) = f(x)$$

whereas indefinite integration is merely finding $F(x)$ when $F'(x) = f(x)$ with $f(x)$ given. Some of the methods of indefinite integration obtain $F(x)$ by making a substitution for x in terms of u, t, z, etc., and then making the inverse substitution to reinstate x in the final step. When definite integrals are involved this "return to x" is not necessary as the next example illustrates.

Example 1. Find $\displaystyle\int_0^3 \frac{dx}{(x+3)\sqrt{x+1}}$.

Solution. Substitute $x + 1 = z^2$, $z > 0$. Hence, as in indefinite integration $x = z^2 - 1$, $dx = 2z\,dz$, and $x + 3 = z^2 + 2$, but now in addition note that $z = 1$ when $x = 0$ and $z = 2$ when $x = 3$. Thus

$$\int_0^3 \frac{dx}{(x+3)\sqrt{x+1}} = \int_{z=1}^{z=2} \frac{2z\,dz}{(z^2+2)z} = 2\int_1^2 \frac{dz}{z^2+2}$$

$$= 2\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}\right]_1^2 = \frac{2}{\sqrt{2}}\left[\tan^{-1}\frac{2}{\sqrt{2}} - \tan^{-1}\frac{1}{\sqrt{2}}\right].$$

The next theorem justifies the use of new limits which depend upon the substitution made.

THEOREM 77. *Let a, b, α, and β be constants and let f and g be functions satisfying the conditions:*

(i) $g(\alpha) = a$, $g(\beta) = b$,

(ii) *f is continuous throughout its domain which includes the closed interval with end points a and b,*

(iii) *for z on the closed interval with end points α and β, $g(z)$ is in the domain of f,*

(iv) *the derived function g' is continuous on the closed interval with end points α and β.*

Then (conditions (ii)–(iv) ensure that the integrals exist)

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(z))g'(z)\,dz. \tag{1}$$

PROOF. Let F and G be the functions defined by

$$F(t) = \int_a^t f(x)\,dx \quad \text{and} \quad G(s) = \int_\alpha^s f(g(z))g'(z)\,dz,$$

for t in the domain of f, but s between α and β. Thus

$$F(b) = \int_a^b f(x)\,dx, \quad G(\beta) = \int_\alpha^\beta f(g(z))g'(z)\,dz, \tag{2}$$

$$F(a) = 0, \quad \text{and} \quad G(\alpha) = 0.$$

Moreover, by the Fundamental Theorem of Calculus,

$$F'(t) = f(t) \quad \text{and} \quad G'(s) = f(g(s))g'(s).$$

From the first of these equations we have $F'(g(s)) = f(g(s))$ and this substituted into the second gives $G'(s) = F'(g(s))g'(s)$ which may be written (by using the chain rule) as

$$D_sG(s) = D_sF(g(s)).$$

But these derivatives being equal, it follows (see Theorem 39) that

$$\begin{aligned} G(\beta) - G(\alpha) &= F(g(\beta)) - F(g(\alpha)) \\ &= F(b) - F(a) \qquad \text{(by condition (i))}. \end{aligned}$$

From this equation and all parts of (2) we therefore have

$$\int_\alpha^\beta f(g(z))g'(z)\,dz = G(\beta) = F(b) = \int_a^b f(x)\,dx$$

which is the equality (1) as we wished to prove.

Example 2. $\displaystyle\int_2^3 \frac{x^2 - 4x + 6}{x^2 - 4x + 5}\,dx = \int_2^3 \frac{(x^2 - 4x + 5) + 1}{x^2 - 4x + 5}\,dx$

$\displaystyle = \int_2^3 dx + \int_2^3 \frac{dx}{(x^2 - 4x + 4) + 1}$ $\left[\begin{array}{l}\text{Set } x - 2 = u \text{ so } dx = du \text{ and } u = 0 \\ \text{when } x = 2, \text{ but } u = 1 \text{ when } x = 3\end{array}\right]$

$\displaystyle = x\Big]_2^3 + \int_0^1 \frac{du}{u^2 + 1}$ $\left[\begin{array}{l}\text{Set } u = \tan t \text{ so } t = 0 \text{ when } u = 0, \\ \text{but } t = \pi/4 \text{ when } u = 1\end{array}\right]$

$\displaystyle = (3 - 2) + \int_0^{\pi/4} \frac{\sec^2 t}{\sec^2 t}\,dt = 1 + \int_0^{\pi/4} dt = 1 + \frac{\pi}{4}.$

The next example illustrates another trigonometric substitution and also emphasizes the danger of setting $\sqrt{a^2} = a$ unless it is known that a is positive.

Example 3. Find the value of $I = \int_{-3}^{0} \sqrt{9 - x^2}\, dx$.

Solution. We first make the substitution $x = 3 \sin t$, since then

$$\sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2 t} = 3\sqrt{\cos^2 t} = 3|\cos t|$$

and the radical is eliminated. Since $dx = 3 \cos t\, dt$, the integrand becomes $9|\cos t| \cos t\, dt$. Now what limits of integration may be used?

> When $x = -3$ then $-3 = 3 \sin t$, $\sin t = -1$ and $t = -\pi/2, 3\pi/2$, etc.,
> and when $x = 0$ then $0 = 3 \sin t$, $\sin t = 0$ and $t = 0, \pi$, etc.

As x changes from -3 to 0 it is natural to have $3 \sin t$ also change from -3 to 0 and thus for t to either increase from $-\pi/2$ to 0 or decrease from $3\pi/2$ to π. Hence, we obtain either

$$I = \int_{-\pi/2}^{0} 9|\cos t| \cos t\, dt \quad \text{or} \quad I = \int_{3\pi/2}^{\pi} 9|\cos t| \cos t\, dt.$$

If $-\pi/2 \leq t \leq 0$ then $|\cos t| = \cos t$, but if $3\pi/2 \geq t \geq \pi$ then $|\cos t| = -\cos t$. Thus, the two integrals are, respectively,

$$I = 9\int_{-\pi/2}^{0} \cos^2 t\, dt = \frac{9}{2}\int_{-\pi/2}^{0} (1 + \cos 2t)\, dt \text{ and}$$

$$I = -9\int_{3\pi/2}^{\pi} \cos^2 t\, dt = -\frac{9}{2}\int_{3\pi/2}^{\pi} (1 + \cos 2t)\, dt$$

and each of these should be worked out to find that in either case $I = 9\pi/4$.

Notice that a careless replacement of $\sqrt{\cos^2 t}$ by $\cos t$ in the second integral would have yielded the incorrect value $-9\pi/4$. As a geometric confirmation that $9\pi/4$ is correct, notice that I is the area (certainly positive) of the region $\{(x,y) \mid -3 \leq x \leq 0, 0 \leq y \leq \sqrt{9 - x^2}\}$ which is the second quadrant quarter of a circle of radius 3.

Actually, any solution of $-3 = 3 \sin t$ may be used for the lower limit of integration, and any solution of $0 = 3 \sin t$ may be used for the upper limit of integration if care is taken to divide the resulting t-interval into subintervals over which $\cos t \geq 0$ or ≤ 0. For example,

$$I = 9\int_{3\pi/2}^{0} |\cos t| \cos t\, dt = 9\int_{3\pi/2}^{\pi/2} |\cos t| \cos t\, dt + 9\int_{\pi/2}^{0} |\cos t| \cos t\, dt$$

$$= -9\int_{3\pi/2}^{\pi/2} \cos^2 t\, dt + 9\int_{\pi/2}^{0} \cos^2 t\, dt$$

and these two should be worked out to see that their sum is again $9\pi/4$.

The following set of problems reviews the applications of definite integrals of Chapter 6 and the evaluation of some of the integrals depends upon methods developed in the present chapter.

PROBLEMS

1. Find the areas of the finite region or regions bounded by the graphs of the given equations.

a. $x^2 + y^2 = 25, \quad 3y^2 = 16x.$

b. $y = \sec x, \quad y = 0, \quad x = -\pi/3, \quad x = \pi/4.$

c. $y = (x^2 + 4)^{-1}, \quad y = 0, \quad x = -3, \quad x = 5.$

d. $y = x^2(x^2 + 4), \quad y = 0, \quad x = -3, \quad x = 5.$

e. $y = \dfrac{a}{2}(e^{x/a} + e^{-x/a}), \quad y = 0, \quad x = -a, \quad x = a.$

f. $y = \dfrac{e^{2x}}{e^x - 1}, \quad y = 0, \quad x = 1, \quad x = 2.$

g. $\sqrt{x} + \sqrt{y} = \sqrt{a}$, and the coordinate axes.

h. $x^{2/3} + y^{2/3} = a^{2/3}$. (Hint: Substitute $x = a \sin^3 t$).

i. $y^2(x^2 + 9) = 25, \quad x = -4, \quad x = 4.$

j. $y^2 = x(2 - x).$

2. In each of the following, the equations of the bounding curves of a region are given. Find the volume of the solid obtained by revolving the region about the axis named after the semicolon.

a. $y = xe^x, \quad y = 0, \quad x = 10;$ x-axis.

b. $y = xe^x, \quad y = 0, \quad x = -10;$ x-axis.

c. The first arch of the sine curve, $y = 0;$ y-axis.

d. $y = \tan^2 x, \quad y = 1;$ y-axis.

e. $y = \dfrac{a}{2}(e^{x/a} + e^{-x/a}), \quad y = 0, \quad x = -a, \quad x = a;$ x-axis.

f. $y = \ln x, \quad y = 0, \quad x = 2;$ x-axis.

g. $y = \ln x, \quad y = 0, \quad x = 2;$ y-axis.

h. $y = \sqrt{(x - 2)(8 - x)}, \quad y = 0;$ y-axis.

3. Find the centroid of the region whose bounding curve is given.

a. $xy = 1, \quad y = 0, \quad x = 1, \quad x = b$ where $b > 1.$

b. $yx^2 = 1, \quad y = 0, \quad x = 1, \quad x = b$ where $b > 1.$

c. $y = \sin x, \quad y = 2, \quad x = 0, \quad x = \pi.$

d. $y = x^3 - 2x^2$, $y = 0$; find $\bar{x}$ only.

e. $y^2(x^2 + 9) = 25$, $x = 0$, $y = 0$, $x = 4$, first quadrant.

f. $x^{1/2} + y^{1/2} = a^{1/2}$, $x = 0$, $y = 0$.

g. $x^{2/3} + y^{2/3} = a^{2/3}$, $x = 0$, $y = 0$, first quadrant.

h. $y = \sqrt{a^2 - x^2}$, $y = \sqrt{b^2 - x^2}$, $a < b$, first quadrant.

i. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $\dfrac{x^2}{(2a)^2} + \dfrac{y^2}{(2b)^2} = 1$, first quadrant.

4. A tank on the surface of the ground is to be filled from a well in which water always stands 20 ft below ground. Find the work done in filling the tank if:

a. The tank is a right circular cylinder of radius 3 ft and altitude 10 ft and is:
(i) Lying horizontal. (ii) Standing upright.

b. The tank has altitude 10 ft and each cross section is an ellipse with semi-axes 5 ft and 3 ft and is:
(i) Lying horizontal with major axis horizontal.
(ii) Standing upright.

5. Find the moment of inertia of the given region with respect to the given line.

a. A circle of radius r; with respect to:
(i) A diameter. (ii) A line tangent to the circle.

b. An ellipse with major and minor axes $2a$ and $2b$; with respect to:
(i) The major axis. (ii) The minor axis.

c. A semicircle; with respect to the line parallel to the bounding diameter and through the centroid.

6. Find the force on an ellipse marked on the vertical face of a dam if the ellipse has its 10-ft major axis horizontal, its 6-ft minor axis vertical, and its center 15 ft below the surface of the water.

7. The cross section of a viaduct is a rectangle surmounted by a semicircle; the rectangle having horizontal base 5 ft and altitude 3 ft. If the viaduct is full of water find the force:

a. On a square foot of the floor.

b. On a vertical wall across the viaduct.

8. Use Table formula 96 to establish what are known as Wallis' formulas.

a. $\displaystyle\int_0^{\pi/2} \sin^{2n} x\, dx = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-3}{2n-2}\cdot\frac{2n-1}{2n}.$

b. $\displaystyle\int_0^{\pi/2} \sin^{2n+1} dx = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n-2}{2n-1}\cdot\frac{2n}{2n+1}.$

CHAPTER 8

Vectors

To an ancient Roman, even a well-educated one, the multiplication of MMDCXLVIII by LXIX was a long and arduous task in contrast to the ease with which we multiply 2648 by 69. The ancient Romans were able, given time, to perform any arithmetic computation, so to them their method of representing numbers was adequate and the learning of Hindu-Arabic numerals might well have seemed superfluous had some visionary advocated changing to this new-fangled arithmetic.

In much the same way vectors may seem superfluous since any quantity represented by a vector could also be described without vectors. It has been found, however, that many geometric and physical facts can be grasped, understood, and represented much more easily via vectors than without them. And isn't a goal of mathematics to make things easy?

Associated with vectors, as geometric directed line segments, there is an algebra involving addition, subtraction, and multiplication (two kinds), but not division. Addition and subtraction of vectors seem fairly natural to most minds, but the multiplications of vectors appear to be more elusive concepts. It would be well, therefore, to accept the definition of the dot product of vectors (Sec. 79) with full confidence that this notion passed the acid test of eventual usefulness.

Even though this chapter has the title "Vectors," the other topics are important in their own rights. Since vectors in the plane are more easily visualized than in three dimensions, this chapter restricts vectors to the plane in order to cultivate thinking in terms of vectors as preparation for their real power in space considerations of the next chapter.

78. Definitions

Two line segments are said to have the same direction if the lines containing the segments are parallel or coincide. A segment is said to be **sensed** if one end point is designated as the **initial** end with the other called the **terminal** end. With A the initial end and B the terminal end, the sensed segment is denoted by $\overrightarrow{AB}$. Hence, $\overrightarrow{BA}$ is the oppositely sensed segment. In terms of a pre-assigned unit, the (positive) length of the segment is called the

modulus or **absolute value** of either $\overrightarrow{AB}$ or of $\overrightarrow{BA}$ and is denoted by $|\overrightarrow{AB}|$ or $|\overrightarrow{BA}|$. The terminology is extended to so-called **point-segments** of **zero modulus**, but no direction or sense is assigned although the notation is extended to $\overrightarrow{AA}$ as designating the "point-segment" consisting of the single point A.

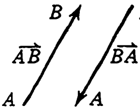


Figure 78.1

Sensed segments and point-segments are classified together as **vectors** with the further understanding that $\overrightarrow{A_1B_1}$ and $\overrightarrow{A_2B_2}$ are the same vector $\vec{v}$ if:

(i) Both $\overrightarrow{A_1B_1}$ and $\overrightarrow{A_2B_2}$ are point-segments, or else
(ii) Both are sensed segments with the same direction, sense, and modulus.

Thus, a sensed segment or a point-segment may be moved by parallel translation and in either position represents the same vector.

The point-vector is called the **zero vector** and is denoted by $\vec{0}$.†

The **product** of a vector $\vec{v} = \overrightarrow{AB}$ and a scalar (i.e., a number) c is denoted either by $c\vec{v}$ or $\vec{v}c$ and is defined by:

(i) $c\vec{v} = \overrightarrow{AA}$ if either $c = 0$ or $\vec{v} = \vec{0}$, but otherwise
(ii) $c\vec{v}$ is the vector of modulus $|c||\vec{v}|$ having the same direction as $\vec{v}$, and with sense the same as $\vec{v}$ if $c > 0$ but opposite to $\vec{v}$ if $c < 0$.

In particular $1\vec{v} = \vec{v}$ and $(-1)\vec{v} = \overrightarrow{BA}$. We define $-\vec{v}$ by setting

$$-\vec{v} = (-1)\vec{v} \quad \text{so that} \quad -\vec{v} = \overrightarrow{BA}.$$

Given two vectors $\vec{v}_1$ and $\vec{v}_2$, the **resultant** or **sum** $\vec{v}_1 + \vec{v}_2$ is obtained by taking any point A and drawing $\vec{v}_1 = \overrightarrow{AB}$, then from B drawing $\vec{v}_2 = \overrightarrow{BC}$, and setting

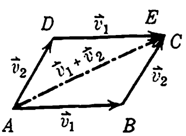


Figure 78.2

$$\vec{v}_1 + \vec{v}_2 = \overrightarrow{AC}.$$

By starting at the same point A and drawing $\vec{v}_2 = \overrightarrow{AD}$ and then $\vec{v}_1 = \overrightarrow{DE}$, notice that $\vec{v}_2 + \vec{v}_1 = \overrightarrow{AE}$ but also $E = C$ so that

1. $$\vec{v}_2 + \vec{v}_1 = \vec{v}_1 + \vec{v}_2.$$

This states that *the commutative law of addition holds for vectors.* In case $\vec{v}_1$ and $\vec{v}_2$ do not have the same direction, or in case neither $\vec{v}_1$ nor $\vec{v}_2$ is $\vec{0}$, then

† On the printed page the usual symbol for a vector is a boldface letter such as $\mathbf{v}$, but in writing on paper or the blackboard the symbolism is a half-headed arrow above a letter. In this book we shall use the script form $\vec{v}$ rather than the print form $\mathbf{v}$.

the resultant is the sensed diagonal $\overrightarrow{AC}$ of the parallelogram having coterminal sides $\vec{v}_1 = \overrightarrow{AB}$ and $\vec{v}_2 = \overrightarrow{AD}$. The **difference** $\vec{v}_1 - \vec{v}_2$ is defined by setting

2. $$\vec{v}_1 - \vec{v}_2 = \vec{v}_1 + (-\vec{v}_2).$$

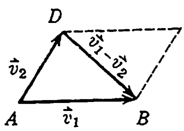


Figure 78.3

Except in the degenerate cases, the parallelogram mentioned above has the other diagonal $\overrightarrow{DB}$, sensed as shown, equal to $\vec{v}_1 - \vec{v}_2$.

Given three vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, draw $\vec{v}_1 = \overrightarrow{AB}$, then $\vec{v}_2 = \overrightarrow{BC}$, and finally $v_3 = \overrightarrow{CD}$. Now $(\vec{v}_1 + \vec{v}_2) = \overrightarrow{AC}$ and then

$$(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}.$$

But also $(\vec{v}_2 + \vec{v}_3) = \overrightarrow{BD}$ and now $\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = \overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$. Thus

3. $$(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$$

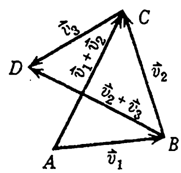


Figure 78.4

which states that *the associative law of addition holds for vectors*. Since the grouping of the addends is immaterial the resultant of three vectors is written without parentheses as

$$\vec{v}_1 + \vec{v}_2 + \vec{v}_3.$$

In addition to 1, 2, and 3 above, properties of vectors which follow directly from the definition are:

4. $-(-\vec{v}) = \vec{v}$.
5. $0\vec{v} = \vec{0}$.
6. $c\vec{0} = \vec{0}$.
7. $\vec{v} - \vec{v} = -\vec{v} + \vec{v} = \vec{0}$.
8. $\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$.
9. $|\vec{v}_1 + \vec{v}_2| \le |\vec{v}_1| + |\vec{v}_2|$, $|\vec{v}_1 - \vec{v}_2| \ge \big|\, |\vec{v}_1| - |\vec{v}_2| \,\big|$.
10. $c(\vec{v}_1 + \vec{v}_2) = c\vec{v}_1 + c\vec{v}_2$, and $(c_1 + c_2)\vec{v} = c_1\vec{v} + c_2\vec{v}$.
11. $a(b\vec{v}) = (ab)\vec{v}$ and $(a + b)\vec{v} = a\vec{v} + b\vec{v}$.
12. If $\vec{v}_1 = \vec{v}_2$, then $\vec{v}_1 \pm \vec{v}_3 = \vec{v}_2 \pm \vec{v}_3$
13. If $a\vec{v} = b\vec{v}$ and $\vec{v} \ne \vec{0}$, then $a = b$.
14. If $a\vec{v}_1 = a\vec{v}_2$ and $a \ne 0$, then $\vec{v}_1 = \vec{v}_2$.

THEOREM 78. *If $\vec{v}_1$ and $\vec{v}_2$ do not have the same direction, if neither $\vec{v}_1$ nor $\vec{v}_2$ is the zero-vector $\vec{0}$, and if*

$$a\vec{v}_1 = b\vec{v}_2, \quad \text{then} \quad a = b = 0.$$

PROOF. Under the given conditions if we assumed in addition that $a \ne 0$, then we would have

$$\frac{1}{a}(a\vec{v}_1) = \frac{1}{a}(b\vec{v}_2), \quad \vec{v}_1 = \frac{b}{a}\vec{v}_2$$

which could hold only if $\vec{v}_1$ and $\vec{v}_2$ had the same direction or were both equal to $\vec{0}$. Consequently $a = 0$. Now $b\vec{v}_2 = 0\vec{v}_1 = \vec{0}$ and since $\vec{v}_2 \neq \vec{0}$ we have $b = 0$.

COROLLARY 78. *If $\vec{v}_1$ and $\vec{v}_2$ do not have the same direction, if $\vec{v}_1 \neq \vec{0}$ and $\vec{v}_2 \neq \vec{0}$, and if*

$$a\vec{v}_1 + b\vec{v}_2 = c\vec{v}_1 + d\vec{v}_2, \quad \text{then} \quad a = c \quad \text{and} \quad b = d.$$

PROOF. Under the stated conditions $(a - c)\vec{v}_1 = (d - b)\vec{v}_2$ so $a - c = 0$ and $d - b = 0$.

Example. Show that the medians of a triangle intersect in a point which is two-thirds of the way from any vertex to the mid-point of the opposite side.

Solution. Let A, B, C be the vertices of a triangle. Hence

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC},$$

and the median vectors from A and B are, respectively

$$\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC} \quad \text{and} \quad \overrightarrow{BA} + \tfrac{1}{2}\overrightarrow{AC} = -\overrightarrow{AB} + \tfrac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = -\tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}.$$

Let Q be the point of intersection of these two medians and let a and b be the scalars such that

$$\overrightarrow{AQ} = a(\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}) \quad \text{and} \quad \overrightarrow{BQ} = b(-\tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}).$$

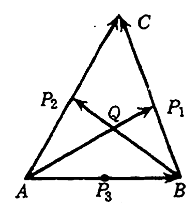


Figure 78.5

In the triangle A, B, Q, we have $\overrightarrow{AQ} = \overrightarrow{AB} + \overrightarrow{BQ}$ so that

$$a(\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}) = \overrightarrow{AB} + b(-\tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}).$$

The coefficients of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ on both sides of this equation are equal (by the corollary) and hence

$$a = 1 - \frac{b}{2} \quad \text{and} \quad \frac{a}{2} = \frac{b}{2} \quad \text{so that} \quad a = b = \frac{2}{3}.$$

Thus, the medians from A and B intersect two-thirds of the way to the mid-points of the opposite sides. The following computations:

$$\begin{aligned} \overrightarrow{CQ} &= \overrightarrow{CA} + \overrightarrow{AQ} = (\overrightarrow{CB} + \overrightarrow{BA}) + \tfrac{2}{3}(\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}) \\ &= \overrightarrow{CB} + \overrightarrow{BA} - \tfrac{2}{3}\overrightarrow{BA} - \tfrac{1}{3}\overrightarrow{CB} = \tfrac{2}{3}\overrightarrow{CB} + \tfrac{1}{3}\overrightarrow{BA} \\ &= \tfrac{2}{3}(\overrightarrow{CB} + \tfrac{1}{2}\overrightarrow{BA}) \end{aligned}$$

show that Q is also two-thirds of the way from C to the mid-point of AB.

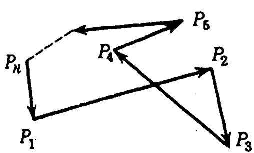


Figure 78.6

Notice that if $P_1, P_2, \cdots, P_n$ are any n points in the plane, then

(1) $\overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \cdots + \overrightarrow{P_{n-1}P_n} + \overrightarrow{P_nP_1} = \vec{0}$,

and thus the negative of any one of these vectors is equal to the sum of all others. The relation (1) states that *the vector sum of the successively sensed sides of a closed polygon is the zero vector.*

PROBLEMS

1. Let A, B, C be the vertices of a triangle and let M be the intersection of the medians.
   a. Show that $\overrightarrow{AM} + \overrightarrow{BM} + \overrightarrow{CM} = \vec{0}$.
   b. With O any point show that $3\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$.
   c. With A', B', C' a second triangle and M' the intersection of its medians show that $3\overrightarrow{MM'} = \overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}$.
   d. Let D, E, and Q be the points such that $\overrightarrow{AD} = \frac{4}{5}\overrightarrow{AB}$, $\overrightarrow{BE} = \frac{1}{3}\overrightarrow{BC}$ and Q the intersection of $\overrightarrow{AE}$ and $\overrightarrow{CD}$. Determine scalars a and b such that $\overrightarrow{AQ} = a\overrightarrow{AE}$ and $\overrightarrow{CQ} = b\overrightarrow{CD}$.
2. Let A, B, C, D be the vertices of a parallelogram so that $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.
   a. Show that the diagonals bisect each other.
   b. Let P and Q be such that $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB}$ and Q is the intersection of $\overrightarrow{AC}$ and $\overrightarrow{PD}$. Determine scalars a and b such that $\overrightarrow{AQ} = a\overrightarrow{AC}$ and $\overrightarrow{PQ} = b\overrightarrow{PD}$.
   c. With O any point and M the intersection of the diagonals show that
   $$4\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}.$$
3. Prove the converse of 2a; that is, show that if A, B, C, D is a quadrilateral whose diagonals bisect each other, then the quadrilateral is a parallelogram. A, B, C, and D are the only points where sides intersect.
4. Let $\vec{v}_1 = \overrightarrow{OA}$, $\vec{v}_2 = \overrightarrow{OB}$, and $\vec{v}_3 = \overrightarrow{OC}$ be non-zero vectors, no two having the same direction.
   a. If $\vec{v}_1 + 2\vec{v}_2 - 3\vec{v}_3 = \vec{0}$ show that A, B, and C lie on a straight line.
   b. Prove that if there are numbers a, b, and c not all zero such that
   $$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0} \quad \text{and} \quad a + b + c = 0$$
   then A, B, and C lie on a straight line. (Hint: A, B, and C lie on a line if there is a number k such that $\vec{v}_1 - \vec{v}_2 = k(\vec{v}_3 - \vec{v}_2)$.)

79. Scalar Product

The **projection** of a point A on a line l is the point A' which is the foot of the perpendicular from A to l. The **vector projection** of a vector $\vec{v} = \overrightarrow{AB}$ on a line l is the vector $\vec{v}_l = \overrightarrow{A'B'}$ where A' and B' are the projections of A and B on l. Hence, if $\vec{v} = \vec{0}$ or if $\vec{v}$ is perpendicular to l, then $\vec{v}_l = \vec{0}$. With $\vec{v} \neq \vec{0}$ and θ any angle between l and the direction of $\vec{v}$, then

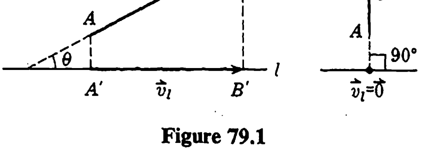


Figure 79.1

(1) $\quad |\vec{v}_l| = |\vec{v}|\,|\cos\theta| = |\vec{v}\cos\theta|.$

Also, given two vectors $\vec{v}_1 = \overrightarrow{AB}$ and $\vec{v}_2 = \overrightarrow{BC}$, then $(\vec{v}_1)_l = \overrightarrow{A'B'}$, $(\vec{v}_2)_l = \overrightarrow{B'C'}$ so that $(\vec{v}_1)_l + (\vec{v}_2)_l = \overrightarrow{A'B'} + \overrightarrow{B'C'} = \overrightarrow{A'C'}$. But $(\vec{v}_1 + \vec{v}_2)_l = (\overrightarrow{AB} + \overrightarrow{BC})_l = (\overrightarrow{AC})_l = \overrightarrow{A'C'}$. Thus

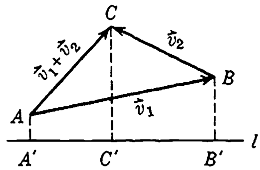


Figure 79.2

(2) $$(\vec{v}_1 + \vec{v}_2)_l = (\vec{v}_1)_l + (\vec{v}_2)_l;$$

that is, *vector projection and vector addition are distributive*. Also, if $\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n = \vec{0}$, then $(\vec{v}_1)_l + (\vec{v}_2)_l + \cdots + (\vec{v}_n)_l = \vec{0}$.

In Sec. 78 the sum and difference of two vectors, and the product of a vector by a scalar were defined. We now define a product of a vector by a vector.

DEFINITION 79. *Given two vectors $\vec{u}$ and $\vec{v}$, the* **scalar product** *or* **dot product** *or* **inner product** *$\vec{u} \cdot \vec{v}$ is defined*† *by*

$$\vec{u} \cdot \vec{v} = \begin{cases} 0 & \text{if } \vec{u} = \vec{0} \text{ or } \vec{v} = \vec{0} \\ |\vec{u}|\,|\vec{v}| \cos\theta & \text{if } \vec{u} \neq \vec{0} \text{ and } \vec{v} \neq \vec{0} \end{cases}$$

where $-180° < \theta \leq 180°$ is the angle through which $\vec{u}$ could be turned to have the direction and sense of $\vec{v}$.

In case $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, it follows that $-\theta$ is the angle through which $\vec{v}$ could be turned to have the direction and sense of $\vec{u}$, and thus

(3) $$\vec{v} \cdot \vec{u} = |\vec{v}|\,|\vec{u}| \cos(-\theta) = |\vec{u}|\,|\vec{v}| \cos\theta = \vec{u} \cdot \vec{v}$$

and in case $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ then $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = 0$ so that *the scalar product is commutative.*

Even though $\vec{u}$ and $\vec{v}$ are vectors it should be noted that $\vec{u} \cdot \vec{v}$ is a scalar; that is, $\vec{u} \cdot \vec{v}$ is merely a number. Also, for $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, then

(4) $$\vec{u} \cdot \vec{v} \text{ is } \begin{cases} >0 & \text{if } |\theta| < 90° \\ =0 & \text{if } \theta = 90° \text{ or } \theta = -90° \\ <0 & \text{if } 90° < |\theta| \leq 180°. \end{cases}$$

Hence, for $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$, then $\vec{u} \cdot \vec{v} = 0$ if and only if $\vec{u}$ and $\vec{v}$ are perpendicular.

Since $\vec{v}$ and $-\vec{v}$ have the same direction but opposite senses, it follows that

(5) $$\vec{u} \cdot (-\vec{v}) = -\vec{u} \cdot \vec{v}.$$

If $\vec{u} = \vec{0}$ then $\vec{u} \cdot \vec{u} = 0$ and if $\vec{u} \neq \vec{0}$ then $\vec{u} \cdot \vec{u} = |\vec{u}||\vec{u}| \cos 0° = |\vec{u}|^2$ and hence in either case

(6) $$\vec{u} \cdot \vec{u} = |\vec{u}|^2.$$

† Later (see Sec. 98), a different product $\vec{u} \times \vec{v}$, called the vector or cross or outer product will be defined.

The **square of a vector** is defined to be the scalar product of the vector by itself so that

$$\vec{u}^2 = \vec{u} \cdot \vec{u} \quad \text{and hence} \quad \vec{u}^2 = |\vec{u}|^2,$$

but no other power of a vector is defined.

A vector is said to be a **unit vector** if its magnitude is 1. Thus, for any vector $\vec{u} \neq \vec{0}$, the vector

$$\frac{1}{|\vec{u}|}\,\vec{u}$$

is a unit vector having the same direction and sense as $\vec{u}$.

The following theorem states a relation between vector projections and scalar products.

THEOREM 79. *Given a vector $\vec{u} \neq \vec{0}$ and any vector $\vec{v}$, then for l the line containing $\vec{u}$*

$$\vec{v}_l = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2}\,\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\,\vec{u}. \tag{7}$$

PROOF. For $\vec{v} = \vec{0}$, then $\vec{v}_l = \vec{0}$ and $\vec{u} \cdot \vec{v} = 0$ so both sides of (7) are equal to $\vec{0}$. Consider then $\vec{v} \neq \vec{0}$ and note for the magnitudes of $\vec{v}_l$ that (see (1))

$$|\vec{v}_l| = |\vec{v}|\,|\cos\theta|.$$

Moreover

$$|\vec{v}|\cos\theta \quad \text{is} \quad \begin{cases} >0 & \text{if} \quad |\theta| < 90^\circ \\ =0 & \text{if} \quad \theta = 90^\circ \quad \text{or} \quad -90^\circ \\ <0 & \text{if} \quad 90^\circ < |\theta| \leq 180^\circ. \end{cases}$$

Also, $\vec{v}_l$ and $\vec{u}$ are both on the line l if $\theta \neq 90^\circ$ or -90° and $\vec{v}_l = \vec{0}$ if $\theta = 90^\circ$ or -90°. Thus, $\vec{v}_l$ is the product of the scalar $|\vec{v}|\cos\theta$ and the unit vector $\frac{1}{|\vec{u}|}\,\vec{u}$ so that

$$\vec{v}_l = (|\vec{v}|\cos\theta)\frac{1}{|\vec{u}|}\,\vec{u} = \frac{|\vec{u}|}{|\vec{u}|}(|\vec{v}|\cos\theta)\frac{1}{|\vec{u}|}\,\vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2}\,\vec{u}.$$

Thus, the first equation in (7) holds and the second follows since $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.

COROLLARY 79. *Given any vectors $\vec{u}$, $\vec{v}_1$, and $\vec{v}_2$, then*

$$\vec{u} \cdot (\vec{v}_1 + \vec{v}_2) = \vec{u} \cdot \vec{v}_1 + \vec{u} \cdot \vec{v}_2. \tag{8}$$

PROOF. If $\vec{u} = \vec{0}$ then both sides of (8) are equal to 0. Consider then $\vec{u} \neq \vec{0}$ and let l be the line containing $\vec{u}$. Hence

$$(\vec{v}_1 + \vec{v}_2)_l = (\vec{v}_1)_l + (\vec{v}_2)_l \qquad \text{(from (2))}$$

$$\frac{\vec{u} \cdot (\vec{v}_1 + \vec{v}_2)}{|\vec{u}|^2}\,\vec{u} = \frac{\vec{u} \cdot \vec{v}_1}{|\vec{u}|^2}\,\vec{u} + \frac{\vec{u} \cdot \vec{v}_2}{|\vec{u}|^2}\,\vec{u} \quad \text{and}$$

$$[\vec{u} \cdot (\vec{v}_1 + \vec{v}_2)]\vec{u} = [\vec{u} \cdot \vec{v}_1 + \vec{u} \cdot \vec{v}_2]\vec{u}$$

since $|\vec{u}|$, $\vec{u} \cdot \vec{v}_1$, $\vec{u} \cdot \vec{v}_2$, and $\vec{u} \cdot (\vec{v}_1 + \vec{v}_2)$ are all scalars. Hence, the numbers which are the coefficients of the vector $\vec{u}$ on both sides are equal (see 13 of Sec. 78), and thus

$$\vec{u} \cdot (\vec{v}_1 + \vec{v}_2) = \vec{u} \cdot \vec{v}_1 + \vec{u} \cdot \vec{v}_2.$$

The equation (8) is therefore established in all cases.

Equation (8) states that *scalar multiplication and vector addition are distributive*. Since scalar multiplication is commutative (see (3)) it follows that

$$(\vec{v}_1 + \vec{v}_2) \cdot \vec{u} = \vec{u} \cdot (\vec{v}_1 + \vec{v}_2) = \vec{u} \cdot \vec{v}_1 + \vec{u} \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{u} + \vec{v}_2 \cdot \vec{u}.$$

Also $\vec{u} \cdot (\vec{v}_1 - \vec{v}_2) = \vec{u} \cdot [\vec{v}_1 + (-\vec{v}_2)] = \vec{u} \cdot \vec{v}_1 + \vec{u} \cdot (-\vec{v}_2) = \vec{u} \cdot \vec{v}_1 - \vec{u} \cdot \vec{v}_2$ by (5); that is,

$$\vec{u} \cdot (\vec{v}_1 - \vec{v}_2) = \vec{u} \cdot \vec{v}_1 - \vec{u} \cdot \vec{v}_2. \tag{9}$$

Thus, many of the usual algebraic rules hold for scalar multiplication. Notice, however, that even if

$$\vec{u} \cdot \vec{v}_1 = \vec{u} \cdot \vec{v}_2 \quad \text{and} \quad \vec{u} \neq \vec{0},$$

we cannot conclude that $\vec{v}_1$ and $\vec{v}_2$ are equal, but can state that $\vec{u} \cdot \vec{v}_1 - \vec{u} \cdot \vec{v}_2 = 0$ so that (from (9)) $\vec{u} \cdot (\vec{v}_1 - \vec{v}_2) = 0$, and hence that either $\vec{v}_1 - \vec{v}_2 = \vec{0}$ (and then $\vec{v}_1 = \vec{v}_2$), or else that $\vec{v}_1 - \vec{v}_2$ and $\vec{u}$ are perpendicular.

Example. Prove for a triangle ABC that the lines through the vertices perpendicular to the opposite sides all meet in a point.

Solution. In case ABC is a right triangle all perpendiculars meet at the vertex of the right triangle. Thus, consider the triangle shown. The lines from A and B perpendicular to the opposite sides are not parallel and hence meet at point O. Let $\overrightarrow{OA} = \vec{v}_1$, $\overrightarrow{OB} = \vec{v}_2$, and $\overrightarrow{OC} = \vec{v}_3$. Then $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\vec{v}_1 + \vec{v}_2$, $\overrightarrow{AC} = -\vec{v}_1 + \vec{v}_3$, and $\overrightarrow{BC} = -\vec{v}_2 + \vec{v}_3$. Since $\overrightarrow{OB}$ and $\overrightarrow{AC}$ are perpendicular

$$\vec{v}_2 \cdot (-\vec{v}_1 + \vec{v}_3) = 0 \quad \text{so that} \quad \vec{v}_2 \cdot \vec{v}_1 = \vec{v}_2 \cdot \vec{v}_3.$$

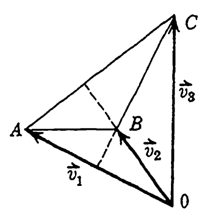


Figure 79.3

Since $\overrightarrow{OA}$ and $\overrightarrow{BC}$ are perpendicular $\vec{v}_1 \cdot (-\vec{v}_2 + \vec{v}_3) = 0$ so that $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3$. Hence $\vec{v}_2 \cdot \vec{v}_3 = \vec{v}_1 \cdot \vec{v}_3$ and therefore $\vec{v}_3 \cdot (\vec{v}_2 - \vec{v}_1) = 0$. Since $\vec{v}_2 - \vec{v}_1 \neq \vec{0}$ ($\vec{v}_2$ and $\vec{v}_1$ are not even parallel) it follows that $\vec{v}_3$ and $\vec{v}_2 - \vec{v}_1$ are perpendicular; i.e., $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{AB}$. Hence, all three perpendiculars meet at O.

80. Scalar and Vector Quantities

Scalar quantities require only a number for complete specification in terms of a pre-assigned unit. Examples are length, area, volume, density, temperature, work, amount of heat or light, and electric charge and potential.

Vector quantities are specified not only by magnitude, but by a direction and sense as well. Examples are displacement, velocity, momentum, force, and electric and magnetic intensities. Arithmetic and ordinary algebra suffice for the discussion of scalar quantities, but vector algebra is required as well to handle problems involving vector quantities.

In the previous discussion of moments and centroids (Sec. 65) the fact that vectors were involved was somewhat hidden. Also, the fact that the centroid of a system depends only on the system itself, and not on the coordinates, was tacitly assumed. The purpose of the next example is to show that the centroid of a system of particles depends only upon the system itself.

Example. Let O, O', P_1, P_2, $\cdots$, P_n be points, let m_1, m_2, $\cdots$, m_n be numbers such that $m_1 + m_2 + \cdots + m_n \neq 0$ and let G and G' be the points defined by

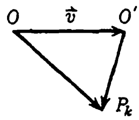


Figure 80.1

$$\overrightarrow{OG} = \frac{\Sigma\, m_k \overrightarrow{OP_k}}{\Sigma\, m_k} \quad \text{and} \quad \overrightarrow{O'G'} = \frac{\Sigma\, m_k \overrightarrow{O'P_k}}{\Sigma\, m_k}$$

where each summation is from $k = 1$ to n. Show that $G = G'$.

Solution. Let $\vec{v} = \overrightarrow{OO'}$. Then $\overrightarrow{OP_k} = \vec{v} + \overrightarrow{O'P_k}$ and hence

$$\overrightarrow{OG} = \frac{\Sigma\, m_k(\vec{v} + \overrightarrow{O'P_k})}{\Sigma\, m_k} = \vec{v}\,\frac{\Sigma\, m_k}{\Sigma\, m_k} + \frac{\Sigma\, m_k \overrightarrow{O'P_k}}{\Sigma\, m_k} = \vec{v} + \overrightarrow{O'G'}$$

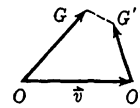


Figure 80.2

But O, O', G', G, O is a closed polygon so that

$$\overrightarrow{OG} = \vec{v} + \overrightarrow{O'G'} + \overrightarrow{G'G}.$$

Hence $\overrightarrow{G'G} = \vec{0}$ and therefore $G' = G$.

With the interpretation that m_k is the mass of a particle located at the point P_k, then the point G of Example 1 (and shown in Example 1 to depend only on the system) is equivalent to the centroid defined earlier.

Vector addition is illustrated by replacing two forces by a single force, or consecutive displacements by a single displacement.

An illustration of the scalar product is furnished by mechanical work of a force $\vec{F}$ whose point of application experiences a straight line displacement represented by a vector $\vec{u}$ not necessarily in line with $\vec{F}$. For the work (being the product of the force *in the direction of motion* times the distance moved) is given by

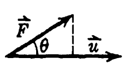


Figure 80.3

$$\text{work} = |\vec{F}|\,|\vec{u}| \cos\theta = \vec{F}\cdot\vec{u}$$

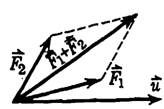


Figure 80.4

where θ is the angle from $\vec{F}$ to $\vec{u}$. If $\vec{F} = \vec{F}_1 + \vec{F}_2$, then the distributive law for scalar multiplication:

$$\vec{F}\cdot\vec{u} = (\vec{F}_1 + \vec{F}_2)\cdot\vec{u} = \vec{F}_1\cdot\vec{u} + \vec{F}_2\cdot\vec{u}$$

says that the work of the resultant force is equal to the sum of the works of the separate forces. Also, if the point of application first follows $\vec{u}_1 = \overrightarrow{AB}$ and then $\vec{u}_2 = \overrightarrow{BC}$, the distributive law

$$\vec{F} \cdot (\vec{u}_1 + \vec{u}_2) = \vec{F} \cdot \vec{u}_1 + \vec{F} \cdot \vec{u}_2$$

shows that the sum of the works during the separate displacements is equal to the work for the single resultant displacement $\vec{u}_1 + \vec{u}_2 = \overrightarrow{AC}$.

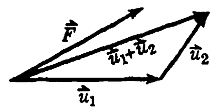


Figure 80.5

PROBLEMS

1. Given that $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors show that:

   a. For any angle α, then $\vec{u}\cos\alpha + \vec{v}\sin\alpha$ and $\vec{u}\sin\alpha - \vec{v}\cos\alpha$ are also perpendicular unit vectors.

   b. $\frac{2}{\sqrt{13}}\vec{u} + \frac{3}{\sqrt{13}}\vec{v}$ and $\frac{3}{\sqrt{13}}\vec{u} - \frac{2}{\sqrt{13}}\vec{v}$ are perpendicular unit vectors.

   c. Find conditions on numbers a, b, c, and d such that

   $$a\vec{u} + b\vec{v} \quad \text{and} \quad c\vec{u} + d\vec{v}$$

   are perpendicular unit vectors.

2. In a triangle ABC let the angles at the vertices be α, β, γ and the sides opposite them have length a, b, c. Let $\overrightarrow{AB} = \vec{v}_1$, $\overrightarrow{BC} = \vec{v}_2$, and $\overrightarrow{AC} = \vec{v}_3$ so that $\vec{v}_1 = -\vec{v}_2 + \vec{v}_3$. Show that

   $$\vec{v}_1^{\,2} = (-\vec{v}_2 + \vec{v}_3)^2$$

   is the vector form of the cosine law $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

3. Show that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

4. a. Show that an isosceles triangle has two medians of equal length.

   b. Prove the converse of the statement in Part a.

5. With $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$ having the same initial point and $0° < \theta < 180°$ the angle between them:

   a. For t a number draw the vector $\vec{v} + t\vec{u}$.

   Find the value of t such that $\vec{v} + t\vec{u}$ is:

   b. Perpendicular to $\vec{u}$.

   c. Perpendicular to $\vec{v}$.

   d. Show geometrically that, for t any number, the vector

   $$\vec{w} = t\left(\frac{1}{|\vec{u}|}\vec{u} + \frac{1}{|\vec{v}|}\vec{v}\right), \quad t \neq 0$$

   with initial end coinciding with those of $\vec{u}$ and $\vec{v}$ has terminal end on the line bisecting the angle between $\vec{u}$ and $\vec{v}$.

6. With $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$ show that $\left(\dfrac{\vec{u}}{\vec{u}^2} - \dfrac{\vec{v}}{\vec{v}^2}\right)^2 = \dfrac{(\vec{u} - \vec{v})^2}{\vec{u}^2\vec{v}^2}$.

7. Show that the perpendicular bisectors of the sides of a triangle meet in a point.

81. Vectors and Coordinates

With a coordinate system established in the plane, the discussion of both vectors and coordinates is facilitated by two special unit vectors $\vec{i}$ and $\vec{j}$ represented by the sensed segments each with initial end at the origin, the first with terminal end at (1,0) and the second with terminal end at (0,1). Since $\vec{i}$ and $\vec{j}$ are perpendicular unit vectors, it follows for scalar products that

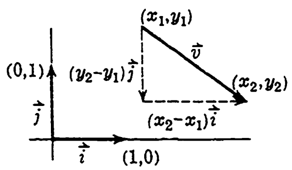


Figure 81.1

$$\text{(1)} \quad \vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{i} = 0 \quad \text{and} \quad \vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = 1$$

The vector $\vec{v}$ having initial end at (x_1,y_1) and terminal end at (x_2,y_2) has x- and y-components

$$\vec{v}_x = (x_2 - x_1)\vec{i} \quad \text{and} \quad \vec{v}_y = (y_2 - y_1)\vec{j}.$$

This vector $\vec{v}$ has magnitude given by the distance formula

$$|\vec{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{or by} \quad |\vec{v}| = \sqrt{\vec{v}_x^2 + \vec{v}_y^2} \quad \text{or by}$$

$$\text{(2)} \qquad |\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\vec{v}_x\cdot\vec{v}_x + \vec{v}_y\cdot\vec{v}_y}.$$

If $\vec{v} \neq \vec{0}$, then any angle α such that

$$\cos\alpha = \frac{x_2 - x_1}{|\vec{v}|} \quad \text{and} \quad \sin\alpha = \frac{y_2 - y_1}{|\vec{v}|}$$

is called an **amplitude** of $\vec{v}$ and the angle α which in addition is such that $-180° < \alpha \leq 180°$ is called the **principal amplitude** of $\vec{v}$. Hence, the vector $\vec{v}$ may also be represented as the product of the number $|\vec{v}|$ and the unit vector $\vec{i}\cos\alpha + \vec{j}\sin\alpha$:

$$\vec{v} = |\vec{v}|\,(\vec{i}\cos\alpha + \vec{j}\sin\alpha).$$

Given any numbers a and b, then $a\vec{i} + b\vec{j}$ is a vector in the plane. Conversely, given a vector $\vec{v}$ in the plane there are unique numbers a and b such that

$$\vec{v} = a\vec{i} + b\vec{j}$$

since $\vec{i}$ and $\vec{j}$ are not parallel while $\vec{i} \neq \vec{0}$ and $\vec{j} \neq \vec{0}$ so that (see Corollary 78) $a\vec{i} + b\vec{j} = c\vec{i} + d\vec{j}$ if and only if $c = a$ and $d = b$.

With $\vec{v}_1 = a_1\vec{i} + b_1\vec{j}$ and $\vec{v}_2 = a_2\vec{i} + b_2\vec{j}$ it follows that

$$\vec{v}_1 + \vec{v}_2 = (a_1 + a_2)\vec{i} + (b_1 + b_2)\vec{j}, \quad \vec{v}_1 - \vec{v}_2 = (a_1 - a_2)\vec{i} + (b_1 - b_2)\vec{j}$$

and $\vec{v}_1 \cdot \vec{v}_2 = a_1a_2\vec{i}\cdot\vec{i} + a_1b_2\vec{i}\cdot\vec{j} + a_2b_1\vec{j}\cdot\vec{i} + b_1b_2\vec{j}\cdot\vec{j}$ so that from (1),

(3) $$\vec{v}_1 \cdot \vec{v}_2 = a_1a_2 + b_1b_2.$$

Since, with θ the angle between $\vec{v}_1$ and $\vec{v}_2$, we also have the scalar product

$$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1|\,|\vec{v}_2| \cos\theta = \sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}\cos\theta, \quad \text{then}$$

(4) $$\cos\theta = \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}$$

is a formula for finding the angle between two vectors or the lines containing the vectors.

Example 1. Find the positive angle less than 180° from the first line whose equation is given to the second:

$$2x - 3y + 4 = 0, \quad 5x + 6y + 7 = 0.$$

Solution. In sketching the lines we found the intercepts $(-2,0)$, $(0,\frac{4}{3})$ for the first and $(-\frac{7}{5},0)$, $(0,-\frac{7}{6})$ for the second. Thus, $\vec{v}_1 = [0 - (-2)]\vec{i} + [\frac{4}{3} - 0]\vec{j} = 2\vec{i} + (\frac{4}{3})\vec{j}$ and $\vec{v}_2 = (-\frac{7}{5})\vec{i} + (\frac{7}{6})\vec{j}$ lie along the lines and are sensed so that when the first is rotated through the angle θ it will have the sense of the second. Thus,

$$\cos\theta = \frac{2(-\frac{7}{5}) + (\frac{4}{3})(\frac{7}{6})}{\sqrt{2^2 + (\frac{4}{3})^2}\sqrt{(-\frac{7}{5})^2 + (\frac{7}{6})^2}}.$$

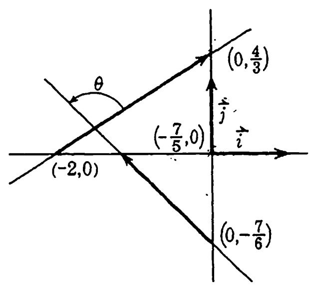


Figure 81.2

Notice that the arithmetic would be simplified by using

$\vec{u}_1 = 3\vec{v}_1 = 6\vec{i} + 4\vec{j}$ and $\vec{u}_2 = \frac{30}{7}\vec{v}_2 = -6\vec{i} + 5\vec{j}$

so that $\cos\theta = \dfrac{6(-6) + 4\cdot 5}{\sqrt{6^2 + 4^2}\sqrt{(-6)^2 + 5^2}} = \dfrac{-16}{\sqrt{(52)(61)}}$, $180° - \theta = 73°\,30'$, and $\theta = 106°\,30'$.

The vectors $\vec{v}_1 = a_1\vec{i} + b_1\vec{j}$ and $\vec{v}_2 = a_2\vec{i} + b_2\vec{j}$ are perpendicular if and only if $\vec{v}_1 \cdot \vec{v}_2 = 0$; that is, from (3)

(5) $\vec{v}_1$ *is perpendicular to* $\vec{v}_2$ *if and only if* $a_1a_2 + b_1b_2 = 0$.

or, provided $a_1 \neq 0$ and $b_2 \neq 0$, if and only if $b_1/a_1 = -a_2/b_2$. Notice that the lines containing $\vec{v}_1$ and $\vec{v}_2$ have respective slopes b_1/a_1 and b_2/a_2 so that:

Two lines, neither parallel to an axis, are perpendicular if and only if their slopes m_1 and m_2 are negative reciprocals of each other:

$$m_2 = -\frac{1}{m_1} \quad \text{or} \quad m_1m_2 = -1.$$

Example 2. Find an equation of the line through the point (1,2) perpendicular to the line whose equation is $3x - y + 1 = 0$.

Solution. The given equation may be written as $y = 3x + 1$ showing that it has slope $m_1 = 3$. The desired perpendicular line thus has slope $m_2 = -\frac{1}{3}$ and equation

$$y - 2 = -\tfrac{1}{3}(x - 1) \quad \text{or} \quad x + 3y = 7.$$

Given a curve C and a point P on C where the tangent to C exists, then the **normal** to C at P is defined to be the line through P perpendicular to the tangent.

Example 3. Find an equation of the normal to the graph of $y = (x - 2)^3 + 3$ at the point on the graph having abscissa 1.

Solution. Upon setting $x = 1$ in the equation, the desired point is (1,2). Since $D_xy]_{x=1} = 3(x - 2)^2]_{x=1} = 3$, the tangent to the curve at the point (1,2) has slope $m_1 = 3$. Thus, the normal to the curve at the point (1,2) has slope $m_2 = -\frac{1}{3}$ and thus equation $x + 3y = 7$.

Given a curve C and a point P on it where the tangent to C exists, then (provided P is not an inflection point of C) a portion of the tangent including P lies on one side of C called the **convex** side of C; the other side is called the **concave** side of C. A vector with initial point at P, lying along the normal at P, and sensed to point into the region on the concave side of C is called an **internal normal** vector for C at P, but oppositely sensed is called an **external normal** vector.

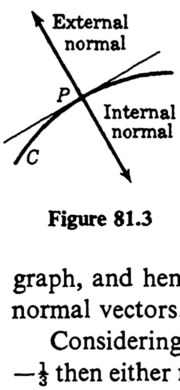


Figure 81.3

For f a function and x in the domain of f'' and such that $f''(x) \neq 0$, then a portion of the graph of f near $(x, f(x))$ lies above the tangent if $f''(x) > 0$, but below the tangent if $f''(x) < 0$. (See Sec. 33.) Thus, for the graph of a function, the sign of the second derivative distinguishes between the convex and concave sides of the graph, and hence together with first derivative determines the senses of the normal vectors.

Considering the graph of Example 3, since the normal at (1,2) has slope $-\frac{1}{3}$ then either normal vector has its y-component divided by its x-component equal to $-\frac{1}{3}$. But $D_x^2y]_{x=1} = 6(x - 2)]_{x=1} < 0$ and thus an internal normal vector $\vec{N}$ has $\vec{N}_x$ of the same sense as $\vec{i}$ and $\vec{N}_y$ of the same sense as $-\vec{j}$. Hence

$$\vec{N} = 3\vec{i} - \vec{j}$$

is an internal normal vector at (1,2) and the same is true of $k\vec{N}$ for $k > 0$. In particular the unit internal normal vector is

$$\vec{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{3}{\sqrt{10}}\vec{i} - \frac{1}{\sqrt{10}}\vec{j}.$$

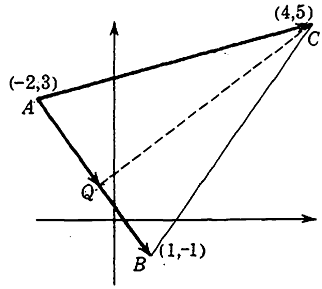


Figure 81.4

Example 4. In the triangle ABC, find the length h of the altitude from C where $A = (-2,3)$, $B = (1,-1)$, and $C = (4,5)$.

Solution. Using A as the origin of vectors, then

$$\overrightarrow{AB} = [1 - (-2)]\vec{i} + [-1 - 3]\vec{j} = 3\vec{i} - 4\vec{j}$$

and
$$\overrightarrow{AC} = 6\vec{i} + 2\vec{j}.$$

For any number t, the point Q where

$$\overrightarrow{AQ} = t\overrightarrow{AB} = 3t\vec{i} - 4t\vec{j}$$

will be on the line through A and B. We wish to locate Q (hence to determine t) so that $\overrightarrow{CQ}$ is perpendicular to $\overrightarrow{AB}$, i.e., so $\overrightarrow{AB} \cdot \overrightarrow{CQ} = 0$. Since

$$\overrightarrow{CQ} = \overrightarrow{CA} + \overrightarrow{AQ} = -\overrightarrow{AC} + \overrightarrow{AQ} = -(6\vec{i} + 2\vec{j}) + (3t\vec{i} - 4t\vec{j})$$
$$= (-6 + 3t)\vec{i} + (-2 - 4t)\vec{j}$$

we have $\overrightarrow{AB} \cdot \overrightarrow{CQ} = 3(-6 + 3t) + (-4)(-2 - 4t) = 0$ and hence $25t - 10 = 0$ and $t = \frac{2}{5}$. Thus

$$\overrightarrow{CQ} = \left(-6 + \frac{6}{5}\right)\vec{i} + \left(-2 - \frac{8}{5}\right)\vec{j} = \frac{-24}{5}\vec{i} - \frac{18}{5}\vec{j} \quad \text{and}$$

$$h = \sqrt{\overrightarrow{CQ} \cdot \overrightarrow{CQ}} = \sqrt{\left(\frac{-24}{5}\right)^2 + \left(\frac{-18}{5}\right)^2} = \frac{6}{5}\sqrt{4^2 + 3^2} = 6.$$

PROBLEMS

1. Express each of the described vectors in the form $a\vec{i} + b\vec{j}$ and represent each vector graphically.

a. $\overrightarrow{P_1P_2}$ for $P_1 = (1,-2)$, $P_2 = (-3,5)$.

b. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ for $A = (7,1)$, $B = (2,5)$, and $C = (4,-3)$.

c. $\overrightarrow{OP}$ with $O = (0,0)$, P the mid-point of $\overrightarrow{P_1P_2}$ where $P_1 = (1,-2)$, $P_2 = (5,7)$.

d. The unit vector of amplitude 30°.

e. The vector obtained by rotating $-2\vec{j}$ through $-120°$.

f. The unit vector having the same direction and sense as $3\vec{i} - 4\vec{j}$.

g. The unit vector tangent to the graph of $y = x^2$ at the point (2,4) and with x-component having the same sense as $\vec{i}$.

h. The unit internal normal vector to the graph of $y = x^2$ at (2,4).

2. Find the magnitude and amplitude of each of the vectors.

a. $\vec{i} - \vec{j}$. c. $-3\vec{i} - 4\vec{j}$. e. $5\vec{i} + 12\vec{j}$.

b. $-\vec{i} + \vec{j}$. d. $\sqrt{3}\vec{i} + \vec{j}$. f. $12\vec{i} - 5\vec{j}$.

3. Find the length of the altitude from C for the triangle ABC:
   a. $A = (1,6)$, $B = (4,2)$, $C = (7,8)$.
   b. $A = (0,0)$, $B = (4,5)$, $C = (9,1)$.
   c. $A = (-1,-7)$, $B = (2,2)$, $C = (5,11)$.
   d. $A = (-2,3)$, $B = (1,4)$, $C = (6,8)$.
   e. In Part a subtract 1 from each abscissa and subtract 6 from each ordinate to obtain triangle $A'B'C'$. Find the length of the altitude from C'.

4. Given a triangle ABC, show that the altitude from C hits the opposite side at Q such that

$$\overrightarrow{CQ} = -\overrightarrow{AC} + t\overrightarrow{AB} \quad \text{where} \quad t = \frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{(\overrightarrow{AB})^2}.$$

Also, show that this altitude has length

$$h = \sqrt{(\overrightarrow{AC})^2 - \frac{(\overrightarrow{AC} \cdot \overrightarrow{AB})^2}{(\overrightarrow{AB})^2}}.$$

5. Find the positive angle less than 180° from the first line to the second.
   a. $y = 0.5x - 3$, $y = 3x + 4$.
   b. $3x - y + 4 = 0$, $x - 2y = 6$.
   c. $x - 2y + 1 = 0$, $9x + 6y = 5$.
   d. $y = 3x$, $y = x - 4$.

6. At a point of intersection of two curves **an angle between the curves** is defined as the corresponding angle between the tangents to the curves. For the graphs of each of the following pairs of equations, find the acute angle at which the graphs intersect.
   a. $xy = 1$, $y = 4x$.
   b. $x^2 + y^2 = 4$, $y^2 = 3x$.
   c. $y = \sin x$, $y = \cos x$.
   d. $y = x^2$, $xy = 1$.
   e. $y = x^2$, $y^2 = x$.
   f. $x^2 + y^2 = 7$, $x^2 - y^2 = 1$.

7. a. For the parabola with vertex at the origin and focus $F = (p,0)$, let $P = (x_1,y_1)$ be any point on the parabola. Show that the angle from $\overrightarrow{PF}$ to the internal normal at P is equal to the angle from this normal to the horizontal line through P.
   b. Let $F_1 = (-c,0)$ and $F_2 = (c,0)$, $c > 0$ be the foci of an ellipse. For $P = (x_1,y_1)$ on the ellipse show that the normal at P bisects the angle F_1PF_2.
   c. Show that an ellipse and a hyperbola having the same foci (are confocal) intersect at right angles.

8. Find the unit internal normal vector to the graph of each of the following equations at the point indicated.
   a. $y = x^3 - 6x^2 + 12x$, $(1,7)$.
   b. $x^2 + y^2 = 25$, $(3,-4)$.
   c. $y = e^{-x^2}$, $(1,e^{-1})$.
   d. $xy + 2x - 5y - 2 = 0$, $(3,2)$.
   e. $y = \sin^2 x$, $(5\pi/6,\frac{1}{4})$.
   f. $y = (\tan^{-1} x)^2$, $(1,\pi^2/16)$.

82. Parametric Equations

Given a vector $\vec{v}$ with amplitude α, then

$$\vec{v} = |\vec{v}|\,(\vec{i}\cos\alpha + \vec{j}\sin\alpha).$$

Upon rotating this vector about its initial end through an angle θ (where rotation is counterclockwise if $\theta > 0$, but clockwise if $\theta < 0$) the result is a vector with the same magnitude as $\vec{v}$ but with amplitude $\alpha + \theta$, and is thus given by

$$|\vec{v}|\,[\vec{i}\cos(\alpha + \theta) + \vec{j}\sin(\alpha + \theta)].$$

In particular $-\vec{j} = \vec{i}\cos(-90°) + \vec{j}\sin(-90°)$ rotated through an angle $-\theta$ is a vector we shall use in the next example and denote temporarily by $\vec{J}$:

(1) $\quad \vec{J} = \vec{i}\cos(-90° - \theta) + \vec{j}\sin(-90° - \theta) = -\vec{i}\sin\theta - \vec{j}\cos\theta.$

Example 1. Let a circle of radius a have its center C at the point $(0,a)$. Roll this circle without slipping along the x-axis and watch the point P on the circumference which started at the origin. The center C moves parallel to the x-axis and the radius vector $\overrightarrow{CP}$ rotates about C. When this rotation is through an angle $-\theta$ (so $\theta > 0$ if the circle rolls to the right but $\theta < 0$ if rolled to the left) the problem is to express the vector from the origin O to P in terms of θ, $\vec{i}$, and $\vec{j}$ with θ measured in radians.

Solution. As illustrated in the figure, let Q be the point of tangency of the circle and the x-axis so that

$$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QC} + \overrightarrow{CP}.$$

Since the circle rolled without slipping, the magnitude of $\overrightarrow{OQ}$ is the length of an arc of the circle subtended by a central angle of $|\theta|$ radians so that $|\overrightarrow{OQ}| = a|\theta|$ and

$$\overrightarrow{OQ} = a\theta\vec{i}.$$

Since $\overrightarrow{QC} = a\vec{j}$ and $\overrightarrow{CP} = a\vec{J}$, where $\vec{J}$ is given by (1), we have

$$\begin{aligned}\overrightarrow{OP} &= a\theta\vec{i} + a\vec{j} + a(-\vec{i}\sin\theta - \vec{j}\cos\theta)\\ &= a(\theta - \sin\theta)\vec{i} + a(1 - \cos\theta)\vec{j}.\end{aligned}$$

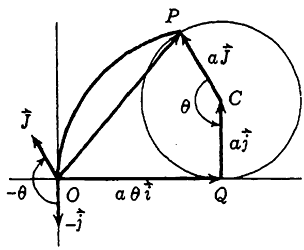


Figure 82.1

With (x,y) the rectangular coordinates of the point P of Example 1, then

(2) $$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

The equations (2) is an example of a pair of parametric equations with parameter θ.

DEFINITION 82. *Given a function F and a function G, then*

(3) $$x = F(t), \quad y = G(t)$$

taken together are said to be ***parametric equations*** *with parameter t.*

The graph of these parametric equations is

$$\{(x,y) \mid x = F(t), \quad y = G(t) \text{ for some number } t\}$$

The graph of the parametric equations (2) is called a **cycloid.**

Other curves may be parametrized in various ways. For example,

$$x = a \cos \theta, \quad y = a \sin \theta$$

is a parametrization of the circle with center at the origin and radius a. The same circle may be represented by

$$x = a \sin t, \quad y = a \cos t, \quad \text{and also by } x = -a \cos 2t, \quad y = -a \sin 2t.$$

The graph of the parametric equations

$$x = a \sec t, \quad y = b \tan t$$

is a hyperbola since a point lies on the graph if and only if

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \sec^2 t - \tan^2 t = 1 \quad \text{or} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The "elimination of the parameter" must be done with caution, however. For example, given the parametric equations

$$(4) \qquad x = \sin t, \quad y = 1 - \cos 2t,$$

then by using trigonometric identities we have

$$(5) \qquad y = 1 - (1 - 2 \sin^2 t) = 2x^2.$$

This might lead one to believe that the graph of (4) is the parabola whose equation is $y = 2x^2$, but this is not so since a point (x,y) on the graph of (4) must have $|x| \leq 1$ and $0 \leq y \leq 2$. If, however, a point is on the graph of (4) then it is also on the graph (5), but not conversely.

It is customary to conserve notation and instead of (3) to write

$$(6) \qquad x = x(t), \quad y = y(t).$$

A parametrization establishes an "order relation" on the curve under the convention that $(x(t_1),y(t_1))$ "precedes" $(x(t_2),y(t_2))$ if $t_1 < t_2$. Thus, for (4) we have that

$$(\sin 0, 1 - \cos 0) \text{ precedes } (\sin 2\pi, 1 - \cos 4\pi),$$

although both are represented geometrically by the origin. A possible concept is of a particle moving in the plane with its position at time t given by (4). For such dynamic considerations, a pair of parametric equations (such as (6)) will be referred to as a **law of motion** in the plane.

Example 2. Describe the motion of a particle moving under the law (4) with $t \geq 0$.

Solution. Although the graph of (4) is part of the graph of (5), in order to visualize the motion we construct the following table:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
y	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0

Hence, with time measured in seconds we can visualize the particle starting at the origin when $t = 0$, moving up along the parabolic arc to $(1,2)$ in $\pi/2$ sec, returning over the same path to the origin during the next $\pi/2$ sec, going up over the other arc of the parabola and arriving at $(-1,2)$ when $t = 3\pi/2$ sec, returning to the origin when $t = 2\pi$ sec, and then repeating this course in each succeeding time interval of 2π sec.

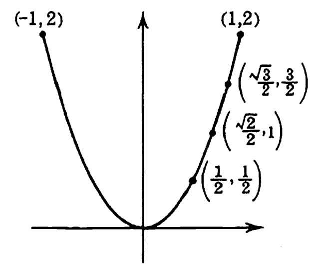


Figure 82.2

Example 3. Given that a particle moves according to the law

$$x = t + 2, \quad y = 2t^2 + 2 \quad \text{for} \quad t \geq 0,$$

how long is the particle inside the circle of center $(2,7)$ and radius $\sqrt{10}$?

Solution. The particle will be within the circle when its distance from $(2,7)$ is less than $\sqrt{10}$, and thus for those values of t satisfying

$$[(t + 2) - 2]^2 + [(2t^2 + 2) - 7]^2 < 10 \quad \text{and} \quad t \geq 0.$$

The inequality simplifies to $4t^4 - 19t^2 + 15 < 0$ and then to

$$(4t^2 - 15)(t^2 - 1) < 0.$$

This inequality is satisfied if and only if $4t^2 - 15 < 0$ and $t^2 - 1 > 0$, so that $1 < t^2 < \frac{15}{4}$ and hence

$$1 < t < \sqrt{15}/2 \quad \text{since} \quad t \geq 0.$$

Hence, the particle is within the circle for $\sqrt{15}/2 - 1$ time units.

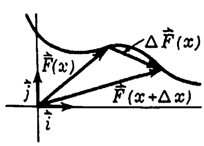


Figure 82.3

Given a function f, one way of parametrizing the graph of f is to set

$$x = t, \quad y = f(t).$$

Also, a vector function $\vec{F}$ may be defined by letting $\vec{F}(x)$ be the vector from the origin to the point $(x, f(x))$ so that

$$\vec{F}(x) = \vec{i}x + \vec{j}f(x).$$

For this function $\vec{F}(x + \Delta x) = \vec{i}(x + \Delta x) + \vec{j}f(x + \Delta x)$ and

$$\Delta\vec{F}(x) = \vec{i}\,\Delta x + \vec{j}[f(x + \Delta x) - f(x)].$$

It is thus natural to define the derivative $\vec{F}'(x)$ by

$$\vec{F}'(x) = \lim_{\Delta x \to 0} \frac{\Delta \vec{F}(x)}{\Delta x} = \lim_{\Delta x \to 0} \left[\vec{i} + \vec{j}\frac{f(x + \Delta x) - f(x)}{\Delta x} \right] = \vec{i} + \vec{j}f'(x).$$

Notice that $|\vec{F}'(x)| = \sqrt{1 + f'^2(x)}$. Also, $\vec{F}'(x)$ has the same slope as the tangent at $(x, f(x))$ and is sensed to point rightward, since its x-component is $\vec{i}$, and

$$\frac{1}{\sqrt{1 + f'^2(x)}} [\vec{i} + \vec{j}f'(x)]$$

is a unit vector with the same direction and sense. By rotating this unit vector through 90° if $f''(x) > 0$ but through $-90°$ if $f''(x) < 0$, then the interior normal unit vector is obtained.

PROBLEMS

1. a. Find parametric equations of the cycloid obtained by rolling the circle along the under side of the x-axis.

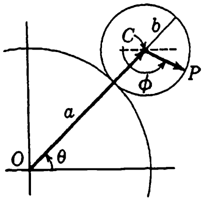


Figure Prob. 1b

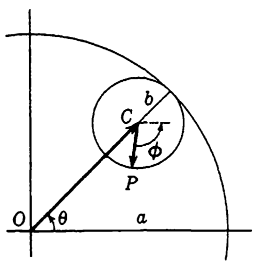


Figure Prob. 1c

b and c. In Figs. Prob. 1b and 1c the circle of radius b rolls without slipping along the circle of radius a with the point P starting at $(a,0)$. Find parametric equations for the path of P in terms of θ. (Hint: In b, $(\phi - \theta)b = \theta a$.) (Note: The curve of Part b is called an **epicycloid**; the one of Part c, a **hypocycloid**.)

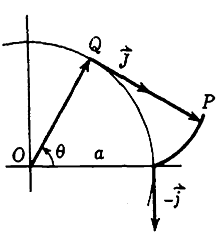


Figure Prob. 1d

d. A string is wound around the circle of radius a and center at the origin, the free end P of the string being at the point $(a,0)$. The string is now unwound keeping it taut as shown in Fig. Prob. 1d. Find parametric equations for the path of P.

e. Show that the parametric equations for the hypocycloid with $b = a/4$ may be written as

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

Obtain an equation in x and y by eliminating θ between these equations.

2. Show that the graph of each of the following pairs of parametric equations is part of a straight line and specify what part.

   a. $x = 2t^2, \quad y = 3t^2 + 1.$
   c. $x = 2 \sin t, \quad y = 3 \sin t + 1.$
   b. $x = 2\,|t|, \quad y = 3\,|t| + 1.$
   d. $x = 2e^t, \quad y = 3e^t + 1.$
   e. $x = 2(1 + \cos t), \quad y = 3 \cos t + 4.$

3. In each case show that both pairs of parametric equations have the same geometric graph, but as laws of motion they are different.

   a. $x = 2t, \quad y = t^2$ and $x = 2t^3, \quad y = t^6.$
   b. $x = 3 \sin t, \quad y = 3 \cos t$ and $x = 3 \cos 2t, \quad y = 3 \sin 2t.$
   c. $x = 1 + 2t, \quad y = 2 - t$ and $x = -1 + 2t, \quad y = 3 - t.$
   d. $x = t, \quad y = t^3$ and $x = 2t + 1, \quad y = (2t + 1)^3.$

4. For each of the following laws of motion, how long will the particle be within $\sqrt{5}$ units of the origin?

   a. $x = 3t^2, \quad y = 2t; \quad t \geq 0.$
   c. $x = 3(t - 1)^2, \quad y = 2(t - 1); \quad t \geq 0.$
   b. $x = 2t, \quad y = 3t^2; \quad t \geq 0.$
   d. $x = 1 + \cos t, \quad y = \sin t; \quad 0 \leq t \leq 2\pi.$

5. A particle has the law of motion $x = 10 \cos t$, $y = 10 \sin t$. How long during each period will it be within the square having vertices $(-5,5)$, $(-5,10)$, $(-10,10)$, $(-10,5)$?

83. Vectors and Lines

THEOREM 83. *Given a line l having equation*

$$ax + by + c = 0,$$

then $\vec{u} = b\vec{i} - a\vec{j}$ is parallel to l, $\vec{v} = a\vec{i} + b\vec{j}$ is perpendicular to l, and

$$\vec{w} = -\frac{c}{a^2 + b^2}(a\vec{i} + b\vec{j}) \tag{1}$$

is the vector from the origin to l and perpendicular to l.

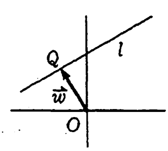


Figure 83.1

PROOF. If $b = 0$ (so $a \neq 0$), then l is perpendicular to the x-axis and so is $\vec{u} = 0 \cdot \vec{i} - a\vec{j} = -a\vec{j}$. If $b \neq 0$, then l has slope $-a/b$ and so does $\vec{u}$. Thus, in either case $\vec{u}$ and l are parallel. Now since

$$\vec{u} \cdot \vec{v} = (b\vec{i} - a\vec{j}) \cdot (a\vec{i} + b\vec{j}) = ba - ab = 0$$

the vector $\vec{v}$ is perpendicular to $\vec{u}$ and hence to l. The line containing $\vec{v}$ passes through the origin O, intersects l at a point Q, and there is a number t_0 such that

$$\overrightarrow{OQ} = t_0\vec{v} = t_0(a\vec{i} + b\vec{j}). \tag{2}$$

Hence $Q = (at_0,bt_0)$ and since it lies on l

$$a(at_0) + b(bt_0) + c = 0 \quad \text{so that} \quad t_0 = \frac{-c}{a^2 + b^2}\,.$$

Upon substituting this expression for t_0 into (2) we obtain (1).

COROLLARY 83. *Given a point $P_0 = (x_0,y_0)$, the perpendicular vector from P_0 to l is*

$$\vec{V} = -\frac{ax_0 + by_0 + c}{a^2 + b^2}(a\vec{i} + b\vec{j}) \tag{3}$$

and the distance from P_0 to l is

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}\,. \tag{4}$$

PROOF. Let l_0 be the line through P_0 parallel to l. Then l_0 has equation $a(x - x_0) + b(y - y_0) = 0$ which may be written as

$$ax + by - (ax_0 + by_0) = 0$$

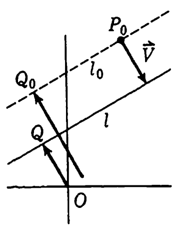


Figure 83.2

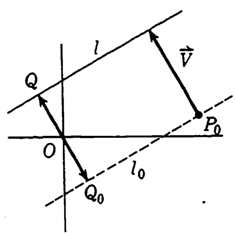


Figure 83.3

Hence, from (1) the vector $\overrightarrow{OQ_0}$ from the origin to l_0 and perpendicular to l_0 is given by

$$\overrightarrow{OQ_0} = -\frac{-(ax_0 + by_0)}{a^2 + b^2}(a\vec{i} + b\vec{j}) = \frac{ax_0 + by_0}{a^2 + b^2}(a\vec{i} + b\vec{j}).$$

Now the vector $\vec{V} = \overrightarrow{Q_0Q}$ is from P_0 to l, perpendicular to l and

$$\overrightarrow{Q_0Q} = \overrightarrow{Q_0O} + \overrightarrow{OQ} = -\overrightarrow{OQ_0} + \overrightarrow{OQ}$$

$$= -\frac{ax_0 + by_0}{a^2 + b^2}(a\vec{i} + b\vec{j}) + \frac{-c}{a^2 + b^2}(a\vec{i} + b\vec{j})$$

from which (3) follows. Thus $|\vec{V}|$ is the distance from P_0 to l and we thus obtain (4) from

$$|\vec{V}|^2 = \vec{V}\cdot\vec{V} = \left(-\frac{ax_0+by_0+c}{a^2+b^2}\right)^2(a\vec{i}+b\vec{j})\cdot(a\vec{i}+b\vec{j})$$

$$= \left(\frac{ax_0+by_0+c}{a^2+b^2}\right)^2(a^2+b^2) = \frac{(ax_0+by_0+c)^2}{a^2+b^2}.$$

Example 1. Is the point $P_0 = (100,70)$ and the origin on the same or opposite sides of the line whose equation is $2x - 3y + 4 = 0$?

Solution. Consider the line through P_0 parallel to the given line. Then

$$\overrightarrow{OQ} = -\frac{4}{2^2+3^2}(2\vec{i}-3\vec{j}) \quad\text{and}\quad \overrightarrow{Q_0Q} = -\frac{2(100)-3(70)+4}{2^2+3^2}(2\vec{i}-3\vec{j}).$$

Since $-4 < 0$ and $-(200 - 210 + 4) > 0$ the vectors $\overrightarrow{OQ}$ and $\overrightarrow{Q_0Q}$ have opposite senses so P_0 and the origin are on opposite sides of the line.

Example 2. Find the distance between the parallel lines having equations

$$3x + 4y - 6 = 0 \quad\text{and}\quad 6x + 8y + 27 = 0.$$

First Solution. The vectors $\overrightarrow{OQ_1}$ and $\overrightarrow{OQ_2}$ from the origin to the first and second lines are

$$\overrightarrow{OQ_1} = -\frac{-6}{3^2+4^2}(3\vec{i}+4\vec{j}) = \frac{6}{25}(3\vec{i}+4\vec{j}) \quad\text{and}$$

$$\overrightarrow{OQ_2} = -\frac{27}{6^2+8^2}(6\vec{i}+8\vec{j}) = -\frac{27}{50}(3\vec{i}+4\vec{j}),$$

respectively. The perpendicular vector from the second to the first line is

$$\overrightarrow{Q_2Q_1} = \overrightarrow{Q_2O} + \overrightarrow{OQ_1} = -\overrightarrow{OQ_2} + \overrightarrow{OQ_1} = (\tfrac{27}{50} + \tfrac{6}{25})(3\vec{i}+4\vec{j}) = \tfrac{39}{50}(3\vec{i}+4\vec{j})$$

and the distance between the lines is

$$\sqrt{\overrightarrow{Q_2Q_1}\cdot\overrightarrow{Q_2Q_1}} = \sqrt{(\tfrac{39}{50})^2(3^2+4^2)} = \tfrac{39}{50}(5) = \tfrac{39}{10}.$$

Second Solution. The distance between the lines is the distance from any point on the second line to the first line. By setting $y = 0$ in the second equation, the point $P_0 = (-\frac{9}{2},0)$ is on the second line and the desired distance is

$$\frac{|3(-\frac{9}{2}) + 4(0) - 6|}{\sqrt{3^2+4^2}} = \frac{|(-27-12)/2|}{5} = \frac{39}{10}.$$

Let P_1 and P_2 be distinct points, let O be any point, and consider the line containing P_1 and P_2. A point P different from P_1 will be on this line if and only if $\overrightarrow{P_1P}$ and $\overrightarrow{P_1P_2}$ have the same direction (but not necessarily the same sense). Thus, P lies on the line if and only if there is a number t such that

$$\overrightarrow{P_1P} = t(\overrightarrow{P_1P_2}).$$

Now add $\overrightarrow{OP_1}$ to both sides of this equation and we have that P lies on the line if and only if

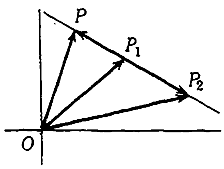


Figure 83.4

$$(5)\qquad \begin{aligned}\overrightarrow{OP_1} + \overrightarrow{P_1P} &= \overrightarrow{OP_1} + t(\overrightarrow{P_1P_2}); \quad \text{that is,}\\ \overrightarrow{OP} &= \overrightarrow{OP_1} + t(\overrightarrow{P_1P_2}).\end{aligned}$$

Thus, (5) is referred to as a **vector equation** relative to O of the line through P_1 and P_2.

Example 3. Let $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$ and $O = (0,0)$. For the line through P_1 and P_2 find a vector equation relative to O.

Solution. Now $\overrightarrow{OP_1} = x_1\vec{i} + y_1\vec{j}$, $\overrightarrow{P_1P_2} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}$, and for $P = (x,y)$, $\overrightarrow{OP} = x\vec{i} + y\vec{j}$. Thus, a vector equation (5) of this line is

$$(6)\qquad \begin{aligned}x\vec{i} + y\vec{j} &= x_1\vec{i} + y_1\vec{j} + t[(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j}]\\ &= [x_1 + t(x_2 - x_1)]\vec{i} + [y_1 + t(y_2 - y_1)]\vec{j}.\end{aligned}$$

In (6) the coefficients of $\vec{i}$ on both sides of the equation must be equal, and also for $\vec{j}$, so that

$$(7)\qquad x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1)$$

which are **parametric equations** of the line through $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$. In case $x_2 \neq x_1$, then by eliminating the parameter we obtain

$$y - y_1 = \frac{x - x_1}{x_2 - x_1}(y_2 - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

which is the ordinary two-point equation of the line.

For A and B numbers not both zero, then

$$(8)\qquad x = x_1 + tA, \quad y = y_1 + tB$$

are parametric equations of a line through $P_1 = (x_1,y_1)$. If $A \neq 0$ the line has slope B/A, but if $A = 0$ the line is perpendicular to the x-axis. Certainly the graph of (8) passes through P_1 as seen by setting $t = 0$. By setting $t = 1$, the graph passes through

$$P_2 = (x_1 + A, y_1 + B).$$

Now the line through P_1 and P_2 has (by (7)) parametric equations

$$x = x_1 + t[(x_1 + A) - x_1], \quad y = y_1 + t[(y_1 + B) - y_1]$$

which are the same as (8) so the graph of (8) is this line. In case $A \neq 0$, then this line passing through P_1 and P_2 has slope

$$m = \frac{(y_1 + B) - y_1}{(x_1 + A) - x_1} = \frac{B}{A},$$

but if $A = 0$, then every point on the line has abscissa x_1 and the line is perpendicular to the x-axis.

Example 4. Find the point of intersection of the line having parametric equations

$$x = 5 + 2t, \quad y = 4 + t \tag{9}$$

and the line having parametric equations

$$x = -3 + t, \quad y = -6 + 2t. \tag{10}$$

Solution. If (9) and (10) are considered as laws of motion such that at any time t the coordinates of the first particle is given by (9) and at the same time those of a second particle by (10), then the particles will never be together. For the particles will have the same abscissa at time t satisfying

$$5 + 2t = -3 + t, \quad \text{i.e., at } t = -8,$$

but at this time the ordinate of the first particle is $4 - 8 = -4$, whereas that of the second is $-6 + 2(-8) = -22$. Nevertheless, the paths of the particles may cross, but the particles go through this intersection at different times.

To see if the paths cross we change the letter designating the parameter in (10) to s, obtain

$$x = -3 + s, \quad y = -6 + 2s \tag{10'}$$

and then ask "Is there a number t and a number s such that (by setting the abscissas equal and the ordinates equal) both of the equations

$$5 + 2t = -3 + s \quad \text{and} \quad 4 + t = -6 + 2s \quad \text{hold?}$$

These simultaneous equations have solution $t = -2$, $s = 4$. Upon setting $t = -2$ in (9) we obtain the point $(5 - 4, 4 - 2)$ whereas $s = 4$ in (10′) yields the point $(-3 + 4, -6 + 8)$. Hence, the lines having parametric equations (9) and (10) intersect at the point $(1,2)$.

PROBLEMS

1. Find a vector equation, then parametric equations, and then an ordinary rectangular equation of the line through the points.

a. $(5,6)$, $(3,4)$. c. $(\frac{1}{2},3)$, $(\frac{1}{2},4)$. e. $(3,-2)$, $(4,-2+m)$.

b. $(250,3)$, $(250,4)$. d. $(3,-5)$, $(4,-5)$. f. (x_0,y_0), (x_0+1,y_0+m).

2. Find the distance between the given point and line. Are the point and the origin on the same or opposite sides of the line?

a. $(-54,71)$; $4x + 3y = 17$. d. $(\frac{5}{6},3)$; $x/2 + y/3 = 1$.

b. $(20,30)$; $6x - 8y + 15 = 0$. e. $(-1,6)$; $3y - 4 = 0$.

c. $(-60,25)$; $3x + 7y + 5 = 0$. f. $(-6,7)$; $x = 1 + 3t$, $y = 2 - 4t$.

3. Find the distance between the parallel lines.

   a. $3x + 4y - 8 = 0$,
   $6x + 8y + 9 = 0$.

   b. $y = 4x + 5$,
   $x - y/4 = 1$.

   c. $0.5x + 1.2y = 7$,
   $2.5x + 6y + 10.5 = 0$.

   d. $x = 1 + 2t, \quad y = 3 - 4t$,
   $2x + y - 15 = 0$.

4. By finding the equation of the line through two points then the distance from the third point to this line, find the area of the triangle with given vertices

   a. $(0,0), (3,4), (-1,6)$.

   b. $(-6,-3), (5,-1), (2,4)$.

   c. $(-5,-6), (-1,4), (2,-3)$.

   d. $(5,-6), (-3,2), (2,-3)$.

5. Find the point of intersection of the two lines represented in each of the following.

   a. $3.75x + 4.35y = 2; \quad x = 2 - 5t, \quad y = -3 + 2t$.

   b. $5x - 4y = 12; \quad x = 4 + 3t, \quad y = 8 - 6t$.

   c. $x = -5 + 4t, \quad y = 3t; \quad x = 1 + 2t, \quad y = 8 + 5t$.

   d. $x = 5 - 3t, \quad y = -5 + t; \quad x = 2 + t, \quad y = -4 + 3t$.

84. Vector Functions

A set of number-vector ordered pairs having the property "If $(a,\vec{V})$ and $(a,\vec{U})$ are in the set, then $\vec{U} = \vec{V}$" is said to be a **vector function** whose **domain** is the set of all numbers which are first elements and whose **range** is the set of all vectors which are second elements. A vector function will be denoted by $\vec{F}$ and, for t in its domain, $\vec{F}(t)$ will denote the "value" of $\vec{F}$ at t. Compare this definition of a vector-valued function with the definition (see Sec. 8) of a real valued function, which will now be called a scalar function.

DEFINITION 84. *For $\vec{F}$ a vector function and $\vec{L}$ a (constant) vector, then $\vec{L}$ is said to be the limit of $\vec{F}$ at t_0 and we write*

$$\lim_{t \to t_0} \vec{F}(t) = \vec{L}$$

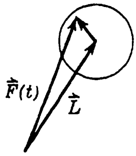


Figure 84.1

if corresponding to each positive number ϵ there is a number $\delta > 0$ such that whenever t is a number satisfying

$$0 < |t - t_0| < \delta, \quad \textit{then} \quad |\vec{F}(t) - \vec{L}| < \epsilon.$$

By considering all vectors of $\vec{F}$ as having the same initial point as $\vec{L}$, then a geometric interpretation of $\vec{L}$ being the limit of $\vec{F}$ at t_0 may be made. Draw any circle with center at the terminal end of $\vec{L}$. Then, depending upon the radius of this circle, all those members of $\vec{F}$ having arguments sufficiently close to t_0 will have directions, senses, and magnitudes so near to the direction, sense, and magnitude of $\vec{L}$ that their terminal ends will lie within this circle.

For $\vec{F}_1$ and $\vec{F}_2$, vector functions, then for each number t in the domain of both, $\vec{F}_1(t) + \vec{F}_2(t)$ is a vector. The vector function consisting of all such vectors is called the **sum** of $\vec{F}_1$ and $\vec{F}_2$. Theorems concerning the limits of sums and differences of vector functions follow corresponding proof for scalar functions.

Given a vector function $\vec{F}$, its **derived function** $\vec{F}'$ is defined by

$$\vec{F}'(t) = \lim_{\Delta t \to 0} \frac{\vec{F}(t + \Delta t) - \vec{F}(t)}{\Delta t} \tag{1}$$

whenever the limit exists. The derivative $\vec{F}'(t)$ is also denoted by

$$D_t\vec{F}(t) \quad \text{or by} \quad \frac{d\vec{F}(t)}{dt}.$$

A vector function may be considered as a law of motion of a particle. With all vectors of $\vec{F}$ having the same initial end, the terminal end of $\vec{F}(t)$ is then the position of the particle at time t. With this interpretation, the **velocity vector** (or merely the **velocity**) $\vec{v}(t)$ at time t is defined by

$$\vec{v}(t) = \vec{F}'(t). \tag{2}$$

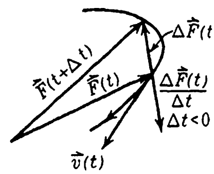


Figure 84.2

The intimate connection between motion and geometry is seen by drawing the path of the particle. Then for a number t where $\vec{F}'(t)$ exists and for $\Delta t \neq 0$, the terminal ends of $\vec{F}(t)$ and $\vec{F}(t + \Delta t)$ are positions of the particle on this path at times t and $t + \Delta t$ and $\vec{F}(t + \Delta t) - \vec{F}(t) = \Delta\vec{F}(t)$ is the vector from the first position to the second. Also, the vectors

$$\Delta\vec{F}(t) \quad \text{and} \quad \frac{\Delta\vec{F}(t)}{\Delta t}$$

have the same direction (along a secant of the path) with senses the same if $\Delta t > 0$ but opposite if $\Delta t < 0$, whereas the magnitude for the second is less than for the first if $|\Delta t| > 1$ but greater than for the first if $|\Delta t| < 1$. Since $\vec{F}'(t)$ exists, the direction of $\Delta\vec{F}(t)/\Delta t$ approaches, as $\Delta t \to 0$, the direction of the tangent to the path at $\vec{F}(t)$ and the magnitude of $\Delta\vec{F}(t)/\Delta t$ approaches a definite value. Also, if $\vec{F}'(t) \neq \vec{0}$ then the velocity vector $\vec{v}(t)$ is pictured with initial end at the terminal end of $\vec{F}(t)$ and lying along the tangent to the path of the particle with sense corresponding to the way the particle is proceeding in its path.

For a vector law of motion $\vec{F}$, the **speed** at time t is defined as the scalar

$$|\vec{v}(t)| = |\vec{F}'(t)|.$$

Given a vector function $\vec{F}$, considered as a law of motion, then the associated vector velocity function $\vec{v}$ is defined and we now define the **acceleration function** $\vec{a}$ by setting

$$\vec{a}(t) = \vec{v}'(t) = \vec{F}''(t) \tag{3}$$

for each number t at which $\vec{v}'(t)$ exists.

Given two vector functions $\vec{F}_1$ and $\vec{F}_2$ and a number t such that $\vec{F}_1'(t)$ and $\vec{F}_2'(t)$ both exist, then the sum and difference functions both have derivatives at t and

$$D_t\{\vec{F}_1(t) \pm \vec{F}_2(t)\} = D_t\vec{F}_1(t) \pm D_t\vec{F}_2(t). \tag{4}$$

The proof follows the pattern of that for scalar functions. Also, for f a scalar function, $\vec{F}$ a vector function, and t a number such that $f'(t)$ and $\vec{F}'(t)$ exist, then

$$D_t f(t)\vec{F}(t) = f(t)\vec{F}'(t) + f'(t)\vec{F}(t) \tag{5}$$

and again the symbolic details of the proof can be copied from the proof of the formula for the derivative of the product of two scalar functions. In particular, if c is a scalar constant, then

$$D_t c\vec{F}(t) = c\vec{F}'(t), \tag{6}$$

whereas if $\vec{c}$ is a vector constant, then

$$D_t\vec{c}f(t) = \vec{c}f'(t). \tag{7}$$

Whenever a vector function is interpreted as a law of motion, then all vectors of $\vec{F}$ (but not of $\vec{v}$ or $\vec{a}$) are considered as **bound vectors**; that is, all vectors of $\vec{F}$ have initial ends at the same point. Also, "$\vec{F}(t)$ is the location of the particle at time t" means that at time t the particle is at the terminal end of $\vec{F}(t)$.

Example 1. Given the vector law of motion

$$\vec{F}(t) = \vec{i}(t^3 - 7t^2 + 17t - 12) + \vec{j}(t^2 - 7t + 11), \tag{8}$$

represent graphically $\vec{F}(2)$, $\vec{v}(2)$, and $\vec{a}(2)$ and then sketch a portion of the graph near the point $\vec{F}(2)$. Also, resolve the acceleration vector $\vec{a}(2)$ into its components along the tangent and along the normal to the path at $\vec{F}(2)$.

Solution. By substituting $t = 2$ in (8) we have $\vec{F}(2) = 2\vec{i} + \vec{j}$. Now

$$\begin{aligned}
\vec{v}(t) &= \vec{F}'(t) = D_t[\vec{i}(t^3 - 7t^2 + 17t - 12] + D_t[\vec{j}(t^2 - 7t + 11)] && \text{(by (4))}\\
&= \vec{i}D_t(t^3 - 7t^2 + 17t - 12) + \vec{j}D_t(t^2 - 7t + 11) && \text{(by (7))}\\
&= \vec{i}(3t^2 - 14t + 17) + \vec{j}(2t - 7), \quad \text{and}\\
\vec{a}(t) &= \vec{v}'(t) = \vec{i}(6t - 14) + \vec{j}(2).
\end{aligned}$$

Thus $\vec{v}(2) = \vec{i} - 3\vec{j}$ and $\vec{a}(2) = -2\vec{i} + 2\vec{j}$.

Upon letting $\vec{a}_T(2)$ be the component of $\vec{a}(2)$ along the tangent and $\vec{a}_N(2)$ the component along the normal, we have (see Theorem 79)

$$\begin{aligned}\vec{a}_T(2) &= \frac{\vec{a}(2)\cdot\vec{v}(2)}{\vec{v}(2)\cdot\vec{v}(2)}\vec{v}(2)\\ &= \frac{(-2\vec{i}+2\vec{j})\cdot(\vec{i}-3\vec{j})}{(\vec{i}-3\vec{j})^2}(\vec{i}-3\vec{j})\\ &= \frac{-2-6}{1+9}(\vec{i}-3\vec{j}) = -\frac{4}{5}(\vec{i}-3\vec{j})\end{aligned}$$

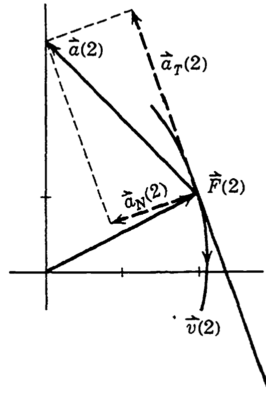


Figure 84.3

and since $3\vec{i}+\vec{j}$ is a vector along the normal, then

$$\begin{aligned}\vec{a}_N(2) &= \frac{(-2\vec{i}+2\vec{j})\cdot(3\vec{i}+\vec{j})}{(3\vec{i}+\vec{j})^2}(3\vec{i}+\vec{j})\\ &= -\frac{2}{5}(3\vec{i}+\vec{j}).\end{aligned}$$

Vector derivative systems involve two scalar derivative systems.

Example 2. Solve the derivative system

$$\vec{F}'(t) = \vec{i}\sin t - 3t^2\vec{j}\,; \quad \vec{F}(0) = 4\vec{i}+\vec{j}.$$

Solution. First $\vec{F}(t) = \vec{i}(-\cos t + c_1) + \vec{j}(-t^3 + c_2)$ since if $x'(t) = \sin t$ then $x(t) = -\cos t + c_1$ and if $y'(t) = -3t^2$ then $y(t) = -t^3 + c_2$. Thus

$$\vec{F}(0) = \vec{i}(-\cos 0 + c_1) + \vec{j}(0 + c_2) = \vec{i}(-1 + c_1) + c_2\vec{j} = 4\vec{i} + \vec{j}$$

The equality of the coefficients of $\vec{i}$ and of $\vec{j}$ gives $-1 + c_1 = 4$ and $c_2 = 1$ so that $\vec{F}(t) = \vec{i}(-\cos t + 5) + \vec{j}(-3t^2 + 1)$.

Example 3. Let a projectile (considered as a particle) be shot at time $t = 0$ from the origin with initial velocity $\vec{v}_0$ having amplitude α. If the force of gravity g is the only other force considered (i.e., air resistance, rifling, etc., are neglected) show that the vector $\vec{F}(t)$ from the origin to the position of the projectile is given by

$$(9)\quad \vec{F}(t) = \vec{i}|\vec{v}_0|\, t\cos\alpha + \vec{j}(|\vec{v}_0| t\sin\alpha - \tfrac{1}{2}gt^2).$$

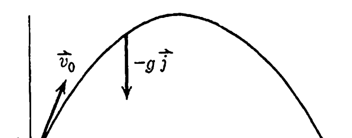


Figure 84.4

Solution. Since the projectile is at the origin when $t = 0$, then

$$(10)\qquad \vec{F}(0) = \vec{0} = 0\vec{i} + 0\vec{j}.$$

The initial velocity $\vec{v}_0$ has amplitude α and

$$(11)\quad \vec{F}'(0) = \vec{v}_0 = \vec{i}|\vec{v}_0|\cos\alpha + \vec{j}|\vec{v}_0|\sin\alpha.$$

For $t > 0$ the only force is gravity. Thus, the acceleration $\vec{a}(t)$ has x-component $\vec{0}$ and y-component downward with magnitude g. Since $\vec{j}$ points upward

$$\vec{F}''(t) = 0\vec{i} - g\vec{j}.$$

Thus $\vec{F}'(t) = c_1\vec{i} + \vec{j}(-gt + c_2)$ so that $\vec{F}'(0) = c_1\vec{i} + c_2\vec{j}$. Hence, from (11), $c_1 = |\vec{v}_0| \cos \alpha$, $c_2 = |\vec{v}_0| \sin \alpha$ and

$$\vec{F}'(t) = \vec{i}|\vec{v}_0| \cos \alpha + \vec{j}(|\vec{v}_0| \sin \alpha - gt).$$

By another integration

$$\vec{F}(t) = \vec{i}(t|\vec{v}_0| \cos \alpha + c_3) + \vec{j}(t|\vec{v}_0| \sin \alpha - \tfrac{1}{2}gt^2 + c_4).$$

Upon setting $t = 0$ and then using (10) we obtain $c_3 = c_4 = 0$, and thus see that (9) holds.

PROBLEMS

1. For each of the following laws of motion, find the vectors of position, velocity, and acceleration and the tangential and normal components of acceleration at the time indicated. Also, make a diagram showing these vectors and a portion of the path.

a. $\vec{F}(t) = \vec{i}(t^2 - t + 3) + \vec{j}(-2 + 4t - \frac{1}{3}t^3)$; $t = 1$.

b. $\vec{F}(t) = \vec{i}(t^2 + 1) + \vec{j}t^3$; $t = 2$.

c. $\vec{F}(t) = \vec{i}e^{1-t} + \vec{j}(t^2 + 1)$; $t = 1$.

d. $\vec{F}(t) = \vec{i} \sin \pi t + \vec{j} \cos \pi t$; $t = \frac{1}{3}$.

2. For each of the following find $\vec{v}(t)$, $\vec{a}(t)$, $\vec{a}_T(t)$, and $\vec{a}_N(t)$.

a. $\vec{F}(t) = \vec{i}\,3 \sin t + \vec{j}\,2 \cos t$.

b. $\vec{F}(t) = \vec{i}e^t + \vec{j}e^{2t}$.

c. $\vec{F}(t) = \vec{i}(2t - 3) + \vec{j}(t^2 + t)$.

d. $\vec{F}(t) = \vec{i} \sin 2t + \vec{j} \cos t$.

e. $\vec{F}(t) = \vec{i} \sinh t + \vec{j} \cosh t$.

f. $\vec{F}(t) = \vec{i} \tanh t + \vec{j} \operatorname{sech} t$.

g. $\vec{F}(t) = \vec{i}(\cos t + t \sin t) + \vec{j}(\sin t - t \cos t)$.

h. $\vec{F}(t) = \vec{i}r(\omega t - \sin \omega t) + \vec{j}r(1 - \cos \omega t)$.

3. With r and ω constants a law of motion is

$$\vec{F}(t) = \vec{i}r \cos \omega t + \vec{j}r \sin \omega t.$$

a. Show that the path is a circle and the speed is constant.

b. Show that $\vec{v}_x(t)$, $\vec{v}_y(t)$, $\vec{a}_x(t)$, and $\vec{a}_y(t)$ are all harmonic.

c. Show that $\vec{a}(t) = -\omega^2\vec{F}(t)$ and explain what this means.

d. Independently of Part c show that $\vec{a}_T(t) = \vec{0}$ and $\vec{a}_N(t) = \vec{a}(t)$.

4. For the law of motion $\vec{F}(t) = \vec{i}(2t + 1) + \vec{j}(t^2 - 2t)$ find when the speed is minimum.

5. For each of the following, find $\vec{F}(t)$.

a. $\vec{a}(t) = 16\vec{j}$, $\vec{v}(0) = 30\vec{i}$, $\vec{F}(0) = 100\vec{j}$.

b. $\vec{a}(t) = -\vec{i} \cos t - \vec{j} \sin t$, $\vec{v}(0) = \vec{j}$, $\vec{F}(0) = \vec{i}$.

c. $\vec{a}(t) = \vec{i}e^t + \vec{j}4e^{-2t}$, $\vec{v}(0) = \vec{i} - 2\vec{j}$, $\vec{F}(0) = 2\vec{i} + \vec{j}$.

d. $\vec{a}(t) = -\vec{i}(t + 1)^{-2} - 2\vec{j} \sec^2 t \tan t$, $\vec{v}(0) = \vec{j}$, $\vec{F}(0) = \vec{i}$.

6. For $\vec{F}(t) = \vec{i}e^t \cos t + \vec{j}e^t \sin t$, show that $\vec{a}(t) \cdot \vec{F}(t) = 0$ and interpret this result geometrically.

7. The 16 lb shot used in athletic field events is relatively so small and travels so slowly that air resistance may be neglected. For a certain athlete the shot leaves his hand $6\frac{1}{2}$ ft above ground at 40 ft/sec. For α the initial angle of elevation, find the law of motion of the shot. Show that the distance along the ground to where the shot hits is (using $g = 32$ ft/sec²)

$$S(\alpha) = 5 \cos \alpha(10 \sin \alpha + \sqrt{(10 \sin \alpha)^2 + 26}) \text{ ft.}$$

Check that $S(45°) = 55$ ft 9.9 in. and $S(43°) = 56$ ft 1.5 in.

85. Curvature

For a particle following the vector law of motion $\vec{F}$, consider the normals to the path at $\vec{F}(t)$ and $\vec{F}(t + \Delta t)$. Unless these normals are parallel they intersect and, if they do, the intersection may approach a definite point as $\Delta t \to 0$. Whenever there is such a limiting point, this point is called the **center of curvature** and the vector from $\vec{F}(t)$ to the center of curvature is called the **vector radius of curvature** $\vec{R}(t)$ of the path at $\vec{F}(t)$. The magnitude of a vector radius of curvature is called a **radius of curvature** $r(t)$.

We now derive formulas for obtaining the center and vector radius of curvature given that $\vec{F}(t)$ has the form

$$\vec{F}(t) = \vec{i}x(t) + \vec{j}y(t)$$

where the scalar functions x and y have second derivatives.

THEOREM 85. *For t a number interior to the domain of both x'' and y'' and such that*

$$x'(t)y''(t) - y'(t)x''(t) \neq 0 \tag{1}$$

then at $\vec{F}(t)$ the center of curvature of the path exists and is the terminal end of the vector

$$\vec{F}(t) + \vec{R}(t) \tag{2}$$

where $\vec{R}(t)$, and then $r(t) = |\vec{R}(t)|$, are given by

$$\vec{R}(t) = \frac{x'^2(t) + y'^2(t)}{x'(t)y''(t) - y'(t)x''(t)}(-\vec{i}y'(t) + \vec{j}x'(t)), \tag{3}$$

$$r(t) = \frac{[x'^2(t) + y'^2(t)]^{3/2}}{|x'(t)y''(t) - y'(t)x''(t)|}.$$

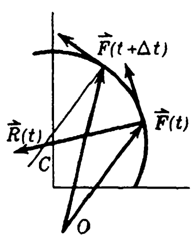


Figure 85.1

PROOF. Select $\Delta t \neq 0$ but so small that, from (1)

$$x'(t)\frac{y'(t + \Delta t) - y'(t)}{\Delta t} - y'(t)\frac{x'(t + \Delta t) - x'(t)}{\Delta t} \neq 0. \tag{4}$$

We shall use the notations $x = x(t)$, $\Delta x = x(t + \Delta t) - x(t)$, $\Delta x' = x'(t + \Delta t) - x'(t)$, etc. Thus (4) becomes, after multiplying by Δt,

$$x'\,\Delta y' - y'\,\Delta x' \neq 0. \tag{5}$$

Now at the points (x,y) and $(x + \Delta x, y + \Delta y)$, normal vectors are

$$-\vec{i}y' + \vec{j}x' \quad \text{and} \quad -\vec{i}(y' + \Delta y') + \vec{j}(x' + \Delta x').$$

The normals themselves intersect if there are numbers a and b such that (think of the vector from the origin to such a point of intersection via $\vec{F}(t)$ and $\vec{F}(t + \Delta t)$)

$$\begin{aligned} \text{(6)} \quad & \vec{i}x + \vec{j}y + a[-\vec{i}y' + \vec{j}x'] \\ & \qquad = \vec{i}(x + \Delta x) + \vec{j}(y + \Delta y) + b[-\vec{i}(y' + \Delta y') + \vec{j}(x' + \Delta x')]. \end{aligned}$$

Such numbers a and b must satisfy (by equating coefficients of $\vec{i}$ and of $\vec{j}$)

$$x - ay' = x + \Delta x - b(y' + \Delta y'), \quad y + ax' = y + \Delta y + b(x' + \Delta x');$$

that is,

$$-ay' = \Delta x - b(y' + \Delta y'), \quad ax' = \Delta y + b(x' + \Delta x').$$

By multiplying the first by $(x' + \Delta x')$, the second by $y' + \Delta y'$, and adding we obtain

$$a[-y'(x' + \Delta x') + x'(y' + \Delta y')] = \Delta x(x' + \Delta x') + \Delta y(y' + \Delta y'), \quad \text{or}$$

$$a(x'\,\Delta y' - y'\,\Delta x') = \Delta x(x' + \Delta x') + \Delta y(y' + \Delta y'). \tag{7}$$

Thus, by (5) there is an appropriate number a, and we need not even find b to see that the vector $\overrightarrow{OC}$ from the origin to the point C of intersection of the normals is

$$\overrightarrow{OC} = \vec{i}x + \vec{j}y + \frac{\Delta x(x' + \Delta x') + \Delta y(y' + \Delta y')}{x'\,\Delta y' - y'\,\Delta x'}(-\vec{i}y' + \vec{j}x')$$

(obtained by substituting the value of a from (7) into the left side of (6)). Now divide both numerator and denominator of the fractional part by Δt and use the facts:

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = x'(t), \quad \lim_{\Delta t \to 0} \frac{\Delta y'}{\Delta t} = y''(t), \quad \lim_{\Delta t \to 0} \Delta x' = \lim_{\Delta t \to 0} \Delta y' = 0, \quad \text{etc.,}$$

to see that as $\Delta t \to 0$ the fractional part approaches $\vec{R}(t)$ as given in (3), and thus that the center of curvature is the terminal end of the vector in (2).

By considering the intersection of the normals at $\vec{F}(t)$ and $\vec{F}(t + \Delta t)$ relative to the convexity of this portion of the path we see that: *The vector* $\vec{R}(t)$ *is an* **internal normal** *to the path at* $\vec{F}(t)$. We now prove:

COROLLARY 85. *The acceleration vector* $\vec{a}(t)$ *is on the concave side of the path at* $\vec{F}(t)$.

PROOF. Let θ, $-180° < \theta \leq 180°$, be the angle from $\vec{a}(t)$ to $\vec{R}(t)$ so that, for the scalar product,

$$\vec{a}(t) \cdot \vec{R}(t) = |\vec{a}(t)|\, |\vec{R}(t)| \cos \theta.$$

But $\vec{a}(t) = \vec{i}x''(t) + \vec{j}y''(t)$, and since

$$[\vec{i}x''(t) + \vec{j}y''(t)] \cdot [-\vec{i}y'(t) + \vec{j}x'(t)] = -x''(t)y'(t) + y''(t)x'(t)$$

we see (using (3)) that also $\vec{a}(t) \cdot \vec{R}(t) = x'^2(t) + y'^2(t)$. Thus

$$|\vec{a}(t)|\, |\vec{R}(t)| \cos \theta = x'^2(t) + y'^2(t) > 0 \quad \text{so} \quad \cos \theta > 0.$$

Hence, the angle θ from $\vec{a}(t)$ to the known internal normal vector $\vec{R}(t)$ is such that $|\theta| < 90°$ and thus $\vec{a}(t)$ also points into the region on the concave side of the path at $\vec{F}(t)$.

The path for the vector law of motion $\vec{F}$ is said to have **circle of curvature** at $\vec{F}(t)$ that circle whose center is the center of curvature and whose radius is $r(t)$.

Example 1. Find an equation of the circle of curvature at $\vec{F}(1)$ given that

$$\vec{F}(t) = t^3\vec{i} + 2t^2\vec{j}.$$

Solution. Here $x(t) = t^3$ and $y(t) = 2t^2$. Thus, we obtain

$$x(1) = 1,\ y(1) = 2,\ x'(1) = 3,\ x''(1) = 6,\ y'(1) = y''(1) = 4 \quad \text{and}$$
$$x'(1)y''(1) - y'(1)x''(1) = 3\cdot 4 - 4\cdot 6 = -12 \neq 0.$$

Hence, the center of curvature at $\vec{F}(1) = \vec{i} + 2\vec{j}$ exists and is the terminal end of the vector

$$\vec{i} + 2\vec{j} + \frac{3^2 + 4^2}{-12}(-4\vec{i} + 3\vec{j}) = \frac{28}{3}\vec{i} - \frac{17}{4}\vec{j}.$$

Thus, the circle of curvature has center $(\frac{28}{3}, -\frac{17}{4})$ and radius

$$r(1) = \frac{(25)^{3/2}}{|-12|} = \frac{125}{12}.$$

A vector and a rectangular equation of the circle of curvature are

$$\left[\left(x - \frac{28}{3}\right)\vec{i} + \left(y + \frac{17}{4}\right)\vec{j}\right]^2 = \left|\frac{3^2 + 4^2}{-12}(-4\vec{i} + 3\vec{j})\right|^2 \quad \text{and}$$

$$\left(x - \frac{28}{3}\right)^2 + \left(y + \frac{17}{4}\right)^2 = \left(\frac{125}{12}\right)^2.$$

To obtain the center of curvature and vector radius of curvature at $(x, f(x))$ for the graph of a scalar function f we may parametrize the graph by setting

$$x = t, \qquad y = f(t)$$

to obtain the vector function $\vec{F}(t) = \vec{i}t + \vec{j}f(t)$. Now (3) and then (2) become

$$\text{(8)} \qquad \vec{R}(t) = \frac{1 + f'^2(t)}{1 \cdot f''(t) - f'(t) \cdot 0}(-\vec{i}f'(t) + \vec{j}) \quad \text{and}$$

$$\vec{C}(t) = \vec{F}(t) + \vec{R}(t) = \vec{i}t + \vec{j}f(t) + \frac{1 + f'^2(t)}{f''(t)}(-\vec{i}f'(t) + \vec{j}).$$

Hence, in the original rectangular coordinates, the center $(h(x), k(x))$ of curvature and the radius of curvature at $(x, f(x))$ are

$$\text{(2')} \quad h(x) = x - f'(x)\frac{1 + f'^2(x)}{f''(x)}, \quad k(x) = f(x) + \frac{1 + f'^2(x)}{f''(x)}, \quad \text{and}$$

$$\text{(3')} \quad r(x) = |R(x)| = \sqrt{\left[\frac{1 + f'^2(x)}{f''(x)}\right]^2 [(-f'(x))^2 + 1]} = \frac{[1 + f'^2(x)]^{3/2}}{|f''(x)|},$$

$$f''(x) \neq 0.$$

The reciprocal of the radius (not the vector radius) of curvature is called the **curvature** κ of the graph at the point considered. Hence, formulas for curvature of graphs of $\vec{F}$ and f are

$$\text{(9)} \qquad \kappa = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{[x'^2(t) + y'^2(t)]^{3/2}} \quad \text{and} \quad \kappa = \frac{|f''(x)|}{[1 + f'^2(x)]^{3/2}}.$$

The notion of curvature is also extended to those graphs for which $x'(t)y''(t) - y'(t)x''(t) = 0$ or $f''(x) = 0$, but in such cases no radius of curvature exists. Thus, every straight line has curvature zero at each of its points.

Example 2. Show that at each point of a circle, the curvature is the reciprocal of the radius of the circle.

Solution. A vector equation of the circle is $\vec{F}(t) = \vec{i}a\cos t + \vec{j}a\sin t$. Now $x'(t) = -a\sin t$, $x''(t) = -a\cos t$, $y'(t) = a\cos t$, $y''(t) = -a\sin t$, and from (3)

$$r(t) = \frac{[(-a\sin t)^2 + (a\cos t)^2]^{3/2}}{|(-a\sin t)(-a\sin t) - (a\cos t)(-a\cos t)|} = \frac{a^3}{a^2} \quad \text{so} \quad \kappa = \frac{1}{a}.$$

A railroad track goes from a straight section to a circular section by means of a transition (or easement) curve whose curvature continuously increases from zero at contact with the straight section to the reciprocal of the radius of the circular section and moreover the transition curve is tangent to both the straight and the circular sections.

Example 3. A section of railroad track coincides with the negative x-axis, then along the graph of $y = \frac{1}{3}x^3$ for $0 \le x \le 1$, and then along the circle with center at $(0,\frac{4}{3})$. Show the curve, the slope function, and the curvature are all continuous.

Solution. Let $f(x) = \frac{1}{3}x^3$. Since $f(0) = 0$ and $f(1) = \frac{1}{3}$, the transition curve makes proper contact and the whole graph is continuous. The straight section has slope 0, the circular section has slope 1 at $(1,\frac{1}{3})$ and, since $f'(x) = x^2$, also $f'(0) = 0$ and $f'(1) = 1$ so the slope function is continuous. The circle has radius $\sqrt{2}$ and thus curvature $1/\sqrt{2}$. Since $f''(x) = 2x$, the curvature for the transition curve is

$$\kappa(x) = \frac{2x}{[1 + (x^2)^2]^{3/2}}, \quad 0 \le x \le 1$$

so κ is continuous on this section. But $\kappa(0) = 0$ and $\kappa(1) = 1/\sqrt{2}$ which are the proper values at both ends of the transition curve. So κ is continuous throughout.

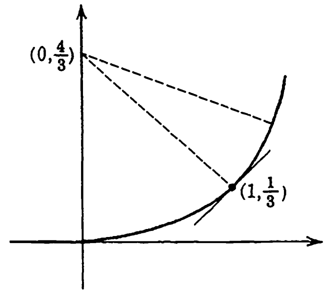


Figure 85.2

PROBLEMS

1. Find an equation of the circle of curvature for the graph of:

   a. $y = x^2$ at $(-2,4)$.
   b. $y = x^2$ at $(0,0)$.
   c. $y = x^3$ at $(1,1)$.
   d. $xy = 2$ at $(1,2)$.
   e. $y = \sin x$ where $x = \pi/4$.
   f. $y = \frac{1}{2}(e^x + e^{-x})$ where $x = 0$.
   g. $y = \ln x$ where $x = 1$.
   h. $y^2 + xy = 1$ at $(0,1)$.
   i. $\vec{F}(t) = (2t - 1)\vec{i} + t^2\vec{j}$ when $t = 2$.
   j. $\vec{F}(t) = \vec{i}t^2 + \vec{j}10^t$ when $t = 0$.
   k. $\vec{F}(t) = \vec{i}\sin t + \vec{j}\cos 2t$ when $t = \pi/6$.
   l. $\vec{F}(t) = \vec{i}t + \vec{j}e^t$ when $t = 0$.

2. Show that the graph of the following pairs of equations intersect at the given point and at this point have common tangents and curvatures.

   a. $y = x, \quad y = x^3 + x; \quad (0,0)$.
   b. $y = x - 1, \quad y = x^3 - 3x^2 + 4x - 2; \quad (1,0)$
   c. $y = \frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{2}x - \frac{7}{6}$,

$$y = -\frac{1}{\pi^2}\sin(x - 1)\pi + x^2 + \left(\frac{1}{\pi} - 1\right)x - \frac{1}{\pi} - 1; \quad (1,-1).$$

3. A railroad track is to go along the negative x-axis, then along a transition curve to the point $(3,1)$, and then on an arc of a circle. Find the center of the circular arc if the transition curve has equation in the form

   a. $y = ax^3$
   b. $y = ax^4$
   c. $y = a_0 + a_1x + a_2x^2 + a_3x^3$.

4. A railroad goes along the negative x-axis, then along a transition curve to the point (4,1), and then along a circular arc with center (1,5). Find an equation of the transition curve in the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5.$$

5. For the curvature function κ of Example 3, show that the derived function κ' is not continuous.

86. Rectifiable Curves

With $a < b$ and f a continuous function on $I[a,b]$, the question of assigning a length to the graph of f between the points $(a, f(a))$ and $(b, f(b))$ is attacked as follows:

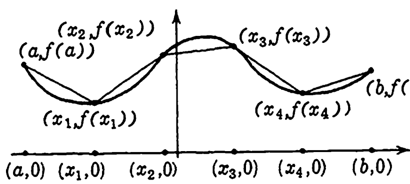


Figure 86.1

(i) Let $\Delta_n x = \dfrac{b-a}{n}$ and $x_k = a + k\dfrac{b-a}{n}$ for $k = 0, 1, 2, \cdots, n$.

(ii) Join the points $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2)), \cdots, (x_{n-1}, f(x_{n-1})), (x_n, f(x_n))$ in succession by line segments, thus forming an inscribed polygon.

(iii) Find the sum of the lengths of these segments:

$$(1)\qquad \sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + [f(x_k) - f(x_{k-1})]^2}$$
$$= \sum_{k=1}^{n} \sqrt{[\Delta_n x]^2 + [f(x_k) - f(x_{k-1})]^2}$$
$$= \sum_{k=1}^{n} \sqrt{1 + \left[\frac{f(x_k) - f(x_{k-1})}{\Delta_n x}\right]^2}\,\Delta_n x.$$

(iv) Whenever the limit as $n \to \infty$ of such sums exists, the value of this limit is denoted by s, is defined to be the **length** of the arc considered, and this arc is said to be **rectifiable**.

THEOREM 86. *For f a function such that f' is continuous on* $I[a,b]$, *then the arc of the graph of f joining the points $(a, f(a))$ and $(b, f(b))$ is rectifiable and has length*

$$(2)\qquad s = \int_a^b \sqrt{1 + f'^2(x)}\, dx.$$

For a proof of this theorem see Appendix A5.

Example 1. Find the length of the arc of the parabola having equation $y = x^2$ between the points having abscissas 1 and 3.

Solution. Here $a = 1, b = 3, f(x) = x^2, f'(x) = 2x, 1 + f'^2(x) = 1 + 4x^2$, and

$$s = \int_1^3 \sqrt{1 + 4x^2}\, dx = 2\int_1^3 \sqrt{\left(\frac{1}{2}\right)^2 + x^2}\, dx$$

$$= 2 \cdot \frac{1}{2}\left\{x\sqrt{\left(\frac{1}{2}\right)^2 + x^2} + \frac{1}{4}\ln\left|x + \sqrt{\frac{1}{4} + x^2}\right|\right\}_1^3$$

$$= \frac{1}{2}\left\{3\sqrt{37} - \sqrt{5} + \frac{1}{2}\ln\frac{6 + \sqrt{37}}{2 + \sqrt{5}}\right\}.$$

A formula which is considered to be equivalent to (2) is

$$(3) \qquad s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

A convenient way of remembering (3) (and thus (2)) is to think of a curve joining two points (a,c) and (b,d) and then to consider, with $\Delta x_k > 0$, that

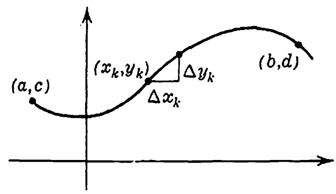


Figure 86.2.

$$\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\, \Delta x_k$$

is the length of one chord of an approximating inscribed polygon.

Should the curve be the graph of $x = g(y)$ with $c < d$, then consider $\Delta y_k > 0$, and form

$$\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{\left(\frac{\Delta x_k}{\Delta y_k}\right)^2 + 1}\, \Delta y_k$$

to obtain the pattern for the integrands in

$$(4) \qquad s = \int_c^d \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy = \int_c^d \sqrt{g'^2(y) + 1}\, dy.$$

Example 2. Find the length of the graph of $y^3 = 8x^2$ joining the points (1,2) and (27,18).

Solution. Now $y = 2x^{2/3}$ so $\dfrac{dy}{dx} = \dfrac{4}{3}x^{-1/3}$ and

$$s = \int_1^{27} \sqrt{1 + \frac{16}{9}x^{-2/3}}\, dx.$$

But upon solving the given equation for x instead of for y, then

$$x = \frac{1}{\sqrt{8}} y^{3/2}, \quad \frac{dx}{dy} = \frac{3}{2\sqrt{8}} y^{1/2}, \quad \left(\frac{dx}{dy}\right)^2 + 1 = \frac{9}{32} y + 1, \quad \text{and}$$

$$s = \int_2^{18} \sqrt{\frac{9}{32} y + 1}\, dy.$$

Since the second integrand is easier to evaluate, we proceed with it:

$$s = \tfrac{32}{9}(\tfrac{9}{32} y + 1)^{3/2} \tfrac{2}{3}\Big]_2^{18} = \tfrac{1}{27} \{(97)^{3/2} - 125\}.$$

We shall not prove (but shall use) the fact that if the law of motion of a particle is

$$x = x(t), \quad y = y(t) \quad \text{for} \quad \alpha \le t \le \beta$$

(the derived function x' and y' being continuous), then the total distance traveled by the particle during this time interval is

$$(5) \qquad s = \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t)}\, dt$$

even if the particle retraces on its path.

Example 3. Find the distance traveled by a particle whose law of motion is

$$x = 2 \sin t, \quad y = \tfrac{1}{2}(1 - \cos 2t), \quad 0 \le t \le \pi.$$

Solution. We could analyze the problem to see that the answer is twice the length of a parabolic arc from (0,0) to (2,1), but proceeding directly have (since $x' = 2 \cos t$ and $y' = \sin 2t$) that

$$s = \int_0^\pi \sqrt{4 \cos^2 t + \sin^2 2t}\, dt = \int_0^\pi \sqrt{4 \cos^2 t + (2 \sin t \cos t)^2}\, dt$$

$$= \int_0^\pi \sqrt{4 \cos^2 t(1 + \sin^2 t)}\, dt = 2\int_0^\pi |\cos t| \sqrt{1 + \sin^2 t}\, dt$$

$$= 2\left\{\int_0^{\pi/2} \cos t \sqrt{1 + \sin^2 t}\, dt + \int_{\pi/2}^\pi (-\cos t) \sqrt{1 + \sin^2 t}\, dt\right\}$$

since $\cos t \ge 0$ for $0 \le t \le \pi/2$, but $\cos t \le 0$ for $\pi/2 \le t \le \pi$. Now by changing the dummy variable to u where $u = \sin t$ and $du = \cos t\, dt$,

$$s = 2\left\{\int_0^1 \sqrt{1 + u^2}\, du - \int_1^0 \sqrt{1 + u^2}\, du\right\}$$

$$= 2\left\{\int_0^1 \sqrt{1 + u^2}\, du + \int_0^1 \sqrt{1 + u^2}\, du\right\} = 4\int_0^1 \sqrt{1 + u^2}\, du$$

$$= \tfrac{4}{2}\left[u\sqrt{1 + u^2} + \ln|u + \sqrt{1 + u^2}|\right]_0^1 = 2[\sqrt{2} + \ln(1 + \sqrt{2})].$$

For the graph of f on $I[a,b]$ let s be the function also on $I[a,b]$ defined by

$$s(x) = \int_a^x \sqrt{1 + f'^2(t)}\, dt, \quad a \le x \le b.$$

By the Fundamental Theorem of Calculus, $s'(x)$ exists and

$$s'(x) = \sqrt{1 + f'^2(x)} \quad \text{for} \quad a \le x \le b.$$

Since (by the definition of differentials) $ds = s'(x)\,dx$, we have

$$\text{(6)} \qquad \boldsymbol{ds = \sqrt{1 + f'^2(x)}\, dx}$$

which is referred to as the **differential of arc length**. We may also write

$$\text{(7)} \qquad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \quad ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

or in the parametric case with s increasing as t increases

$$\text{(8)} \qquad \boldsymbol{ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{x'^2(t) + y'^2(t)}\, dt.}$$

PROBLEMS

1. For each of the following, find the length of the graph between the points indicated.
   a. $y = \frac{1}{2}(e^x + e^{-x})$ points with abscissas -1 and 1.
   b. $y = \ln \sin x$, points with abscissas $\pi/4$ and $\pi/3$.
   c. $y = \ln(1 - x^2)$, points with abscissas 0 and 0.9.
   d. $y = e^x$ points $(0,1)$ and $(1,e)$.
   e. $y = \ln(x + \sqrt{x^2 - 1})$, points with abscissas 2 and 3.
   f. $y = \frac{1}{3}x^3 + \frac{1}{4}x^{-1}$ points with abscissas 1 and 2.

2. Find the circumference of a circle (a) by using rectangular coordinates, (b) by using parametric equations.

3. For each of the following laws of motion, find the distance traveled by the particle during the given time interval.
   a. $x = 2t + 1, \quad y = t^2; \quad 1 \le t \le 3.$
   b. $x = \cos t, \quad y = \cos^2 t; \quad 0 \le t \le 2\pi.$
   c. $x = 2t^2, \quad y = t^3; \quad -1 \le t \le 1.$
   d. $x = t^2, \quad y = \frac{2}{5}t^{5/2}; \quad 0 \le t \le 4.$
   e. $x = 2t - t^2, \quad y = (t - 1)^2; \quad 0 \le t \le 2.$

4. a. Find the length of one arch of the cycloid. (See (2) Sec. 82).
   b. For the special hypocycloid of Prob. 1e, Sec. 82, find the length for $0 \le \theta \le 2\pi$.

87. Parametric Derivatives

The parametric equations

$$x = x(t), \quad y = y(t) \tag{1}$$

and the vector function $\vec{F}$ defined by $\vec{F}(t) = \vec{i}x(t) + \vec{j}y(t)$ have identical graphs. Moreover, assuming the existence of derivatives, the velocity function $\vec{v}$ is such that $\vec{v}(t) = \vec{i}x'(t) + \vec{j}y'(t)$ is a vector along the tangent to the graph at the point $(x(t),y(t))$ so that this tangent has slope

$$\frac{y'(t)}{x'(t)} = \frac{D_t y(t)}{D_t x(t)} \qquad \text{provided } x'(t) \neq 0.$$

With $x'(t) \neq 0$, then the ordinate of any point on a portion of the graph may be determined from the abscissa of the point; that is, there is a function expressing y in terms of x for (x,y) on the graph. Whether or not an explicit expression can be found for y in terms of x, a function does exist and has a derivative, denoted by $D_x y$, whose value at a point is the slope of the graph at the point so that

$$D_x y = \frac{D_t y(t)}{D_t x(t)}. \tag{2}$$

Example 1. For $x(t) = \sin t$ and $y(t) = \cos 2t$, then

$$D_x y = \frac{D_t \cos 2t}{D_t \sin t} = \frac{-2 \sin 2t}{\cos t} = -4 \frac{\sin t \cos t}{\cos t} = -4 \sin t, \quad \cos t \neq 0.$$

Formula (2) may be used repeatedly to find derivatives of higher order. Since $D_x^2 y = D_x[D_x y]$, replace $[D_x y]$ by the right side of (2) to obtain

$$D_x^2 y = D_x\left[\frac{D_t y(t)}{D_t x(t)}\right] = \frac{D_t\left[\dfrac{D_t y(t)}{D_t x(t)}\right]}{D_t x(t)}$$

$$= \frac{\dfrac{D_t x(t) D_t[D_t y(t)] - D_t y(t) D_t[D_t x(t)]}{[D_t x(t)]^2}}{D_t x(t)}$$

$$= \frac{D_t x(t) D_t^2 y(t) - D_t y(t) D_t^2 x(t)}{[D_t x(t)]^3}$$

It is usually better, however, when specific expressions for $x(t)$ and $y(t)$ are given, to first express $D_x y$ specifically in terms of t.

Example 2. Find $D_x^2 y$ given $x(t) = \sin t$, $y(t) = \cos 2t$.

Solution. Had we not already done so in Example 1, we would first find $D_x y = -4 \sin t$. Now

$$D_x^2 y = D_x(D_x y) = D_x(-4 \sin t) = \frac{D_t(-4 \sin t)}{D_t x} \quad \begin{pmatrix}\text{from (2) with } y \text{ replaced by}\\ -4 \sin t\end{pmatrix}$$

$$= \frac{-4 \cos t}{D_t \sin t} = \frac{-4 \cos t}{\cos t} = -4.$$

A formal procedure is to start with (1), use differentials, write $dx = x'(t)\,dt$, $dy = y'(t)\,dt$, and then divide to obtain

$$\frac{dy}{dx} = \frac{y'(t)\,dt}{x'(t)\,dt} = \frac{y'(t)}{x'(t)} = \frac{D_t y(t)}{D_t x(t)},$$

which is equivalent to (2). A further equivalent expression entirely in terms of differentials is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{3}$$

Given a rectifiable graph and a fixed point P_0 on it, then any point $P = (x,y)$ of the graph is at a definite distance *along the graph* from P_0. Considering s as measured along the graph positively in one direction from P_0 but negatively in the other direction, it is usual to write

$$x = x(s), \quad y = y(s)$$

and to say "The graph is parametrized in terms of arc length."

Example 3. For a function f show that $\left|\dfrac{d}{ds} \tan^{-1} f'(x)\right|$, where $f'(x) = D_x f(x)$, is the formula for curvature. (See (9) of Sec. 85)

Solution. Schematically $\dfrac{d[\ \ ]}{ds} = \dfrac{d[\ \ ]}{dx}\dfrac{dx}{ds}$ so that, with $\tan^{-1} f'(x)$ placed inside both brackets,

$$\frac{d[\tan^{-1} f'(x)]}{ds} = \frac{d[\tan^{-1} f'(x)]}{dx}\frac{dx}{ds}$$

$$= \frac{\dfrac{d}{dx} f'(x)}{1 + f'^2(x)} \frac{dx}{\sqrt{1 + f'^2(x)}\,dx}.$$

The last step followed from the formula for the derivative of the inverse tangent function and $ds = \sqrt{1 + f'^2(x)}\,dx$ as given in (6) Sec. 86. Thus

$$\frac{d}{ds} \tan^{-1} f'(x) = \frac{f''(x)}{[1 + f'^2(x)]^{3/2}}$$

and the absolute value of the right side is the formula (9) of Sec. 85 for the curvature† function κ.

PROBLEMS

1. Find $D_x y$ and $D_x^2 y$ given that:

a. $x = 2t, \quad y = \frac{1}{3}t^3$.
b. $x = 3\cos t, \quad y = 2\sin t$.
c. $x = \sin 2\theta, \quad y = \sin\theta$.
d. $x = a\cos^3\theta, \quad y = a\sin^3\theta$.
e. $x = te^{-t}, \quad y = e^t$.
f. $x = e^t, \quad y = 1 + t^2$.
g. $x = 1 + t^2, \quad y = e^{2t}$.
h. $x = \ln t, \quad y = t$.

2. On the graphs of each of the following pairs of parametric equations find the points where the tangents are parallel to the coordinate axes.

a. $x = \frac{1}{2}t, \quad y = \frac{1}{8}(12t - t^3)$.
b. $x = 2 + 5\cos t, \quad y = 3 + 5\sin t$.
c. $x = \sin\theta, \quad y = \cos 2\theta$.
d. $x = t^2 - 1, \quad y = t^3 - t$.

3. For the law of motion $\vec{F}(t) = \vec{i}x(t) + \vec{j}y(t)$, and with s increasing as t increases show that

a. $\dfrac{ds}{dt}$ is the speed.

b. $\vec{\tau} = \dfrac{d\vec{F}}{ds}$ is unit tangent vector to the path.

c. Find $\dfrac{d\vec{\tau}}{dt}, \dfrac{d\vec{\tau}}{ds}$ and show that $\dfrac{d\vec{\tau}}{ds} \cdot \vec{R} = 1$.

(Note: For this reason $\dfrac{1}{\vec{R}}$ is sometimes defined to be the expression obtained for $\dfrac{d\vec{\tau}}{ds}$ and is called the curvature.)

88. Rotation of Axes

Given a coordinate system in the plane and a vector $\vec{v}$ with modulus a and amplitude α, then

$$\vec{v} = a(\vec{i}\cos\alpha + \vec{j}\sin\alpha), \quad a > 0.$$

Upon rotating $\vec{v}$ about its initial end through an angle θ (where rotation is counterclockwise if $\theta > 0$, but clockwise if $\theta < 0$), then the vector $\vec{V}$ is obtained where

$$\vec{V} = a[\vec{i}\cos(\alpha + \theta) + \vec{j}\sin(\alpha + \theta)].$$

† Some books use the expression on the right as it stands for the curvature in rectangular coordinates so the curvature will be positive on concave upward portions, but negative on concave downward portions. It is possible (although more involved) to provide for negative curvatures for graphs defined parametrically, but we do not bother here since the vector radius of curvature gives full information about the concavity properties of the graph.

In particular $\vec{i} = \vec{i}\cos 0° + \vec{j}\sin 0°$ and $\vec{j} = \vec{i}\cos 90° + \vec{j}\sin 90°$ yield

$$\vec{I} = \vec{i}\cos\theta + \vec{j}\sin\theta \quad \text{and}$$

$$\vec{J} = \vec{i}\cos(90° + \theta) + \vec{j}\sin(90° + \theta)$$

$$= -\vec{i}\sin\theta + \vec{j}\cos\theta.$$

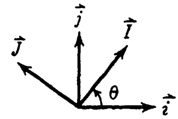


Figure 88.1

Consider now a second coordinate system in which $\vec{I}$ and $\vec{J}$ are the basic unit vectors. A point P has coordinates (x,y) in the first system and (X,Y) in the second system. The vector $\overrightarrow{OP}$, with O the common origin, may be written both as

$$\overrightarrow{OP} = \vec{i}x + \vec{j}y \quad \text{and as} \quad \overrightarrow{OP} = \vec{I}X + \vec{J}Y.$$

Thus $\vec{i}x + \vec{j}y = \vec{I}X + \vec{J}Y$ and from the above expressions for $\vec{I}$ and $\vec{J}$

$$\vec{i}x + \vec{j}y = (\vec{i}\cos\theta + \vec{j}\sin\theta)X + (-\vec{i}\sin\theta + \vec{j}\cos\theta)Y$$

$$= \vec{i}(X\cos\theta - Y\sin\theta) + \vec{j}(X\sin\theta + Y\cos\theta).$$

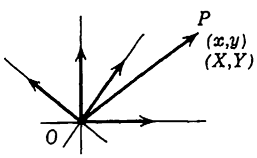


Figure 88.2

Hence x, y, X, Y, and θ are related by

$$x = X\cos\theta - Y\sin\theta,$$

(1)

$$y = X\sin\theta + Y\cos\theta.$$

Upon solving these equations for X and Y in terms of x, y, and θ we obtain

$$X = x\cos\theta + y\sin\theta$$

(2)

$$Y = -x\sin\theta + y\cos\theta.$$

Both (1) and (2) may be remembered by the schematic array

(3)

$$\begin{array}{cc} & \begin{array}{cc} X & \quad Y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{array}$$

The XY-system is said to be a **rotation** of the xy-system through the angle θ.

Example 1. A graph has equation $3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y = 0$. Find an equation of the same graph referred to an XY-system obtained by rotating the xy-system through the angle $\theta = 30°$.

Solution. By setting $\cos\theta = \cos 30° = \sqrt{\frac{3}{2}}$ and $\sin\theta = \frac{1}{2}$ into the array (3) (or into (1)) we obtain

(4)

$$x = \tfrac{1}{2}(\sqrt{3}X - Y), \quad y = \tfrac{1}{2}(X + \sqrt{3}Y).$$

Now these expressions substituted into the given equation yield

$$\frac{3}{4}(\sqrt{3}X - Y)^2 + \frac{2\sqrt{3}}{4}(\sqrt{3}X - Y)(X + \sqrt{3}Y) + \frac{1}{4}(X + \sqrt{3}Y)^2$$
$$+ (\sqrt{3}X - Y) - \sqrt{3}(X + \sqrt{3}Y) = 0.$$

The algebra should be carried out to show that this expression simplifies to $X^2 = Y$. Thus, the graph of the given expression is a parabola.

Had we been asked for the xy-coordinates of the focus and the xy-equation of the directrix of this parabola, we would proceed as follows. Since the equation $X^2 = Y$ may be written as

$$X^2 = 4(\tfrac{1}{4})Y,$$

the XY-coordinates of the focus are $(0,\frac{1}{4})$ and the XY-equation of the directrix is $Y = -\frac{1}{4}$. Hence, upon setting $X = 0$ and $Y = \frac{1}{4}$ in (4) we obtain $x = -\frac{1}{8}$ and $y = \sqrt{3}/8$ so the focus in the xy-system is the point $(-\frac{1}{8},\sqrt{3}/8)$. Now for the directrix, set $Y = -\frac{1}{4}$ in both equations of (4) to obtain

$$x = \frac{1}{2}\left(\sqrt{3}X + \frac{1}{4},\right) \quad y = \frac{1}{2}\left(X - \frac{\sqrt{3}}{4}\right).$$

These are parametric equations of the directrix and upon eliminating X between them we obtain the xy-equation of the directrix as

$$2x - 2\sqrt{3}y = 1.$$

Another way of arriving at this equation of the directrix is to set $\cos\theta = \sqrt{3}/2$ and $\sin\theta = \frac{1}{2}$ in the second equation of (2) to obtain $Y = (-x + \sqrt{3}y)/2$ and then substitute $Y = -\frac{1}{4}$.

Example 2. Given the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ with $b \neq 0$. (a) Rotate the axes through an angle θ and find the resulting equation. (b) Obtain a condition in terms of $a, b, \cdots$, that θ must satisfy in order that the XY term of the equation of Part (a) have coefficient zero.

Solution (a). Upon substituting x and y as given by (1) into the given equation and then collecting the X^2, XY, $\cdots$ terms we obtain

$$\{a\cos^2\theta + b\sin\theta\cos\theta + c\sin^2\theta\}X^2 + \{b(\cos^2\theta - \sin^2\theta) - 2(a - c)\sin\theta\cos\theta\}XY$$
$$+ \{a\sin^2\theta - b\sin\theta\cos\theta + c\cos^2\theta\}Y^2 + \{d\cos\theta + e\sin\theta\}X$$
$$+ \{-d\sin\theta + e\cos\theta\}Y + f = 0.$$

Solution (b). The coefficient of XY is zero if and only if θ satisfies the equation $b(\cos^2\theta - \sin^2\theta) - (a - c)\,2\sin\theta\cos\theta = 0$; that is

$$b\cos 2\theta = (a - c)\sin 2\theta. \tag{5}$$

Now $\sin 2\theta \neq 0$. [For if we set $\sin 2\theta = 0$ in (5) we obtain $\cos 2\theta = 0$ (since $b \neq 0$) which cannot be, since no angle has both its sine and its cosine equal to zero.] We therefore divide both sides of (5) by the non-zero term $b\sin 2\theta$ to obtain

$$\frac{\cos 2\theta}{\sin 2\theta} = \frac{a - c}{b}; \quad \text{that is,} \quad \cot 2\theta = \frac{a - c}{b}.$$

The following theorem is a statement of the facts established in Example 2.

THEOREM 88. *The graph in the xy-system of the equation*

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \qquad \text{with } b \neq 0 \tag{6}$$

will be the graph of an equation of the form

$$AX^2 + CY^2 + DX + EY + F = 0 \quad (\textit{with no } XY \textit{ term})$$

in an XY-system obtained by rotating the xy-system through any angle θ such that

$$\cot 2\theta = \frac{a - c}{b}. \tag{7}$$

Example 3. Name the conic which is the graph of

$$2x^2 - 3xy - 2y^2 - 5 = 0. \tag{8}$$

Solution. Here $a = 2$, $b = -3$, and $c = -2$ so a rotation through any angle θ such that

$$\cot 2\theta = \frac{2 - (-2)}{-3} = -\frac{4}{3}$$

will yield an equation we can classify. There is an angle in the second quadrant whose cotangent is $-\frac{4}{3}$ and we choose this angle for 2θ so θ will be in the first quadrant. Now $\cot 2\theta = -\frac{4}{3}$, 2θ is in the second quadrant, $\cos 2\theta = -\frac{4}{5}$, $\cos\theta > 0$, $\sin\theta > 0$, so that

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 - \frac{4}{5}}{2}} = \frac{1}{\sqrt{10}} \quad \text{and} \quad \sin\theta = \frac{3}{\sqrt{10}}.$$

These values substituted into (1) give the rotations

$$x = \frac{1}{\sqrt{10}}(X - 3Y), \quad y = \frac{1}{\sqrt{10}}(3X + Y).$$

The substitution of these expressions for x and y into (8) yields an equation which simplifies to $Y^2 - X^2 = 2$ which (of course) has no XY term and moreover shows that the graph of (8) is a hyperbola.

(Note: We never found θ itself. Should we wish to draw the graph of (8), we would draw the X-axis through the xy-point (1,3), since $\tan\theta = \sin\theta/\cos\theta = 3$, then draw the Y-axis and relative to these axes sketch the graph of $Y^2 - X^2 = 2$.)

PROBLEMS

1. Draw the graph, if one exists, of each of the following equations by first rotating the axes through the angle given.

   a. $x^2 + 4xy + y^2 = 1$; $45°$.

   b. $2x^2 - 2xy + 2y^2 = 1$; $45°$.

   f. $|4x - 3y| + |3x + 4y| = 5$; $\tan^{-1}(\frac{4}{3})$.

   g. $7x^2 - 2\sqrt{3}xy + 5y^2 + 4 = 0$; $-30°$.

c. $2x^2 + 2\sqrt{3}xy = 1$; $30°$.
d. $x^2 - y^2 = 1$; $45°$.
e. $|x - y| + |x + y| = \sqrt{2}$; $45°$.
h. $x^2 + y^2 + 2(xy - x + y) = 0$; $-45°$.
i. $16x^2 - 24xy + 9y^2 = 25$; $\cot^{-1}(-\frac{4}{3})$.
j. $x^2 - 2\sqrt{3}xy - y^2 = 4$; $\tan^{-1}\sqrt{3}$.

2. Let A, B, and C be the coefficients of X^2, XY, and Y^2 in the equation of Example 2(a), with no restriction on θ. Show that

$$B^2 - 4AC = b^2 - 4ac.$$

(Note: For this reason $b^2 - 4ac$ is said to be an **invariant** under rotation.)

3. Find the foci, vertices, etc., of the conics having equations
a. $4(x^2 + y^2) + xy = 63$.
b. $12xy - 5y^2 = 1$.
c. $12xy + 5y^2 = 36$.
d. $x^2 + xy + y^2 = 0$.
e. $16x^2 - 24xy + 9y^2 - 15x - 20y = 0$.

89. Polar Coordinates

With $a > 0$ and α any angle, the vector

$$a(\vec{i}\cos\alpha + \vec{j}\sin\alpha)$$

has magnitude a and α is an amplitude. Thus the vector

$$-2(\vec{i}\cos 30° + \vec{j}\sin 30°)$$

does not have magnitude -2 nor is $30°$ its amplitude.

For ρ a positive, negative, or zero number and for θ any angle, the vector

$$\rho(\vec{i}\cos\theta + \vec{j}\sin\theta)$$

with initial end at the origin has its terminal end at a point which is said to have **polar coordinates** (ρ,θ). The origin is termed the **pole** and has coordinates $(0,\theta)$ for any angle θ. As well as the pole, any point has indefinitely many pairs of polar coordinates. For example,

$$-2(\vec{i}\cos 30° + \vec{j}\sin 30°) = 2(\vec{i}\cos -150° + \vec{j}\sin -150°)$$
$$= 2(\vec{i}\sin 210° + j\sin 210°) = \text{etc.}$$

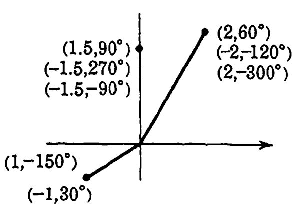


Figure 89.1

and the terminal end of this vector has polar coordinates

$$(-2,30°),\quad (2,-150°),\quad (2,210°),\quad \text{etc.}$$

Figure 89.1 illustrates some other points and some of their polar coordinates. On the other hand, given any number ρ and any angle θ, then there is one and only one point having polar coordinates (ρ,θ).

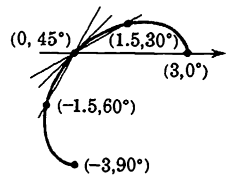
The half line consisting of all points with polar coordinates $(\rho, 0^\circ)$ with $\rho \geq 0$ is called the **polar axis.**

The **polar graph** of an equation in ρ and θ is, by definition, the set of points such that a point belongs to the set if and only if at *least one of the pairs of polar coordinates* of the point satisfies the equation.

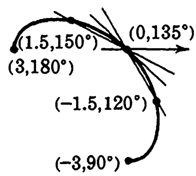
Example 1. Sketch the polar graph of $\rho = 3 \cos 2\theta$.

Solution. Quadrant-by-quadrant tables of corresponding values of ρ and θ together with the related section of the graph are given below:

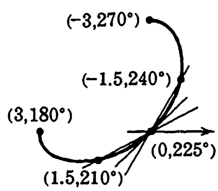
θ	0°	30°	45°	60°	90°
2θ	0	60°	90°	120°	180°
$\cos 2\theta$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1
ρ	3	1.5	0	-1.5	-3



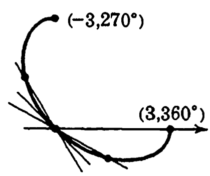
θ	90°	120°	135°	150°	180°
2θ	180°	240°	270°	300°	360°
$\cos 2\theta$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
ρ	-3	-1.5	0	1.5	3



θ	180°	210°	225°	240°	270°
2θ	360°	420°	450°	480°	540°
$\cos 2\theta$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1
ρ	3	1.5	0	-1.5	-3



θ	270°	300°	315°	330°	360°
2θ	540°	600°	630°	660°	720°
$\cos 2\theta$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
ρ	-3	-1.5	0	1.5	3



The curve then proceeds to repeat each section. The pieces are put together in Fig. 89.2 of pg. 284.

When the equation involves trigonometric functions, then after some experience the whole graph can be obtained from a portion of it. For example, since

$$\cos 2(\theta + 90^\circ) = \cos (2\theta + 180^\circ) = -\cos 2\theta$$

the portion of the graph in Example 1 corresponding to angles in the second quadrant can be predicted from the first quarter. Then since $\cos 2(\theta + 180°) = \cos 2\theta$ (or $\cos(-\theta) = \cos\theta$) the graph is symmetric to the line containing the polar axis so the rest of the graph is obtained by symmetry.

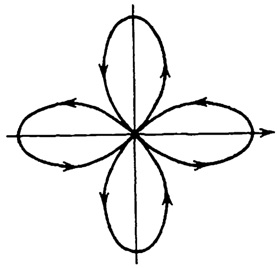


Figure 89.2

Example 2. Sketch the polar graph of $\rho = \frac{1}{2}\theta$.

Solution. This is interpreted to mean that for each point the number ρ is one-half of the number of radians in the angle. Hence, the points

$$(0,0),\quad (\pi/12,\pi/6),\quad (\pi/6,\pi/3),\quad (\pi/4,\pi/2),\quad (\pi/3,2\pi/3)$$

are on the graph. For purposes of graphing, these points are listed (with values of ρ approximated from $\pi = 3.14$ and angles changed to degree measure) as

$$(0,0°),\quad (0.26,30°),\quad (0.52,60°),\quad (0.79,90°),\quad (1.05,120°),\quad \text{etc.}$$

Figure 89.3 shows part of the graph, with the dotted portion corresponding to negative values of θ. The graph is called an **Archimedes spiral.**

Notice, for example, that the point $(\pi/12,390°)$ lies on the graph even though the coordinates given as $(\pi/12,13\pi/6)$ do not satisfy the equation $\rho = \frac{1}{2}\theta$.

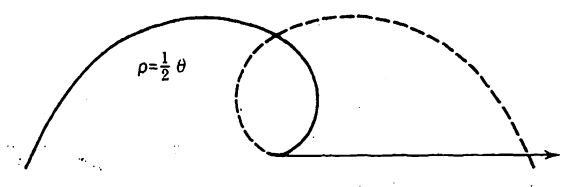


Figure 89.3

With a rectangular coordinate system in the plane, and a polar coordinate system having pole at the origin and polar axis along the positive x-axis, the rectangular coordinates (x,y) of a point and any of the polar coordinates (ρ,θ) of the same point are related. For the vector from the origin-pole to the point may be written either as

$$\vec{i}x + \vec{j}y \quad \text{or as} \quad \rho(\vec{i}\cos\theta + \vec{j}\sin\theta)$$

so that $\vec{i}x + \vec{j}y = \vec{i}\rho\cos\theta + \vec{j}\rho\sin\theta$ and therefore

(1) $$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad \text{and}$$

$$\rho = \sqrt{x^2+y^2} \quad \text{if} \quad \rho \geq 0, \quad \text{but} \quad \rho = -\sqrt{x^2+y^2} \quad \text{if} \quad \rho < 0.$$

Example 3. Find the rectangular equation of the graph having polar equation $\rho = 3 \sin 4\theta$.

Solution. By using trigonometric identities

$$\begin{aligned}\rho &= 3 \sin 4\theta = 3 \sin 2(2\theta) = 6 \sin 2\theta \cos 2\theta \\ &= 6(2 \sin \theta \cos \theta)(\cos^2 \theta - \sin^2 \theta) = 12(\sin \theta \cos^3 \theta - \sin^3 \theta \cos \theta), \\ \rho^5 &= 12[\rho \sin \theta(\rho \cos \theta)^3 - (\rho \sin \theta)^3 \rho \cos \theta], \quad \text{and}\end{aligned}$$

$$\pm (x^2 + y^2)^{5/2} = 12(x^3y - xy^3).$$

PROBLEMS

1. Describe the polar graphs of the equations:
   a. $\rho = 3 \sin 2\theta$. c. $\rho = \cos 3\theta$. e. $\rho = 2\theta$.
   b. $\rho = \sin \theta$. d. $\rho = \cos 4\theta$. f. $2 = \theta$.

2. Sketch the rectangular graph of $y = f(x)$ and the polar graph of $\rho = f(\theta)$ given:
   a. $f(t) = \cos t$. c. $f(t) = 1 + \cos t$. e. $f(t) = \sin \frac{1}{2}t$.
   b. $f(t) = \sin^2 t$. d. $f(t) = \sin 3t$. f. $f(t) = 4$.

3. Draw the graph of $\rho = a + 2 \cos \theta$ for:
   a. $a = 1$. b. $a = -1$. c. $a = 2$.

4. Draw the graph of $\rho = \sin (\theta/2)$ for $0 \leq \theta \leq 720°$. Notice that the same graph is obtained for $\rho = \cos (\theta/2)$.

5. Find an equation whose rectangular graph is the same as the polar graph of:
   a. $\rho = 4 \cos 3\theta$. c. $\rho = 3 \sin \theta + 4 \cos \theta$.
   b. $\rho \cos 3\theta = 4$. d. $\rho = 3 \csc \theta + 4 \sec \theta$.

6. Transform to polar coordinates:
   a. $(x - 2)^2 + (y + 3)^2 = 4$. c. $(x^2 + y^2)(x - a)^2 = b^2x^2$.
   b. $xy = 4$. d. $y^2(2a - x) = x^3$.

90. Polar Analytic Geometry

A. ALTERNATIVE EQUATIONS. Given a number ρ and an angle θ, then a single point has all of the designations

$$(\rho,\theta),\ (-\rho,\theta + 180°),\ (-\rho,\theta - 180°),\ (\rho,\theta + 360°),\ (\rho,\theta - 360°),\ \cdots$$

all of which may be written as $((-1)^n\rho,\ \theta + n \cdot 180°)$ for $n = 0, \pm 1, \pm 2, \cdots$. As a point has alternative designations, so a graph may have alternative equations. In fact, given a function f, then the equations

$$(1) \qquad \rho = f(\theta) \quad \text{and} \quad \rho = (-1)^n f(\theta + n \cdot \pi)$$

have the same graph. For if (ρ_1, θ_1) is a designation of a point on the first graph so that $\rho_1 = f(\theta_1)$, then this point is also on the second graph with designation

$$((-1)^n\rho_1, \theta_1 - n\pi).$$

Conversely, any point on the second graph is on the first graph.

Example 1. Find the different equations of the graph one of whose equations is $\rho = \sin \frac{1}{2}\theta$.

Solution. From (1), possible equations of this graph are

$$\rho = (-1)^n \sin \tfrac{1}{2}(\theta + n \cdot 180°)$$

for n an integer. For $n = 0, 1, 2, 3$, these equations are

$$\begin{aligned}
\rho &= (-1)^0 \sin \tfrac{1}{2}(\theta + 0 \cdot 180°) = \sin \tfrac{1}{2}\theta,\\
\rho &= (-1) \sin \tfrac{1}{2}(\theta + 180°) = -\sin (\tfrac{1}{2}\theta + 90°) = -\cos \tfrac{1}{2}\theta,\\
\rho &= (-1)^2 \sin \tfrac{1}{2}(\theta + 2 \cdot 180°) = \sin (\tfrac{1}{2}\theta + 180°) = -\sin \tfrac{1}{2}\theta,\\
\rho &= (-1)^3 \sin \tfrac{1}{2}(\theta + 3 \cdot 180°) = -\sin (\tfrac{1}{2}\theta + 270°) = \cos \tfrac{1}{2}\theta.
\end{aligned}$$

For $n = 4, 5, 6, 7$, etc. (or $n = 0, -1, -2, -3$, etc.) the same equations are obtained. Thus

$$\rho = \sin \tfrac{1}{2}\theta, \quad \rho = -\cos \tfrac{1}{2}\theta, \quad \rho = -\sin \tfrac{1}{2}\theta, \quad \text{and} \quad \rho = \cos \tfrac{1}{2}\theta$$

all are equations of the same graph.

B. INTERSECTIONS OF GRAPHS. The graph of $\rho = \frac{1}{2}\theta$ is an Archimedes spiral (see Example 2, Sec. 89) and the graph of $\theta = \pi/6$ is a straight line. Upon solving $\rho = \frac{1}{2}\theta$ and $\theta = \pi/6$ simultaneously, the only point obtained is $(\pi/12,\pi/6)$, although both the spiral and the line pass through the pole and the line intersects the spiral repeatedly. The spiral and line also have equations

$$\rho = (-1)^n\tfrac{1}{2}(\theta + n\pi) \quad \text{and} \quad \theta = \frac{\pi}{6} + m\pi.$$

Upon solving these equations simultaneously, the points of intersection are

$$\left((-1)^n\tfrac{1}{2}(\frac{\pi}{6} + m\pi + n\pi), \frac{\pi}{6} + m\pi\right)$$

for all possible combinations of integers m and n. Also, the pole is on both graphs. The principle illustrated is:

To find the intersection of two polar graphs,

(1) *Check whether the pole is on both graphs, and then*

(2) *Solve each equation of one graph simultaneously with each equation of the other graph.*

Example 2. Find the points of intersection of the graphs of

$$\rho = \frac{1}{1 + \cos\theta} \quad \text{and} \quad \rho = \frac{-2}{2 + \cos\theta}.$$

Solution. Neither graph passes through the pole since in either case $\rho = 0$ leads to an incompatible equation.

Since alternate equations of these graphs are, respectively,

$$\rho = \frac{-1}{1 - \cos\theta} \quad \text{and} \quad \rho = \frac{2}{2 - \cos\theta},$$

we must check the following four equations for possible solutions

$$\frac{1}{1 + \cos\theta} = \frac{-2}{2 + \cos\theta}, \qquad \frac{1}{1 + \cos\theta} = \frac{2}{2 - \cos\theta},$$

$$\frac{-1}{1 - \cos\theta} = \frac{-2}{2 + \cos\theta}, \qquad \frac{-1}{1 - \cos\theta} = \frac{2}{2 - \cos\theta}.$$

The first equation leads to $2 + \cos\theta = -2 - 2\cos\theta$, $3\cos\theta = -4$, $\cos\theta = -\frac{4}{3}$ which has no solution.

From the second equation $2 - \cos\theta = 2 + 2\cos\theta$ and hence $\theta = 90° + m \cdot 180°$. These values substituted in either of the pertinent equations gives $\rho = 1$ if m is 0 or an even integer but $\rho = -1$ if m is an odd integer. Since $(1,90°)$ and $(-1,270°)$ designate the same point we give $(1,90°)$ as the simplest designation of the point.

From the third equation we also first obtain $\cos\theta = 0$, then $\theta = 90° + m \cdot 180°$, but this time $\rho = -1$ if m is zero or an even integer, but $\rho = 1$ if m is an odd integer. Thus, the simplest designation is $(-1,90°)$.

The fourth equation simplifies to $\cos\theta = \frac{4}{3}$, which has no solution.

Thus, these graphs intersect at the points $(1,90°)$ and $(-1,90°)$.

PROBLEMS

1. Find the equations of the graph of:

a. $\rho = 4\cos\theta$. c. $\rho = \cos\frac{3}{2}\theta$. e. $\rho = |\sin\theta|$.

b. $\rho = 4\sin 2\theta$. d. $\rho = 1$. f. $\rho = \theta$.

2. Find the points of intersection of the graphs of the pair of equations:

a. $\rho = \dfrac{1}{1 - \sin\theta}, \rho = \dfrac{-2}{2 - \sin\theta}$. d. $\rho = \dfrac{1}{1 - \sin\theta}, \rho = \dfrac{1}{1 + \sin\theta}$.

b. $\rho = 1 - \sin\theta, \rho = \cos 2\theta$. e. $\rho = \cos\theta, \rho = \sqrt{3}\sin\theta$.

c. $\rho = 1, \rho = 2\sin 3\theta$. f. $\rho = \cos\theta, \rho = \cos 2\theta$.

3. Find the points at which the graph of the given equation intersects itself.

a. $\rho = \sin\frac{1}{2}\theta$. b. $\rho = \cos\frac{3}{2}\theta$. c. $\rho = 1 + 2\cos 2\theta$.

C. DISTANCE FORMULA. The polar coordinate points (ρ_1,θ_1) and (ρ_2,θ_2) are the terminal ends of the vectors

$$\vec{v}_1 = \rho_1(\vec{i}\cos\theta_1 + \vec{j}\sin\theta_1) \quad\text{and}\quad \vec{v}_2 = \rho_2(\vec{i}\cos\theta_2 + \vec{j}\sin\theta_2).$$

Thus $\vec{v} = \vec{v}_2 - \vec{v}_1 = \vec{i}(\rho_2\cos\theta_2 - \rho_1\cos\theta_1) + \vec{j}(\rho_2\sin\theta_2 - \rho_1\sin\theta_1)$ is the vector from the first point to the second, and the square of the distance between the points is

$$\begin{aligned}|\vec{v}|^2 = \vec{v}\cdot\vec{v} &= (\rho_2\cos\theta_2 - \rho_1\cos\theta_1)^2 + (\rho_2\sin\theta_2 - \rho_1\sin\theta_1)^2\\ &= \rho_2^2(\cos^2\theta_2 + \sin^2\theta_2) + \rho_1^2(\cos^2\theta_1 + \sin^2\theta_1)\\ &\quad - 2\rho_2\rho_1(\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1)\\ &= \rho_2^2 + \rho_1^2 - 2\rho_2\rho_1\cos(\theta_2 - \theta_1).\end{aligned}$$

A formula for the distance between the points (ρ_1,θ_1) and (ρ_2,θ_2) is therefore

$$\sqrt{\rho_2^2 + \rho_1^2 - 2\rho_2\rho_1\cos(\theta_2 - \theta_1)}. \tag{2}$$

Also, a point (ρ,θ) lies on a circle with center (ρ_1,θ_1) and radius $|a|$ if and only if

$$a^2 = \rho^2 + \rho_1^2 - 2\rho\rho_1\cos(\theta - \theta_1). \tag{3}$$

Thus (3) is an equation of this circle. In particular for $\rho_1 = a$, equation (3) becomes $a^2 = \rho^2 + a^2 - 2\rho a\cos(\theta - \theta_1)$; that is, $\rho^2 = 2\rho a\cos(\theta - \theta_1)$. From this we obtain (even whenever $\rho = 0$) that

$$\rho = 2a\cos(\theta - \theta_1) \tag{4}$$

is an equation of the circle with center (a,θ_1) and radius $|a|$.

D. LINES. Any line through the pole has $\theta = k$ as an equation.

Let l be a line not through the pole, and let $\vec{v}_1$ be the vector from the pole to l and perpendicular to l. Hence, for (ρ_1,θ_1) any designation of the terminal end of $\vec{v}_1$, then

$$\vec{v}_1 = \rho_1(\vec{i}\cos\theta_1 + \vec{j}\sin\theta_1).$$

Now let (ρ,θ) be a point in the plane and let

$$\vec{v} = \rho(\vec{i}\cos\theta + \vec{j}\sin\theta).$$

The point (ρ,θ) will be on l if and only if $\vec{v} - \vec{v}_1$ is perpendicular to $\vec{v}_1$; that is, if and only if

$$\begin{aligned}0 &= (\vec{v} - \vec{v}_1)\cdot\vec{v}_1\\ &= [\vec{i}(\rho\cos\theta - \rho_1\cos\theta_1) + \vec{j}(\rho\sin\theta - \rho_1\sin\theta_1)]\cdot[\rho_1(\vec{i}\cos\theta_1 + \vec{j}\sin\theta_1)]\\ &= (\rho\cos\theta - \rho_1\cos\theta_1)(\rho_1\cos\theta_1) + (\rho\sin\theta - \rho_1\sin\theta_1)(\rho_1\sin\theta_1)\\ &= \rho\rho_1(\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1) - \rho_1^2(\cos^2\theta_1 + \sin^2\theta_1)\\ &= \rho\rho_1\cos(\theta - \theta_1) - \rho_1^2.\end{aligned}$$

Hence, since $\rho_1 \neq 0$ (because l does not pass through the pole) an equation of l is

$$\rho \cos(\theta - \theta_1) = \rho_1. \tag{5}$$

Notice also for (ρ,θ) on l that $\rho \neq 0$ so that $\cos(\theta - \theta_1) \neq 0$ and hence it is safe to write the equation as

$$\rho = \frac{\rho_1}{\cos(\theta - \theta_1)} \quad \text{or as} \quad \rho = \frac{\rho_1}{\cos(\theta_1 - \theta)}. \tag{6}$$

Example 3. Find the distance from the point $P = (5,30°)$ to the line having equation $\rho \cos(150° - \theta) = 3$.

Solution. Since $\cos(-A) = \cos A$, another equation of the line is $\rho \cos(\theta - 150°) = 3$. Any parallel line has equation $\rho \cos(\theta - 150°) = a$ and the one through P must be such that

$$a = 5 \cos(30° - 150°) = 5 \cos(-120°) = -\tfrac{5}{2}.$$

Hence, the point $(3,150°)$ is on l, $(-\frac{5}{2},150°)$ is on the line parallel to l through P, both are on the perpendicular to l so the distance between them is the distance from P to l, and since they are on opposite sides of the pole this distance is $3 + \frac{5}{2} = \frac{11}{2}$ units.

E. CONICS. With $e > 0$ and $q \neq 0$, there is a conic having eccentricity e, focus at the pole, and directrix the line perpendicular to the polar axis at the point $(q,0°)$. A point (ρ,θ) is on this conic if and only if the vector

$$\vec{v} = \rho(\vec{i} \cos \theta + \vec{j} \sin \theta)$$

from the focus to the point (ρ,θ) and the horizontal vector

$$\vec{u} = -\vec{i}q + \vec{i}\rho \cos \theta = \vec{i}(-q + \rho \cos \theta)$$

from the directrix to the point (ρ,θ) have their moduli related by $|\vec{v}| = e|\vec{u}|$. Hence

$$|\rho| = e|-q + \rho \cos \theta|$$

is an equation of the conic. Without absolute values, the two equations

$$\rho = e(-q + \rho \cos \theta) \quad \text{and} \quad \rho = -e(-q + \rho \cos \theta)$$

result and these may be written, respectively as,

$$\rho = \frac{-eq}{1 - e \cos \theta} \quad \text{and} \quad \rho = \frac{eq}{1 + e \cos \theta}. \tag{7}$$

The graph of either equation is the whole conic and neither equation has preference over the other. For upon starting with the first equation, an alternate equation for its graph is (see Division A)

$$\rho = (-1)\frac{-eq}{1 - e\cos(\theta + 180°)} = \frac{eq}{1 + e\cos\theta}$$

which is the second equation.

Note that for ϕ any given angle then the graph of

$$\text{(8)} \qquad \rho = \frac{-eq}{1 - e\cos(\theta - \phi)} \quad \text{and} \quad \rho = \frac{eq}{1 + e\cos(\theta - \phi)}$$

is the rotation through the angle ϕ of the conic which is the graph of (7).

Example 4. Show that the graph of $\rho(3 + 4\cos\theta) = -5$ is a hyperbola; find its center, foci, vertices, directrices, and length of semi-conjugate axis.

Solution. The equation may be written as

$$\rho = \frac{-\frac{5}{3}}{1 + (\frac{4}{3})\cos\theta}$$

which (because of the $+$ sign in the denominator) is the second form in (7) with $e = \frac{4}{3}$. Since $e > 1$ the graph is a hyperbola. Also, $eq = -\frac{5}{3}$ and hence $q = -\frac{5}{4}$. Hence, a directrix (the one associated with the focus at the pole) is perpendicular to the line of the polar axis at the point $(-\frac{5}{4},0°)$ which is preferably designated as $(\frac{5}{4},180°)$. This graph being a hyperbola (and all hyperbolas having their directrices passing between their foci) there is a number $h > \frac{5}{4}$ such that the center is at $(h,180°)$. In the usual conic-notation, the distance from the center to a vertex is a, from the center to a focus is ae, from the center to a directrix is a/e, and hence from a focus to the corresponding directrix is $a/e - ae$ for an ellipse but $ae - a/e$ for a hyperbola. Hence, for this hyperbola in which the distance from a focus to its directrix is $|q| = \frac{5}{4}$

$$ae - \frac{a}{e} = a(\tfrac{4}{3} - \tfrac{3}{4}) = \tfrac{7}{12}a = \tfrac{5}{4} \quad \text{and} \quad a = \tfrac{15}{7}.$$

Thus $ae = \frac{20}{7}$ and, measuring from the focus at the pole,

$(\frac{20}{7},180°)$ is the center,

and $\frac{20}{7}$ units farther along is the other focus at $(\frac{40}{7},180°)$. The vertices are at the distance $a = \frac{15}{7}$ on both sides of the center so

$(\frac{5}{7},180°)$ and $(5,180°)$ are the vertices.

Since the length of the semi-conjugate axis is

$$b = a\sqrt{e^2 - 1} = \frac{15}{7}\sqrt{\frac{16}{9} - 1} = \frac{5}{\sqrt{7}}$$

the asymptotes may be drawn and the hyperbola sketched.

PROBLEMS

1. Name the graph having equation:

   a. $\rho(3 \cos \theta + 4 \sin \theta) = 10.$
   b. $\rho = 10(3 \cos \theta + 4 \sin \theta).$
   c. $4 \sin \theta = 10\rho.$
   d. $3\rho \cos \theta = 10.$
   e. $\rho[3 + 4(\cos \theta + \sin \theta)] = 10.$
   f. $\rho[3(\cos \theta - \sin \theta) + 4] = 10.$
   g. $10 = \rho^2 - \rho(3 \cos \theta + 4 \sin \theta).$

2. Find the usual information about the conic having equation:

   a. $\rho(2 + \cos \theta) = 6.$
   b. $\rho(1 + 2 \cos \theta) = 12.$
   c. $\rho(4 \sin \theta + 4) = 5.$
   d. $\rho(3 + 3 \cos \theta) = -5.$
   e. $\rho(3 + 5 \cos \theta) = 32.$
   f. $\rho(3 \cos \theta - 5) = 32.$

3. Find a polar coordinate equation of the conic having focus at the pole,

   a. Center $(3,0°)$, and eccentricity $\frac{1}{2}$.
   b. Center $(3,0°)$, and eccentricity 2.
   c. Vertex $(3,0°)$, and eccentricity 1.
   d. Corresponding vertex $(3,0°)$ and center $(5,0°)$.
   e. Corresponding vertex $(3,0°)$ and center $(5,180°)$.

91. Polar Calculus

A. DERIVATIVES. For f a function and t in the domain of f', then

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

There are different interpretations of a derivative, each depending upon how the function f is represented; by a rectangular graph, by a particle moving on a line, etc. In the present context f will be represented by a polar coordinate graph; i.e., the graph whose equation is $\rho = f(\theta)$ with θ measured in radians. The graph is also the graph of the vector function $\vec{F}$ defined by

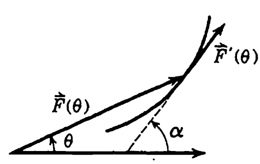


Figure 91.1

$$\vec{F}(\theta) = f(\theta)(\vec{i} \cos \theta + \vec{j} \sin \theta).$$

Now any value of $\vec{F}$ is the product of a scalar and a vector so that, under the assumption that $f'(\theta)$ exists, $\vec{F}'(\theta)$ exists and

$$\begin{aligned} \vec{F}'(\theta) &= [D_\theta f(\theta)][\vec{i} \cos \theta + \vec{j} \sin \theta] + f(\theta) D_\theta[\vec{i} \cos \theta + \vec{j} \sin \theta] \\ &= f'(\theta)[\vec{i} \cos \theta + \vec{j} \sin \theta] + f(\theta)[-\vec{i} \sin \theta + \vec{j} \cos \theta] \\ &= \vec{i}[f'(\theta) \cos \theta - f(\theta) \sin \theta] + \vec{j}[f'(\theta) \sin \theta + f(\theta) \cos \theta]. \end{aligned}$$

As with any vector function, $\vec{F}'(\theta)$ is a tangent to the graph of $\vec{F}$ at $\vec{F}(\theta)$. This tangent vector $\vec{F}'(\theta)$ has modulus

$$\sqrt{[f'(\theta)\cos\theta - f(\theta)\sin\theta]^2 + [f'(\theta)\sin\theta + f(\theta)\cos\theta]^2} = \sqrt{f'^2(\theta) + f^2(\theta)}$$

and with α an amplitude of this tangent vector, then

$$\cos\alpha = \frac{f'(\theta)\cos\theta - f(\theta)\sin\theta}{\sqrt{f'^2(\theta)+f^2(\theta)}} \quad \text{and} \quad \sin\alpha = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{\sqrt{f'^2(\theta)+f^2(\theta)}}.$$

Upon multiplying both sides of the first equation by $\cos\theta$, then both sides of the second by $\sin\theta$, and then adding we have

$$\cos\alpha\cos\theta + \sin\alpha\sin\theta = \frac{f'(\theta)}{\sqrt{f'^2(\theta)+f^2(\theta)}} = \cos(\alpha-\theta)$$

and in a similar way

$$-\cos\alpha\sin\theta + \sin\alpha\cos\theta = \frac{f(\theta)}{\sqrt{f'^2(\theta)+f^2(\theta)}} = \sin(\alpha-\theta).$$

Thus, $\cot(\alpha - \theta) = f'(\theta)/f(\theta)$ so that, as usually written

(1) $$f'(\boldsymbol{\theta}) = f(\boldsymbol{\theta})\cot(\boldsymbol{\alpha} - \boldsymbol{\theta}).$$

Hence, in polar coordinates, $f'(\theta)$ does not have the interpretation of "the slope of the tangent," but from the values of $f'(\theta)$ and $f(\theta)$ the inclination α of the tangent line to the graph at $(f(\theta),\theta)$ may be computed.

Example 1. For the graph of $\rho = \sin 2\theta$, find the angle α with $0 \le \alpha \le 180°$ from the polar axis to the tangent at the point where $\theta = 60°$.

Solution. Since $\rho' = 2\cos 2\theta$, the angle α is such that $2\cos(2\cdot 60°) = \sin(2\cdot 60°)\cot(\alpha - 60°)$ so that $\cot(\alpha - 60°) = -2/\sqrt{3}$. From tables, $\cot(40°\,50') = 2/\sqrt{3}$ so that $\alpha - 60°$ is either $-40°\,50'$ or else $180° - (40°\,50')$. Because of the requirement $0 \le \alpha \le 180°$, the first value is used to obtain $\alpha = 60° - (40°\,50') = 19°\,10'$.

Example 2. Find the angles between 0° and 180° of intersection of the graphs of

$$\rho = \frac{1}{1+\cos\theta} \quad \text{and} \quad \rho = \frac{-2}{2+\cos\theta}.$$

Solution. These equations have no simultaneous solution (see Example 2, Sec. 90), but the first graph has alternate equation

$$\rho = \frac{-1}{1-\cos\theta}$$

and this with the second equation has solution $(-1, 90°)$. Since

$$\frac{d}{d\theta}\left(\frac{-1}{1-\cos\theta}\right) = \frac{\sin\theta}{(1-\cos\theta)^2} \quad \text{and} \quad \frac{d}{d\theta}\left(\frac{-2}{2+\cos\theta}\right) = \frac{-2\sin\theta}{(2+\cos\theta)^2}$$

then, upon letting α_1 and α_2 be inclinations of the tangents to the graphs at $\theta = 90°$,

$$\frac{\sin 90°}{(1 - \cos 90°)^2} = \frac{-1}{1 - \cos 90°} \cot(\alpha_1 - 90°) \quad \text{and}$$

$$\frac{-2 \sin 90°}{(2 + \cos 90°)^2} = \frac{-2}{2 + \cos 90°} \cot(\alpha_2 - 90°)$$

so that $\cot(\alpha_1 - 90°) = -1$ and $\cot(\alpha_2 - 90°) = \frac{1}{2}$. An angle at which the curves intersect is $\alpha_2 - \alpha_1$ and

$$\cot(\alpha_2 - \alpha_1) = \cot[(\alpha_2 - 90°) - (\alpha_1 - 90°)] = \frac{\cot(\alpha_2 - 90°)\cot(\alpha_1 - 90°) + 1}{\cot(\alpha_1 - 90°) - \cot(\alpha_2 - 90°)}$$

$$= \frac{(\frac{1}{2})(-1) + 1}{-1 - \frac{1}{2}} = -\frac{1}{3}.$$

From the tables, $\cot 71° 34' = \frac{1}{3}$ so that $71° 34'$ and its supplement $108° 26'$ are angles between $0°$ and $180°$ of intersection at the point $(-1,90°)$. Since $\cos(-\theta) = \cos\theta$ both curves are symmetric to the polar line and thus have their other intersection at $(1,90°)$ and here intersect at the same angles $71° 34'$ and $108° 26'$.

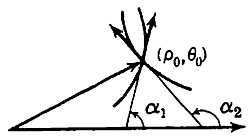


Figure 91.2

Upon generalizing the procedure in Example 2, it follows that:

If the graphs of $\rho = f(\theta)$ and $\rho = g(\theta)$ intersect at (ρ_0,θ_0) so that $\rho_0 = f(\theta_0) = g(\theta_0)$, then an angle $\alpha_2 - \alpha_1$ at which they intersect is such that

$$\cot(\alpha_2 - \alpha_1) = \frac{\cot(\alpha_2 - \theta_0)\cot(\alpha_1 - \theta_0) + 1}{\cot(\alpha_1 - \theta_0) - \cot(\alpha_2 - \theta_0)} = \frac{(g'(\theta_0)/\rho_0)(f'(\theta_0)/\rho_0) + 1}{f'(\theta_0)/\rho_0 - g'(\theta_0)/\rho_0}$$

$$(2) \qquad = \frac{g'(\theta_0)f'(\theta_0) + \rho_0^2}{\rho_0[f'(\theta_0) - g'(\theta_0)]}, \qquad \text{provided } f'(\theta_0) - g'(\theta_0) \neq 0 \text{ and } \rho_0 \neq 0.$$

PROBLEMS

1. Find the acute angle of intersection of the graphs of:

a. $\rho = 4 \sin\theta$ and $\rho = \dfrac{1}{\sin\theta}$. c. $\rho = 2 + \cos\theta$ and $\rho = 5\cos\theta$.

b. $\rho = \dfrac{9}{1 - \cos\theta}$ and $\rho = 4(1 - \cos\theta)$. d. $\rho = \sin 3\theta$ and $2\rho = 1$.

2. Show that the graphs of the two equations intersect at right angles.

a. $\rho = \dfrac{1}{\cos\theta - 1}$ and $\rho = \dfrac{1}{1 - \cos\theta}$.

b. $\rho = 2a\cos\left(\theta - \dfrac{\pi}{6}\right)$ and $\rho = \dfrac{a}{\cos\left(\theta - \dfrac{\pi}{6}\right)}$.

3. Find the acute angle of intersection of the rectangular graph of $y = f(x)$ and $y = g(x)$ and also the acute angle of intersection of the polar graph of $\rho = f(\theta)$ and $\rho = g(\theta)$ where f and g are the functions defined by:
   a. $f(t) = \sin t, g(t) = \cos t$.
   b. $f(t) = \sin t, g(t) = \frac{1}{2}$.
   c. $f(t) = -t + 3, g(t) = t + 2$.
   d. $f(t) = \sin t, g(t) = \sin 2t$.

4. Find the acute angle at which the graph of the given equation intersects itself
   a. $\rho = \sin \frac{1}{2}\theta$.
   b. $\rho = \cos \frac{3}{2}\theta$.
   c. $\rho = 1 + 2\cos 2\theta$.

B. AREA AND LENGTH. With angles measured in radians, let $\alpha < \beta$ be angles, let f be a continuous function and consider the vector

$$\vec{F}(\theta) = f(\theta)(\vec{i}\cos\theta + \vec{j}\sin\theta), \quad \alpha \leq \theta \leq \beta \tag{3}$$

from the pole to the point $(f(\theta),\theta)$. As θ is visualized to increase from α to β the body of this vector sweeps over a portion of the plane, possibly some portions more than once. For example, the vector

$$2a\cos\theta\,(\vec{i}\cos\theta + \vec{j}\sin\theta), \quad 0 \leq \theta \leq 2\pi,\ a > 0 \tag{4}$$

sweeps twice over the circular disk with center $(a,0^\circ)$ and radius a.

The **area swept out** by the vector (3), counting repetitions if any, is defined to be

$$\int_\alpha^\beta \tfrac{1}{2}f^2(\boldsymbol{\theta})\,d\boldsymbol{\theta} \text{ units}^2. \tag{5}$$

As an illustration of (5) using (4)

$$\int_0^{2\pi} \tfrac{1}{2}(2a\cos\theta)^2\,d\theta = a^2\int_0^{2\pi} 2\cos^2\theta\,d\theta = a^2\int_0^{2\pi}(\cos 2\theta + 1)\,d\theta$$
$$= a^2\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{2\pi} = 2\pi a^2$$

which is twice the area of the circle of radius a.

Also, the tip of the vector (3) traces out a curve as θ increases from α to β and may repeat some portions of the curve. Given that f' is also continuous, the **total distance traveled** by the tip is defined to be

$$\int_\alpha^\beta \sqrt{f'^2(\boldsymbol{\theta}) + f^2(\boldsymbol{\theta})}\,d\boldsymbol{\theta}. \tag{6}$$

Again using (4), since $D_\theta(2a\cos\theta) = -2a\sin\theta$,

$$\int_0^{2\pi}\sqrt{(-2a\sin\theta)^2 + (2a\cos\theta)^2}\,d\theta = 2a\int_0^{2\pi}\sqrt{\sin^2\theta + \cos^2\theta}\,d\theta$$
$$= 2a\int_0^{2\pi} d\theta = 2a\theta\Big]_0^{2\pi} = 4\pi a$$

which is twice the circumference of the circle.

In support of the above definitions, recall that a circular sector of radius r units and angle h radians has:

$$\text{area} = \tfrac{1}{2}r^2h \text{ units}^2 \text{ and circular arc of length } = rh \text{ units.}$$

Now with n a positive integer, let $\Delta\theta = (\beta - \alpha)/n$ and

$$\theta_k = \alpha + k\,\Delta\theta \quad \text{for} \quad k = 1, 2, \cdots, n.$$

A circular sector of radius $|f(\theta_k)|$ units and central angle $\Delta\theta$ radians has

$$\text{area} = \tfrac{1}{2}f^2(\theta)\,\Delta\theta \text{ units}^2,\ \text{circular arc length} = |f(\theta_k)|\,\Delta\theta \text{ units.}$$

For such sectors drawn appropriately relative to the graph of $\rho = f(\theta)$, the sectors themselves seem to approximate the sweep of the body of the vector $\vec{F}(\theta)$ (even accounting for overlapping if there is any) but the arcs of the sectors do not follow closely the course of the tip of the vector. Since, regardless of the interpretation,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \tfrac{1}{2}f^2(\theta_k)\,\Delta\theta = \int_\alpha^\beta \tfrac{1}{2}f^2(\theta)\,d\theta$$

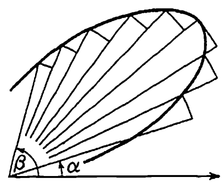


Figure 91.3

we settle for (5) as a reasonable definition of "area swept out," but even though

$$\lim_{n\to\infty} \sum_{k=1}^{n} |f(\theta_k)|\,\Delta\theta = \int_\alpha^\beta |f(\theta)|\,d\theta$$

we do not agree that the value of this integral fits our intuitive concept of "distance traveled by the tip." In fact the tip of the vector

$$\frac{1}{\cos\theta}(\vec{i}\cos\theta + \vec{j}\sin\theta), \quad 0 \le \theta \le \frac{\pi}{4}$$

traces once over the line segment from $(1,0°)$ to $(\sqrt{2},45°)$, travels $\sqrt{2}$ unit, but

$$\int_0^{\pi/4}\left|\frac{1}{\cos\theta}\right|d\theta = \int_0^{\pi/4}\sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4}$$
$$= \ln|\sqrt{2}+1| - \ln|1+0| = \ln|\sqrt{2}+1| \ne \sqrt{2}.$$

Upon letting Δs_k be the length of the line segment joining the points $(f(\theta_{k-1}),\theta_{k-1})$ and $(f(\theta_k),\theta_k)$, then

$$\sum_{k=1}^{n}\Delta s_k$$

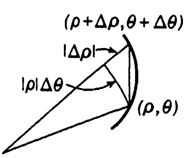


Figure 91.4

is the length of a polygon inscribed in the path. It is proved (f' being continuous) that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Delta s_k = \int_\alpha^\beta \sqrt{f'^2(\theta) + f^2(\theta)}\,d\theta,$$

but there are too many preliminaries to include a proof here. All we will do is to present an intuitive way of remembering the form of the integrand in (6). For (ρ,θ) and $(\rho + \Delta\rho, \theta + \Delta\theta)$ two points on the path, a circular sector of radius $|\rho|$ units and central angle $\Delta\theta$ radians has circular arc length $= |\rho|\,\Delta\theta$ units which with the difference $|(\rho + \Delta\rho) - \rho| = |\Delta\rho|$ units are lengths of sides of a right triangular shaped region whose "hypotenuse" is visualized as having length

$$\sqrt{|\Delta\rho|^2 + (|\rho|\,\Delta\theta)^2} = \sqrt{\left(\frac{\Delta\rho}{\Delta\theta}\right)^2 + \rho^2}\,\Delta\theta \text{ units.}$$

This expression is merely supposed to remind one of the form of the integrand in (6).

Upon letting ϕ be a dummy variable of integration, let s be the function defined by

$$s(\theta) = \int_\alpha^\theta \sqrt{f'^2(\phi) + f^2(\phi)}\, d\phi \quad \text{for} \quad \alpha \le \theta \le \beta.$$

Thus, first in derivative and then differential notation,

$$s'(\theta) = \sqrt{f'^2(\theta) + f^2(\theta)} \quad \text{and} \quad ds = \sqrt{\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2}\, d\theta = \sqrt{(d\rho)^2 + (\rho\, d\theta)^2}.$$

The latter is referred to as the **polar differential of arc length.**

PROBLEMS

1. Find the area swept out and the distance traveled by the tip of the vector $\vec{F}(\theta) = 6 \sin\theta(\vec{i}\cos\theta + \vec{j}\sin\theta)$, $0 \le \theta \le 2\pi$. Explain why the answers are twice the area and circumference of a circle of radius 3.

2. Find the area swept out and the distance traveled by the tip of:
   a. $\vec{F}(\theta) = \vec{i}\theta\cos\theta + \vec{j}\theta\sin\theta, \quad -\pi \le \theta \le \pi.$
   b. $\vec{F}(\theta) = (3\sin\theta + 4\cos\theta)(\vec{i}\cos\theta + \vec{j}\sin\theta), \quad 0 \le \theta \le 2\pi.$
   c. $\vec{F}(\theta) = (1 + \cos\theta)(\vec{i}\cos\theta + \vec{j}\sin\theta), \quad 0 \le \theta \le 2\pi.$
   d. $\vec{F}(\theta) = a\sec^2\frac{\theta}{2}(\vec{i}\cos\theta + \vec{j}\sin\theta), \quad \pi/3 \le \theta \le 2\pi/3.$

3. Find the area of the region enclosed by the graph of
   a. $\rho = 10\sin\theta.$ b. $\rho = 2 - \cos\theta.$ c. $\rho = 2 + \sin 3\theta.$

4. Find the length of the polar graph of the given equation between the points indicated:
   a. $\rho = \dfrac{2}{1 + \cos\theta}$; $(1,0)$, $\left(2, \dfrac{\pi}{2}\right)$. b. $\rho = 1 - \cos\theta$; $(0,0)$, $(0,2\pi)$.

5. Find the area of the region enclosed by:
   a. The inside loop of the graph of $\rho = 1 + 2\cos\theta$.
   b. One loop of the graph of $\rho^2 = \sin\theta$.
   c. The graph of $\rho^2 = \cos 2\theta$.
   d. One loop of the graph of $\rho = a \sin n\theta$, with n a positive integer.

Let f and f' be continuous on $I[a,b]$. Revolve the graph of $y = f(x)$, $a \le x \le b$, about the x-axis thus creating a surface of revolution. This surface is said, by definition, to have

$$\text{area} = \int_a^b 2\pi\,|f(x)|\,\sqrt{1 + f'^2(x)}\,dx.$$

Show how this definition might have been arrived at.

Use this formula to obtain:

a. The lateral area of a right circular cone.
b. The area of a sphere.

Revolve the graph of $xy = 1$, $x \ge 1$ about the x-axis (unit 1 ft) thus forming an infinitely long funnel-shaped surface. To paint the inside surface, merely fill the funnel with paint (show that 24 gal will do), then pour out what does not stick. Now show that there is not enough paint in the world to paint the inside surface of this funnel.

(Hint: Use $(1/x)\sqrt{1 + (1/x)^4} > 1/x$ for $x > 0$.)

CHAPTER 9

Solid Geometry

Having stood aloof for eight chapters and looked down on the flatland of plane geometry, we now consider what is perceived as more nearly the space we live in. On first blush it might seem that the more natural space of three dimensions should be easier to study than the lifeless plane, but the freedom of one more dimension carries with it, as do most freedoms, additional responsibilities and complications. First off, a pencil is yet to be invented whose point will leave a visible trace in space or shade in a twisted surface; hence to portray characteristics of three-dimensional objects we rely upon perspective sketches of their profiles in a few strategically placed planes. Although we seldom attempt artistic effects, we are at one with the artist who paints a view on a flat canvas so the eye will transmit a spacial concept to the brain.

The limitations of plane drawings of three-dimensional objects is largely overcome in mathematics by even more abstractly identifying analytic expressions with solids and surfaces. For this we need frames of reference; see Sec. 93 for one of these and Sec. 103 for two others. Now with geometry represented analytically we turn the tables and visualize analytically expressed relationships geometrically. More of this in Chapters 10 and 11.

92. Preliminaries

The following properties of ordinary solid geometry will be used.

A. A set of points is a plane if:
   a. The set contains three non-collinear points.
   b. If two points of a line are in the set, the whole line is in the set.
   c. The set is not the whole space.

B. Two planes are either parallel or they intersect in a line.

C. Three different planes which have a common point either have one and only one point or one and only one line in common.

D. A line is perpendicular to a plane if and only if it intersects the plane and is perpendicular to every line in the plane through this point of intersection.

E. Through a given point there is one and only one plane perpendicular to a given line.

F. Through a given point there is one and only one line perpendicular to a given plane.

G. With a unit length assigned, a rectangular parallelopiped with sides of lengths a, b, and c units has each diagonal of length

$$\sqrt{a^2 + b^2 + c^2} \text{ units.}$$

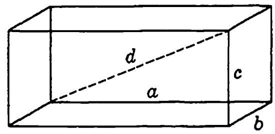


Figure 92

Vectors in space, the resultant of two vectors, and the product of a vector by a scalar are defined as when vectors were restricted to the plane. In particular if $P_1, P_2, \cdots, P_n$ are points in space, then

$$\overrightarrow{P_1P_2} + \overrightarrow{P_2P_3} + \cdots + \overrightarrow{P_{n-1}P_n} + \overrightarrow{P_nP_1} = \vec{0}.$$

The projection of a point P on a line l is the intersection P_l of l and the plane through P perpendicular to l, and the vector projection of a vector $\vec{u} = \overrightarrow{AB}$ on l is the vector $\vec{u}_l = \overrightarrow{A_lB_l}$. Hence

$$(\vec{u} + \vec{v})_l = \vec{u}_l + \vec{v}_l$$

since with $\vec{v} = \overrightarrow{BC}$ then $\vec{u} + \vec{v} = \overrightarrow{AC}$ and $\vec{u}_l + \vec{v}_l = \overrightarrow{A_lB_l} + \overrightarrow{B_lC_l} = \overrightarrow{A_lC_l}$.

With $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$ any given vectors, the **angle from $\vec{u}$ to $\vec{v}$** is defined to be the angle θ from $\overrightarrow{AB}$ to $\overrightarrow{AC}$ where $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are any directed segments with the same initial points such that $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{AC}$. The scalar product

$$\vec{u} \cdot \vec{v} = \begin{cases} 0 & \text{if} \quad \vec{u} = \vec{0} \text{ or } \vec{v} = \vec{0} \\ |\vec{u}|\,|\vec{v}| \cos\theta & \text{if} \quad \vec{u} \neq \vec{0} \quad \text{and} \quad \vec{v} \neq \vec{0} \end{cases}$$

is the same as for plane vectors. Hence, the vector projection of $\vec{v}$ on the line l containing $\vec{u} \neq \vec{0}$ is

$$\vec{v}_l = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\,\vec{u}.$$

An angle between two lines l_1 and l_2 in space (whether the lines intersect or not) is defined to be the angle from any vector $\vec{u} \neq \vec{0}$ on l_1 and any vector $\vec{v} \neq \vec{0}$ on l_2. Also, two lines are said to be **perpendicular** (whether they intersect or not) if an angle between them is 90°. Thus lines l_1 and l_2 are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$ where $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$ are any vectors on l_1 and l_2, respectively.

93. Coordinates

Select two unit vectors $\vec{i}$ and $\vec{j}$ having the same initial point and perpendicular to each other. With the same initial point, let $\vec{k}$ be the unit vector perpendicular to the plane of $\vec{i}$ and $\vec{j}$ and so sensed that a rotation from $\vec{i}$

toward $\vec{j}$ would advance a right-handed screw from the initial toward the terminal end of $\vec{k}$. This is called a **right-handed** system. Hence

(1) $\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1$ and $\vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{i}=\vec{i}\cdot\vec{k}=\vec{k}\cdot\vec{i}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{j}=0.$

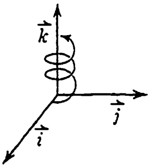


Figure 93.1

Now, for example, $2\vec{i}$, $-4\vec{j}$, and $3\vec{k}$ are vectors whose resultant

$$\vec{u} = 2\vec{i} - 4\vec{j} + 3\vec{k}$$

is a vector in space. The proof that vector addition and scalar multiplication are distributive in no way depended upon the dimension of the space (see Corollary 79). Thus, with $\vec{v} = \vec{i} + 5\vec{j} + 6\vec{k}$ the scalar product of $\vec{u}$ and $\vec{v}$ is

$$\begin{aligned}\vec{u}\cdot\vec{v} &= (2\vec{i} - 4\vec{j} + 3\vec{k})\cdot(\vec{i} + 5\vec{j} + 6\vec{k})\\ &= 2\vec{i}\cdot(\vec{i} + 5\vec{j} + 6\vec{k}) - 4\vec{j}\cdot(\vec{i} + 5\vec{j} + 6\vec{k}) + 3\vec{k}\cdot(\vec{i} + 5\vec{j} + 6\vec{k})\end{aligned}$$

and by using the distributive law again together with (1),

$$\vec{u}\cdot\vec{v} = 2(1) - 4(5) + 3(6) = 2 - 20 + 18 = 0.$$

Hence, these vectors $\vec{u}$ and $\vec{v}$ are perpendicular to each other. Also

$$|\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{2(2) - 4(-4) + 3(3)} = \sqrt{4 + 16 + 9} = \sqrt{29}.$$

Let $\vec{w}$ be a vector with the same initial end as $\vec{i}$, $\vec{j}$, and $\vec{k}$. The vector projection of $\vec{w}$ on the line of $\vec{i}$ is a scalar multiple of $\vec{i}$ so there is a unique number x (which may be positive, negative, or zero) such that this projection is $\vec{i}x$. In the same way there are numbers y and z such that $\vec{j}y$ and $\vec{k}z$ are the vector projections of $\vec{w}$ on the lines of $\vec{j}$ and $\vec{k}$. Hence†

(2) $\vec{w} = \vec{i}x + \vec{j}y + \vec{k}z$ and $|\vec{w}| = \sqrt{\vec{w}^2} = \sqrt{x^2 + y^2 + z^2}.$

The terminal end of $\vec{w}$ is assigned the **coordinates** (x,y,z). The common initial point of $\vec{i}$, $\vec{j}$, and $\vec{k}$ then has coordinates $(0,0,0)$ and is called the **origin** of the **rectangular coordinate system** thus established. It is customary to call the line of $\vec{i}$ the x-axis, the line of $\vec{j}$ the y-axis, the line of $\vec{k}$ the z-axis, the plane of $\vec{i}$ and $\vec{j}$ the xy-plane, the plane of $\vec{j}$ and $\vec{k}$ the yz-plane, and the plane of $\vec{i}$ and $\vec{k}$ the xz-plane.

Let $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$ be two points in space. Then the vector $\overrightarrow{P_1P_2} = \vec{i}(x_2 - x_1) + \vec{j}(y_2 - y_1) + \vec{k}(z_2 - z_1)$ has initial end at P_1, terminal end at P_2, and $|\overrightarrow{P_1P_2}|$ is the distance between P_1 and P_2. Hence

(3) $|\overrightarrow{P_1P_2}| = \sqrt{(\overrightarrow{P_1P_2})^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

is a formula for the distance between two points.

† Recall that $\vec{w}^2$, by definition, $= \vec{w}\,.\,\vec{w}$ and that the square of a vector is the only power that is defined.

Example. Show that the points $(-1,-1,9)$, $(4,-2,6)$, and $(5,-5,11)$ are vertices of an isosceles triangle, find the area of the triangle, and the equal angles of the triangle.

Solution. Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be the vectors from the first point to the second, from the first to the third, and from the second to the third. Hence

$$\vec{u} = \vec{i}[4-(-1)] + \vec{j}[-2-(-1)] + \vec{k}[6-9] = 5\vec{i} - \vec{j} - 3\vec{k},$$

$$\vec{v} = 6\vec{i} - 4\vec{j} + 2\vec{k}, \quad \text{and} \quad \vec{w} = \vec{i} - 3\vec{j} + 5\vec{k}.$$

The squares of the lengths of the sides are

$$|\vec{u}|^2 = \vec{u}^2 = 5^2 + (-1)^2 + (-3)^2 = 35, \quad \vec{v}^2 = 56, \quad \text{and} \quad \vec{w}^2 = 35.$$

Since $\vec{u}^2 = \vec{w}^2$ the triangle is isosceles. Considering $\vec{v}$ the base of the triangle, the altitude h and the area A are

$$h = \sqrt{\vec{u}^2 - (\tfrac{1}{2}\vec{v})^2} = \sqrt{35 - \tfrac{56}{4}} = \sqrt{21} \quad \text{and}$$

$$A = \tfrac{1}{2}\sqrt{56}\sqrt{21} = \sqrt{14}\sqrt{21} = 7\sqrt{6}.$$

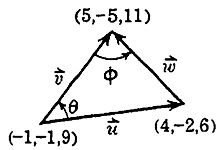


Figure 93.2

One of the equal angles is the angle θ from $\vec{u}$ to $\vec{v}$ and

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{5(6) + (-1)(-4) + (-3)2}{\sqrt{35}\sqrt{56}} = \frac{30+4-6}{\sqrt{(35)(56)}} = \frac{28}{\sqrt{(35)(56)}} = \frac{2}{\sqrt{10}}.$$

From tables $\theta = 50°\,46'$. As a check, for ϕ the angle from $\vec{v}$ to $\vec{w}$,

$$\cos\phi = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\,|\vec{w}|} = \frac{6(1) + (-4)(-3) + 2(5)}{\sqrt{56}\sqrt{35}} = \frac{6+12+10}{\sqrt{(56)(35)}} = \frac{28}{\sqrt{(56)(35)}}.$$

PROBLEMS

1. For the given point, find the point symmetrical with respect to the given plane or line.

   a. $(1,2,5)$, xz-plane.
   b. $(-3,2,-1)$, yz-plane.
   c. $(2,-1,-3)$, yz-plane.
   d. $(-1,-2,3)$, z-axis.
   e. $(5,6,3)$, x-axis.
   f. $(3,-1,-5)$, y-axis.

2. Show that the three points are vertices of a right triangle. Find the other angles of the triangle.

   a. $(0,0,0)$, $(1,2,-3)$, $(3,3,3)$.
   b. $(1,-2,3)$, $(3,-1,2)$, $(3,2,11)$.
   c. $(4,-1,3)$, $(8,1,1)$, $(5,1,7)$.
   d. $(9,15,0)$, $(4,20,25)$, $(-1,10,5)$.

3. What is the distance from the point (x,y,z) to:

   a. The x-axis? b. The origin? c. The yz-plane? d. The point $(3,-1,6)$?

4. Show that the three points are vertices of an isosceles triangle and find the angle between the sides of equal length.

   a. (0,1,2), (−1,−2,7), (5,0,−1). b. (−8,17,8), (−2,13,10), (1,18,−3).
   c. (−8,17,8), (−2,13,10), (−1,17,1).

5. Show that the four points are vertices of a parallelogram.

   a. (2,6,8), (−4,2,7), (1,−2,3), (−3,10,12).
   b. (−1,2,3,), (4,−3,2), (6,4,3), (11,−1,2).

6. Find a fourth point which together with the given three points forms a parallelogram. There are three solutions in each case.

   a. (1,2,3), (−1,4,5), (3,6,2). b. (−1,3,4), (−6,2,5), (−7,5,9).

94. Direction Cosines and Numbers

For plane vectors the amplitude determines (or is determined by) the direction and sense of a vector. For vectors in three-dimensional space the **direction cosines** l, m, n of a vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ are used where

$$l = \cos\alpha = \frac{\vec{i}\cdot\vec{v}}{|\vec{i}|\,|\vec{v}|} = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}},$$

$$n = \cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

Thus α, β, and γ are the angles from the coordinate axes to the vector and

$$l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2+b^2+c^2}{(\sqrt{a^2+b^2+c^2})^2} = 1.$$

With $\vec{v}$ as above, then $-\vec{v}$ has direction cosines $-l$, $-m$, $-n$.

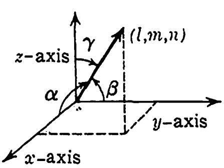


Figure 94

If numbers l, m, n are given with $l^2 + m^2 + n^2 = 1$, then l, m, and n are not all zero and there is a vector having these numbers as direction cosines. The vector

$$l\vec{i} + m\vec{j} + n\vec{k}$$

from the origin to the point (l,m,n) is such a vector and for $\lambda > 0$ so is the vector

$$(\lambda l)\vec{i} + (\lambda m)\vec{j} + (\lambda n)\vec{k}.$$

For l, m, n the direction cosines of a vector $\vec{v}$ and for any number λ such that $\lambda \neq 0$, then

$$a = \lambda l, \quad b = \lambda m, \quad c = \lambda n$$

are said to be **direction numbers** of $\vec{v}$. Thus, any vector has one and only one set of direction cosines, but has indefinitely many sets of direction numbers.

Also, if a, b, and c are numbers not all zero, then there is a vector having a, b, c as a set of direction numbers; namely the vector

$$a\vec{i} + b\vec{j} + c\vec{k}$$

from the origin to the point (a,b,c). Notice that this vector has direction cosines

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

A line in three dimensions has direction, but a sense on the line is usually not assigned. Direction cosines and numbers of a line are defined to be the direction cosines or numbers of any vector on the line. Thus, if l, m, n are direction cosines of a line, then this line also has direction cosines $-l$, $-m$, $-n$ (since if $\vec{v}$ is on the line then so is $-\vec{v}$) and direction numbers λl, λm, λn for any number $\lambda \neq 0$.

Let $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$ be distinct points. Then $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ are not all zero, $\vec{v} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}$ is on the line joining the points, and this vector and the line have

$$x_2 - x_1, \quad y_2 - y_1, \quad z_2 - z_1$$

as a set of direction numbers. Also, these numbers divided by the constant $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ are direction cosines of the vector and line.

95. Parametric Equations of Lines

Let $P_1 = (x_1,y_1,z_1)$ and $P_2 = (x_2,y_2,z_2)$ be distinct points. Then the vector $\vec{v}$ from P_1 to P_2 is given by

$$\vec{v} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}.$$

A point $P = (x,y,z)$ lies on the line through P_1 and P_2 if and only if the vector $\vec{u} = (x - x_1)\vec{i} + (y - y_1)\vec{j} + (z - z_1)\vec{k}$ from P_1 to P is a scalar multiple of $\vec{v}$; i.e., if and only if there is a number t such that

$$(x - x_1)\vec{i} + (y - y_1)\vec{j} + (z - z_1)\vec{k} = t[(x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k}].$$

By equating the coefficients of $\vec{i}$, $\vec{j}$, and $\vec{k}$ the three equations

(1) $\quad x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1), \quad z = z_1 + t(z_2 - z_1)$

are obtained and are **parametric equations** of the line passing through P_1 and P_2. This line thus has direction numbers $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

Also, with a, b, and c not all zero and with $P_1 = (x_1,y_1,z_1)$, then

(2) $$x = x_1 + at, \quad y = y_1 + bt, \quad z = z_1 + ct$$

are parametric equations of the line through P_1 with direction numbers a, b, c. For the set of points satisfying (2) contains P_1, as seen by setting $t = 0$, and also contains the point $P_2 = (x_1 + a, y_1 + b, z_1 + c)$, by setting $t = 1$. Now by (1) the line through P_1 and this point P_2 has parametric equations

$$x = x_1 + t[(x_1 + a) - x_1], \quad y = y_1 + t[(y_1 + b) - y_1],$$
$$z = z_1 + t[(z_1 + c) - z_1]$$

(which are the same as the equations in (2)) and thus direction numbers a, b, c.

In considering more than one straight line, it is generally better to use a different letter for the parameter on different lines.

Example 1. Show that the lines having parametric equations

$$x = -1 + 3t, \quad y = -3 + 2t, \quad z = 4t$$
$$x = -2 + 2s, \quad y = -13 + 6s, \quad z = -2 + 3s$$

intersect by finding the point of intersection.

Solution. A point on the first line will have the same first two coordinates as a point on the second line if t and s simultaneously satisfy

$$-1 + 3t = -2 + 2s \quad \text{and} \quad -3 + 2t = -13 + 6s.$$

These values are $t = 1$ and $s = 2$. At $t = 1$ the point on the first line is $(2, -1, 4)$ and at $s = 2$ the point on the second line is also $(2, -1, 4)$ so the lines intersect at this point.

Example 2. Show that the two sets of parametric equations

$$x = -1 + 3t, \quad y = -3 + 2t, \quad z = 4t, \quad \text{and}$$
$$x = -4 + 6s, \quad y = -5 + 4s, \quad z = -4 + 8s$$

both represent the same straight line.

Solution. Setting the x values equal: $-1 + 3t = -4 + 6s$, leads to the equation $t - 2s = -1$. Also, the y-expressions set equal: $-3 + 2t = -5 + 4s$, leads to the same expression $t - 2s = -1$. Now $t = 2s - 1$ substituted into $z = 4t$ gives $z = 4(2s - 1) = -4 + 8s$, which is the expression for z in the second set of equations.

THEOREM 95. *A line having direction numbers a_1, b_1, c_1 and a line having direction numbers a_2, b_2, c_2 are:*

(i) *Perpendicular if and only if*

$$a_1a_2 + b_1b_2 + c_1c_2 = 0. \tag{3}$$

(ii) *Parallel (or coincide) if and only if $a_2{:}a_1 = b_2{:}b_1 = c_2{:}c_1$; that is, if and only if there is a constant $\lambda \neq 0$ such that*

$$a_2 = \lambda a_1, \quad b_2 = \lambda b_1, \quad c_2 = \lambda c_1. \tag{4}$$

PROOF. Select vectors

$$\vec{v}_1 = a_1\vec{i} + b_1\vec{j} + c_1\vec{k} \quad \text{and} \quad \vec{v}_2 = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$$

on the first and second lines. The angle θ from $\vec{v}_1$ to $\vec{v}_2$ is such that

$$\cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1|\,|\vec{v}_2|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \tag{5}$$

and is an angle between the lines. The lines are perpendicular if and only if $\cos\theta = 0$, and thus if and only if (3) holds.

Next, let it be given that (4) holds. Thus, from (5)

$$\cos\theta = \frac{a_1\lambda a_1 + b_1\lambda b_1 + c_1\lambda c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{\lambda^2(a_1^2 + b_1^2 + c_1^2)}} = \frac{\lambda}{|\lambda|}\,\frac{a_1^2 + b_1^2 + c_1^2}{a_1^2 + b_1^2 + c_1^2}$$

$$= \begin{cases} 1 & \text{if } \lambda > 0 \\ -1 & \text{if } \lambda < 0. \end{cases}$$

Hence, either $\theta = 0°$ or $\theta = 180°$ and in either case the lines are parallel (or coincide).

Finally, let the lines be parallel (or coincide) so that either $\theta = 0°$ or $\theta = 180°$ and

$$\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \pm 1. \tag{6}$$

By clearing of fractions and squaring both sides, the terms may be collected in the form

$$(a_1b_2 - a_2b_1)^2 + (a_1c_2 - a_2c_1)^2 + (b_1c_2 - b_2c_1)^2 = 0.$$

Consequently $a_1b_2 = a_2b_1$, $a_1c_2 = a_2c_1$, and $b_1c_2 = b_2c_1$. From (6) it follows that $a_1a_2 + b_1b_2 + c_1c_2 \neq 0$ so at least one of the numbers a_1a_2, or b_1b_2, or c_1c_2 is not zero. Considering that $a_1a_2 \neq 0$, let $\lambda = a_2/a_1$. From $a_1b_2 = a_2b_1$ and $a_1c_2 = a_2c_1$ it then follows that also $b_2 = \lambda b_1$ and $c_2 = \lambda c_1$, and hence that (4) holds. If $a_1a_2 = 0$ then either $b_1b_2 \neq 0$ or $c_1c_2 \neq 0$ and whichever holds (4) follows by similar reasoning.

Example 3. Find parametric equations of the line through the point $(7,-3,5)$ which is perpendicular to and intersects the line having parametric equations

$$x = -1 - 6t, \quad y = -1 + 2t, \quad z = 4 + 3t.$$

Solution. With t undetermined, let $\vec{v}$ be the vector from the point $(7,-3,5)$ to the point $(-1 - 6t, -1 + 2t, 4 + 3t)$ on the line, so that

$$\vec{v} = (-8 - 6t)\vec{i} + (2 + 2t)\vec{j} + (-1 + 3t)\vec{k}.$$

Next determine t so that this vector (having direction numbers $-8 - 6t, 2 + 2t, -1 + 3t$) and the line (having direction numbers $-6,2,3$) are perpendicular. Hence

$$(-6)(-8 - 6t) + 2(2 + 2t) + 3(-1 + 3t) = 0 \quad \text{and}$$

$$t = \frac{-48 - 4 + 3}{6^2 + 2^2 + 3^2} = \frac{-49}{49} = -1.$$

The point $\big(-1 - 6(-1), -1 + 2(-1), 4 + 3(-1)\big) = (5,-3,1)$ is therefore on the given line, the line joining this point and the given point $(7,-3,5)$ is perpendicular to the given line, and parametric equations of the line joining these points are

$$\begin{aligned} x &= 7 + (5 - 7)s, & y &= -3 + \big(-3 - (-3)\big)s, & z &= 5 + (1 - 5)s \\ &= 7 - 2s & &= -3 & &= 5 - 4s. \end{aligned}$$

Example 4. Given the parametric equation

$$x = 4 + 2t, \quad y = 4 + t, \quad z = -3 - t, \quad \text{and}$$
$$x = -2 + 3s, \quad y = -7 + 2s, \quad z = 2 - 3s$$

of two lines, find a point P_1 on the first line and a point P_2 on the second line such that the line joining P_1 and P_2 is perpendicular to both of the given lines.

Solution. For t and s any numbers whatever the points

$$(4 + 2t, 4 + t, -3 - t), \quad \text{and} \quad (-2 + 3s, -7 + 2s, 2 - 3s)$$

are on the first and second given lines, and the line joining these points has direction numbers

$$4 + 2t - (-2 + 3s), 4 + t - (-7 + 2s), -3 - t - (2 - 3s); \quad \text{that is,}$$
$$2t - 3s + 6, t - 2s + 11, -t + 3s - 5.$$

Since direction numbers of the first line are 2, 1, -1 and of the second line are 3, 2, -3 we want t and s to simultaneously satisfy

$$2(2t - 3s + 6) + 1(t - 2s + 11) + (-1)(-t + 3s - 5) = 0 \quad \text{and}$$
$$3(2t - 3s + 6) + 2(t - 2s + 11) + (-3)(-t + 3s - 5) = 0; \quad \text{that is,}$$
$$6t - 11s = -28 \quad \text{and} \quad 11t - 22s = -55.$$

The solution is $t = -1$, $s = 2$ and these values substituted in the given equations yield the desired points $(2,3,-2)$ and $(4,-3,-4)$. (Note: Since the solutions are unique, it follows that there is one and only one line perpendicular to and intersecting the given lines.)

PROBLEMS

1. Find a set of direction cosines of the line joining the points

a. $(0,0,0)$ and $(1,1,1)$.

b. $(-1,2,1)$ and $(3,1,-1)$.

c. $(2,0,2)$ and $(-1,2,3)$.

d. $(4,5,-2)$ and $(4,6,-2)$.

2. Find direction cosines of the line having parametric equations
   a. $x = 1 + 6t, \quad y = 1 - 3t, \quad z = 3 + 2t.$
   b. $x = -1 - t, \quad y = 2 + t, \quad z = 1 - 3t.$
   c. $x = 2t, \quad y = 4, \quad z = 1 + t.$
   d. $x = -14 + 25t, \quad y = 11 + 25t, \quad z = -43 + 25t.$

3. A line has direction angles α, β, γ. Given
   a. $\alpha = 45°, \quad \beta = 60°;$ find γ.
   b. $\cos\alpha = 0.1, \quad \cos\beta = 0.7;$ find γ.
   c. $\cos\alpha = 0.3, \quad \cos\gamma = 0.4;$ find β.
   d. $\cos\beta = 0.6, \quad \cos\gamma = 0.8;$ find α.

4. Find the acute angle between the pair of lines:
   a. One joining $(3,-1,2)$ to $(2,4,1)$, the other joining $(5,8,-3)$ to $(6,4,1)$.
   b. One joining $(1,2,0)$ to $(3,-2,4)$, the other having parametric equations $x = 4 - 3t, y = 6 + 2t, z = 3 - 2t$.
   c. Having parametric equations
   $x = 3 - 2t, y = -10 + t, z = 25 + 2t$, and
   $x = 5 - t\cos 60°, y = 3 + t\cos 45°, z = 8 + t\cos 60°$.

5. Find the point where the perpendicular line from the given point hits the given line.
   a. $(2,4,-2); \quad x = 6 + t, \quad y = -5 - 4t, \quad z = -1 + 2t.$
   b. $(5,-9,6); \quad x = -3 + 2t, \quad y = 2 + t, \quad z = -5 + 3t.$
   c. $(x_1,y_1,z_1); \quad x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$
   d. $(7,-8,9); \quad x = 13 + 2t, \quad y = -37 - 7t, \quad z = 10 + t.$

6. Show that the pair of lines intersect, find the point of intersection, and find parametric equations of the line through this point of intersection perpendicular to both lines.
   a. $x = 8 + 3t, \quad y = -2 - t, \quad z = 5 + 2t,$
   $x = 11 + 3s, \quad y = -11 - 5s, \quad z = 5 + s.$
   b. $x = 3 + 2t, \quad y = -3 - 6t, \quad z = 1,$
   $x = 7 + s, \quad y = 5 + s, \quad z = 6 + s.$

7. Two lines are said to be **skew** if they are neither parallel nor intersect. Show that the pair of lines is skew. Find the point on each line such that the line joining this pair of points is perpendicular to both of the given lines.
   a. $x = 8 + t, \quad y = -1 - 3t, \quad z = -3 - t,$
   $x = -5 + s, \quad y = -5 + 2s, \quad z = -9 + 4s.$
   b. $x = 5 + t, \quad y = 2 + t, \quad z = 6 + 3t,$
   $x = -1, \quad y = -5 + s, \quad z = -1 + s.$

96. Planes

Given a line L and a point $P_0 = (x_0,y_0,z_0)$ there is one and only one plane through P_0 perpendicular to L (see Sec. 92E). To find an equation of this plane let A, B, C be direction numbers of L so that

$$\vec{v} = A\vec{i} + B\vec{j} + C\vec{k}$$

is parallel to (or on) L. A point $P = (x,y,z)$ is on the plane if and only if the vector

$$\vec{u} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$$

from P_0 to P is such that $\vec{v} \cdot \vec{u} = 0$ (that is, either $\vec{u} = \vec{0}$ in case $P = P_0$ or else $\vec{u}$ is perpendicular to $\vec{v}$ in case $P \neq P_0$). Hence

$$(1) \qquad \vec{v} \cdot \vec{u} = A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

is an equation of the plane through P_0 perpendicular to L.

THEOREM 96. *Given numbers A, B, C not all zero and any number D, then the graph of*

$$(2) \quad Ax + By + Cz = D; \quad \textit{that is,} \quad \{(x, y, z) \mid Ax + By + Cz = D\}$$

is a plane and any line normal (i.e., perpendicular) to this plane has direction numbers A, B, C.

PROOF. Let x_0, y_0, z_0 be numbers such that $Ax_0 + By_0 + Cz_0 = D$. (If, for example, $A \neq 0$ then y_0 and z_0 may be chosen arbitrarily and $x_0 = (D - By_0 - Cz_0)/A$.) Hence, $P_0 = (x_0,y_0,z_0)$ is a point on the graph of (2). Let L be any line having direction numbers A, B, C. Then a point (x,y,z) lies on the plane through P_0 perpendicular to L if and only if

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0;$$

i.e.,
$$Ax + By + Cz = Ax_0 + By_0 + Cz_0 = D$$

which is the same as equation (2). Thus, the graph of (2) is this plane and any line normal to this plane is parallel to L and hence has direction numbers A, B, C.

Example 1. Find an equation of the plane which contains the three points $(-3,1,1)$, $(0,-1,-2)$, and $(4,3,-1)$.

Solution. The following three equations are obtained by substituting in (2):

$$\begin{aligned} -3A + B + C &= D \\ -B - 2C &= D \\ 4A + 3B - C &= D. \end{aligned}$$

The solution of these equations for A, B, and C in terms of D is

$$A = -\frac{2}{5}D, \quad B = \frac{3}{5}D, \quad C = -\frac{4}{5}D.$$

The substitution of these expressions into (2) yields

$$-\frac{2}{5}Dx + \frac{3}{5}Dy - \frac{4}{5}Dz = D,$$

and upon multiplying by $-5/D$ we obtain the equation

$$2x - 3y + 4z = -5.$$

Example 2. Find the point where the line having parametric equations

$$x = 5 - 3t, \quad y = 6 + 2t, \quad z = -4 + t$$

pierces the plane having equation $2x - 3y + 4z = -5$.

Solution. The point $(5 - 3t, 6 + 2t, -4 + t)$ will be on the plane whenever t is such that

$$2(5 - 3t) - 3(6 + 2t) + 4(-4 + t) = -5 \quad \text{so that} \quad t = -\frac{19}{8}.$$

Hence, the point of intersection has $x = 5 - 3\left(-\frac{19}{8}\right) = \frac{97}{8}$, $y = 6 + 2\left(-\frac{19}{8}\right) =$ etc., and the point is $\left(\frac{97}{8}, \frac{5}{4}, -\frac{51}{8}\right)$.

Example 3. Show that the line having parametric equations

$$x = 2 + t, \quad y = 6 + 2t, \quad z = 4 + t$$

lies on the plane having equation $3x - 4y + 5z = 2$.

Solution. By proceeding as if we were solving for the intersection of the line and plane we obtain

$$3(2 + t) - 4(6 + 2t) + 5(4 + t) = 2; \quad t(3 - 8 + 5) = 2 - 6 + 24 - 20$$

and therefore $t \cdot 0 = 0$ which is satisfied for all values of t. Thus, the point $(2 + t, 6 + 2t, 4 + t)$ lies both on the line and on the plane for all values of t and hence the line lies on the plane.

Example 4. Find an equation of the plane containing the point $(2, -2, -7)$ and the line having parametric equations

$$x = 1 - 2t, \quad y = 2 + t, \quad z = 7 + 7t.$$

Solution 1. First find two points on the line; then pass a plane through these two points and the given point. Convenient points on the line are $(1,2,7)$ and

(5,0,−7) obtained by using $t = 0$ and $t = -2$. Hence, the plane having equation (2) passes through these three points if

$$\begin{aligned} A + 2B + 7C &= D \\ 5A \qquad - 7C &= D \\ 2A - 2B - 7C &= D. \end{aligned}$$

We obtain $A = 2D/3$, $B = -D$, $C = D/3$ and thus the equation

$$2x - 3y + z = 3. \tag{3}$$

Solution 2. A plane with equation (2) contains the line if

$$A(1 - 2t) + B(2 + t) + C(7 + 7t) = D$$

is satisfied for all t; that is, if the coefficient of t is zero for all t:

$$-2A + B + 7C = 0$$

and in addition the sum of terms not containing t is equal to D:

$$A + 2B + 7C = D.$$

Also, the plane must pass through the point (2, −2, −7) and hence

$$2A - 2B - 7C = D.$$

From these three equations we again obtain (3) as an equation of the plane.

An **angle between two planes** is defined as an angle between the normals to the planes. Thus, for planes having equations

$$A_1x + B_1y + C_1z = D_1 \quad \text{and} \quad A_2x + B_2y + C_2z = D_2,$$

an angle θ between the planes is such that

$$\cos \theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

In particular the planes are parallel if and only if

$$A_1 : A_2 = B_1 : B_2 = C_1 : C_2.$$

PROBLEMS

1. Find parametric equations of the line through the given point perpendicular to the given plane.

a. $(-1,2,3)$; $-2x + 3y - z = -1$.
b. $(5,6,-4)$; $3x - y + 2z = 10$.
c. $(0,0,0)$; $2x + 4y = 15$.
d. $(2,-1,1)$; $x + 2z = 3$.

2. Find an equation of the plane through the given point and perpendicular to the given line.

a. $(2,-1,3)$; $x = 5 + 3t$, $y = -4 + 8t$, $z = 16 - 7t$.
b. $(0,5,2)$; line joining $(3,6,-1)$, $(5,-8,4)$.

c. $(0,0,-4)$; line which is the intersection of the planes having equations $2x + y + z = 12$ and $3x - y + 2z = 7$.

d. $(7,8,-5)$; line through the point $(2,-6,8)$ and having direction numbers $2, -3, 4$.

3. Find an equation of the plane containing the three points:

a. $(-2,-1,2)$, $(3,9,3)$, $(1,1,1)$.

b. $(5,3,-1)$, $(3,-2,3)$, $(2,0,2)$.

c. $(1,1,1)$, $(5,1,1)$, $(3,2,5)$.

d. $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

4. Find an equation of the plane containing the given point and line.

a. $(2,1,3)$; $x = 3 + 2t, \quad y = 3 + t, \quad z = 8 - 2t$.

b. $(0,0,5)$; $x = 1 - 2t, \quad y = 2 - t, \quad z = 10 + 2t$.

c. $(4,3,5)$; $x = -1 + t, \quad y = 1 + \frac{1}{2}t, \quad z = 12 - t$.

d. $(1,2,10)$; $x = 1 + 5t, \quad y = 1 + 2t, \quad z = 6 - 7t$.

5. Show that the two lines intersect and find an equation of the plane containing these lines.

a. $x = 2 + 3t, \quad y = -t, \quad z = 1 + 2t$.
$x = 8 + 3s, \quad y = -6 - 5s, \quad z = 4 + s$.

b. $x = 2t, \quad y = 6 - 6t, \quad z = 1$.
$x = 2 + s, \quad y = s, \quad z = 1 + s$.

c. $x = 2 + 4t, \quad y = 0, \quad z = 1 + 3t$.
$x = 0, \quad y = 2s, \quad z = -\frac{1}{2} + \frac{1}{2}s$.

d. $x = 1 + 99t, \quad y = 3 - 97t, \quad z = 1 + 50t$.
$x = 1 - 101s, \quad y = 3 + 99s, \quad z = 1 - 51s$.

6. A plane and line are said to be **parallel** if they have no point in common. Notice that if a plane and line are parallel, then an angle from any normal of the plane to the line is 90°. Find an equation of the plane through the given point and parallel to both of the given lines.

a. $(2,-3,4)$; $\begin{cases} x = 10 + t, \quad y = 15 + t, \quad z = 20 - t. \\ x = 4 + 3s, \quad y = -8 + 5s, \quad z = 7 - 2s. \end{cases}$

b. $(2,-3,4)$; $\begin{cases} x = 10 + t, \quad y = 15 + t, \quad z = 20 - t. \\ x = 10 + 3s, \quad y = 15 + 5s, \quad z = 20 - 2s. \end{cases}$

c. $(0,0,0)$; $\begin{cases} \text{line joining } (2,-1,6) \text{ and } (3,2,-4). \\ \text{line having direction numbers } 3, -5, 4. \end{cases}$

97. Determinants

With a_1, b_1, a_2, and b_2 representing numbers, then

$$(1) \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is a **determinant** of order 2 (or second order) whose **value**, by definition, is the number $a_1b_2 - b_1a_2$. For example:

$$\begin{vmatrix} 5 & -3 \\ 4 & 2 \end{vmatrix} = 5 \cdot 2 - (-3)4 = 10 + 12 = 22 \quad \text{and}$$

$$\begin{vmatrix} 4 & 2 \\ 5 & -3 \end{vmatrix} = 4(-3) - 2 \cdot 5 = -12 - 10 = -22.$$

Since a determinant represents a number, then "a number times a determinant" has meaning. For example

$$2\begin{vmatrix} 5 & -3 \\ 4 & 2 \end{vmatrix} = 2(22) = 44.$$

Third-order determinants, of which

$$\begin{vmatrix} 2 & -7 & 5 \\ -4 & 5 & -3 \\ 3 & 4 & 2 \end{vmatrix} \tag{2}$$

is an example, will also be considered. The determinant (2) has the value -104 which is found, according to the definition below, by computing

$$2\begin{vmatrix} 5 & -3 \\ 4 & 2 \end{vmatrix} - (-7)\begin{vmatrix} -4 & -3 \\ 3 & 2 \end{vmatrix} + 5\begin{vmatrix} -4 & 5 \\ 3 & 4 \end{vmatrix} \tag{3}$$

$$= 2[5 \cdot 2 - (-3)4] + 7[(-4)2 - (-3)3] + 5[(-4)4 - 5 \cdot 3]$$

$$= 2[22] + 7[1] + 5[-31] = 44 + 7 - 155 = -104.$$

The value of the third-order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{4}$$

is defined, in terms of second-order determinants, as

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \tag{5}$$

Hence, since each second-order determinant may be evaluated, the value of the third order determinant (4) is

$$
\begin{aligned}
&a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3) \\
(6)\qquad &= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1.
\end{aligned}
$$

The manipulation of (5) and (6) are referred to as **expanding** the determinant (4). Also, (6) is called "the expansion" of (4), and (5) is called "the expansion of (4) in terms of elements of the first row."

The expansion (6) will be used in establishing some properties of determinants, but it is not recommended that (6) be memorized. The exact form of (5) should, however, be kept in mind. In particular notice that the middle term of (5) is $-b_1$ (instead of $+b_1$) times the second-order determinant consisting of those elements of (4) which are not in the same horizontal row and not in the same vertical column with b_1.

We now point out three properties of determinants.

PROPERTY 1. *If all elements in any row or column of a determinant are* 0, *then the value of the determinant is* 0.

For example, if the second row of (4) consists entirely of zeros, then $a_2 = 0$, $b_2 = 0$, and $c_2 = 0$ and these substituted into (6) yield the value 0.

PROPERTY 2. *If any two rows of a determinant are the same (or if any two columns of a determinant are the same), then the value of the determinant is* 0.

For example, if the second and third rows of (4) are the same, then $a_3 = a_2$, $b_3 = b_2$, and $c_3 = c_2$ and (6) with every subscript 3 changed to 2 becomes

$$a_1b_2c_2 - a_1b_2c_2 - a_2b_1c_2 + a_2b_1c_2 + a_2b_2c_1 - a_2b_2c_1 = 0.$$

Example. Show that in the plane the graph of the equation

$$\begin{vmatrix} x & y & 1 \\ 3 & -2 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 0$$

is a straight line passing through the points $(3, -2)$ and $(4,5)$.

Solution. Without actually expanding the determinant, one should visualize that the equation could be written in the form $Ax + By + C = 0$ in which

$$A = \begin{vmatrix} -2 & 1 \\ 5 & 1 \end{vmatrix}, \qquad B = -\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix}, \qquad C = \begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix}.$$

Since A, B, and C are constants, the equation is of first degree in the variables x and y so its graph is a straight line.

Also, the equation is satisfied by $x = 3$ and $y = -2$ (since the substitution of these values yields a determinant with two rows identical) so the line passes through the point $(3, -2)$. In the same way the line passes through the point $(4,5)$.

By following the reasoning of the example it should be seen that

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

is a two-point-determinant equation of a line in the xy-plane.

PROPERTY 3. *By multiplying each element of a row (or column) of a given determinant by a number k, then a new determinant is created whose value is k times the value of the given determinant.*

For example
$$\begin{vmatrix} a_1 & b_1 & kc_1 \\ a_2 & b_2 & kc_2 \\ a_3 & b_3 & kc_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

since the left-hand determinant has expanded form obtained from (6) upon replacing c_1 by kc_1, c_2 by kc_2, and c_3 by kc_3:

$$a_1b_2kc_3 - a_1b_3kc_2 - a_2b_1kc_3 + a_3b_1kc_2 + a_2b_3kc_1 - a_3b_2kc_1,$$

and this is k times the expression (6) itself.

The Property 3 may be used to "remove a common factor" from any row or column. Thus

$$\begin{vmatrix} 2 & 4 & -6 \\ 1 & 3 & 9 \\ 2 & -1 & 0 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & -3 \\ 1 & 3 & 9 \\ 2 & -1 & 0 \end{vmatrix} = 6\begin{vmatrix} 1 & 2 & -1 \\ 1 & 3 & 3 \\ 2 & -1 & 0 \end{vmatrix}$$

wherein 2 was "factored" from the first row (to obtain the middle expression) and then 3 was "factored" from the resulting third column.

Determinant notation and terminology are also extended to allow some quantities other than numbers. For example, with $\vec{i}$, $\vec{j}$, and $\vec{k}$ the basic unit vectors, then

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ 5 & 2 & -3 \end{vmatrix}$$

is used to represent the vector

$$\vec{i}\begin{vmatrix} -3 & 4 \\ 2 & -3 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 4 \\ 5 & -3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -3 \\ 5 & 2 \end{vmatrix}$$
$$= \vec{i}(9 - 8) - \vec{j}(-6 - 20) + \vec{k}(4 + 15) = \vec{i} + 26\vec{j} + 19\vec{k}.$$

PROBLEMS

1. Find the value of each of the determinants

a. $\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}$

b. $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix}$

c. $\begin{vmatrix} 3 & -2 \\ 4 & -5 \end{vmatrix}$

d. $\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$

e. $\begin{vmatrix} 10 & 20 \\ 5 & 15 \end{vmatrix}$

f. $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

g. $\begin{vmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{vmatrix}$

h. $\begin{vmatrix} b & 2a \\ 2c & b \end{vmatrix}$

i. $\begin{vmatrix} 2 & 4 & -3 \\ -1 & 3 & 2 \\ 6 & 4 & 5 \end{vmatrix}$

j. $\begin{vmatrix} 2 & -1 & 6 \\ 4 & 3 & 4 \\ -3 & 2 & 5 \end{vmatrix}$

k. $\begin{vmatrix} 5 & 20 & 78 \\ 0 & 3 & 49 \\ 0 & 0 & 2 \end{vmatrix}$

2. Show that $\begin{vmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & c \end{vmatrix} = abc$ and thus that the value of this determinant is independent of x, y, and z.

3. Draw the graph of each of the following equations:

a. $\begin{vmatrix} x & y \\ 2 & 3 \end{vmatrix} = 4.$

b. $\begin{vmatrix} x - 2 & y - 3 \\ 4 & 1 \end{vmatrix} = 0.$

c. $\begin{vmatrix} x^2 & y^2 \\ a^2 & b^2 \end{vmatrix} = a^2b^2.$

d. $\begin{vmatrix} x & y & 1 \\ 5 & 2 & 1 \\ -3 & 4 & 1 \end{vmatrix} = 0.$

e. $\begin{vmatrix} 1 & 1 & 1 \\ x & 2 & 4 \\ y & 3 & 0 \end{vmatrix} = 0.$

4. Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x - y)(y - z)(z - x).$

5. Show that the triangle with vertices (x_1,y_1), (x_2,y_2), and (x_3,y_3) has area the absolute value of

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(Hint: First use the corollary of Sec. 83 to find a determinant expression for the distance from the point (x_1,y_1) to the line passing through the points (x_2,y_2) and (x_3,y_3).)

6. Express each of the following vectors in the usual vector form:

a. $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 4 & 2 & -5 \end{vmatrix}$ b. $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 4 & -3 & 2 \end{vmatrix}$ c. $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 3 & 4 & 0 \end{vmatrix}$

7. Obtain the following relation for the scalar (dot) product of two vectors

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1' & b_1' & c_1' \\ a_2' & b_2' & c_2' \end{vmatrix} \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1'' & b_1'' & c_1'' \\ a_2'' & b_2'' & c_2'' \end{vmatrix}$$

$$= \begin{vmatrix} a_1' & b_1' \\ a_2' & b_2' \end{vmatrix}\begin{vmatrix} a_1'' & b_1'' \\ a_2'' & b_2'' \end{vmatrix} + \begin{vmatrix} a_1' & c_1' \\ a_2' & c_2' \end{vmatrix}\begin{vmatrix} a_1'' & c_1'' \\ a_2'' & c_2'' \end{vmatrix} + \begin{vmatrix} b_1' & c_1' \\ b_2' & c_2' \end{vmatrix}\begin{vmatrix} b_1'' & c_1'' \\ b_2'' & c_2'' \end{vmatrix}.$$

The review of second- and third-order determinants given above is sufficient for the use to which determinants will be put in the remainder of this book. Since, however, determinants appear in a wide variety of applications, it is well to be acquainted with the further properties of determinants given below.

It has been found that discussions of determinants are facilitated by means of a double subscript notation. Thus (4), (5), and (6) above are replaced by

$$(4') \qquad A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$(5') \qquad = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$(6') \qquad = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

(Note: The A in (4′) stands either for the array in (4′) or the value as given in (6′).)

The first subscript indicates the **row**, whereas the second subscript indicates the **column** in which an element is found. Thus a_{23} is in row 2 and column 3.

Each of the three second-order determinants of (5′) requires two multiplications to evaluate. Thus, the evaluation of a third-order determinant by this definition requires more than

$$(7) \qquad 3 \cdot 2 = 3! \text{ multiplications.}$$

Given an element a_{rs} of the determinant A of (4′), then A_{rs} will be used to denote the determinant (or its value) obtained by deleting all of row r and column s from A. Also A_{rs} is called the **minor** of a_{rs}. Hence, in this notation (5′) becomes

$$(5'') \qquad a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$$

and is said to be "the expansion of A in terms of minors of the first row."

With λ a number, note that

$$(8)\qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21}+\lambda a_{11} & a_{22}+\lambda a_{12} \end{vmatrix} = a_{11}(a_{22}+\lambda a_{12}) - a_{12}(a_{21}+\lambda a_{11})$$

$$= a_{11}a_{22} - a_{12}a_{21} + \lambda(a_{11}a_{12} - a_{12}a_{11}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

since the coefficient of λ is $a_{11}a_{12} - a_{12}a_{11} = 0$. Also

$$(9)\qquad \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{21} & a_{22} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{21}+\lambda a_{21} & a_{22}+\lambda a_{22} & a_{22}+\lambda a_{22} \end{vmatrix}$$

This is seen by noting that in the right-hand determinant the minor of a_{11} is (by using (8))

$$\begin{vmatrix} a_{22} & a_{22} \\ a_{22}+\lambda a_{22} & a_{22}+\lambda a_{22} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{22} \\ a_{22} & a_{22} \end{vmatrix} = A_{11}$$

and in the same way both determinants of (9) have A_{12} as the minor of a_{12} and A_{13} as the minor of a_{12}.

These equalities (8) and (9) indicate how the following property may be established.

PROPERTY 4. *If each element of any row of a determinant is multiplied by a number λ and the product is added to the corresponding element of another row, then the resulting determinant has the same value as the original determinant. The same is true if "row" is replaced throughout by "column."*

The use of Property 4 is facilitated by the notational device illustrated below:

$$\begin{vmatrix} 1 & -3 & 4 \\ 2 & 1 & 5 \\ -5 & 4 & 2 \end{vmatrix}\begin{array}{c} \boxed{-2} \\ \longleftarrow \\ \ \end{array} = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 7 & -3 \\ -5 & 4 & 2 \end{vmatrix}\begin{array}{c} \boxed{5} \\ \ \\ \longleftarrow \end{array} = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 7 & -3 \\ 0 & -11 & 22 \end{vmatrix}$$

This means that in the determinant on the left each element of the first row is multiplied by -2 and added to the corresponding element of the second row to form the second row of the middle determinant; the other two rows remaining the same. Then in the middle determinant each element of the first row is multiplied by 5 and added to the corresponding element of the third row to obtain the determinant on the right. The middle determinant may be omitted by writing

$$(10)\qquad \begin{vmatrix} 1 & -3 & 4 \\ 2 & 1 & 5 \\ -5 & 4 & 2 \end{vmatrix}\begin{array}{c} \boxed{-2} \\ \longleftarrow \\ \overset{5}{\longleftarrow} \end{array} = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 7 & -3 \\ 0 & -11 & 22 \end{vmatrix} \quad \text{and this is} = \begin{vmatrix} 7 & -3 \\ -11 & 22 \end{vmatrix}$$

since, in the second determinant, the first-row elements -3 and 4 both have minors with only zeros in a column. The point of such a manipulation is "to reduce a third-order determinant to a second-order determinant."

If the upper left element a_{11} is not the number 1, then such a reduction should be preceded by "dividing each element of the first row by a_{11}." Thus (see Property 3)

$$\begin{vmatrix} 4 & -2 & 6 \\ 3 & 0 & -4 \\ 7 & 5 & 2 \end{vmatrix} = 4\begin{vmatrix} 1 & -0.5 & 1.5 \\ 3 & 0 & -4 \\ 7 & 5 & 2 \end{vmatrix}$$

and now the reduction may proceed as in (10). Even though all elements of the given determinant are integers this "preparing for reduction" may introduce fractions. With modern electronic computers, however, decimal fractions are handled as easily as integers.

By extending the notion of minors to fourth-order determinants, then a fourth-order determinant is evaluated, by definition, as below

$$(11) \qquad A = \begin{vmatrix} a_{11} & a_{13} & a_{13} & a_{14} \\ a_{21} & a_{33} & a_{23} & a_{34} \\ a_{31} & a_{33} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{43} & a_{44} \end{vmatrix} = a_{11}A_{11} - a_{12}A_{13} + a_{13}A_{13} - a_{14}A_{14}$$

where each A_{1s} is the third-order determinant obtained by deleting row 1 and column s from A. Since each third-order determinant requires more than 3! multiplications (see (7)), then to evaluate a fourth-order determinant by the above definition, requires more than

$$(12) \qquad 4 \cdot 3! = 4! \text{ multiplications.}$$

Properties 1–4 also hold for fourth-order determinants, as may be seen by using (11) and the knowledge that these properties hold for third-order determinants. In particular a fourth-order determinant may be "reduced to a single third-order determinant," then this third-order determinant "reduced to a second-order determinant."

When computations are with pencil and paper, then fractions may be avoided in the "preparing for reduction" by using a little ingenuity as illustrated below:

$$\begin{vmatrix} 2 & 3 & -2 & 6 \\ 4 & 1 & 0 & 2 \\ -1 & 2 & 1 & 4 \\ 4 & 3 & 5 & 4 \end{vmatrix} = 2\begin{vmatrix} -1 & 3 & -2 & 3 \\ 3 & 1 & 0 & 1 \\ -3 & 2 & 1 & 2 \\ 1 & 3 & 5 & 2 \end{vmatrix} = -2\begin{vmatrix} 1 & 3 & -2 & 3 \\ -3 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ -1 & 3 & 5 & 2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 3 & -2 & 3 \\ 0 & 10 & -6 & 10 \\ 0 & -7 & 7 & -7 \\ 0 & 6 & 3 & 5 \end{vmatrix} = -2\begin{vmatrix} 10 & -6 & 10 \\ -7 & 7 & -7 \\ 6 & 3 & 5 \end{vmatrix} = -28\begin{vmatrix} 5 & -3 & 5 \\ -1 & 1 & -1 \\ 6 & 3 & 5 \end{vmatrix}$$

$$= -28\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 6 & 3 & 5 \end{vmatrix} = -28\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & -3 & -1 \end{vmatrix}$$

$$= -28\begin{vmatrix} 2 & 0 \\ -3 & -1 \end{vmatrix} = -28(-2) = 56.$$

PROBLEMS (cont.)

8. Evaluate each of the determinants

a. $\begin{vmatrix} 3 & 2 & 6 & -2 \\ 1 & 4 & 2 & 0 \\ 2 & -1 & 4 & 1 \\ 3 & 4 & 4 & 5 \end{vmatrix}$ b. $\begin{vmatrix} 5 & 2 & 1 & -2 \\ 3 & 0 & 1 & 4 \\ 2 & -2 & 3 & 0 \\ 4 & -3 & 0 & 2 \end{vmatrix}$ c. $\begin{vmatrix} 1 & 2 & -3 & 1 \\ 2 & 3 & -2 & -3 \\ -1 & -2 & 2 & 1 \\ 4 & -3 & 1 & 2 \end{vmatrix}$

9. Evaluate each of the determinants by first performing the indicated operations.

a. $$\begin{array}{c} \begin{array}{cccc} \downarrow -3 & \downarrow 2 & & \end{array} \\ \begin{vmatrix} 3 & -2 & 1 & 0 \\ 10 & 4 & 3 & -4 \\ 10 & 0 & 5 & 3 \\ 20 & -16 & 8 & 1 \end{vmatrix} \end{array}$$

b. $$\begin{array}{cc} \begin{vmatrix} 5 & 2 & 9 & 1 \\ 2 & 2 & 3 & 0 \\ 7 & 7 & 10 & 2 \\ 4 & 3 & 5 & 1 \end{vmatrix} & \begin{array}{l} \\ -2 \\ \leftarrow \\ -1 \ \leftarrow \end{array} \end{array}$$

10. a. Show that the graph of the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

is the plane determined by the points (x_1,y_1,z_1), (x_2,y_2,z_2), and (x_3,y_3,z_3).

b. Use the result of Part a to find the equation of the plane passing through $(1,-2,3)$, $(0,2,1)$, and $(5,0,3)$.

A determinant of order n has n rows, n columns, and is written

(13) $$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{31} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n3} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

By deleting row r and column s, the resulting determinant of order $n - 1$ is represented by A_{rs} and is called the **minor** of the element a_{rs}. By definition the **value** of the determinant in (13) is

(14) $$A = \sum_{s=1}^{n} (-1)^{s+1} a_{1s}A_{1s}.$$

Hence, the value of a determinant of order n is obtained from the values of determinants of order $n - 1$ which in turn are obtained from values of determinants of order $n - 2$, etc., until the order is reduced to 2. It should thus be seen, by extending (7) and (12), that the evaluation of a determinant of order n from the definition requires more than

$$n! \text{ multiplications.}$$

Determinants of orders 20 or more are frequently evaluated by modern electronic computers. A medium-fast computer will perform multiplications at the rate of 1000 per sec, and the fastest projected computer will perform multiplications at 50,000 per sec. Thus, merely to perform the multiplication to evaluate a determinant of order 20 from the definition (14) would require this fastest computer $\frac{20!}{50{,}000}$ sec $= \frac{20!}{(50{,}000)(60)(60)(24)(365)}$ years, and even this number (check by using logarithms) is greater than

$$10^{10} \text{ years.}$$

Clearly some method other than the definition is used to evaluate determinants of high order.

Properties 1–4 also hold for determinants of order n. In particular, a determinant of order n may be "reduced" to a single determinant of order $n - 1$ (as a fourth-order determinant was reduced to a third-order determinant). The "preparation for reduction" consists of dividing each element of the first row by a_{11} and hence n divisions are performed. Now, the n-elements of the new first row are multiplied by a_{21} (hence n multiplication) and the result in each case is subtracted from the corresponding element of the second row. The same is done for $a_{21}, a_{41}, \cdots, a_{n1}$ so in all there are $(n - 1) \cdot n$ such multiplications. By counting the n divisions of the "preparation" as multiplications, a reduction from order n to order $n - 1$ requires

$$n + (n - 1) \cdot n = n^2 \text{ multiplications.}$$

To reduce the $n - 1$ order determinant to an $n - 2$ order determinant requires $(n - 1)^2$ multiplications, etc.

If we also reduce the final second-order determinant to a single number by the same method, then (see Formula (2) of Sec. 56)

$$n^2 + (n - 1)^2 + \cdots + 3^2 + 2^2 = \tfrac{1}{6}(2n^2 + 3n^2 + n) - 1 < n^2 \text{ multiplications}$$

are required to evaluate a determinant of order n by the "reduction" method. A medium-fast (1000 multiplications per sec) computer could, therefore, perform the multiplications of a complete reduction of a determinant of order 20 in less than

$$\frac{(20)^2}{1000} = 8 \text{ sec.}$$

Since additions and subtractions are performed on an electronic computer at a much faster rate than multiplications and divisions, 16 sec would be ample time for all of the arithmetic operations necessary to evaluate a determinant of order 20.

For efficient use of determinants, a few more properties are desirable.

PROPERTY 5. *By interchanging any two rows (or any two columns) of a determinant, the new determinant formed has value the negative of the value of the original determinant.*

A method of establishing this property is illustrated below for the interchange of the first and third columns of a third-order determinant:

$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = \begin{vmatrix} a & d & g - a \\ b & e & h - b \\ c & f & i - c \end{vmatrix} = \begin{vmatrix} g & d & g - a \\ h & e & h - b \\ i & f & i - c \end{vmatrix} = \begin{vmatrix} g & d & -a \\ h & e & -b \\ i & f & -c \end{vmatrix} = -\begin{vmatrix} g & d & a \\ h & e & b \\ i & f & c \end{vmatrix}$$

PROPERTY 6. *With $r > 1$, if row r is moved to the top position and each row above the* rth *is lowered one position, the resulting determinant has value $(-1)^{r-1}$ times the value of the original determinant.*

This may be seen by interchanging row r with the row immediately above it, then interchanging with the next row above its present position, etc., until (after $r - 1$ such interchanges) the original row r arrives at the top. For each of these $r - 1$ interchanges there is a change of sign (by Property 5), and thus the factor $(-1)^{r-1}$ appears.

In the example below, the original row 3 arrives at top position in 2 moves:

$$\begin{vmatrix} 3 & 4 & 6 & 2 \\ 5 & 0 & 3 & 4 \\ 1 & 0 & 0 & 3 \\ 4 & 3 & 7 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 3 & 4 & 6 & 2 \\ 1 & 0 & 0 & 3 \\ 5 & 0 & 3 & 4 \\ 4 & 3 & 7 & 2 \end{vmatrix} = (-1)^2\begin{vmatrix} 1 & 0 & 0 & 3 \\ 3 & 4 & 6 & 2 \\ 5 & 0 & 3 & 4 \\ 4 & 3 & 7 & 2 \end{vmatrix}$$

PROPERTY 7. *The determinant A of order n (see (13)) also has value*

$$A = \sum_{s=1}^{n} (-1)^{r+s} a_{rs}A_{rs} \tag{15}$$

which is called the expansion of A in terms of minors of row r.

To see that A has value given by (15), let B be the determinant obtained from A by moving row r to first position so that (by Property 6)

$$A = (-1)^{r-1} B.$$

An element a_{rs} has minor A_{rs} in A and this same element in its new position in B has the same minor in B so that (by expanding B in terms of minors of its first row; that is, by using (14) applied to B)

$$B = \sum_{s=1}^{n} (-1)^{s+1} a_{rs}A_{rs}$$

Hence $A = (-1)^{r-1} B = (-1)^{r-1} \sum_{s=1}^{n} (-1)^{s+1} a_{rs}A_{rs} = \sum_{s=1}^{n} (-1)^{r-1}(-1)^{s+1} a_{rs}A_{rs}$

$$= \sum_{s=1}^{n} (-1)^{r+s} a_{rs}A_{rs} \qquad \text{which is (15).}$$

In the illustration below, expansion is in terms of minors of row 3:

$$\begin{vmatrix} 4 & -1 & 2 & 3 \\ 6 & 2 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 2 & -3 & 4 & 5 \end{vmatrix} = (-)^4 \cdot 0 \begin{vmatrix} & & \\ & & \\ & & \end{vmatrix} + (-)^5 \cdot 0 \begin{vmatrix} & & \\ & & \\ & & \end{vmatrix} + (-)^6 \cdot 1 \begin{vmatrix} 4 & -1 & 3 \\ 6 & 2 & 4 \\ 2 & -3 & 4 \end{vmatrix} + (-)^7 \cdot 0 \begin{vmatrix} & & \\ & & \\ & & \end{vmatrix}$$

where three of the minors are not filled in since each is multiplied by zero.

Notice that the signs which appear whenever (15) is used may be remembered by the array

$$\begin{vmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

We now mention that throughout all the discussion of determinants the word "row" may be replaced by "column." In particular we check that at the very beginning the value of a third-order determinant could have been defined in terms of elements of the first column (instead of first row) since, for the third-order determinant (4), this expansion is (instead of (5))

$$a_1 \begin{vmatrix} b_3 & c_3 \\ b_3 & c_3 \end{vmatrix} - a_3 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$
$$= a_1(b_3c_3 - b_3c_3) - a_3(b_1c_3 - b_3c_1) + a_3(b_1c_3 - b_3c_1)$$
$$= a_1b_2c_3 - a_1b_3c_3 - a_3b_1c_3 + a_3b_3c_1 + a_3b_1c_3 - a_3b_3c_1$$

which has exactly the same terms (but in slightly different order) as (6).

After checking through all details, it should be seen that the nth-order determinant (13) has expansion in terms of minors of column s given by

$$A = \sum_{r=1}^{n} (-1)^{r+s}a_{rs}A_{rs} \tag{16}$$

(which has summation with respect to r). Formula (16) is a companion to (15) (in which summation is with respect to s).

Another way of phrasing the equivalence of row and column expansion (as given in (15) and (16)) is:

PROPERTY 8. *If the rows* 1, 2, 3, $\cdots$ *of a determinant* A *are made the columns* 1, 2, 3, $\cdots$ *of a determinant* B, *then*

$$A = B$$

or, as sometimes stated, "Interchanging rows and columns of a determinant does not change the value."

PROBLEMS (cont.)

11. Notice how the "little determinants" of the second determinant are made up of the elements of the first determinant.

$$A = \begin{vmatrix} 5 & -4 & 1 \\ 2 & 3 & -1 \\ 1 & 6 & 2 \end{vmatrix}, \qquad B = \frac{1}{5}\begin{Vmatrix} \begin{vmatrix} 5 & -4 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 2 & -1 \end{vmatrix} \\ \\ \begin{vmatrix} 5 & -4 \\ 1 & 6 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix} \end{Vmatrix}$$

Show that $A = 89$. Next evaluate the "little determinants" in B and then evaluate the resulting second-order determinant and find that also $B = 89$.

12. Prove that

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \frac{1}{a_{11}^{n-2}} \begin{Vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{21} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{31} & a_{3n} \end{vmatrix} \\ \cdot & \cdot & \cdot & \cdot \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{n1} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} \end{Vmatrix}$$

(Hint: Use the "reduction method" in which the "preparation for reduction" is done by dividing each element of the first row by a_{11}.)

13. Use the method of Problem 12 to evaluate each of the determinants. Notice that it is not necessary to write down the "little determinants."

a. $\begin{vmatrix} 2 & 1 & -3 \\ 1 & 0 & 2 \\ 3 & 1 & 0 \end{vmatrix}$ b. $\begin{vmatrix} 3 & 2 & 0 & 1 \\ 4 & 1 & -2 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & -5 \end{vmatrix}$

98. Cross Products

Let $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$ be non-parallel vectors. If $\vec{u}$ and $\vec{v}$ do not have the same initial point, then move $\vec{v}$ so they do. In this position $\vec{u}$ and $\vec{v}$ lie in a plane. Perpendicular to this plane and with the same initial point as $\vec{u}$ and $\vec{v}$, select the unit vector $\vec{n}$ from whose terminal end the angle θ *from* $\vec{u}$ *to* $\vec{v}$ appears counterclockwise with $0 < \theta < 180°$.

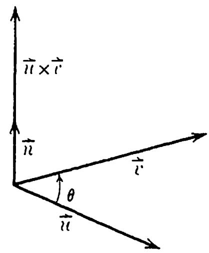


Figure 98.1

DEFINITION. *Given any two vectors $\vec{u}$ and $\vec{v}$, the cross (or* **vector** *or* **outer***) product of $\vec{u}$ into $\vec{v}$ is denoted by $\vec{u} \times \vec{v}$ and is defined by*

$$(1) \qquad \vec{u} \times \vec{v} = \begin{cases} \vec{0} \text{ for } \vec{u} \text{ and } \vec{v} \text{ parallel or either} = \vec{0} \\ \vec{n}|\vec{u}|\,|\vec{v}| \sin\theta \quad \text{otherwise.} \end{cases}$$

From this definition and the relative positions of the basic unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ it follows that

$$(2) \quad \vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j} \quad \text{and} \quad \vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}.$$

The first three equations may be remembered by writing $\vec{i}, \vec{j}, \vec{k}, \vec{i}, \vec{j}$ and noting that the result of a cross between any two is equal to the next in line.

If $\vec{u} \times \vec{v} = \vec{0}$ then also $\vec{v} \times \vec{u} = \vec{0}$. If, however, $\vec{u} \times \vec{v} \neq \vec{0}$, then the rotation of $\vec{v}$ into $\vec{u}$ is opposite the rotation of $\vec{u}$ into $\vec{v}$ and hence $\vec{v} \times \vec{u}$ and $\vec{u} \times \vec{v}$ have opposite senses, but their magnitudes are the same. Hence, in any case

$$(3) \qquad \vec{v} \times \vec{u} = -(\vec{u} \times \vec{v}).$$

Thus, *cross multiplication of two vectors is* **not** *commutative* and the order of the factors must be carefully observed. In particular in contrast to (2),

$$(4) \qquad \vec{j} \times \vec{i} = -\vec{k}, \quad \vec{k} \times \vec{j} = -\vec{i}, \quad \text{and} \quad \vec{i} \times \vec{k} = -\vec{j}.$$

With c a scalar, we shall show that

$$(5) \qquad (c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}).$$

If $c = 0$ then both sides are $\vec{0}$. If $c > 0$ then $|c| = c$ so that

$$\begin{aligned}(c\vec{u}) \times \vec{v} &= \vec{n}\,|c\vec{u}|\,|\vec{v}| \sin\theta = \vec{n}\,|c|\,|\vec{u}|\,|\vec{v}| \sin\theta \\ &= c(\vec{n}\,|\vec{u}|\,|\vec{v}| \sin\theta) = c(\vec{u} \times \vec{v}).\end{aligned}$$

If, however, $c < 0$ then $c\vec{u}$ has opposite sense to $\vec{u}$ so the unit normal vector must be reversed (but the angle less than a straight angle from $c\vec{u}$ to $\vec{v}$ is the same size as θ) so that

$$\begin{aligned}(c\vec{u}) \times \vec{v} &= -\vec{n}\,|c\vec{u}|\,|\vec{v}| \sin\theta = -\,|c|\,\vec{n}\,|\vec{u}|\,|\vec{v}| \sin\theta \\ &= c\vec{n}\,|\vec{u}|\,|\vec{v}| \sin\theta \quad \text{since } |c| = -c.\end{aligned}$$

It then follows from (3) and (5) that

$$(6) \qquad \vec{u} \times (c\vec{v}) = -(c\vec{v}) \times \vec{u} = -c(\vec{v} \times \vec{u}) = c(\vec{u} \times \vec{v}).$$

Hence, for a and b scalars, we have that

$$(a\vec{u}) \times (b\vec{v}) = ab(\vec{u} \times \vec{v}),$$

and in particular $(-\vec{u}) \times \vec{v} = \vec{u} \times (-\vec{v}) = -(\vec{u} \times \vec{v})$.

In the next section it will be proved that "Cross is distributive over plus"; i.e.

(7)
$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$
$$\text{and} \quad (\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}.$$

Example 1. Given $\vec{u}_1 = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$ and $\vec{u}_2 = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}$, use the distributivity of cross over plus to show that

(8)
$$\vec{u}_1 \times \vec{u}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Solution. Each step of the following algebraic manipulations may be justified by one of the above formulas:

$$\begin{aligned}
\vec{u}_1 \times \vec{u}_2 &= \vec{u}_1 \times (a_2\vec{i} + b_2\vec{j} + c_2\vec{k}) = a_2\vec{u}_1 \times \vec{i} + b_2\vec{u}_1 \times \vec{j} + c_2\vec{u}_1 \times \vec{k} \\
&= a_2(a_1\vec{i} + b_1\vec{j} + c_1\vec{k}) \times \vec{i} + b_2(a_1\vec{i} + b_1\vec{j} + c_1\vec{k}) \times \vec{j} \\
&\qquad + c_2(a_1\vec{i} + b_1\vec{j} + c_1\vec{k}) \times \vec{k} \\
&= a_2(a_1\vec{i} \times \vec{i} + b_1\vec{j} \times \vec{i} + c_1\vec{k} \times \vec{i}) \\
&\qquad + b_2(a_1\vec{i} \times j + b_1\vec{j} \times \vec{j} + c_1\vec{k} \times \vec{j}) \\
&\qquad\qquad + c_2(a_1\vec{i} \times \vec{k} + b_1\vec{j} \times \vec{k} + c_1\vec{k} \times \vec{k}) \\
&= a_2(a_1\vec{0} - b_1\vec{k} + c_1\vec{j}) + b_2(a_1\vec{k} + b_1\vec{0} - c_1\vec{i}) + c_2(-a_1\vec{j} + b_1\vec{i} + c_1\vec{0}) \\
&= \vec{i}(b_1c_2 - b_2c_1) - \vec{j}(a_1c_2 - a_2c_1) + \vec{k}(a_1b_2 - a_2b_1)
\end{aligned}$$

which is the formal expansion of the determinant of (8) in terms of minors of the first row.

A plane having equation $Ax + By + Cz = D$ and a line L are perpendicular if and only if A, B, C is a set of direction numbers of L. (See Theorem 96.) This fact is used in the next example.

Example 2. Find an equation of the plane passing through the point $P = (-3,1,6)$ and perpendicular to the line L of intersection of the planes having equations

$$2x - 3y + 4z = 7 \quad \text{and} \quad 5x + 2y - 3z = 4.$$

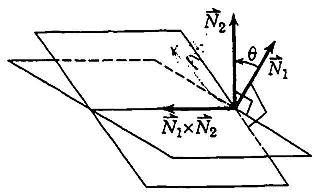


Figure 98.2

Solution. For A, B, and C not all zero the graph of

$$A(x + 3) + B(y - 1) + C(z - 6) = 0$$

is a plane passing through $P = (-3,1,6)$. Such a plane will be perpendicular to L if A, B, C are direction numbers of L. To determine these direction numbers, visualize any point on L as the initial point of vectors $\vec{N}_1$ and $\vec{N}_2$ normal to the given planes (see Fig. 98.2). Then $\vec{N}_1 \times \vec{N}_2$ lies on L and direction numbers of $\vec{N}_1 \times \vec{N}_2$ are also direction numbers of L. Noting the coefficients in the equations of the given planes, it follows that

$$\left.\begin{aligned} \vec{N}_1 &= 2\vec{i} - 3\vec{j} + 4\vec{k} \\ \vec{N}_2 &= 5\vec{i} + 2\vec{j} - 3\vec{k} \end{aligned}\right\} \quad \text{so}$$

$$\vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ 5 & 2 & -3 \end{vmatrix} \begin{aligned} &= \vec{i}(9 - 8) - \vec{j}(-6 - 20) + \vec{k}(4 + 15) \\ &= \vec{i} + 26\vec{j} + 19\vec{k}. \end{aligned}$$

Hence $\vec{N}_1 \times \vec{N}_2$ and L have direction numbers 1, 26, 19 and the plane has equation

$$1(x + 3) + 26(y - 1) + 19(z - 6) = 0 \quad \text{or} \quad x + 26y + 19z = 137.$$

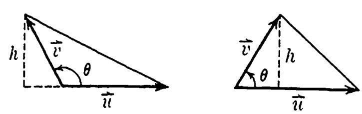


Figure 98.3

Given that $\vec{u} \times \vec{v} \neq \vec{0}$, the vectors $\vec{u}$ and $\vec{v}$ are *adjacent sides of a triangle whose area is*

(9) $$\tfrac{1}{2}\,|\vec{u} \times \vec{v}| \text{ units}^2.$$

The triangle has base $\vec{u}$ units and the altitude (see Fig. 98.3) from the terminal end of $\vec{v}$ is $h = |\vec{v}| \sin\theta$ since $0 < \theta < 180°$. Thus the area is $\frac{1}{2}\,|\vec{u}|\,|\vec{v}| \sin\theta$. But

$$|\vec{u} \times \vec{v}| = |\vec{n}|\,|\vec{u}|\,|\vec{v}| \sin\theta = |\vec{u}|\,|\vec{v}| \sin\theta,$$

since $|\vec{n}| = 1$, and thus (9) follows.

Example 3. Given $P_0 = (2,-6,7)$, $P_1 = (-3,1,4)$, and $P_2 = (3,2,4)$, find the area of T of the triangle having these vertices. Also, find the area t of the projection of this triangle on the xy-plane.

Solution. Let $\vec{u} = \overrightarrow{P_0P_1} = \vec{i}(-3 - 2) + \vec{j}(1 + 6) + \vec{k}(4 - 7) = -5\vec{i} + 7\vec{j} - 3\vec{k}$ and $\vec{v} = \overrightarrow{P_0P_2} = \vec{i} + 8\vec{j} - 3\vec{k}$. Then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 7 & -3 \\ 1 & 8 & -3 \end{vmatrix} = \vec{i}\begin{vmatrix} 7 & -3 \\ 8 & -3 \end{vmatrix} - \vec{j}\begin{vmatrix} -5 & -3 \\ 1 & -3 \end{vmatrix} + \vec{k}\begin{vmatrix} -5 & 7 \\ 1 & 8 \end{vmatrix}$$
$$= 3\vec{i} - 18\vec{j} - 47\vec{k}, \quad \text{and}$$

$$T = \tfrac{1}{2}|\vec{u} \times \vec{v}| = \tfrac{1}{2}\sqrt{3^2 + (-18)^2 + (-47)^2} = \tfrac{1}{2}\sqrt{2542} \text{ units}^2.$$

The projection of the triangle has vertices $P_0' = (2,-6,0)$, $P_1' = (-3,1,0)$, and $P_2' = (3,2,0)$ and two sides $\vec{u}' = -5\vec{i} + 7\vec{j} + 0\vec{k}$, $\vec{v}' = \vec{i} + 8\vec{j} + 0\vec{k}$. Since

$$\vec{u}' \times \vec{v}' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 7 & 0 \\ 1 & 8 & 0 \end{vmatrix} = -47\vec{k}, \quad \text{then} \quad t = \tfrac{1}{2}(47) \text{ units}^2.$$

The following theorem, for use later, may be proved by generalizing the results of Example 3.

THEOREM 98. *If T is the area of a triangle in a plane whose normal has direction angles α, β, γ and if t is the area of the projection of this triangle on the xy-plane, then $t = T|\cos \gamma|$ and if $t \neq 0$, then $\gamma \neq 90°$ and*

$$T = t|\sec \gamma|.$$

PROBLEMS

1. Find $\vec{u} \times \vec{v}$ given that:

   a. $\vec{u} = 2\vec{i} + 3\vec{j} - 4\vec{k}$
   $\vec{v} = -3\vec{i} - 5\vec{j} + 6\vec{k}$.

   b. $\vec{u} = 3\vec{i} + 5\vec{j} - 6\vec{k}$
   $\vec{v} = 2\vec{i} + 3\vec{j} - 4\vec{k}$.

   c. $\vec{u} = \vec{i} - 2\vec{j} + 3\vec{k}$
   $\vec{v} = -2\vec{i} + 4\vec{j} - 6\vec{k}$.

   d. $\vec{u} = 4\vec{i} + 6\vec{j} + 9\vec{k}$
   $\vec{v} = 2\vec{i} + 3\vec{j}$.

2. Compute $(\vec{u} \times \vec{v}) \times \vec{w}$ and $(\vec{u} \cdot \vec{w})\vec{v} - (\vec{v} \cdot \vec{w})\vec{u}$ given that:

   a. $\vec{u} = 3\vec{i} + 5\vec{j} - 6\vec{k}, \quad \vec{v} = \vec{i} + 3\vec{j} - 4\vec{k}, \quad \vec{w} = 3\vec{i} + 2\vec{j} + \vec{k}$.

   b. $\vec{u} = 2\vec{i} + 3\vec{j} - 4\vec{k}, \quad \vec{v} = -3\vec{i} - 5\vec{j} + 6\vec{k}, \quad \vec{w} = \vec{i} + 3\vec{j} - 2\vec{k}$.

3. For the given point P, and L the line of intersection of the planes whose equations are given, find an equation of the plane through P perpendicular to L.

   a. $P = (2,-3,4)$; $x - 4y + 5z = 2, \quad 3x + y - z = -4$.

   b. $P = (0,0,0)$; $x - 4y = 2, \quad 3x + y - z = -4$.

   c. $P = (10,3,-20)$; $3x - 4y = 2, \quad 5x + y = 12$.

   d. $P = (-6,2,18)$; $3x - 4y = 2, \quad 5z = 12$.

4. For P and L as given in Prob. 3, find parametric equations of the line through P parallel to L.

5. Find the area T of the triangle having vertices P_0, P_1, and P_2 where:

   a. $P_0 = (1,0,2), \quad P_1 = (3,1,4), \quad P_2 = (-1,5,1)$.

   b. $P_0 = (0,0,0), \quad P_1 = (4,0,3), \quad P_2 = (0,5,6)$.

   c. $P_0 = (-1,-1,2), \quad P_1 = (1,3,5), \quad P_2 = (4,9,1)$.

6. Project the triangle of Prob. 5 on the coordinate planes, let t_1, t_2, and t_3 be the areas of these projections on the *xy*-, *xz*-, and *yz*-planes, respectively. Find t_1, t_2, and t_3 and show that $T = \sqrt{t_1^2 + t_2^2 + t_3^2}$.

99. Triple Products

The scalar (dot) product of two vectors is a scalar, whereas the vector (cross) product of two vectors is a vector. Thus

$$(\vec{u} \times \vec{v}) \cdot \vec{w} \text{ is a scalar,}$$
$$(\vec{u} \cdot \vec{v})\vec{w} \text{ is a vector, and}$$
$$(\vec{u} \times \vec{v}) \times \vec{w} \text{ is a vector, but}$$
$$(\vec{u} \cdot \vec{v}) \cdot \vec{w} \quad \text{and} \quad (\vec{u} \cdot \vec{v}) \times \vec{w} \quad \text{have no meaning.}$$

The last "have no meaning" since a dot or cross may be placed only between two vector symbols and $(\vec{u} \cdot \vec{v})$ is a scalar. The triple product $(\vec{u} \times \vec{v}) \cdot \vec{w}$ has a geometric interpretation given below. A relation between triple products of the form $(\vec{u} \cdot \vec{v})\vec{w}$ and $(\vec{u} \times \vec{v}) \times \vec{w}$ is given in Problem 1.

The vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ with the same initial point are adjacent edges of a tetrahedron. The volume V of this tetrahedron is given by

(1) $$V = \tfrac{1}{6}\,|(\vec{u} \times \vec{v}) \cdot \vec{w}|$$

as we now show. The triangular face with $\vec{u}$ and $\vec{v}$ as adjacent sides has area $\frac{1}{2}\,|\vec{u} \times \vec{v}|$ as shown in Sec. 98. The vector $\vec{u} \times \vec{v}$ is perpendicular to this face and the projection of $\vec{w}$ on the line of $\vec{u} \times \vec{v}$ has the same length as the altitude H of the tetrahedron. See Fig. 99.1. Thus

$$H = \frac{|(\vec{u} \times \vec{v}) \cdot \vec{w}|}{|\vec{u} \times \vec{v}|}.$$

Consequently

$$V = \tfrac{1}{3}\,(\text{area of base})H = \tfrac{1}{3}[\tfrac{1}{2}\,|\vec{u} \times \vec{v}|]\,\frac{|(\vec{u} \times \vec{v}) \cdot \vec{w}|}{|\vec{u} \times \vec{v}|}$$

from which (1) follows.

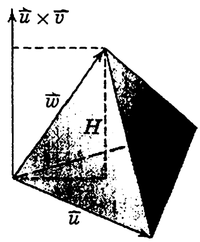


Figure 99.1

Example 1. Find the volume V of the tetrahedron having vertices at the points $P_0 = (-1,2,4)$, $P_1 = (2,-3,4)$, $P_2 = (0,-3,5)$ and $P_3 = (5,-8,9)$.

Solution. Let $\vec{u} = \overrightarrow{P_0P_1}$, $\vec{v} = \overrightarrow{P_0P_2}$, and $\vec{w} = \overrightarrow{P_0P_3}$, then

$$\vec{u} = \overrightarrow{P_0P_1} = 3\vec{i} - 5\vec{j} + 0\vec{k}, \quad \vec{v} = \vec{i} - 5\vec{j} + \vec{k}, \quad \vec{w} = 6\vec{i} - 10\vec{j} + 5\vec{k},$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -5 & 0 \\ 1 & -5 & 1 \end{vmatrix} \begin{aligned} &= \vec{i}(-5) - \vec{j}(3) + \vec{k}(-15 + 5) \\ &= -5\vec{i} - 3\vec{j} - 10\vec{k}, \end{aligned}$$

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (-5\vec{i} - 3\vec{j} - 10\vec{k}) \cdot (6\vec{i} - 10\vec{j} + 5\vec{k}) = -30 + 30 - 50 \quad \text{and}$$

$$V = \tfrac{1}{6}\,|-50| = \tfrac{25}{3} \text{ units}^3.$$

Consider the situation in which $\vec{w}$ and $\vec{u} \times \vec{v}$ are on the same side of the $(\vec{u},\vec{v})$-plane (as shown in Fig. 99.1). Then the scalar $(\vec{u} \times \vec{v}) \cdot \vec{w}$ is positive and the absolute value in (1) may be omitted to give

$$V = \tfrac{1}{6}(\vec{u} \times \vec{v}) \cdot \vec{w}.$$

The same tetrahedron may be considered as having base in the $(\vec{v},\vec{w})$-plane. Then $\vec{u}$ and $\vec{v} \times \vec{w}$ lie on the same side of the $(\vec{v},\vec{w})$-plane and the same volume V of the tetrahedron is

$$V = \tfrac{1}{6}(\vec{v} \times \vec{w}) \cdot \vec{u}.$$

Consequently $(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{w}) \cdot \vec{u}$. The right side may be written as $\vec{u} \cdot (\vec{v} \times \vec{w})$ since the dot product is commutative. Thus

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w}). \tag{2}$$

We thus have the:

RULE. *In $(\vec{u} \times \vec{v}) \cdot \vec{w}$ the cross and dot may be interchanged* **provided** *the terms are grouped so as to have meaning.*

This rule has been proved only for $\vec{w}$ and $\vec{u} \times \vec{v}$ on the same side of the $(\vec{u},\vec{v})$-plane. If, however, $\vec{w}$ and $\vec{u} \times \vec{v}$ are on opposite sides, then $-\vec{w}$ and $\vec{u} \times \vec{v}$ are on the same side of the $(\vec{u},\vec{v})$-plane and

$$\begin{aligned}(\vec{u} \times \vec{v}) \cdot \vec{w} &= -(\vec{u} \times \vec{v}) \cdot (-\vec{w}) \\ &= -\vec{u} \cdot [\vec{v} \times (-\vec{w})] \text{ by the Rule as proved} \\ &= -\vec{u} \cdot [-(\vec{v} \times \vec{w})] = \vec{u} \cdot (\vec{v} \times \vec{w}).\end{aligned}$$

Thus the Rule holds whether $\vec{u}$, $\vec{v}$, $\vec{w}$ is a right-handed or a left-handed system.

We now use the Rule to prove (as promised on page 324):

THEOREM 99. *The cross product is distributive over addition; i.e.*

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w} \text{ and} \tag{3}$$

$$(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}. \tag{4}$$

PROOF. We shall first prove (3) by showing that the vector $\vec{p}$ defined by

$$\vec{p} = \vec{u} \times (\vec{v} + \vec{w}) - \vec{u} \times \vec{v} - \vec{u} \times \vec{w} \tag{5}$$

is the zero vector $\vec{0}$. This will be done by showing that $\vec{p} \cdot \vec{p} = 0$. We write $\vec{p} \cdot \vec{p}$ as

$$\begin{aligned}\vec{p} \cdot \vec{p} &= \vec{p} \cdot [\vec{u} \times (\vec{v} + \vec{w}) - \vec{u} \times \vec{v} - \vec{u} \times \vec{w}] \\ &= \vec{p} \cdot [\vec{u} \times (\vec{v} + \vec{w})] - \vec{p} \cdot (\vec{u} \times \vec{v}) - \vec{p} \cdot (\vec{u} \times \vec{w})\end{aligned}$$

since the dot product is distributive over addition (and subtraction). Next we use the above Rule to interchange cross and dot:

$$\vec{p}\cdot\vec{p} = (\vec{p}\times\vec{u})\cdot(\vec{v}+\vec{w}) - (\vec{p}\times\vec{u})\cdot\vec{v} - (\vec{p}\times\vec{u})\cdot\vec{w}.$$

Again the dot is distributive over addition so that

$$\vec{p}\cdot\vec{p} = (\vec{p}\times\vec{u})\cdot\vec{v} + (\vec{p}\times\vec{u})\cdot\vec{w} - (\vec{p}\times\vec{u})\cdot\vec{v} - (\vec{p}\times\vec{u})\cdot\vec{w} = 0.$$

Thus $\vec{p}\cdot\vec{p} = 0$ so that $\vec{p} = \vec{0}$ and (5) becomes

$$\vec{0} = \vec{u}\times(\vec{v}+\vec{w}) - \vec{u}\times\vec{v} - \vec{u}\times\vec{w}$$

from which (3) follows.

It is left as an exercise to prove that (4) follows from (3).

Many books use $(\vec{u}\,\vec{v}\,\vec{w})$ to mean either side of (2); i.e. by definition

$$(\vec{u}\,\vec{v}\,\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w} = \vec{u}\cdot(\vec{v}\times\vec{w}). \tag{6}$$

Then $(\vec{u}\,\vec{v}\,\vec{w})$ is called the **scalar triple product** of $\vec{u}$, $\vec{v}$, and $\vec{w}$ **in this order.** Hence upon seeing $(\vec{u}\,\vec{v}\,\vec{w})$ the reader is to insert mentally a cross and a dot at his pleasure, but then to group the terms to have meaning.

PROBLEMS

1. Let $\vec{u}_n = \vec{i}a_n + \vec{j}b_n + \vec{k}c_n$ for $n = 1, 2,$ and 3.

a. Show that $(\vec{u}_1 \times \vec{u}_2) \times \vec{u}_3$ equals

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{aligned} &= \vec{i}(-a_1c_2c_3 + a_2c_1c_3 - a_1b_2b_3 + a_2b_1b_3) \\ &\quad - \vec{j}(b_1c_2c_3 - b_2c_1c_3 - a_1a_3b_2 + a_2a_3b_1) \\ &\quad + \vec{k}(b_1b_3c_2 - b_2b_3c_1 + a_1a_3c_2 - a_2a_3c_1). \end{aligned}$$

b. Show that $(\vec{u}_1\cdot\vec{u}_3)\vec{u}_2 - (\vec{u}_2\cdot\vec{u}_3)\vec{u}_1$ and $(\vec{u}_1 \times \vec{u}_2) \times \vec{u}_3$ both expand to the same expression; that is, $(\vec{u}_1 \times \vec{u}_2) \times \vec{u}_3 = (\vec{u}_1\cdot\vec{u}_3)\vec{u}_2 - (\vec{u}_2\cdot\vec{u}_3)\vec{u}_1$.

c. Show that $(\vec{u}_1 \times \vec{u}_2) \times (\vec{u}_3 \times \vec{u}_4) = (\vec{u}_1\vec{u}_3\vec{u}_4)\vec{u}_2 - (\vec{u}_2\vec{u}_3\vec{u}_4)\vec{u}_1$.

2. Find the volume of the tetrahedron having vertices:

a. $(-1,2,-3)$, $(4,-1,2)$, $(7,4,-2)$, $(2,5,8)$.

b. $(-2,4,7)$, $(-3,2,-1)$, $(8,5,2)$, $(2,-1,4)$.

c. $(0,0,0)$, $(1,2,3)$, $(3,1,2)$, $(2,3,1)$.

d. $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

3. Show that:

a. $(\vec{u}\vec{v}\vec{w}) = (\vec{w}\vec{u}\vec{v}) = (\vec{v}\vec{w}\vec{u})$.

b. $(\vec{u}\vec{v}\vec{w}) = -(\vec{v}\vec{u}\vec{w})$.

100. Space Curves

Consider a particle moving in space in such a way that at time $t \geq 0$ it is at the terminal end of the bound vector

$$\vec{F}(t) = 3\vec{i}\cos \pi t + 3\vec{j}\sin \pi t + \tfrac{1}{2}\vec{k}t \tag{1}$$

whose initial end is at the origin. The particle is then at the point $(3\cos \pi t, 3\sin \pi t, \frac{1}{2}t)$ which is 3 units from the z-axis and $\frac{1}{2}t$ units above the xy-plane. Hence, the particle moves on the surface of a right circular cylinder of radius 3 in a path called a **right circular helix.**

For a general discussion, a vector function $\vec{F}$ defined by

$$\vec{F}(t) = \vec{i}f(t) + \vec{j}g(t) + \vec{k}h(t) \tag{2}$$

(where f, g, and h are continuous functions) is called a **law of motion** and the path of a particle obeying this law has parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t). \tag{3}$$

The vector from the terminal end of $\vec{F}(t)$ to the terminal end of $\vec{F}(t + \Delta t)$ is expressed by

$$\Delta\vec{F}(t) = \vec{i}\,\Delta f(t) + \vec{j}\,\Delta g(t) + \vec{k}\,\Delta h(t),$$

and is a chord of the path. The vector

$$\frac{\Delta\vec{F}(t)}{\Delta t} = \vec{i}\frac{\Delta f(t)}{\Delta t} + \vec{j}\frac{\Delta g(t)}{\Delta t} + \vec{k}\frac{\Delta h(t)}{\Delta t}$$

lies along the line containing this chord. Assuming $f'(t)$, $g'(t)$, and $h'(t)$ exist, then

$$\vec{v}(t) = \frac{d\vec{F}(t)}{dt} = \vec{i}\frac{df(t)}{dt} + \vec{j}\frac{dg(t)}{dt} + \vec{k}\frac{dh(t)}{dt} \tag{4}$$

is defined, kinematically, as the **velocity** at time t and is represented geometrically by a tangent vector of length $\sqrt{f'^2(t) + g'^2(t) + h'^2(t)}$ at the point $(f(t),g(t),h(t))$. If $f'(t)$, $g'(t)$, and $h'(t)$ are not all zero (so $\vec{v}(t) \neq \vec{0}$), the line containing the vector $\vec{v}(t)$ and the plane perpendicular to $\vec{v}(t)$ at the point $(f(t),g(t),h(t))$ are called the **tangent line** and **normal plane** to the path at this point. The **acceleration** at time t is defined by

$$\vec{a}(t) = \vec{v}'(t) = \vec{F}''(t) = \vec{i}f''(t) + \vec{j}g''(t) + \vec{k}h''(t). \tag{5}$$

Example 1. For the law of motion $\vec{F}(t) = 3\vec{i}t - \vec{j}t^2 + \vec{k}t^3$, find equations of the tangent line and normal plane to the path at the point corresponding to $t = 2$. Also, find the tangential component of acceleration when $t = 2$.

Solution. $\vec{F}(2) = 6\vec{i} - 4\vec{j} + 8\vec{k}$ so $(6,-4,8)$ is the point in question.

$$\vec{v}(t) = 3\vec{i} - 2\vec{j}t + 3\vec{k}t^2, \quad \vec{v}(2) = 3\vec{i} - 4\vec{j} + 12\vec{k}$$

so 3, -4, 12 is a set of direction numbers of the tangent line. Parametric equations of the tangent line and an equation of the normal plane are therefore

$$x = 6 + 3s,\ y = -4 - 4s,\ z = 8 + 12s; \quad \text{and}$$
$$3(x - 6) - 4(y + 4) + 12(z - 8) = 0.$$

Since $\vec{a}(t) = 0\vec{i} - 2\vec{j} + 6\vec{k}t$ and $\vec{a}(2) = -2\vec{j} + 12\vec{k}$, the tangential component when $t = 2$ is (see Theorem 79)

$$\frac{\vec{v}(2) \cdot \vec{a}(2)}{\vec{v}(2) \cdot \vec{v}(2)}\vec{v}(2) = \frac{3(0) - 4(-2) + 12(12)}{9 + 16 + 144}(3\vec{i} - 4\vec{j} + 12\vec{k}) = \frac{152}{169}(3\vec{i} - 4\vec{j} + 12\vec{k}).$$

If $f'(t)$, $g'(t)$, and $h'(t)$ are continuous for $\alpha \le t \le \beta$, then the integral

$$\int_\alpha^\beta \sqrt{f'^2(t) + g'^2(t) + h'^2(t)}\, dt \tag{6}$$

exists and is defined to be the **length of the path** from $\vec{F}(\alpha)$ to $\vec{F}(\beta)$. The **differential of arc length** is defined by

$$ds = \sqrt{f'^2(t) + g'^2(t) + h'^2(t)}\, dt \tag{7}$$

which carries the implication that a point $P_0 = (f(t_0), g(t_0), h(t_0))$ has been selected and that s is the increasing function for which $|s(t)|$ is the length of the path from P_0 to the point $(f(t), g(t), h(t))$. Hence s has an inverse function which, to conserve notation, will be denoted by t and is such that

$$dt = \frac{1}{\sqrt{f'^2 + g'^2 + h'^2}}\, ds.$$

A value of this inverse function is denoted by $t(s)$ and the path is parameterized in terms of arc length by setting

$$x = x(s) = f(t(s)), \quad y = y(s) = g(t(s)), \quad z = z(s) = h(t(s)).$$

Geometric properties of curves are studied (in a branch of mathematics called Differential Geometry) by means of a right-handed system of three unit vectors at each point of the curve. With the independent variables suppressed to shorten the formulas, the first of these unit vectors is denoted by $\vec{\tau}$ and is defined by

$$\vec{\tau} = \vec{i}\frac{dx}{ds} + \vec{j}\frac{dy}{ds} + \vec{k}\frac{dz}{ds}. \tag{8}$$

Hence $\vec{\tau}$ is tangent to the path. Since

$$\frac{dx}{ds} = \frac{df}{dt}\frac{dt}{ds} = \frac{f'}{\sqrt{f'^2 + g'^2 + h'^2}}, \quad \frac{dy}{ds} = \frac{g'}{\sqrt{f'^2 + g'^2 + h'^2}},$$
$$\frac{dz}{ds} = \frac{h'}{\sqrt{f'^2 + g'^2 + h'^2}}$$

it follows that $\vec{\tau} \cdot \vec{\tau} = 1$ so that $\vec{\tau}$ is a unit vector. Also

$$\frac{d(\vec{\tau} \cdot \vec{\tau})}{ds} = \frac{d1}{ds} \quad \text{so} \quad \vec{\tau} \cdot \frac{d\vec{\tau}}{ds} + \frac{d\vec{\tau}}{ds} \cdot \vec{\tau} = 0 \quad \text{and} \quad 2\vec{\tau} \cdot \frac{d\vec{\tau}}{ds} = 0.$$

Upon setting $\vec{N} = d\vec{\tau}/ds$, then $\vec{\tau} \cdot \vec{N} = 0$ so either $\vec{N} = \vec{0}$ or else $\vec{N}$ is a definite vector perpendicular to $\vec{\tau}$. In case $\vec{N} \neq \vec{0}$ let $\kappa > 0$ be the number and $\vec{\eta}$ the unit vector such that $\vec{N} = \kappa\vec{\eta}$ and if $\vec{N} = \vec{0}$ let $\kappa = 0$ and $\vec{\eta}$ any unit vector perpendicular to $\vec{\tau}$ so that

(9) $\vec{N} = \kappa\vec{\eta}$ in both cases.

With $\vec{\tau}$ and $\vec{\eta}$ defined, let $\vec{\beta} = \vec{\tau} \times \vec{\eta}$. Then $\vec{\tau}$, $\vec{\eta}$, and $\vec{\beta}$ form a right-handed system of three unit vectors in which $\vec{\tau}$ is called the **unit tangent**, $\vec{\eta}$ the **unit principal normal**, $\vec{\beta}$ the **unit binormal**, and the scalar κ is called the **curvature** of the curve at the point considered.

Example 2. For the path of a particle following the law of Example 1, find $\vec{\tau}$, $\vec{\eta}$, $\vec{\beta}$, and κ at the point corresponding to $t = 1$.

Solution. $\vec{\tau}(t) = \dfrac{d\vec{F}}{dt}\dfrac{dt}{ds} = (3\vec{i} - 2\vec{j}t + 3\vec{k}t^2)\dfrac{1}{\sqrt{3^2 + (-2t)^2 + (3t^2)^2}}.$

$$\vec{N}(t) = \frac{d\vec{\tau}}{ds} = \frac{d\vec{\tau}}{dt}\frac{dt}{ds}$$

$$= \left[\frac{(3\vec{i} - 2\vec{j}t + 3\vec{k}t^2)}{-2(9 + 4t^2 + 9t^4)^{3/2}}(8t + 36t^3) + \frac{(0\vec{i} - 2\vec{j} + 6\vec{k}t)}{\sqrt{9 + 4t^2 + 9t^4}}\right]\frac{1}{\sqrt{9 + 4t^2 + 9t^4}} = \text{etc.}$$

$$= \frac{1}{(9 + 4t^2 + 9t^4)^2}[(-12t - 54t^3)\vec{i} + (-18 + 18t^4)\vec{j} + (54t + 12t^3)\vec{k}].$$

$$\vec{\tau}(1) = \frac{1}{\sqrt{22}}(3\vec{i} - 2\vec{j} + 3\vec{k}),$$

$$\vec{N}(1) = \frac{1}{(22)^2}(-66\vec{i} + 0\vec{j} + 66\vec{k}) = \frac{66}{(22)^2}(-\vec{i} + \vec{k}) = \frac{3\sqrt{2}}{22}\left(\frac{-\vec{i} + \vec{k}}{\sqrt{2}}\right).$$

Therefore $\vec{\eta}(1) = \dfrac{-\vec{i} + \vec{k}}{\sqrt{2}}$ and $\kappa(1) = \dfrac{3\sqrt{2}}{22}$. Consequently

$$\vec{\beta}(1) = \frac{1}{\sqrt{22}}\frac{1}{\sqrt{2}}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 3 \\ -1 & 0 & 1 \end{vmatrix} = \frac{1}{2\sqrt{11}}(-2\vec{i} - 6\vec{j} - 2\vec{k}) = -\frac{1}{\sqrt{11}}(\vec{i} + 3\vec{j} + \vec{k}).$$

PROBLEMS

1. Find $\vec{\tau}(t)$, $\vec{\eta}(t)$, $\kappa(t)$, and $\vec{\beta}(t)$ for the path of the particle following the given law. Also, find the distance traveled by the particle from $t = 0$ to $t = 2\pi$.

   a. $\vec{F}(t) = \vec{i}3\cos\pi t + \vec{j}3\sin\pi t + \vec{k}\frac{1}{2}t.$

b. $\vec{F}(t) = \vec{i}e^t \cos t + \vec{j}e^t \sin t + \vec{k}e^t$.

c. $\vec{F}(t) = \vec{i}2 \cosh 3t + \vec{j}2 \sinh 3t + \vec{k}6t$.

d. $\vec{F}(t) = \vec{i}3t \cos t + \vec{j}3t \sin t + \vec{k}4t$, but find only $\vec{\tau}(0)$, $\vec{\eta}(0)$, $\kappa(0)$, and $\vec{\beta}(0)$.

2. For the general law of motion (2) prove that the angle θ between the acceleration vector $\vec{a}(t)$ and the principal normal vector $\vec{N}(t)$ is such that $|\theta| \leq 90°$.

101. Surfaces and Solids

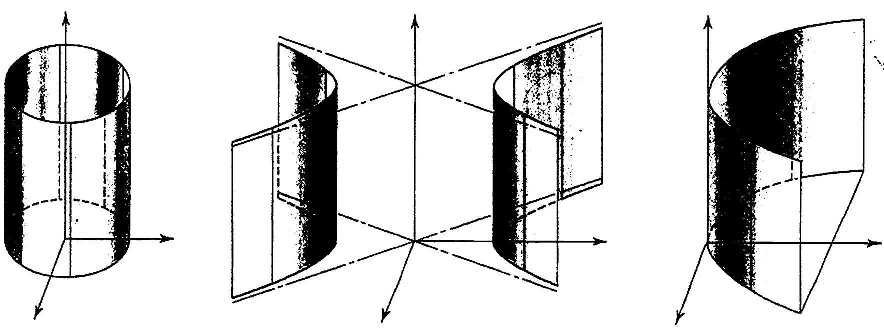
In terms of plane rectangular coordinates the set $\{(x,y) \mid 3x + 2y - 6 = 0\}$ is a line. In terms of three-dimensional coordinates the set $\{(x,y,z) \mid 3x + 2y - 6 = 0\}$ is a plane (it may be considered as the set $\{(x,y,z) \mid 3x + 2y + 0z - 6 = 0\}$). The set $\{(x,y,0) \mid 3x + 2y - 6 = 0\}$ is a line but is imbedded in three-dimensional space. The two sets

$$\{(x,y,0) \mid x^2 + y^2 = 4\} \quad \text{and} \quad \{(x,y,z) \mid x^2 + y^2 = 4\}$$

have graphs in three-dimensional space; the first is a circle in the xy-plane whereas the second is unbounded, but each plane section parallel to the xy-plane is a circle of radius 2 with center on the z-axis.

Given a curve C in a plane and a line L not in or parallel to this plane, the set of all points on all lines parallel to L through points of C is called a **cylindrical surface**. In particular the sets

$$\left\{(x,y,z) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}, \quad \left\{(x,y,z) \,\middle|\, \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1\right\}, \quad \text{and} \quad \left\{(x,y,z) \,\middle|\, x^2 = 2py\right\}$$



A. B. C.

are **elliptic**, **hyperbolic**, and **parabolic cylinders**, respectively.

The rectangular graphs of sets whose defining equations are of second degree in x, y, and z are called **quadrics**. Thus $\{(x,y,z) \mid (x - 3)^2 + (y - 2)^2 + (z + 4)^2 = 0\}$ is the single point $(3,2,-4)$; the set $\{(x,y,z) \mid (x - 3)^2 + (y - 2)^2 = 0\}$ is the line through the point $(3,2,0)$ perpendicular to the xy-plane, and since

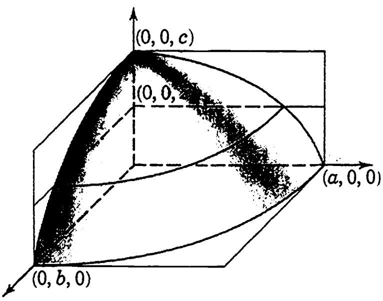
$$\{(x,y,z) \mid (x - 4)^2 = 9\} = \{(x,y,z) \mid x = 7\} \cup \{(x,y,z) \mid x = 1\}$$

this set is two parallel planes. Also, the set

$$\left\{(x,y,z) \,\middle|\, \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0\right\} = \left\{(x,y,z) \,\middle|\, y = \frac{b}{a}x\right\} \cup \left\{(x,y,z) \,\middle|\, y = -\frac{b}{a}x\right\}$$

is two intersecting planes. These four graphs typify the **degenerate quadrics.**

Names of the general conics follow and the conics are designated without using set notation, as is the usual practice. In each case the set of all points with coordinates (x,y,z) satisfying the equation is intended. Herein $a > 0$, $b > 0$, and $c > 0$.



Ellipsoid

(i) **Ellipsoid,** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

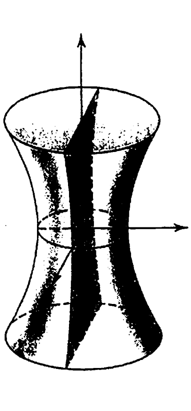
Upon writing $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 - \dfrac{z^2}{c^2}$

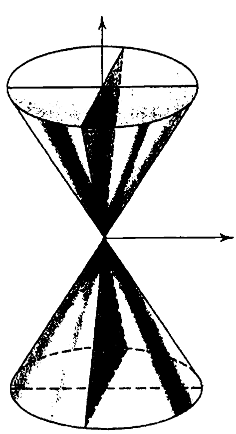
or $$\frac{x^2}{a^2(1 - z^2/c^2)} + \frac{y^2}{b^2(1 - z^2/c^2)} = 1,$$

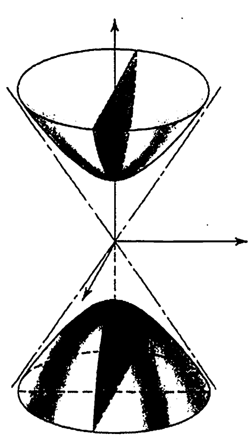
$-c < z < c$, it follows that the plane perpendicular to the z-axis at the point $(0,0,z)$ with $-c < z < c$ intersects the ellipsoid in an ellipse having semiaxes $a\sqrt{1 - (z/c)^2}$ and $b\sqrt{1 - (z/c)^2}$ units long.

(ii) **Hyperboloid of one sheet,** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1.$

(iii) **Elliptic cone,** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0.$


Hyperboloid of one sheet

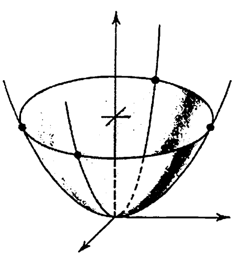

Elliptic Cone


Hyperboloid of two sheets

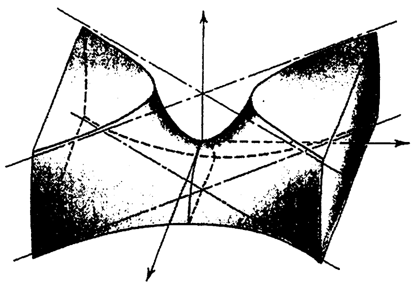
(iv) **Hyperboloid of two sheets,** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1.$

(v) **Elliptic paraboloid,** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = z.$

(vi) **Hyperbolic paraboloid,** $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = z.$



Elliptic paraboloid



Hyperbolic paraboloid

Example 1. Prove that through each point of the hyperbolic paraboloid (vi) there are two lines which lie completely on the surface. (Note: Such a surface is called a **doubly ruled surface.**)

Solution. Let (x_1,y_1,z_1) be a point on the surface so that

$$\frac{y_1^2}{a^2} - \frac{x_1^2}{b^2} = z_1. \tag{1}$$

For A, B, and C not all zero the line having parametric equations

$$x = x_1 + At, \quad y = y_1 + Bt, \quad z = z_1 + Ct$$

passes through the point (x_1,y_1,z_1). This line lies on the surface if

$$\frac{(y_1 + Bt)^2}{a^2} - \frac{(x_1 + At)^2}{b^2} = z_1 + Ct$$

is satisfied for all values of t. This equation may be written as

$$\frac{y_1^2}{a^2} + \frac{(2y_1Bt + B^2t^2)}{a^2} - \frac{x_1^2}{b^2} - \frac{(2x_1At + A^2t^2)}{b^2} = z_1 + Ct \quad \text{or by (1) as}$$

$$\left(\frac{2y_1B}{a^2} - \frac{2x_1A}{b^2} - C\right)t + \left(\frac{B^2}{a^2} - \frac{A^2}{b^2}\right)t^2 = 0,$$

which is satisfied for all values of t if and only if the coefficients of both t and t^2 are zero:

$$\frac{2y_1B}{a^2} - \frac{2x_1A}{b^2} - C = 0 \quad \text{and} \quad \frac{B^2}{a^2} - \frac{A^2}{b^2} = 0; \quad \text{that is,}$$

$$B = \pm\frac{a}{b}A, \quad C = \pm\frac{2y_1}{a^2}\frac{a}{b}A - \frac{2x_1}{b^2}A = \frac{2}{b}\left(\pm\frac{y_1}{a} - \frac{x_1}{b}\right)A.$$

Now $A \neq 0$ (for if $A = 0$ then also $B = 0$ and $C = 0$) but otherwise A is arbitrary and we choose $A = 1$. Hence, the lines having parametric equations

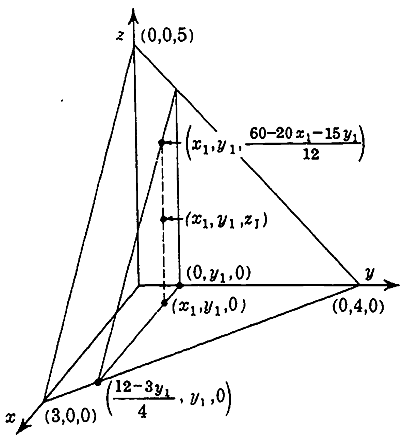


Figure 101.1

$$x = x_1 + t, \quad y = y_1 + \frac{a}{b}t, \quad z = z_1 + \frac{2}{b}\left(\frac{y_1}{a} - \frac{x_1}{b}\right)t,$$

and

$$x = x_1 + t, \quad y = y_1 - \frac{a}{b}t, \quad z = z_1 - \frac{2}{b}\left(\frac{y_1}{a} + \frac{x_1}{b}\right)t$$

are distinct, both pass through the point (x_1,y_1,z_1) and both lie completely on the hyperbolic paraboloid (vi).

The plane having equation

$$20x + 15y + 12z = 60$$

cuts the axes at the points (3,0,0), (0,4,0), and (0,0,5). These three points together with the origin are vertices of a tetrahedron which as a solid may be characterized as

$$\left\{(x,y,z) \,\middle|\, 0 \le y \le 4, \quad 0 \le x \le \frac{12 - 3y}{4}, \quad 0 \le z \le \frac{60 - 20x - 15y}{12}\right\}.$$

For consider a point (x_1,y_1,z_1) of this solid. Then the plane through this point perpendicular to the y-axis cuts the y-axis at the point $(0,y_1,0)$ with $0 \le y_1 \le 4$. The given plane cuts the xy-plane in the line

$$\{(x,y,0) \mid 20x + 15y = 60\} = \left\{(x,y,0) \,\middle|\, x = \frac{12 - 3y}{4}\right\}$$

so that $0 \le x_1 \le \dfrac{12 - 3y_1}{4}$. Now the point (x_1,y_1,z_1) is on the segment joining the point $(x_1,y_1,0)$ and that point (x_1,y_1,z) for which z satisfies the equation

$$20x_1 + 15y_1 + 12z = 60 \quad \text{so that} \quad 0 \le z_1 \le \frac{60 - 20x_1 - 15y_1}{12}.$$

The same solid may also be characterized (by first limiting x, then y) as

$$\left\{(x,y,z) \,\middle|\, 0 \le x \le 3, \quad 0 \le y \le \frac{12-4x}{3}, \quad 0 \le z \le \frac{60-20x-15y}{12}\right\}$$

and there are other ways of designating the solid according to the order in which the variables are limited.

Example 2. Give a characterization of the solid which lies above the xy-plane and inside both the surfaces characterized by the equations

$$(x-1)^2 + (y-2)^2 = 4 \quad \text{and} \quad \frac{x^2}{25} + \frac{y^2}{25} + \frac{z^2}{16} = 1.$$

Solution. The base of this solid is the circular disc

$$\{(x,y,0) \mid (x-1)^2 + (y-2)^2 \le 4\}$$

having center $(1,2,0)$ and radius 2. Thus, a plane perpendicular to the y-axis and intersecting the solid cuts the y-axis at a point $(0,y,0)$ having $0 \le y \le 4$ and (with y so limited) a point $(x,y,0)$ is in the base if $(x-1)^2 + (y-2)^2 \le 4$ and hence $(x-1)^2 \le 4 - (y-2)^2$ so that

$$1 - \sqrt{4-(y-2)^2} \le x \le 1 + \sqrt{4-(y-2)^2}.$$

Now with x and y so limited, z must be such that $0 \le z$ and

$$\frac{x^2}{25} + \frac{y^2}{25} + \frac{z^2}{16} \le 1 \quad \text{so that} \quad 0 \le z \le 4\sqrt{1 - \frac{x^2}{25} - \frac{y^2}{25}} = \frac{4}{5}\sqrt{25-x^2-y^2}$$

Thus (since $4 - (y-2)^2 = 4y - y^2$), a characterization of the solid is

$$\{(x,y,z) \mid 0 \le y \le 4, \quad 1 - \sqrt{4y-y^2} \le x \le 1 + \sqrt{4y-y^2},$$
$$0 \le z \le \tfrac{4}{5}\sqrt{25-x^2-y^2}\}.$$

It should be seen that another way of expressing the same solid is

$$\{(x,y,z) \mid -1 \le x \le 3, \quad 2 - \sqrt{-x^2+2x+3} \le y \le 2 + \sqrt{-x^2+2x+3},$$
$$0 \le z \le \tfrac{4}{5}\sqrt{25-x^2-y^2}\}.$$

With f a function of one variable, a portion of the yz-plane is characterized by

$$\{(0,y,z) \mid 0 \le z \le |f(y)|\}.$$

By revolving this plane section about the y-axis a solid is generated which is characterized by

$$\{(x,y,z) \mid \sqrt{x^2+z^2} \le |f(y)|\}$$
$$= \{(x,y,z) \mid x^2 + z^2 \le f^2(y)\}.$$

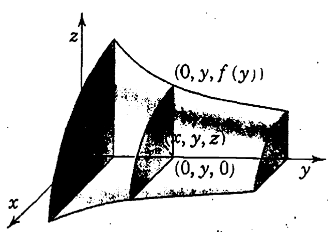


Figure 101.2

For, with y a given number, a point (x,y,z) belongs to this solid of revolution if and only if the distance between the points (x,y,z) and $(0,y,0)$ is less than or equal the distance between the points $(0,y,f(y))$ and $(0,y,0)$. The surface of this solid is the set

$$\{(x,y,z) \mid x^2 + z^2 = f^2(y)\}.$$

Example 3. Characterize the solid obtained by revolving the circular disc $\{(x,0,z) \mid (x-2)^2 + z^2 \le 1\}$ about the z-axis.

Solution. $\{(x,y,z) \mid -1 \le z \le 1,\ 2 - \sqrt{1-z^2} \le \sqrt{x^2+y^2} \le 2 + \sqrt{1-z^2}\}$. For with z a number such that $-1 \le z \le 1$, the line in the xz-plane parallel to the x-axis intersects the circle at the points $(x_1,0,z)$ and $(x_2,0,z)$ where x_1 and x_2 are the x-roots of the equation $(x-2)^2 + z^2 = 1$ so that $x_1 = 2 - \sqrt{1-z^2}$ and $x_2 = 2 + \sqrt{1-z^2}$. Hence, a point (x,y,z), with $-1 \le z \le 1$, belongs to the solid whenever its distance to the point $(0,0,z)$ is between x_1 and x_2.

PROBLEMS

1. A solid is bounded below by the xy-plane and on the sides and above by the given sets. Characterize the solid by limiting x, then y, then z and again by limiting y, then x, then z.
   a. $\{(x,y,z) \mid x = 1 \text{ or } x = 2\}$, $\{(x,y,z) \mid y = 0 \text{ or } y = 2.\}$, $\{(x,y,z) \mid z = x + y\}$.
   b. $\{(x,y,z) \mid y^2 = 16x\}$, $\{(x,y,z) \mid z = -x + 4\}$.
   c. $\{(x,y,z) \mid x^2 + y^2 = 16\}$, $\{(x,y,z) \mid 2y = z\}$.
   d. $\{(x,y,z) \mid x^2 + y^2 = 16\}$, $\{(x,y,z) \mid z = x + y + 6\}$.
   e. $\{(x,y,z) \mid y^2 = 4x\}$, $\{(x,y,z) \mid x^2 = 4y\}$, $\{(x,y,z) \mid z = x^2 + y\}$.
   f. $\{(x,y,z) \mid y^2 = x\}$, $\{(x,y,z) \mid x^3 = y\}$, $\{(x,y,z) \mid z = 2 + xy^2\}$.
2. Describe the solid:
   a. $\{(x,y,z) \mid -4 \le x \le 4,\ -\sqrt{16-x^2} \le y \le \sqrt{16-x^2},\ 0 \le z \le x + 5\}$.
   b. $\{(x,y,z) \mid -a \le x \le a,\ -\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2},\ -\sqrt{a^2-x^2-y^2} \le z \le \sqrt{a^2-x^2-y^2}\}$.
   c. $\{(x,y,z) \mid 0 < (x-1)^2 + (y-2)^2 < 1,\ 5 \le z \le 7\}$.
   d. $\{(x,y,z) \mid 1 \le (x-1)^2 + (y+2)^2 \le 4,\ 3 \le z \le 8\}$.
   e. $\{(x,y,z) \mid |x| + |y| \le 1,\ 0 \le z \le 4 - x^2 - y^2\}$.
   f. $\{(x,y,z) \mid x^2 + y^2 \le 4,\ 0 \le z \le 4 - x^2 - y^2\}$.
3. Characterize the solid common to two cylinders each of radius a, one with axis along the x-axis, the other with axis along the y-axis.
4. Revolve each of the given plane regions about the axis indicated and characterize the solid generated. Also find the volume of the solid.
   a. $\{(x,0,z) \mid 0 \le x \le 2, 0 \le z \le x^2\}$; x-axis.
   b. $\{(x,0,z \mid 0 \le z \le 4, \sqrt{z} \le x \le 2\}$; x-axis.
   c. $\{(x,0,z) \mid 0 \le x \le 2, 0 \le z \le x^2\}$; z-axis.
   d. $\{(0,y,z) \mid 0 \le y \le \pi, 0 \le z \le \sin y\}$; y-axis.

102. Functions of Two Variables

As defined earlier, a function is a set of ordered pairs such that if (a,b) and (a,c) are in the set, then $b = c$ with the set of all first elements called the **domain** and the set of all second elements called the **range**. The elements of the domain and/or range of a function need not be numbers. For example, a telephone directory may be considered as a function since it gives a set of ordered pairs; the first element of each pair being a name and address, whereas the second element is a call signal (which may contain letters as well as numbers). A function in which each element of the domain is an ordered pair of numbers is said to be a function of **two real variables** and is said to be **numerical valued** if each element of the range is a number. Such functions will now be considered. Hence, a numerical valued function of two real variables is a set of ordered triples of numbers such that if (p,q,r) and (p,q,s) are in the set then $r = s$. For f a function of two variables and (x,y) in the domain of f, the **value** of f at (x,y) is represented by $f(x,y)$.

Relative to a three-dimensional coordinate system, the rectangular graph of a numerical valued function f of two real variables is the set of all points each having its coordinates in f. A perspective drawing of the graph of

(1) $f = \{(x,y,z) \mid z = y^2 - x^2\}$

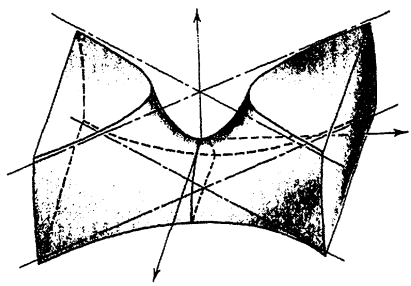


Figure 102.1

is given in Fig. 102.1. Such a surface is visualized by sketching intersections of the surface and planes; these intersections being called **profiles** of the surface. For example, the plane perpendicular to the z-axis at the point (0,0,1) cuts the surface (1) in

$$\{(x,y,1) \mid y^2 - x^2 = 1\},$$

which is a hyperbola shown at the top of the figure together with its asymptotes (the dot-dash lines). The profile in the xz-plane is

$$\{(x,0,z) \mid z = -x^2\}$$

and is a parabola with the vertex up and at the origin. The set

$$\{(0,y,z) \mid z = y^2\}$$

is the profile in the yz-plane and is a parabola with its vertex down. The profile in the xy-plane is

$$\{(x,y,0) \mid y^2 - x^2 = 0\} = \{(x,y,0) \mid y = x\} \cup \{(x,y,0) \mid y = -x\}$$

and thus consists of two straight lines (not shown on the figure).

Example 1. Discuss the set $f = \left\{(x,y,z) \,\middle|\, z = \dfrac{2xy}{x^2 + y^2}\right\}$.

Solution. The profiles of this set in the planes perpendicular to the y-axis at the points (0,1,0), and (0,3,0) are the sets

$$\left\{(x,1,z) \,\middle|\, z = \frac{2x}{x^2 + 1}\right\} \quad \text{and} \quad \left\{(x,3,z) \,\middle|\, z = \frac{6x}{x^2 + 9}\right\},$$

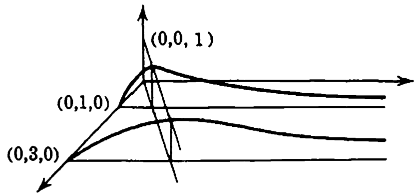


Figure 102.2

portions of which are shown in Fig. 102.2. Since

$$D_x \frac{2x}{x^2 + 1} = \frac{2(1 - x^2)}{(x^2 + 1)^2} \quad \text{and}$$

$$D_x^2 \frac{2x}{x^2 + 1} = \frac{4(x^3 - 3x)}{(x^2 + 1)^3},$$

the maximum and minimum points of the first of these profiles are (1,1,1) and (−1,1,−1). In the same way the maximum and minimum points of the second profile are (3,3,1), and (−3,3,−1).

For m any number

$$f(x,mx) = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2m}{1 + m^2}, \quad x \neq 0$$

and hence in the plane $\{(x,y,z) \mid y = mx\}$ the profile of f is all of the line $\left\{(x,y,z) \,\middle|\, y = mx,\, z = \dfrac{2m}{1 + m^2}\right\}$ except the point $\left(0, 0, \dfrac{2m}{1 + m^2}\right)$. Notice that if z_0 is any number such that $-1 \leq z_0 \leq 1$, then

$$\frac{2m}{1 + m^2} = z_0$$

has a solution for m. Hence, the set f has no point on the z-axis, but we can find a point of f as close as we please to any designated point of the interval of the z-axis joining the points (0,0,−1) and (0,0,1).

Definition. *A function f of two variables is said to have* **limit** *L at (a,b) and we write*

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

if corresponding to each number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\{(x,y,z) \mid 0 < (x - a)^2 + (y - b)^2 < \delta^2 \quad \text{and} \quad z = f(x,y)\}$$

is a non-empty subset of

$$\{(x,y,z) \mid 0 < (x - a)^2 + (y - b)^2 < \delta^2 \quad \text{and} \quad L - \epsilon < z < L + \epsilon\};$$

that is, whenever $0 < (x - a)^2 + (y - b)^2 < \delta^2$, *then* $|f(x,y) - L| < \epsilon$.

A better conception of the above definition may be gained by a specific example of non-existence of a limit. The function f defined by

$$(2) \qquad f(x,y) = \frac{2xy}{x^2 + y^2}, \quad x^2 + y^2 \neq 0$$
$$f(0,0) = 0$$

(see Example 1) fails to have a limit at (0,0) as the following argument shows. The graph of f has only the origin on the z-axis, but every right circular cylinder about the z-axis contains additional points of the graph of f in its interior and, moreover, some such points at every level from 1 unit below to 1 unit above the xy-plane. Thus, for L any number, for ϵ any number such that $0 < \epsilon < 1$, and for δ any positive number, there are ordered pairs (x_1,y_1) and (x_2,y_2) in the domain of f such that $0 < (x_1 - 0)^2 + (y_1 - 0)^2 < \delta^2$, $0 < (x_2 - 0)^2 + (y_2 - 0)^2 < \delta^2$ and such that at least one of $f(x_1,y_1)$ or $f(x_2,y_2)$ differs from L by more than ϵ. Specifically

$$f\left(\frac{\delta}{2}, \frac{\delta}{2}\right) = \frac{2(\delta/2)(\delta/2)}{(\delta/2)^2 + (\delta/2)^2} = 1 \quad \text{whereas} \quad f\left(\frac{\delta}{2}, -\frac{\delta}{2}\right) = -1$$

and any given number L differs from either 1 or -1 by more than ϵ since $0 < \epsilon < 1$. Hence, given any number L, this number cannot be the limit of f at (0,0). It is customary to say, "The limit of this function at (0,0) does not exist."

With the same function f given by (2), note that

$$\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{2x(x)}{x^2 + x^2} = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1.$$

Suggestive geometric terminology is used to express this fact by saying, "At (0,0) the **limit along the line** $\{(x,y) \mid y = x\}$ exists and is equal to 1." In the same way, the limit at (0,0) of this function along the line $\{(x,y) \mid y = 2x\}$ exists and is equal to $\frac{4}{5}$ since

$$\lim_{x\to 0} f(x,2x) = \lim_{x\to 0} \frac{2x(2x)}{x^2 + (2x)^2} = \frac{4}{5}.$$

Also, the limit at (0,0) of this function along the parabola $\{(x,y) \mid y = x^2\}$ exists and is 0; that is,

$$\lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{2x(x^2)}{x^2 + x^4} = \lim_{x\to 0} \frac{2x}{1 + x^2} = 0.$$

In general terms it should be seen that:

If a function f is such that its limits at (a,b) exist along two curves and these limits are different, then the limit of f at (a,b) does not exist. Also, if the limit at (a,b) along a curve does not exist, then the limit of f at (a,b) does not exist.

Example 2. Show the limit at $(1,-1)$ does not exist for the function f defined by

$$f(x,y) = \frac{\sin(x+y)}{x-1}, \quad x \neq 1.$$

Solution. Recall that $\lim\limits_{h\to 0} \dfrac{\sin h}{h} = 1$. In the plane, the line through $(1,-1)$ with slope m has equation $y + 1 = m(x-1)$. Now

$$\lim_{x\to 1} f(x, -1 + m(x-1)) = \lim_{x\to 1} \frac{\sin[x - 1 + m(x-1)]}{x-1}$$

$$= \lim_{x\to 1} \frac{\sin[(x-1)(1+m)]}{x-1} = \lim_{x\to 1} \frac{[\sin(x-1)(1+m)]}{(x-1)(1+m)} \cdot (1+m) = 1 + m.$$

By assigning different values to m, we see that this function has different limits along different lines through $(1,-1)$ and thus

$$\lim_{(x,y)\to(1,-1)} f(x,y) \text{ does not exist.}$$

A proof of the following theorem may be patterned after the proof of Theorem 17.

THEOREM 102. *If f and g are functions of two variables such that the limit of each exists at (a,b) then*

$$\text{(i)} \quad \lim_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y),$$

$$\text{(ii)} \quad \lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right]\left[\lim_{(x,y)\to(a,b)} g(x,y)\right],$$

and if the limit of g at (a,b) is not zero, then also

$$\text{(iii)} \quad \lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right] \Big/ \left[\lim_{(x,y)\to(a,b)} g(x,y)\right].$$

DEFINITION. *A function f is said to be* **continuous** *at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists and equals $f(a,b)$.*

Thus, a function f is continuous at (x,y) if and only if

$$\lim_{(h,k)\to(0,0)} f(x+h, y+k) = f(x,y).$$

It may be possible to use Theorem 102 and known limits of functions of a single variable to establish a limit of a function of two variables.

Example 3. For the function defined by (2) show that

$$\lim_{(x,y)\to(3,4)} f(x,y) = \tfrac{24}{25}.$$

Solution. Since

$$\lim_{(x,y)\to(3,4)} x = \lim_{x\to 3} x = 3 \quad \text{and} \quad \lim_{(x,y)\to(3,4)} y = \lim_{y\to 4} y = 4$$

it follows from Theorem 102(ii) that

$$\lim_{(x,y)\to(3,4)} 2xy = \left(\lim_{(x,y)\to(3,4)} 2\right)\left(\lim_{(x,y)\to(3,4)} x\right)\left(\lim_{(x,y)\to(3,4)} y\right) = 2\cdot 3\cdot 4 = 24.$$

Also, from Theorem 102(ii) and (i) we have

$$\lim_{(x,y)\to(3,4)} (x^2 + y^2) = \left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} x\right) + \left(\lim_{y\to 4} y\right)\left(\lim_{y\to 4} y\right) = 3^2 + 4^2 = 25.$$

Since the limit in the given denominator is not zero, we then use (iii) to obtain the existence and value of the stated limit.

The method of Example 3 may be extended to show that the function f defined by (2) is continuous at any point other than the origin (and we have already seen that this function is not continuous at the origin).

Problem 4 below illustrates a situation not covered by Theorem 102.

PROBLEMS

1. For the function f whose definition is given show that f does not have a limit at any of the designated points.

   a. $f(x,y) = \dfrac{x}{x+y}$, $x + y \neq 0$; points (0,0), (1,−1), (−1,1).

   b. $f(x,y) = \dfrac{x+y}{x-y}$, $x \neq y$; points (0,0), (2,2).

   c. $f(x,y) = \dfrac{x-4}{x-y^2}$, $y^2 \neq x$; points (4,2), (4,−2).

   d. $f(x,y) = \dfrac{1-\cos(x+y)}{(x-1)^2}$, $x \neq 1$; point (1,−1).

   e. $f(x,y) = \dfrac{(x-1)(y+2)}{(x-1)^2 + (y+2)^2}$, $(x-1)^2 + (y+2)^2 \neq 0$; point (1,−2).

2. Let the function f be defined by $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$, $x^2 + y^2 \neq 0$.

   a. Show that at (0,0) the limit along any line exists.
   b. Show that the limit of this function does not exist at (0,0).

3. Replace the definition of f in Prob. 2 by

$$f(x,y) = \frac{x^2 \sin y}{x^4 + y^2}, \quad x^2 + y^2 \neq 0.$$

4. Show that the function f defined below is continuous at the origin:

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & \text{if} \quad x^2 + y^2 \neq 0 \\ 0 & \text{if} \quad x^2 + y^2 = 0. \end{cases}$$

(Hint: $|xy| \leq \frac{1}{2}(x^2 + y^2)$ which follows from $(|x| - |y|)^2 \geq 0$.)

5. Establish each of the following limits:

a. $\displaystyle\lim_{(x,y)\to(2,3)} \frac{xy - x}{x^2 - y^2} = -\frac{4}{5}.$

b. $\displaystyle\lim_{(x,y)\to(-1,3)} \frac{x^3y^2 - y^3}{xy + 1} = 18.$

c. $\displaystyle\lim_{(x,y)\to(\pi,\pi/6)} \frac{\sin(x - y)}{x - y} = \frac{3}{5\pi}$

(Hint: $\sin(x - y) = \sin x \cos y - \cos x \sin y$)

d. $\displaystyle\lim_{(x,y)\to(\pi,\pi/4)} \frac{\tan(x - y)}{x - y} = -\frac{4}{3\pi}.$

103. Cylindrical and Spherical Coordinates

In addition to a rectangular coordinate system for locating points in space there are two other systems in common use.

A. CYLINDRICAL COORDINATES. Given a rectangular coordinate system in space, establish a polar coordinate system in the xy-plane with polar axis coinciding with the positive x-axis. A point P which has rectangular coordinates (x,y,z) will now be assigned cylindrical coördinates (ρ,θ,z) where ρ and θ are the polar coordinates of the projection of P on the xy-plane.

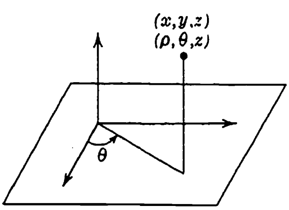


Figure 103.1

Thus, equations for transforming from rectangular coordinates (x,y,z) to cylindrical coordinates (ρ,θ,z) are

(1) $\quad x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = z.$

In cylindrical coordinates $\{(\rho,\theta,z) \mid \rho = 2\}$ is the surface of a right circular cylinder of radius 2. Also, from (1) and $x^2 + y^2 + z^2 = a^2$ it follows that

$$\{(\rho,\theta,z) \mid \rho^2 + z^2 = a^2\}$$

is a sphere with center at the origin and radius a.

In the xz-plane the graph of $z = x^2$ is a parabola. Upon revolving this parabola about the z-axis the surface designated first in cylindrical coordinates and then in rectangular coordinates is

$$\{(\rho,\theta,z) \mid z = \rho^2\} \quad \text{and} \quad \{(x,y,z) \mid z = x^2 + y^2\}.$$

B. SPHERICAL COORDINATES. Let P be a point in space and consider the vector $\overrightarrow{OP}$, where O is the origin of a rectangular coordinate system. Then

$$\overrightarrow{OP} = \overrightarrow{OP}_1 + \overrightarrow{OP}_2$$

where P_1 and P_2 are the projections of P on the xy-plane and the z-axis, respectively. With $r = |\overrightarrow{OP}|$, θ the angle such that $0° \leq \theta \leq 360°$ from the positive x-axis to $\overrightarrow{OP}_1$, and ϕ the angle such that $0° \leq \phi \leq 180°$ from the positive z-axis to $\overrightarrow{OP}$, then P is assigned the spherical coordinates (r,θ,ϕ). Hence

(2) $\overrightarrow{OP}_2 = \vec{k}r \cos \phi$ and $|\overrightarrow{OP}_1| = r \sin \phi$.

P also has rectangular coordinates (x,y,z) where

$$\overrightarrow{OP} = \vec{i}x + \vec{j}y + \vec{k}z.$$

Since $\vec{i} \cos \theta + \vec{j} \sin \theta$ is unit vector along $\overrightarrow{OP}_1$, then

$$\begin{aligned}\overrightarrow{OP} &= |\overrightarrow{OP}_1|(\vec{i} \cos \theta + \vec{j} \sin \theta) + \overrightarrow{OP}_2 \\ &= r \sin \phi(\vec{i} \cos \theta + \vec{j} \sin \theta) + \vec{k}r \cos \phi \text{ from (2).}\end{aligned}$$

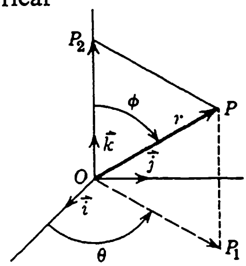


Figure 103.2

Hence, upon equating coefficients of $\vec{i}$, $\vec{j}$, and $\vec{k}$, the formulas

(3) $\qquad x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi$

are obtained for transforming between rectangular and spherical coordinates.

Formulas for transforming between cylindrical coordinates (ρ,θ,z) with $\rho > 0$ and spherical coordinates (r,θ,ϕ) are†

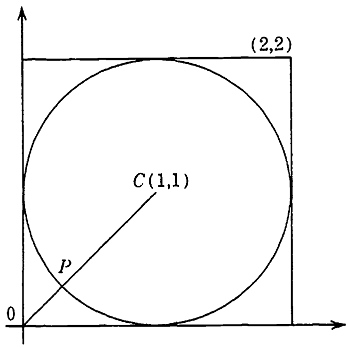
(4) $\qquad \rho = r \sin \phi, \quad \theta = \theta, \quad z = r \cos \phi.$

PROBLEMS

1. Describe each of the following sets (where the designations indicate the coordinate system to be used):
   a. $\{(r,\theta,\phi) \mid r \leq 2\}$.
   b. $\{(r,\theta,\phi) \mid \phi = 30°\}$.
   c. $\{(r,\theta,\phi) \mid \theta = 30°\}$.
   d. $\{(\rho,\theta,z) \mid 0 \leq z \leq 2\}$.
   e. $\{(r,\theta,\phi) \mid \phi \leq 15°\} \cap \{(r,\theta,\phi) \mid r \leq 2\}$.
   f. $\{(r,\theta,\phi) \mid r \leq 2\} \cap \{(\rho,\theta,z) \mid \rho > 1\}$.
   g. $\{(x,y,z) \mid x = 1\} \cap \{(r,\theta,\phi) \mid \phi = 30° \text{ or } 150°\}$.
   h. $\{(x,y,z) \mid z = x + 1\} \cap \{(r,\theta,\phi) \mid \phi = 30°\}$.
2. Express each of the following sets in cylindrical and in spherical coordinates.
   a. $\{(x,y,z) \mid 2x - 3y + 4z = 6\}$.
   b. $\{(x,y,z) \mid 4z = x^2 + y^2\}$.
   c. $\left\{(x,y,z) \,\middle|\, \frac{x^2}{9} + \frac{y^2}{9} - \frac{z^2}{4} = 1\right\}$.
   d. $\{(x,y,z) \mid z^2 = x^2 - y^2\}$.
   e. $\{(x,y,z) \mid x^2 + y^2 = 4 - 4z^2\}$.
   f. $\left\{(x,y,z) \,\middle|\, \frac{x^2}{9} - \frac{y^2}{9} + \frac{z^2}{4} = 1\right\}$.

† Most books use ρ as the first coordinate in spherical coordinates, as well as for polar and cylindrical coordinates; we shall do the same after this section.

3. Find the rectangular coordinates of the point whose cylindrical coordinates are:
   a. $(3,\frac{1}{4}\pi,6)$. b. $(4,150°,-2)$. c. $(3,330°,4)$.
4. Find the rectangular coordinates of the point whose spherical coordinates are:
   a. $(2,45°,60°)$. b. $(2,150°,150°)$. c. $(3,210°,30°)$.
5. Find the direction angles of the vector from the origin to the point whose
   a. Cylindrical coordinates are $(3\sqrt{6},\tan^{-1}\sqrt{2}, -3\sqrt{2})$.
   b. Spherical coordinates are $(4,135°,45°)$.



Find the distance OP as shown in the figure.

Consider an analogous figure in three-dimensional space (unit sphere with center $C(1,1,1)$ inscribed in a cube of edge 2, etc.). Find the distance OP for this 3-dimensional figure.

In n-dimensional Euclidean space, points are given in terms of n coordinates as $(x_1,x_2, \cdots, x_n)$ and $(y_1,y_2, \cdots, y_n)$. The distance between such points is defined by

$$\text{dist.} = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}.$$

The generalization of the figure is an n-sphere of radius 1, center $C(1,1, \cdots, 1)$, inscribed in an n-cube of edge 2, and P is the point where the segment from the origin to C pierces the n-sphere.

Is there any dimension n in which OP is longer than the edge of the circumscribed cube?

CHAPTER 10

Multiple Integrals

The solid geometry of Chapter 9 is called upon heavily to illustrate the extension of definite integrals (now called single integrals) to another dimension. The resulting double integral is intrinsically three dimensional. True to the tradition of reducing space concepts to plane sections, double integrals are then evaluated by repeated use of single integrals. After double integrals become familiar through practice and application, triple integrals are introduced without the crutch of geometry. Multiple integrals of orders higher than three are used, but their definitions would be repetitious.

104. Double and Iterated Integrals

Let S be a function whose domain is the set of all pairs (m,n) of integers. The notation

$$\lim_{(m,n)\to(\infty,\infty)} S(m,n) = L \tag{1}$$

is used if corresponding to each positive number ϵ there are integers M and N such that

$$\text{whenever } m > M \text{ and } n > N, \text{ then } |S(m,n) - L| < \epsilon.$$

Also (1) is read "The limit of $S(m,n)$, as m and n become infinite independently, exists and is L."

Consider a simple closed curve† surrounding a portion of the plane and let R be the set of points of this portion of the plane together with all points of the curve itself. Such a set R will be called a **closed region** of the plane. The projection of this region R on the x- and y-axes consist of the closed intervals with end points $(a,0)$, $(b,0)$ and $(0,c)$, $(0,d)$ so that R is a subset of the rectangular region with lower left-hand corner at (a,c) and upper right-hand corner (b,d).

† For a definition of "simple closed curve" see Appendix A7.

Let f be a function which is continuous on R. With n and m integers, let

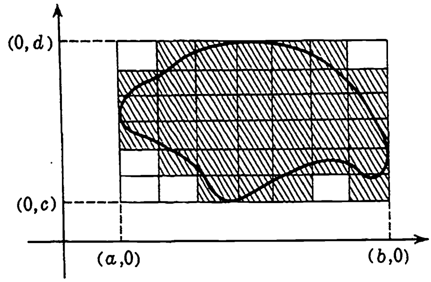


Figure 104.1

(2) $\Delta_n x = \dfrac{b-a}{n}, \quad x_i = a + i\,\Delta_n x$

for $i = 0, 1, 2, \cdots, n$

$\Delta_m y = \dfrac{d-c}{m}, \quad y_j = c + j\,\Delta_m y$

for $j = 0, 1, 2, \cdots, m$.

For those indices i and j for which the set

(3) $$\{(x,y) \mid x_{i-1} \le x \le x_i, \quad y_{j-1} \le y \le y_j\}$$

intersects R, let $f(\underline{x}_i,\underline{y}_j)$ and $f(\bar{x}_i,\bar{y}_j)$ be the minimum and maximum of f on this intersection. Then form the following sums

(4). $$\underline{S}(n,m) = \sum_i \sum_j f(\underline{x}_i,\underline{y}_j)\,\Delta_n x\,\Delta_m y \quad \text{and}$$

(5) $$\bar{S}(n,m) = \sum_i \sum_j f(\bar{x}_i,\bar{y}_j)\,\Delta_n x\,\Delta_m y.$$

where the sums in (4) are taken over those indices i and j for which (3) lies in R, but the sums in (5) are taken over those indices i and j for which the set (3) intersects R. In Appendix A6 it is proved that as $(n,m) \to (\infty,\infty)$ each of $\underline{S}(n,m)$ and $\bar{S}(n,m)$ has a limit and that both have the same limit. This limit is denoted by

$$\iint_R f(x,y)\,dR$$

and is called the **double integral** of f over R, so that

(6) $$\iint_R f(x,y)\,dR = \lim_{(n,m)\to(\infty,\infty)} \underline{S}(n,m) = \lim_{(n,m)\to(\infty,\infty)} \bar{S}(n,m).$$

In case the continuous function f is also positive on R then

(7) $$\{(x,y,z) \mid (x,y,0) \text{ on } R, \quad 0 \le z \le f(x,y)\}$$

is a three-dimensional solid and it is natural to define the volume V of this solid by

(8) $$V = \iint_R f(x,y)\,dR \text{ units}^3.$$

For the solid column of altitude $f(\underline{x}_i,\underline{y}_j)$ units standing on the base (3) of area $\Delta_n x\,\Delta_m y$ units^2 has volume $f(\underline{x}_i,\underline{y}_j)\,\Delta_n x\,\Delta_m y$ units^3 is within the solid (7) so $\underline{S}(n,m)$ units^3 is, if anything, too small for the volume of the solid (7). Similarly $\bar{S}(n,m)$ units^3 is, if anything, too large for the volume of the solid (7).

Since $\underline{S}(n,m)$ and $\bar{S}(n,m)$ have the same limit as $(n,m) \to (\infty,\infty)$, the definition (8) is a natural one to make.

Double integrals have many applications wherein it is desirable to evaluate a double integral of a specific function over a specific region. For example, we may wish to find

$$(9) \qquad \iint_R x^2y\,dR \quad \text{where} \quad R = \left\{(x,y) \mid 0 \le x \le 2,\ \frac{x}{2} \le y \le \sqrt{\frac{x}{2}}\right\},$$

but this evaluation from the definition alone would be difficult.

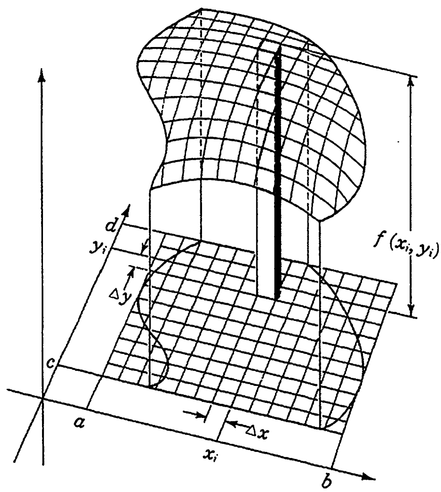


Figure 104.2

As a process relative to (9), but quite different from double integration, let F be the function defined for each number x by

$$F(x) = \int_{x/2}^{\sqrt{x/2}} x^2y\,dy = x^2\frac{y^2}{2}\Bigg]_{y=x/2}^{y=\sqrt{x/2}} = \frac{x^2}{2}\left[\left(\sqrt{\frac{x}{2}}\right)^2 - \left(\frac{x}{2}\right)^2\right] = \frac{1}{8}(2x^3 - x^4)$$

and then take

$$\int_0^2 F(x)\,dx = \int_0^2 \frac{1}{8}(2x^3 - x^4)\,dx = \frac{1}{8}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = \frac{1}{8}\left[\frac{16}{2} - \frac{32}{5}\right] = \frac{1}{5}.$$

A standard notation for these two single integrations, *performed in this order*, is

$$(10) \qquad \int_0^2\int_{x/2}^{\sqrt{x/2}} x^2y\,dy\,dx \qquad \text{which has value } \frac{1}{5}$$

and is called **iterated** (or **repeated**) integration† (as distinct from double integration) of the function f defined by $f(x,y) = x^2y$ over the region $R = \{(x,y) \mid 0 \le x \le 2,\ x/2 \le y \le \sqrt{x/2}\}$. The "inside" integration is to be performed first so that in (10) "integration with respect to y" is done first.

In the following example, the order of integration is reversed.

Example 1. Find the iterated integral of the function f defined by $f(x,y) = x^2y$ over the region

$$(11) \qquad R = \{(x,y) \mid 0 \le y \le 1,\quad 2y^2 \le x \le 2y\}.$$

† For the definition of "iterated integral" see Appendix A6.

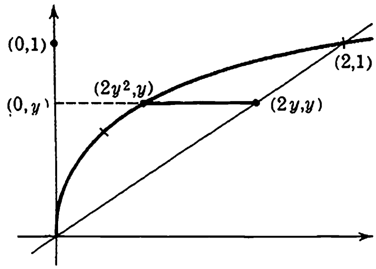


Figure 104.3

Solution.

$$(12)\quad \int_0^1\int_{2y^2}^{2y} x^2y\,dx\,dy = \int_0^1\left[\frac{x^3}{3}y\right]_{x=2y^2}^{x=2y} dy$$

$$= \int_0^1 \frac{y}{3}\left[(2y)^3 - (2y^2)^3\right] dy$$

$$= \frac{8}{3}\int_0^1 (y^4 - y^7)\,dy$$

$$= \frac{8}{3}\left[\frac{y^5}{5} - \frac{y^8}{8}\right]_0^1 = \frac{8}{3}\left[\frac{1}{5} - \frac{1}{8}\right] = \frac{1}{5}.$$

The iterated integrals in both (10) and (12) have $f(x,y) = x^2y$, both have value $\frac{1}{5}$, and note furthermore that the region R in (11) is merely another way of expressing the region R of (9). This fact is an illustration of part of the results of the following theorem.

THEOREM 104. *If a function f is continuous on a bounded region R which includes its rim, then both iterated integrals of f over R have the same value which is also the value of the double integral of f over R.*

This theorem is proved in Appendix A6 for R rectangular.

Thus, to find the value of a double integral, we shall merely evaluate either of the associated iterated integrals.

PROBLEMS

In each of the following problems find the value of the double integral

$$\iint_R f(x,y)\,dR$$

by evaluating one of the appropriate repeated integrals.

1. $f(x,y) = xy^2 + 1,\quad R = \{(x,y) \mid 0 \le x \le 2,\quad x/2 \le y \le \sqrt{x/2}\}$.
2. $f(x,y) = xy^2 + 1,\quad R = \{(x,y) \mid 0 \le y \le 1,\quad 2y^2 \le x \le 2y\}$.
3. $f(x,y) = x + y,\quad R = \{(x,y) \mid 0 \le y \le \pi,\quad 0 \le x \le \sin y\}$.
4. $f(x,y) = x + y,\quad R = \{(x,y) \mid -\pi/2 \le x \le \pi/2,\quad 0 \le y \le \cos x\}$.
5. $f(x,y) = \sqrt{xy - y^2},\quad R = \{(x,y) \mid 1 \le y \le 2,\quad y \le x \le 10y\}$.
6. $f(x,y) = x^2y - xy^2,\quad R = \{(x,y) \mid 0 \le x \le 2,\quad 1 \le y \le 3\}$.
7. $f(x,y) = (x + 1)y$, R is the triangle with vertices (0,0), (2,0), and (1,1).
8. $f(x,y) = x\sqrt{y}$, R is the first quadrant of the circle with center at the origin and radius 4.

105. Volumes of Solids

Let R be a region in the xy-plane and let f be function which is continuous on R and such that $f(x,y) \geq 0$ for $(x,y,0)$ any point of R. Then as discussed in Sec. 104

(1) $$\{(x,y,z) \mid (x,y,0) \text{ is in } R \text{ and } 0 \leq z \leq f(x,y)\}$$

is a solid and the volume V of this solid is defined by

$$V = \iint_R f(x,y)\, dR \text{ units}^3.$$

Example. Find the volume V of the solid above the xy-plane and directly below the portion of the elliptic paraboloid

$$\left\{(x,y,z) \,\middle|\, x^2 + \frac{y^2}{4} = z\right\}$$

which is cut off by the plane $\{(x,y,z) \mid z = 9\}$.

Solution. By symmetry considerations, V is 4 times the volume of the portion of the solid in the first octant. The plane 9 units above the xy-plane intersects the given surface in the ellipse

$$\left\{(x,y,9) \,\middle|\, x^2 + \frac{y^2}{4} = 9\right\}$$
$$= \left\{(x,y,9) \,\middle|\, \frac{x^2}{3^2} + \frac{y^2}{6^2} = 1\right\}.$$

The plane region within this ellipse and in the first octant projects onto the xy-plane into the region

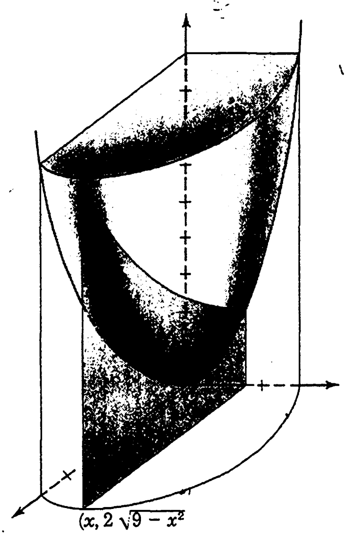


Figure 105.1

$$R = \left\{(x,y,0) \,\middle|\, 0 \leq x,\ 0 \leq y,\ \frac{x^2}{3^2} + \frac{y^2}{6^2} \leq 1\right\} \text{ and } V = 4\iint_R \left(x^2 + \frac{y^2}{4}\right) dR.$$

To evaluate this double integral by means of an iterated integral, write the same region R as

$$R = \{(x,y,0) \mid 0 \leq x \leq 3,\ 0 \leq y \leq 2\sqrt{9 - x^2}\} \text{ and have}$$

(3) $$V = 4\int_0^3 \int_0^{2\sqrt{9-x^2}} \left(x^2 + \frac{y^2}{4}\right) dy\, dx = \text{etc.} = 81\pi.$$

A reason for suspecting the equality of the double integral and an iterated integral may be gained, at least if f is positive over R, by considering volumes. Let X_1 and X_2 be functions of one variable such that†

$$R = \{(x,y,0) \mid c \le y \le d, \quad X_1(y) \le x \le X_2(y)\}$$

and

$$(4) \qquad V = \iint_R f(x,y)\,dR.$$

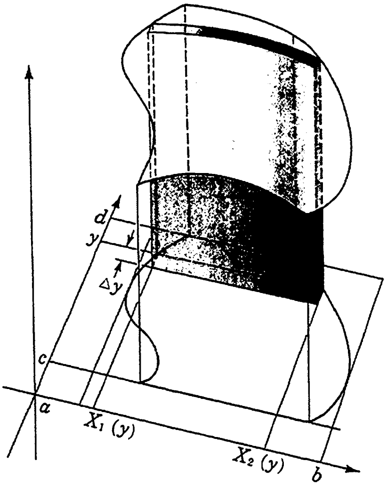


Figure 105.2

The function F defined by

$$F(y) = \int_{X_1(y)}^{X_2(y)} f(x,y)\,dx, \quad c \le y \le d$$

is such that each value of F is the area of a plane section of the solid. Let m be a positive integer, let $\Delta y = (d - c)/m$ and let $y_j = c + j\,\Delta y$ for $j = 1, 2, \cdots, m$. Now visualize the solid as laminated by planes perpendicular to the y-axis at the points $(0,y_j,0)$ for $j = 1, 2, \cdots, m$. The jth lamina appears to have volume close to $F(y_j)\,\Delta y$. Then

$$(5) \qquad \sum_{j=1}^{m} F(y_j)\,\Delta y$$

seems to approximate V in the sense that (5) approaches V as the limit as $m \to \infty$. Since

$$(6) \qquad \lim_{m\to\infty} \sum_{j=1}^{m} F(y_j)\,\Delta y = \int_c^d F(y)\,dy = \int_c^d \int_{X_1(y)}^{X_2(y)} f(x,y)\,dx\,dy,$$

if we could prove all conjectural statements in this discussion we would know the equality of the double integral in (4) and the iterated integral in (6). For further details see Appendix A6.

A heuristic approach is to place a net on the xy-plane of lines spaced Δx and Δy units apart and parallel to the axes. On a mesh within R stand an upright $f(x,y)$ units high where $(x,y,0)$ is a point of the mesh so that

$$f(x,y)\,\Delta x\,\Delta y = f(x,y)\,\Delta y\,\Delta x$$

is the volume of this upright. The forming of a slab from such uprights typifies the "inside" integral and then the fabrication of a solid from such slabs typifies the "outside" integral.

† In Fig. 105.1 the y-axis is visualized as extending out from the page, but in Fig. 105.2 as away from the reader. The first is called a **left-hand** system, the second a **right-hand** system. Whenever vectors are involved, the right-hand system must be used (so $\vec{i} \times \vec{j} = \vec{k}$), but otherwise we use whichever system seems better for the situation under consideration.

PROBLEMS

1. Evaluate the iterated integral (3).
2. Find the volumes of the solids described in Prob. 1, Sec. 101.
3. Find the volume of each of the solids described in Prob. 2, Sec. 101.

106. Mass, Moments, Centroids

Think of the xy-plane covered by a thin sheet of material which may or may not be of uniform density. The **density function** δ is that function for which

(1) $$\mu = \iint_R \delta(x,y)\, dR$$

is the mass of the sheet lying over any region R.

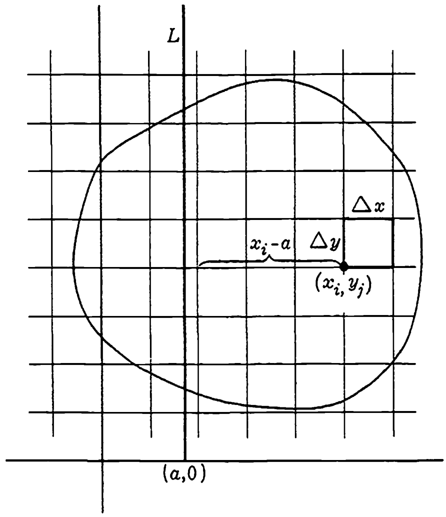


Figure 106.1

Let L be a line perpendicular to the x-axis at $(a,0)$ and let R be a region of the plane. Place a net of lines on the plane made up of lines Δx units apart perpendicular to the x-axis and Δy units apart perpendicular to the y-axis. Index the points of intersection of these lines within R. With (x_i,y_j) one of these points of intersection within R, then

$$(x_i - a)\, \delta(x_i,y_j)\, \Delta x\, \Delta y$$

is in a sense "an arm times a mass" and hence "a first moment of a small mass with respect to L." By summing such terms and then considering the limit of such sums as $(\Delta x, \Delta y) \to (0,0)$, it is natural to define

$$\iint_R (x - a)\, \delta(x,y)\, dR$$

to be the **first moment** with respect to the line L of the portion of the sheet lying over the region R.

In particular if $a = 0$, then

(2) $$M_y = \iint_R x\, \delta(x,y)\, dR$$

is defined to be the first moment with respect to the y-axis of this portion of the sheet.

A similar definition is given for first moments with respect to lines perpendicular to the y-axis so that

$$(3) \qquad M_x = \iint_R y\,\delta(x,y)\,dR$$

is the first moment with respect to the x-axis of the portion of the sheet lying over the region R.

With constants $\bar{x}$ and $\bar{y}$ such that

$$(4) \qquad \iint_R (x - \bar{x})\,\delta(x,y)\,dR = 0 \quad \text{and} \quad \iint_R (y - \bar{y})\,\delta(x,y)\,dR = 0$$

the point $(\bar{x},\bar{y})$ is defined to be the **centroid** of the portion of the sheet lying over the region R. From the first equation of (4), since $\bar{x}$ is constant, then

$$0 = \iint_R x\,\delta(x,y)\,dR - \bar{x}\iint_R \delta(x,y)\,dR = M_y - \bar{x}\mu$$

from (2) and (1). Similarly $M_x - \bar{y}\mu = 0$ so that

$$\bar{x} = \frac{M_y}{\mu} \quad \text{and} \quad \bar{y} = \frac{M_x}{\mu}$$

are expressions for the coordinates of the centroid.

Second moments (or **moments of inertia**) with respect to the x-axis, the y-axis, and the origin are defined, respectively, by

$$I_x = \iint_R y^2\,\delta(x,y)\,dR, \quad I_y = \iint_R x^2\,\delta(x,y)\,dR, \quad \text{and}$$

$$I_0 = \iint_R (x^2 + y^2)\,\delta(x,y)\,dR.$$

Sometimes I_0 is called the **polar moment** of inertia.

Computations of mass and moments of a sheet require equations of bounding curves of the region R over which the sheet lies and then the double integrals involved are evaluated by means of iterated integrals.

Example 1. Find the coordinates of the centroid of a sheet lying over the region R bounded by the graphs of $y = -x$ and $y^2 = 2 - x$ where $\delta(x,y) = y^2$ for each point (x,y) of R.

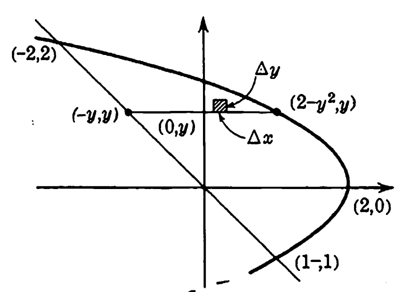


Figure 106.2

Solution. Since $R = \{(x,y) \mid -1 \le y \le 2, -y \le x \le 2 - y^2\}$ we have

$$\mu = \int_{-1}^{2}\int_{-y}^{2-y^2} y^2\,dx\,dy,$$

$$M_y = \int_{-1}^{2}\int_{-y}^{2-y^2} xy^2\,dx\,dy, \text{ and}$$

$$M_x = \int_{-1}^{2}\int_{-y}^{2-y^2} yy^2\,dx\,dy.$$

Hence $\mu = \frac{63}{20}$, $M_y = -\frac{9}{7}$, $M_x = \frac{18}{5}$, $\bar{x} = -\frac{20}{49}$, $\bar{y} = \frac{8}{7}$.

Example 2. Given that the density function is the constant $\delta = 1$ and R is bounded by the graphs of $y = \sin x, y = 0$, and $x = \pi/2$, find I_x and I_y for the sheet lying over R.

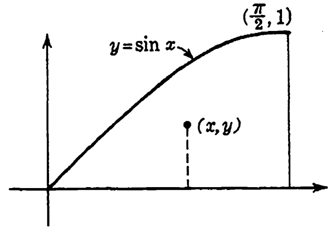


Figure 106.3

Solution. Since $R = \{(x,y) \mid 0 \le x \le \pi/2, 0 \le y \le \sin x\}$ we have

$$I_x = \int_0^{\pi/2}\int_0^{\sin x} y^2\,dy\,dx = \int_0^{\pi/2} \frac{y^3}{3}\Big]_0^{\sin x} dx$$

$$= \frac{1}{3}\int_0^{\pi/2} \sin^3 x\,dx$$

$$= \frac{1}{3}\int_0^{\pi/2} (1 - \cos^2 x)\sin x\,dx = \frac{1}{3}\Big[-\cos x + \tfrac{1}{3}\cos^3 x\Big]_0^{\pi/2} = \frac{2}{9},$$

$$I_y = \int_0^{\pi/2}\int_0^{\sin x} x^2\,dy\,dx = \int_0^{\pi/2} x^2 y\Big]_0^{\sin x} dx = \int_0^{\pi/2} x^2 \sin x\,dx$$

$$= \Big[x^2\int \sin x\,dx\Big]_0^{\pi/2} - \int_0^{\pi/2} 2x(-\cos x)\,dx \qquad \text{(by integration of a product)}$$

$$= -x^2\cos x\Big]_0^{\pi/2} + 2\Big\{\Big[x\int\cos x\,dx\Big]_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx\Big\}$$

$$= 0 + 2\Big[x\sin x + \cos x\Big]_0^{\pi/2} = \pi - 2.$$

Whenever the density function is the constant 1, we shall speak of the "moments and centroid of the region" rather than of the sheet lying on the region. The area of a region R is denoted by $|R|$.

PROBLEMS

1. Find the centroid of the region bounded by the graphs of the given equations.

a. $y = 9 - x^2, y = x + 3.$

b. $x = 9 - y^2, x = y - 3.$

c. $xy = 6, x + y = 5.$

d. $y^2 = -x + 4, y^2 = x - 2.$

e. $x = \sqrt{10 - y^2}, y^2 = 9x.$

f. $y = \sin x, y = \cos x; -\frac{3}{4}\pi \le x \le \frac{\pi}{4}.$

2. Find the area of the shaded region. The graphs have equations

$$x^2 + y^2 = 25, \quad 4x^2 = 9y, \quad x - 2y + 5 = 0.$$

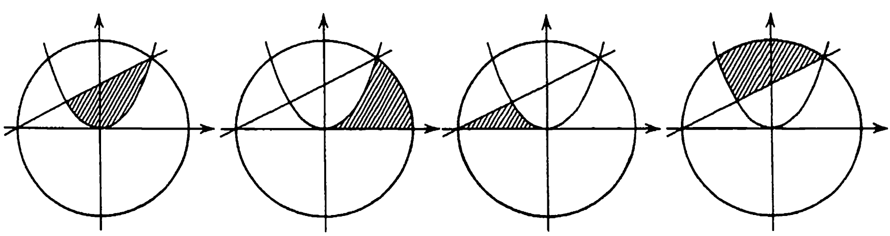


Figure Prob. 2

3. Find I_x and I_y for the sheet lying on the given region and having the given density function
   a. $y = \sin^{-1} x$, $y = \pi/2$, $x = 0$; $\delta(x,y) = x$.
   b. $y = \sin x$, $x = 0$, $y = 1$, in first quadrant; $\delta(x,y) = y$.
   c. $y = e^x$, $y = 0$, $x = 0$, $x = 2$; $\delta(x,y) = xy$.
   d. $(x - 1)^2 + (y - 1)^2 = 1$; $\delta(x,y) = (x^2 + y^2)^{-1}$ (Hint: Find I_0 first.)
   e. $y^2 = 2x$, $x = 2$, first quadrant; $\delta(x,y) = |x - y|$.
   f. Triangle with vertices (0,0), $(a,0)$, (b,c); $\delta(x,y) = 1$, with a, b, and c all positive.
4. For t a number let $I_y(t)$ be the second moment of a sheet about the line perpendicular to the x-axis at the point $(t,0)$. Show that
   a. $I_y(t) = I_y(0) - 2tM_y + t^2\mu$. b. $I_y(t) \geq I_y(\bar{x})$.

107. Polar Coordinates

In a circle of radius r, a sector with central angle of α radians has area $\frac{1}{2}r^2\alpha$. Thus, the portion of a ring between two circles with polar equations $\rho = r$ and $\rho = r + \Delta r$, $\Delta r > 0$ which lies between radial lines of angles θ and $\theta + \Delta\theta$, $\Delta\theta > 0$ has area (see Fig. 107.1).

$$\tfrac{1}{2}(r + \Delta r)^2\,\Delta\theta - \tfrac{1}{2}r^2\,\Delta\theta = r\,\Delta r\,\Delta\theta + \tfrac{1}{2}(\Delta r)^2\,\Delta\theta$$

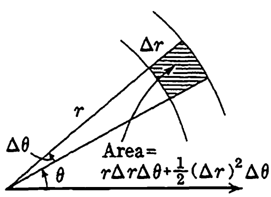


Figure 107.1

This area is approximated (not by $\Delta r\,\Delta\theta$, but) by $r\,\Delta r\,\Delta\theta$ since

$$\lim_{(\Delta r,\Delta\theta)\to(0,0)} \frac{r\,\Delta r\,\Delta\theta + \frac{1}{2}(\Delta r)^2\,\Delta\theta}{r\,\Delta r\,\Delta\theta} = \lim_{(\Delta r,\Delta\theta)\to(0,0)} \left(1 + \frac{\Delta r}{2r}\right) = 1.$$

The area and first and second moments of a plane region bounded by polar graphs may be defined by procedures leading to double integrals which in turn may be evaluated by means of iterated integrals. For example, let R_1 and R_2 be functions such that $0 \leq R_1(\theta) \leq R_2(\theta)$ for $\alpha < \theta < \beta$ and let G be the region expressed in polar coordinates as

$$G = \{(\rho,\theta) \mid \alpha \leq \theta \leq \beta,\ \ R_1(\theta) \leq \rho \leq R_2(\theta)\}.$$

With G a region we shall use the convenient notation $|G|$ = area of G but this should not be confused with the absolute value of a number. Also the intermingling of (r,θ) and (ρ,θ) should cause no confusion.

The area of this region G is defined by the double integral

$$|G| = \iint_G dG.$$

A heuristic method of arriving at an iterated integral evaluation of this double integral is to place a network on the plane consisting of concentric circles $\Delta\rho$ units apart and radial lines at consecutive angles of $\Delta\theta$ radians. Each mesh then has area approximated by $\rho\,\Delta\rho\,\Delta\theta$ and leads to the expectation that

$$\iint_G dG = \int_\alpha^\beta \int_{R_1(\theta)}^{R_2(\theta)} \rho\, d\rho\, d\theta$$

which is true but cannot be proved here.

Also, for F a function of two variables, then the double integral of F over G and an equivalent iterated integral in terms of polar coordinates is

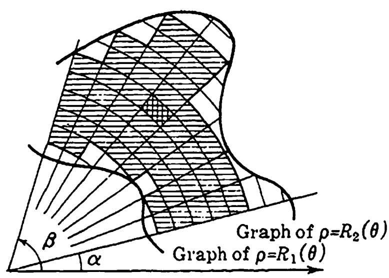


Figure 107.2

$$\iint_G F(\rho,\theta)\, dG = \int_\alpha^\beta \int_{R_1(\theta)}^{R_2(\theta)} F(\rho,\theta)\rho\, d\rho\, d\theta.$$

A way of remembering to use $\rho\, d\rho\, d\theta$ (and not just $d\rho\, d\theta$, as is often mistakenly done) is to think of $d\rho$ and $d\theta$ as small positive numbers and to consider a small "almost" rectangle with opposite corners (ρ,θ) and $(\rho + d\rho,\ \theta + d\theta)$. This small "almost" rectangle has two opposite (though not parallel) sides of length $d\rho$ and one other side (actually arc) has length $\rho\, d\theta$ (and not $d\theta$) and area "almost"

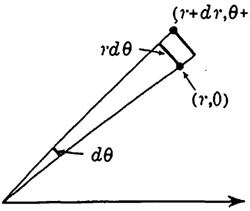


Figure 107.3

$$(\rho\, d\rho)\, d\theta = \rho\, d\rho\, d\theta$$

Example 1. Find the polar coordinates of the centroid of the region G enclosed by one loop of the polar graph of

$$\rho = 4 \sin 2\theta.$$

Solution. First draw one loop of this graph and note that

$$G = \{(\rho,\theta) \mid 0 \le \theta \le \pi/2,$$
$$0 \le \rho \le 4 \sin 2\theta\}.$$

A point (ρ,θ) within this region is $\rho \sin \theta$ units from the initial line (see Fig. 107.4). Hence, by using a rectangular system in conjunction with the polar system we have

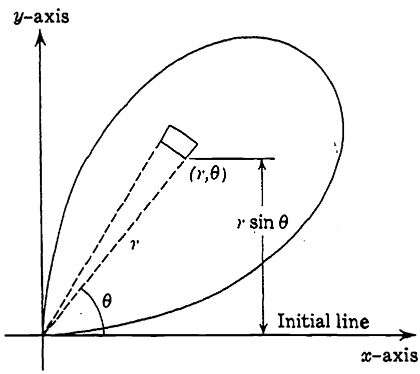


Figure 107.4

$$M_x = \iint_G (\rho \sin \theta)\, dG$$
$$= \int_0^{\pi/2} \int_0^{4 \sin 2\theta} (\rho \sin \theta)\rho\, d\rho\, d\theta$$

for the first moment of G about the x-axis. Therefore

$$M_x = \int_0^{\pi/2} \frac{\rho^3}{3}\sin\theta\Big]_0^{4\sin 2\theta} d\theta = \tfrac{1}{3}\int_0^{\pi/2} 64\sin^3 2\theta \sin\theta\, d\theta$$

$$= \tfrac{64}{3}\int_0^{\pi/2} (2\sin\theta\cos\theta)^3 \sin\theta\, d\theta = \tfrac{64}{3}\, 8\int_0^{\pi/2} \sin^4\theta\cos^3\theta\, d\theta$$

$$= \tfrac{512}{3}\int_0^{\pi/2} \sin^4\theta(1-\sin^2\theta)\cos\theta\, d\theta = \text{etc.} = \tfrac{512}{3}\,\tfrac{2}{35}.$$

Also

$$|G| = \int_0^{\pi/2}\int_0^{4\sin 2\theta} \rho\, d\rho\, d\theta = \int_0^{\pi/2} \frac{\rho^2}{2}\Big]_0^{4\sin 2\theta} d\theta = \text{etc.} = 2\pi.$$

Since $\bar{y} = M_x/|G|$ and since $\bar{x} = \bar{y}$ by symmetry, we may transform to obtain the polar coordinates $(\bar{\rho},\bar{\theta})$ of the centroid:

$$\bar{x} = \bar{y} = \frac{512}{105\pi}; \quad \bar{\rho} = \frac{512}{105\pi}\sqrt{2}, \quad \bar{\theta} = \frac{\pi}{4}.$$

Even though a solid is defined in terms of rectangular coordinates, it may be advantageous to translate to polar coordinates to find the volume.

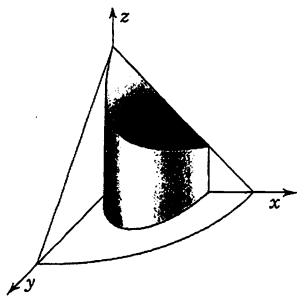


Figure 107.5

Example 2. Find the volume V of the solid in the first octant under the graph of $z = 3 - \sqrt{x^2+y^2}$ and inside the right circular cylinder having equation $(x-1)^2 + y^2 = 1$.

Solution. In the xy-plane $(x-1)^2 + y^2 = 1$ is the equation of a circle, which may be written as $x^2 + y^2 - 2x = 0$, and the portion in the first quadrant as $y = \sqrt{2x - x^2}$. Then

$$V = \iint_R (3 - \sqrt{x^2+y^2})\, dR$$

where $R = \{(x,y) \mid 0 \le x \le 2, \quad 0 \le y \le \sqrt{2x - x^2}\}$

so
$$V = \int_0^2\int_0^{\sqrt{2x-x^2}} (3 - \sqrt{x^2+y^2})\, dy\, dx.$$

This integral would be very hard to evaluate. By the transformation formulas $x = \rho\cos\theta$, $y = \rho\sin\theta$ we have $z = 3 - \sqrt{x^2+y^2} = 3 - \sqrt{\rho^2} = 3 - |\rho|$ so that

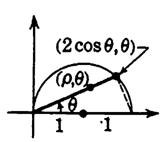


Figure 107.6

$$V = \iint_G (3 - |\rho|)\, dG$$

where G is the region R expressed in polar coordinates:

$$G = \{(\rho,\theta) \mid 0 \le \theta \le \pi/2, \quad 0 \le \rho \le 2\cos\theta\}$$

wherein $\rho > 0$ so that $|\rho| = \rho$ and

$$V = \int_0^{\pi/2}\int_0^{2\cos\theta}(3 - \rho)\rho \, d\rho \, d\theta = \int_0^{\pi/2}\left[\frac{3\rho^2}{2} - \frac{\rho^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/2}[\tfrac{3}{2}(2^2\cos^2\theta) - \tfrac{1}{3}(2^3\cos^3\theta)]\, d\theta = \text{etc.} = \frac{3\pi}{2} - \frac{16}{9}.$$

PROBLEMS

1. Find the area of the region enclosed by the graph of:
   a. $\rho = 4\cos\theta$.
   b. $\rho = 5\cos 2\theta$.
   c. $\rho = 2\cos 3\theta$.
   d. $\rho = 4(1 + \cos\theta)$.
   e. $\rho = 2 + \cos\theta$.
   f. $\rho = 4(1 - \cos\theta)$.
2. Find the centroid of the region G where
   a. $G = \left\{(\rho,\theta) \,\middle|\, 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \rho \le 2a\cos\theta\right\}$.
   b. $G = \left\{(\rho,\theta) \,\middle|\, -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, \quad 0 \le \rho \le a\sec\theta\right\}$.
   c. $G = \left\{(\rho,\theta) \,\middle|\, -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, \quad 0 \le \rho \le a\cos 2\theta\right\}$.
   d. $G = \{(\rho,\theta) \mid 0 \le \theta \le \frac{2}{3}\pi, \quad 0 \le \rho \le 1 + 2\cos\theta\}$.
3. Find the volume of the solid:
   a. In the first octant below the graph of $z = \sqrt{x^2 + y^2}$ and inside the cylinder having equation $(x - 1)^2 + y^2 = 1$.
   b. Above the xy-plane, below the graph of $z = 3 - \sqrt{x^2 + y^2}$ and inside the cylinder having equation $x^2 + y^2 = 4$.
   c. Common to a sphere of radius a and a right circular cylinder of radius b, $b < a$, with axis along a diameter of the sphere.
   d. Common to a sphere of radius a and a right circular cylinder of radius $a/2$ having an element along a diameter of the sphere.
4. Show the existence and obtain the value of the improper integral

$$\int_0^{\infty} e^{-x^2} dx = \tfrac{1}{2}\sqrt{\pi}$$

   by proceeding as follows:

   a. Let $I(r) = \int_0^r e^{-x^2}\, dx$ so that $I(r) = \int_0^r e^{-y^2}\, dy$ and hence

$$I^2(r) = \int_0^r e^{-x^2}\, dx \int_0^r e^{-y^2}\, dy = \int_0^r\int_0^r e^{-(x^2+y^2)}\, dx\, dy.$$

b. Let R_1 and R_2 be the first quadrant regions bounded by the axes and circles with centers at the origin and radii r and $r\sqrt{2}$, respectively. In terms of double integrals see that

$$\iint_{R_1} e^{-(x^2+y^2)}\, dR_1 \leq I^2(r) \leq \iint_{R_2} e^{-(x^2+y^2)}\, dR_2.$$

c. Use iterated integrals in terms of polar coordinates to see that

$$\iint_{R_1} e^{-(x^2+y^2)}\, dR_1 = \int_0^{\pi/2}\int_0^r e^{-\rho^2}\, \rho\, d\rho\, d\theta$$

with a similar expression for the integral over R_2, and obtain

$$\frac{\pi}{4}(1 - e^{-r^2}) \leq I^2(r) \leq \frac{\pi}{4}(1 - e^{-2r^2}).$$

d. Use these inequalities to show that $\lim\limits_{r\to\infty} I^2(r) = \dfrac{\pi}{4}$. Complete the proof by noting that $I(r) > 0$ so $\lim\limits_{r\to\infty} I(r) = \frac{1}{2}\sqrt{\pi}$.

108. Reversing Order Transformations

Iterated integrals sometimes arise in differential equations or applied mathematics through considerations in which no region of integration is evident. If such an integral is not readily evaluated, a method worth trying is to examine the equivalent iterated integral in reverse order.

Example 1. Evaluate $\displaystyle\int_0^1\int_{2x}^2 e^{y^2}\, dy\, dx.$

Solution. From the order of integration and the limits of integration, this integral may be interpreted as a double integral over the region

$$R = \{(x,y) \mid 0 \leq x \leq 1,\ 2x \leq y \leq 2\}.$$

By limiting y first and then x, we have $R = \{(x,y) \mid 0 \leq y \leq 2,\ 0 \leq x \leq y/2\}$ and the given integral is equal to

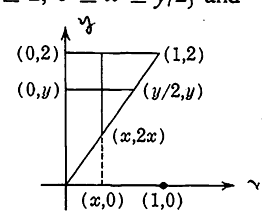


Figure 108

$$\int_0^2\int_0^{y/2} e^{y^2}\, dx\, dy = \int_0^2 e^{y^2}x\Big]_0^{y/2} dy = \int_0^2 (e^{y^2})\frac{y}{2}\, dy$$

$$= \frac{1}{4}\int_0^2 e^{y^2}\, dy^2 = \frac{1}{4}\, e^{y^2}\Big]_0^2 = \frac{1}{4}(e^4 - 1).$$

The evaluation of an iterated integral may be facilitated by transformation to polar coordinates.

Example 2. Evaluate $I = \displaystyle\int_0^a\int_0^{\sqrt{a^2-x^2}} \sin(x^2 + y^2)\, dy\, dx.$

Solution. In terms of the double integral

$$I = \iint_R \sin(x^2 + y^2)\, dR \quad \text{where} \quad R = \{(x,y) \mid 0 \leq x \leq a,\ 0 \leq y \leq \sqrt{a^2 - x^2}\},$$

so that R is the first quadrant portion of a circle expressed in polar coordinates as

$$R = \{(\rho,\theta) \mid 0 \leq \theta \leq \pi/2, \quad 0 \leq \rho \leq a\}.$$

By the translation formulas $x = \rho \cos \theta$, $y = \rho \sin \theta$ we have $x^2 + y^2 = \rho^2$ and

$$I = \iint_R \sin \rho^2 \, dR.$$

Thus, remembering to use $\rho \, d\rho \, d\theta$ (and not merely $d\rho \, d\theta$), we have

$$I = \int_0^{\pi/2} \int_0^a (\sin \rho^2)\rho \, d\rho \, d\theta = \tfrac{1}{2}\int_0^{\pi/2} \int_0^a \sin \rho^2 \, d\rho^2 \, d\theta$$

$$= \tfrac{1}{2}\int_0^{\pi/2} \Big[-\cos \rho^2 \Big]_0^a d\theta = \text{etc.} = \frac{\pi}{4}(1 - \cos a^2).$$

PROBLEMS

1. Evaluate each of the following by first reversing the order of integration:

a. $\displaystyle\int_0^1 \int_{2x}^2 \sin y^2 \, dy \, dx.$ d. $\displaystyle\int_2^5 \int_{y-2}^3 \sin (\pi x^2) \, dx \, dy.$

b. $\displaystyle\int_0^2 \int_{x/2}^1 \cos y^2 \, dy \, dx.$ e. $\displaystyle\int_0^1 \int_0^{\tan^{-1} x} x \, dy \, dx.$

c. $\displaystyle\int_1^3 \int_{y-1}^2 e^{-x^2} \, dx \, dy.$ f. $\displaystyle\int_0^1 \int_x^1 \frac{dy \, dx}{1 + y^2}.$

g. $\displaystyle\int_0^4 \int_{x/2}^{\sqrt{x}} f(y) \, dy \, dx$ where $f(y) = \begin{cases} \dfrac{\sin y}{y} & \text{if } y \neq 0 \\ 1 & \text{if } y = 0. \end{cases}$

2. Transform to polar coordinates to evaluate:

a. $\displaystyle\int_0^a \int_0^{\sqrt{a^2 - x^2}} e^{(x^2+y^2)} \, dy \, dx.$ c. $\displaystyle\int_0^2 \int_0^{\sqrt{2x - x^2}} y \, dy \, dx.$

b. $\displaystyle\int_0^a \int_{-\sqrt{a^2-y^2}}^0 x\sqrt{x^2 + y^2} \, dx \, dy.$ d. $\displaystyle\int_{-a}^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} x^2 \, dx \, dy.$

3. Evaluate each of the following:

a. $\displaystyle\int_0^1 \int_0^{\cos^{-1} x} y \, dy \, dx.$ c. $\displaystyle\int_0^{\pi/4} \int_0^{2 \sec \theta} e^{\rho \cos \theta} \rho \, d\rho \, d\theta.$

b. $\displaystyle\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x \, dy \, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} x \, dy \, dx.$ d. $\displaystyle\int_{-1}^1 \int_{-\pi/2}^{\sin^{-1} y} e^x \, dx \, dy.$

109. Triple Integrals

Given a function f of three variables and a solid S then, by extending the discussion of double integrals to one more dimension, the concept of the triple integral of f over S is obtained and denoted by

$$\iiint_S f(x,y,z) \, dS.$$

Moreover, according to the designations of the bounding surfaces of S, a triple integral is equal to any of $3! = 6$ iterated integrals. The volume of the solid S is denoted by $|S|$ where

$$|S| = \iiint_S dS.$$

The first moments of S relative to the three coordinate planes are

$$M_{yz} = \iiint_S x\, dS, \quad M_{xz} = \iiint_S y\, dS, \quad M_{xy} = \iiint_S z\, dS$$

and the centroid $(\bar{x},\bar{y},\bar{z})$ of S is given by

$$\bar{x} = \frac{M_{yz}}{|S|}, \quad \bar{y} = \frac{M_{xz}}{|S|}, \quad \bar{z} = \frac{M_{xy}}{|S|}.$$

Example 1. Find $\bar{x}$ for $S = \{(x,y,z) \mid 0 \le x \le 1, x^2 \le y \le 1, 0 \le z \le 1 - y\}$.

Solution. $|S| = \int_0^1\int_{x^2}^1\int_0^{1-y} dz\, dy\, dx = \int_0^1\int_{x^2}^1 z\Big]_0^{1-y} dy\, dx$

$$= \int_0^1\int_{x^2}^1 (1 - y)\, dy\, dz = \int_0^1\left[-\frac{(1-y)^2}{2}\right]_{x^2}^1 dx$$

$$= \tfrac{1}{2}\int_0^1 (1 - x^2)^2\, dx = \text{etc.} = \tfrac{4}{15},$$

$$M_{yz} = \int_0^1\int_{x^2}^1\int_0^{1-y} x\, dz\, dy\, dx = \int_0^1\left[x\int_{x^2}^1\int_0^{1-y} dz\, dy\right] dx$$

$$= \tfrac{1}{2}\int_0^1 x(1 - x^2)^2\, dx = \text{etc.} = \tfrac{1}{12} \quad \text{and} \quad \bar{x} = \tfrac{5}{16}.$$

The second moments with respect to the axes are

$$I_x = \iiint_S (y^2 + z^2)\, dS, \quad I_y = \iiint_S (x^2 + z^2)\, dS, \quad I_z = \iiint_S (x^2 + y^2)\, dS.$$

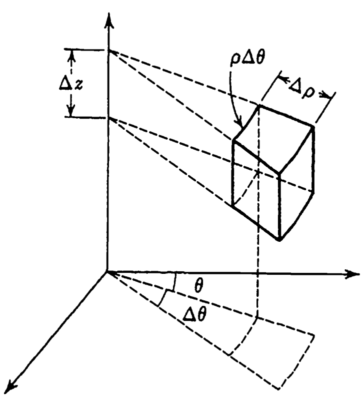


Figure 109.1

If the bounding surfaces of S are expressed in cylindrical coordinates (ρ,θ,z) and a function f is also expressed in these coordinates, then for the triple integral and threefold iterated integrals

$$\iiint_S f(\rho,\theta,z)\, dS = \iiint f(\rho,\theta,z)\rho\, d\rho\, d\theta\, dz$$

(with appropriate limits on each of integral signs of the iterated integral) or any of the other five possible orders of $d\rho$, $d\theta$, and dz. A way of remembering to use $\rho\, d\rho\, d\theta\, dz$ (and not $d\rho\, d\theta\, dz$) is to visualize the box-like solid of Fig. 109.1 having meeting edges of lengths $\rho\,\Delta\theta$, $\Delta\rho$, and Δz and hence volume approximately $\rho\,\Delta\theta\,\Delta\rho\,\Delta z = \rho\,\Delta\rho\,\Delta\theta\,\Delta z =$ etc.

Example 2. A homogeneous solid right circular cylinder has radius a and altitude h. Find the second moment of the cylinder about a line which contains one of its elements.

Solution. With $S = \{(\rho,\theta,z) \mid 0 \le \rho \le a,\ 0 \le \theta \le 2\pi,\ 0 \le z \le h\}$ let L be the line through the point $(a,0,0)$ perpendicular to the (ρ,θ)-plane (or the xy-plane). In rectangular coordinates the perpendicular distance from a point (x,y,z) to L is the distance between (x,y,z) and $(a,0,z)$ so is

$$\sqrt{(x-a)^2 + y^2} = \sqrt{x^2 + y^2 - 2ax + a^2}.$$

In cylindrical coordinates this distance is $\sqrt{\rho^2 - 2a\rho\cos\theta + a^2}$ so that

$$\begin{aligned} I_L &= \int_0^a\int_0^{2\pi}\int_0^h (\rho^2 - 2a\rho\cos\theta + a^2)\rho\,dz\,d\theta\,d\rho \\ &= h\int_0^a \Big[(\rho^2 + a^2)\theta - 2a\,\rho\sin\theta\Big]_0^{2\pi}\rho\,d\rho \\ &= 2\pi h\int_0^a (\rho^3 + a^2\rho)\,d\rho = \tfrac{3}{2}\pi h a^4. \end{aligned}$$

In spherical coordinates

$$\iiint_S f(\rho,\theta,\phi)\,dS = \iiint f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

(with appropriate limits on the threefold iterated integral) where the presence of ρ^2 and $\sin\phi$ in addition to $d\rho\,d\theta\,d\phi$ may be seen as follows:

1. With $\rho > 0$ consider a sphere of radius ρ and center at the pole, and visualize a point (ρ,θ,ϕ) on the sphere.

2. With $\Delta\phi > 0$ proceed from (ρ,θ,ϕ) to $(\rho,\theta,\phi + \Delta\phi)$ along an arc each point of which has ρ and θ as its first and second coordinates. This arc is on a circle of radius ρ and center at the pole so the arc has length $\rho\,\Delta\phi$.

3. With $\Delta\theta > 0$ proceed from (ρ,θ,ϕ) to $(\rho,\theta + \Delta\theta,\phi)$ along an arc having ρ and ϕ as first and third coordinates of each of its points. This arc is on a circle which is parallel to the xy-plane with its center on the z-axis and radius $\rho\sin\phi$. Thus, the arc has length $\rho\sin\phi\,\Delta\theta$.

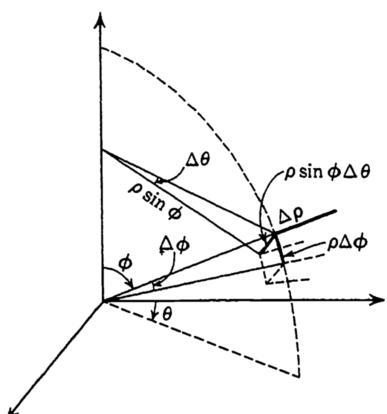


Figure 109.2

4. With $\Delta\rho > 0$, extend the radius of the sphere $\Delta\rho$ units. Now the box-like solid having these two arcs and segment as edges should have volume approximately

$$(\rho\,\Delta\phi)(\rho \sin \phi\,\Delta\theta)\,\Delta\rho = \rho^2 \sin \phi\,\Delta\rho\,\Delta\theta\,\Delta\phi.$$

PROBLEMS

1. Evaluate the triple integral over the solid given

a. $\iiint_S x\,dS$, $S = \{(x,y,z) \mid 2 \le x \le 4,\ \ 1 \le y \le x,\ \ 0 \le z \le x\}$.

b. $\iiint_S xy\,dS$, $S = \{(x,y,z) \mid 1 \le x \le 2,\ \ 1 \le z \le x,\ \ 1 \le y \le z\}$.

c. $\iiint_S \cos\theta\,dS$, $S = \{(\rho,\theta,z) \mid 0 \le \theta \le \pi,\ \ 0 \le \rho \le \sin\theta,\ \ 0 \le z \le \rho\sin\theta\}$.

d. $\iiint_S \rho\sin\theta\,dS$, $S = \{(\rho,\theta,z) \mid 0 \le \rho \le 1,\ \ 0 \le \theta \le \sin^{-1}\rho,\ \ 0 \le z \le \cos\theta\}$.

e. $\iiint_S z\,dS$, $S = \{(\rho,\theta,\phi) \mid 0 \le \phi \le \pi/6,\ \ 0 \le \theta \le 2\pi,\ \ 0 \le \rho \le a\sec\phi\}$.

f. $\iiint_S |x|\,dS$, S a sphere of radius a and center at the origin.

2. Find the centroid of each of the following:

a. Hemisphere with density varying as the distance from the great circle base.

b. Hemisphere with density varying as the distance from the center of the base.

c. Cylinder with density varying as the square of the distance from one base.

d. Cylinder with density varying as the square of the distance from the center of one base.

110. Attraction

According to Newton's law of gravitation, a particle of mass m_1 attracts a particle of mass m by a force (acting along the line joining the particles) of magnitude

$$K\frac{mm_1}{r_1^2}$$

where r_1 is the distance between the particles and K depends upon the units used in measuring distance, mass, and time. In the C.G.S. system $K = (6.664)10^{-8}$. This force may be represented by a vector directed along the line of the particles. By drawing the attraction vectors from a particle of mass m toward two particles of masses m_1 and m_2, the vector sum is the resultant attraction.

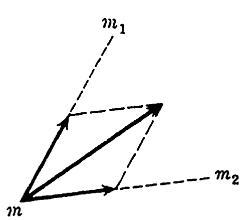


Figure 110.1

In finding the resultant of the attractions of several particles of masses $m_1, m_2, \cdots, m_n$ on a particle of mass m it is better to use vector components. With $\vec{l}$ a unit vector at m, with $\alpha_1, \alpha_2, \cdots, \alpha_n$ the angles from $\vec{l}$ to the various attraction vectors, and with $r_1, r_2, \cdots, r_n$ the distances from the particle of mass m to the other particles, then the component in the direction $\vec{l}$ of the resultant attraction is

$$\sum_{i=1}^{n} \vec{l}K \frac{mm_i}{r_i^2} \cos \alpha_i = \vec{l}Km \sum_{i=1}^{n} \frac{m_i}{r_i^2} \cos \alpha_i.$$

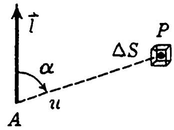


Figure 110.2

Let a solid S with density function δ be given, let a particle of mass m be located at a point A not in the solid, and let $\vec{l}$ be a unit vector at A. Let P be a point in the solid and let ΔV be a small portion of the solid including P, let $u(P)$ be the distance between A and P, and let $\alpha(P)$ be the angle from $\vec{l}$ to a vector from A toward P. Then the mass of the portion is approximately $\delta(P)\,\Delta V$ and its attraction on m has component in the direction $\vec{l}$ approximately

$$\vec{l}K \frac{m\,\delta(P)\,\Delta V}{u^2(P)} \cos \alpha(P) = \vec{l}Km \frac{\delta(P)}{u^2(P)} \cos \alpha(P)\,\Delta V.$$

By the usual extension we define the component in the direction $\vec{l}$ of the attraction of the solid S on the particle of mass m to be the triple integral

$$\vec{l}\iiint_S Km \frac{\delta}{u^2} \cos \alpha \, dS = \vec{l}Km \iiint_S \frac{\delta}{u^2} \cos \alpha \, dS.$$

If δ is constant, then it may also be taken outside the integral.

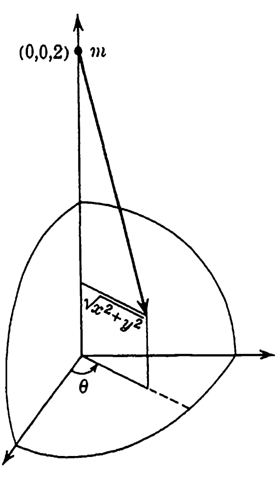


Figure 110.3

Example. Find the attraction of a homogeneous solid sphere of radius 1 on a particle of mass m situated 2 units from the center of the sphere.

Solution. Locate the sphere with its center at the origin and m at the point (0,0,2). Then by symmetry, the x- and y-components of attraction are $\vec{0}$. To find the z-component, let (x,y,z) be a point of the solid sphere, note that the square of the distance between this point and (0,0,2) is $x^2 + y^2 + (2 - z)^2$, and the angle from the negative z-axis to the line of these points has its cosine equal to

$$\frac{z - 2}{\sqrt{x^2 + y^2 + (2 - z)^2}}.$$

Thus, with $\vec{k}$ the usual unit vector on the z-axis, the z-component of attraction is

$$-\vec{k}Km\,\delta \iiint_S \frac{1}{x^2 + y^2 + (2 - z)^2} \cdot \frac{z - 2}{\sqrt{x^2 + y^2 + (2 - z)^2}} \, dS.$$

The equivalent iterated integral (leaving off the multiplicative constant $-\vec{k}Km\,\delta$) in rectangular coordinates is

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{2-z}{[x^2+y^2+(2-z)^2]^{3/2}}\,dz\,dy\,dx.$$

This is a formidable looking integral, so we try transforming to spherical coordinates:

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1}\frac{(2-\rho\cos\theta)}{[\rho^2-4\rho\cos\phi+4]^{3/2}}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

Even though the limits of integration are constants, the integrand is complicated so we try cylindrical coordinates with

$$S=\{(\rho,\theta,z)\mid 0\le\theta\le 2\pi,\quad -1\le z\le 1,\quad 0\le\rho\le\sqrt{1-z^2}\}:$$

$$\int_0^{2\pi}\int_{-1}^{1}\int_0^{\sqrt{1-z^2}}\frac{2-z}{[\rho^2+(2-z)^2]^{3/2}}\rho\,d\rho\,dz\,d\theta$$

$$=\int_0^{2\pi}\int_{-1}^{1}\left\{(2-z)\int_0^{\sqrt{1-z^2}}[\rho^2+(2-z)^2]^{-3/2}\rho\,d\rho\right\}dz\,d\theta$$

$$=\int_0^{2\pi}\int_{-1}^{1}(2-z)\left\{\tfrac{1}{2}(-2)[\rho^2+(2-z)^2]^{-1/2}\Big]_0^{\sqrt{1-z^2}}\right\}dz\,d\theta$$

$$=\int_0^{2\pi}\int_{-1}^{1}(2-z)\left[\frac{-1}{\sqrt{1-z^2+(2-z)^2}}+\frac{1}{\sqrt{(2-z)^2}}\right]dz\,d\theta$$

$$=\int_0^{2\pi}\int_{-1}^{1}\left[\frac{-(2-z)}{\sqrt{5-4z}}+1\right]dz\,d\theta=\text{etc.}=\frac{-11}{3}\pi+4\pi=\frac{\pi}{3}.$$

Thus, the total attraction vector is $-\vec{k}\,Km\,\delta\,\pi/3$.

PROBLEMS

1. Given a solid homogeneous right circular cylinder of radius a, altitude h, and on the extension of its axis c units from the nearest base is a particle of mass m. Find the attraction of the cylinder on the particle with the cylinder expressed as
   a. $\{(\rho,\theta,z)\mid 0\le\theta\le 2\pi,\ 0\le\rho\le a,\ 0\le z\le h\}$ particle at $(0,0,h+c)$.
   b. $\{(\rho,\theta,z)\mid 0\le\theta\le 2\pi,\ 0\le z\le h,\ 0\le\rho\le a\}$ particle at $(0,0,h+c)$.
   c. Cylinder same as in b, but particle at $(0,0,-c)$.
   d. $\{(\rho,\theta,z)\mid 0\le\rho\le a,\ c\le z\le c+h,\ 0\le\theta\le 2\pi\}$ particle at $(0,0,0)$.
2. Find the attraction of the solid cone inside the graph of $z^2=x^2+y^2$ and below the plane having equation $z=1$ on a particle of mass m at the point:
   a. $(0,0,-1)$. b. $(0,0,2)$. c. Turn the cone over so its base is in the xy-plane with its vertex at $(0,0,1)$ and put the particle at the point $(0,0,-1)$.

CHAPTER 11

Partial Derivatives

In this chapter, more than any other, the inherent complications of advancing from two to three or more dimensions are in evidence. For example, the most reasonable way of extending the definition of arc length to surface area is exploded by the so-called "Schwarz Paradox." Also, it may seem an anomaly that a function can have a differential at a point and still not be differentiable there.

Line integrals and Green's theorem are traditionally postponed to a course on advanced calculus, but are included here because their usefulness has enticed physicists at some schools to rely upon them early.

111. Definitions

For f a function of two variables, the mental image of $\{(x,y,z) \mid z = f(x,y)\}$ is a surface and of

$$\{(x,y,z) \mid y = c\} \cap \{(x,y,z) \mid z = f(x,y)\} = \{(x,c,z) \mid z = f(x,c)\}$$

is the curve on this surface in the plane perpendicular to the y-axis at the point $(0,c,0)$. The points $(x,c,f(x,c))$ and $(x + \Delta x,\ c,\ f(x + \Delta x,\ c))$ on this curve determine a line of "slope"

$$\frac{f(x + \Delta x,c) - f(x,c)}{\Delta x}$$

and the limit of this ratio as $\Delta x \to 0$ is visualized as the slope of the line tangent to the curve at the point $(x,c,f(x,c))$.

Such a limit of a ratio is reminiscent of a derivative and we define

$$(1) \qquad \lim_{\Delta x \to 0} \frac{f(x + \Delta x,y) - f(x,y)}{\Delta x}$$

as the partial derivative, with respect to the first variable, of f at (x,y). The partial derivative, with respect to the second variable, of f at (x,y) is

$$(2) \qquad \lim_{\Delta y \to 0} \frac{f(x,y + \Delta y) - f(x,y)}{\Delta y}.$$

The two functions whose values at (x,y) are given by (1) and (2) could be denoted, respectively by

(3) $$\frac{\partial f}{\partial \text{ first}} \quad \text{and} \quad \frac{\partial f}{\partial \text{ second}}.$$

The symbols designating a value of the first or second variables in defining f are ordinarily used instead of "first" and "second" in (3). Thus

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x} \quad \text{and}$$

$$\frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}.$$

In $\partial/\partial x$ the x is merely part of the symbolism and does not represent a value of the first variable just as neither x in "x-axis" represents a number. For example

$$\frac{\partial \sin^{-1} \frac{y}{x}}{\partial x} = \lim_{\Delta x \to 0} \frac{\sin^{-1} \frac{y}{x + \Delta x} - \sin^{-1} \frac{y}{x}}{\Delta x}.$$

Partial derivatives are obtained by applying known derivative formulas with one variable held constant.

Example.

$$\frac{\partial \sin^{-1}(y/x)}{\partial x} = \frac{1}{\sqrt{1 - (y/x)^2}} \frac{\partial (y/x)}{\partial x} = \frac{|x|}{\sqrt{x^2 - y^2}} \left(-\frac{y}{x^2} \right) = \frac{-y}{|x| \sqrt{x^2 - y^2}},$$

$$\frac{\partial \sin^{-1}(y/x)}{\partial y} = \frac{1}{\sqrt{1 - (y/x)^2}} \frac{\partial (y/x)}{\partial y} = \frac{|x|}{\sqrt{x^2 - y^2}} \frac{1}{x} = \begin{cases} (x^2 - y^2)^{-1/2} & \text{if } x > 0 \\ -(x^2 - y^2)^{-1/2} & \text{if } x < 0. \end{cases}$$

There are accepted alternative notations for the partial derivatives, such as

$$f_x \text{ for } \frac{\partial f}{\partial x} \quad \text{and} \quad f_y \text{ for } \frac{\partial f}{\partial y}.$$

For example, given the equation $z = x^2 e^{3y}$, we write either

$$\frac{\partial z}{\partial x} = 2xe^{3y} \quad \text{or} \quad z_x = 2xe^{3y} \quad \text{and either}$$

$$\frac{\partial z}{\partial y} = 3x^2 e^{3y} \quad \text{or} \quad z_y = 3x^2 e^{3y}.$$

PROBLEMS

1. For the given definition of a function f, find the partial derivatives of f with respect to the first and with respect to the second variable.

   a. $f(x,y) = x^2y^3 + xy^2$.   d. $f(s,t) = \sin(s^2t^3)$.

b. $f(x,y) = \tan^{-1}\dfrac{y}{x}$. e. $f(u,v) = u^2 \ln\left|\dfrac{v}{u}\right|$.

c. $f(x,y) = \tan^{-1}\dfrac{x}{y}$. f. $f(x,y) = y^x, \quad y > 0$.

2. Find each of the indicated partial derivatives.

a. $\dfrac{\partial\sqrt{x^2+y^2}}{\partial x}$ and $\dfrac{\partial\sqrt{x^2+y^2}}{\partial y}$. d. $\dfrac{\partial}{\partial u}\cos(2u^2 - 3v^2)$.

b. $\dfrac{\partial}{\partial x}(e^{y/x})$ and $\dfrac{\partial}{\partial y}(e^{y/x})$. e. $\dfrac{\partial}{\partial t}\cos(2s^2 - 3t^2)$.

c. $\dfrac{\partial}{\partial x}\sin\dfrac{y}{x}$ and $\dfrac{\partial}{\partial y}\sin\dfrac{y}{x}$. f. $\dfrac{\partial}{\partial\theta}\cos^{-1}\left(\dfrac{\theta}{\phi}\right)$.

3. Find z_x and z_y given

a. $z = \dfrac{x+y}{x-y}$. c. $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. e. $z = xe^{y/x}$.

b. $z = \sin x \cos y$. d. $z = xy + \ln|xy|$. f. $z = \ln|x + y|$.

4. For the functional relationship between x, y, and z as given by the first equation, check the second equation.

a. $z = x^2 + xy + y^2; \quad xz_x + yz_y = 2z$.

b. $z = e^{xy}; \quad xz_x - yz_y = 0$.

c. $z = \tan^{-1}\dfrac{y}{x}; \quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$.

d. $z = \sin x \cos y; \quad z_x + z_y = \cos(x + y)$.

112. Normals and Tangents to a Surface

The line through a point (x_0,y_0,z_0) and having direction numbers A, B, C has parametric representation

$$x = x_0 + At, \quad y = y_0 + Bt, \quad z = z_0 + Ct \tag{1}$$

and (see Sec. 98) is perpendicular at (x_0,y_0,z_0) to the plane having equation

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{2}$$

Consider now a surface having equation

$$z = f(x,y), \quad \text{where } f_x \text{ and } f_y \text{ exist,}$$

and let (x_0,y_0,z_0) be a point on this surface so that $z_0 = f(x_0,y_0)$. With this point as initial end the vector $\vec{u}$ to the point $(x_0, y_0 + 1, z_0 + f_y(x_0,y_0))$ is tangent to the profile of the surface in the plane perpendicular to the x-axis at $(x_0,0,0)$. Thus

$$\vec{u} = \vec{i}\cdot 0 + \vec{j}\cdot 1 + \vec{k}f_y(x_0,y_0).$$

In the same way the vector

$$\vec{v} = \vec{i}\cdot 1 + \vec{j}\cdot 0 + \vec{k}f_x(x_0,y_0)$$

is tangent at (x_0,y_0,z_0) to the profile of the surface in the plane having equations $y = y_0$. The vector $\vec{w} = \vec{u}\times\vec{v}$ is then perpendicular to these tangent vectors $\vec{u}$ and $\vec{v}$.

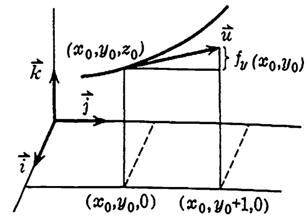


Figure 112.1

DEFINITION. *The line containing $\vec{w}$ and the plane through (x_0,y_0,z_0) perpendicular to $\vec{w}$ are said to be the **normal line** and **tangent plane** to the surface at (x_0,y_0,z_0).*

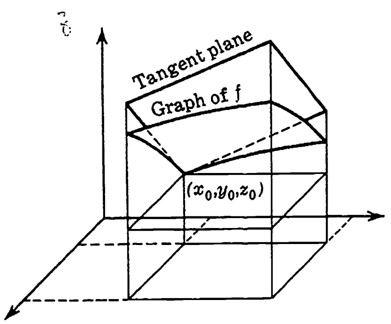


Figure 112.2

THEOREM 112. *The normal line has parametric representation*

$$(3)\quad x = x_0 + tf_x(x_0,y_0),\qquad y = y_0 + tf_y(x_0,y_0),\qquad z = z_0 - t$$

and the tangent plane has equation

$$(4)\quad z - f(x_0,y_0) = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

PROOF. First note (see Sec. 98) that

$$\vec{w} = \vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & f_y(x_0,y_0) \\ 1 & 0 & f_x(x_0,y_0) \end{vmatrix} = \vec{i}f_x(x_0,y_0) + \vec{j}f_y(x_0,y_0) - \vec{k}.$$

Thus the normal line has direction numbers $f_x(x_0,y_0)$, $f_y(x_0,y_0)$, -1 and hence (3) follows from (1). Then from (2) the tangent plane has equation

$$f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) - (z - z_0) = 0$$

which is equivalent to (4) since $z_0 = f(x_0,y_0)$.

Example 1. Let f be the function of two variables defined by

$$f(x,y) = xy^2 + y.$$

Obtain an equation of the tangent plane and equations of the normal line to the graph of $z = f(x,y)$ at the point having $x_0 = 3$ and $y_0 = -2$.

Solution. $z_0 = f(3,-2) = 3(-2)^2 + (-2) = 10$, $f_x(3,-2) = y^2\Big]_{(3,-2)} = 4$, $f_y(3,-2) = 2xy + 1\Big]_{(3,-2)} = -11$ so that

$$x = 3 + 4t, \quad y = -2 - 11t, \quad z = 10 - t \quad \text{and}$$

$$z - 10 = 4(x - 3) - 11(y + 2).$$

are equations of the normal line and tangent plane, respectively.

By eliminating t from the parametric equations of the normal line we obtain a symmetric representation of this normal:

$$\frac{x-3}{4} = \frac{y+2}{-11} = \frac{z-10}{-1}.$$

Notice that if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ are both different from zero, then from (3) the normal line has the symmetric representation

$$\frac{x - x_0}{f_x(x_0,y_0)} = \frac{y - y_0}{f_y(x_0,y_0)} = \frac{z - f(x_0,y_0)}{-1}. \tag{3}$$

Example 2. The equation of a surface is $z = \sqrt{3x^2 - xy + y^2}$. Find a representation of the normal line and an equation of the tangent plane at the point of the surface having $x = 1$ and $y = -2$.

Solution. Since $\sqrt{3(1)^2 - (1)(-2) + (-2)^2} = 3$ the point is $(1,-2,3)$. From

$$z_x = \frac{6x - y}{2\sqrt{3x^2 - xy + y^2}} \quad \text{and} \quad z_y = \frac{-x + 2y}{2\sqrt{3x^2 - xy + y^2}}$$

a set of direction numbers of the normal line is

$$\frac{6 - (-2)}{2(3)} = \frac{4}{3}, \quad \frac{-1 + 2(-2)}{2(3)} = \frac{-5}{6}, \quad -1$$

but a simpler set to use in (1) is $A = 8$, $B = -5$, $C = -6$. Hence

$$\text{normal line:} \quad x = 1 + 8t, \quad y = -2 - 5t, \quad z = 3 - 6t$$

$$\text{tangent plane:} \quad 8(x - 1) - 5(y + 2) - 6(z - 3) = 0.$$

A symmetric representation of the normal line and a simplified equation of the tangent plane are

$$\frac{x-1}{8} = \frac{y+2}{-5} = \frac{z-3}{-6} \quad \text{and} \quad 8x - 5y - 6z = 0$$

Example 3. Find the cosine of an angle between the tangent plane to the graph of $z = \ln(x^2 + y^2)$ at the point having $x = 2$, $y = 3$ and the tangent plane to the graph of $z = x^2y + y^2$ where $x = -1$, $y = 1$.

Solution. For the first surface

$$z_x\Big]_{(2,3)} = \frac{2x}{x^2 + y^2}\Big]_{(2,3)} = \frac{4}{13}, \quad z_y\Big]_{(2,3)} = \frac{2y}{x^2 + y^2}\Big]_{(2,3)} = \frac{6}{13}$$

so $\frac{4}{13}\vec{i} + \frac{6}{13}\vec{j} - \vec{k}$ is a normal vector to this surface at the given point, but an easier normal vector to use is

$$\vec{n}_1 = 4\vec{i} + 6\vec{j} - 13\vec{k}.$$

Similarly, a normal vector to the second surface at the given point is

$$\vec{n}_2 = -2\vec{i} + 3\vec{j} - \vec{k}.$$

Hence, an angle θ between these normals, and thus between the tangent planes, is such that

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{(4\vec{i} + 6\vec{j} - 13\vec{k}) \cdot (-2\vec{i} + 3\vec{j} - \vec{k})}{\sqrt{4^2 + 6^2 + 13^2}\,\sqrt{2^2 + 3^2 + 1^2}}$$

$$= \frac{4(-2) + 6 \cdot 3 + (-13)(-1)}{\sqrt{221}\,\sqrt{14}} = \frac{23}{\sqrt{(221)(14)}}.$$

PROBLEMS

1. Find an equation of the tangent plane and a representation of the normal line to the surface whose equation is given, at the point of the surface having the given numbers for x_0 and y_0.

   a. $z = x^2y$; $2, -1$.

   b. $z = x^2 + y^2$; $1, 2$.

   c. $z = \tan^{-1}\dfrac{y}{x}$; $-2, 2$.

   d. $z = \sin(xy)$; $\dfrac{\pi}{6}, 1$.

   e. $z = \sqrt{8 - x^2 + 5y^2}$; $3, -1$.

   f. $z = \ln(x^2 + y^2)$; $1, 0$.

2. Find the volume of the tetrahedron bounded by the coordinate planes and the plane tangent to the graph of $z = 16 - x^2 + 3xy$ at the point $(1, -4, 3)$.

3. Show that the graph of $z = \sqrt{x^2 + y^2}$ contains the origin, but the graph has no tangent plane at the origin. Show that the tangent plane at any point other than the origin contains the whole line joining the origin and that point.

4. For what values of D will the plane with the given equation be tangent to the surface whose equation is given. Find the points of tangency.

   a. $20x - 12y + 2z + D = 0$, $z = x^2y + xy - 8$.

   b. $4x + 6y - 3z + D = 0$, $18z = 4x^2 + 9y^2$.

   c. $x + 2y + z + D = 0$, $z(x + y) = x$.

5. Find the point (or points) on the graph of the given equation where the normal to the graph is parallel to the described line.

   a. $z = xy + 2x - y$; line with direction numbers $7, -2, -2$.

   b. $z = y(x + y)^{-1}$; line joining points $(9,2,-7)$ and $(3,-6,5)$.

   c. $z = x^2y + y$; line with two-plane representation

   $$5x - 11y + 5z + 11 = 0, \quad 5x - 13y - 5z + 3 = 0.$$

6. Let P be a point on the surface, let Q be the point where the normal to the surface pierces the xy-plane, and let P_1 be the projection of P on the xy-plane. The segment P_1Q is called the **xy-subnormal** of the surface at P.

   Show that at a point (x_0,y_0,z_0) on a surface represented by $z = f(x,y)$, the length of the subnormal is
$$|f(x_0,y_0)|\sqrt{f_x^2(x_0,y_0) + f_y^2(x_0,y_0)}.$$

7. Prove: An angle θ between the xy-plane and the tangent plane to the graph of $z = f(x,y)$ at a point (x_0,y_0,z_0) is such that
$$|\cos\theta| = \frac{1}{\sqrt{f_x^2(x_0,y_0) + f_y^2(x_0,y_0) + 1}}.$$

113. The Schwarz Paradox

Given a curved surface, take many points rather evenly distributed over the surface. By judiciously selecting triplets of these points as vertices of plane triangles, a crinkled surface may be made which in some sense should approximate the given surface. The area of such a crinkled surface is the sum of the areas of its triangular parts. As more and more points are taken closer and closer together, it seems that the areas of the resulting crinkled surfaces should approach a limit and that this limit would be a reasonable definition of area for the given surface.

The following example shows, however, that even for the lateral surface of a right circular cylinder, the limit does not exist. *Thus, some other definition of area for a curved surface is necessary.* (See Sec. 114.)

The following facts are used in the example.

$$\text{(1)}\quad \lim_{m\to\infty} m\sin\frac{\pi}{m} = \lim_{m\to\infty}\pi\frac{\sin(\pi/m)}{\pi/m} = \pi \quad\text{since}\quad \lim_{h\to 0}\frac{\sin h}{h} = 1.$$

$$\text{(2)}\quad \lim_{m\to\infty} m^2\sin^4\frac{\pi}{2m} = \lim_{m\to\infty}\left(\frac{\pi}{2}\right)^2\left[\frac{\sin\pi/2m}{\pi/2m}\right]^2\sin^2\frac{\pi}{2m} = \left(\frac{\pi}{2}\right)^2\cdot 1\cdot 0 = 0.$$

$$\text{(3)}\quad \lim_{m\to\infty} m^4\sin^4\frac{\pi}{2m} = \lim_{m\to\infty}\left(\frac{\pi}{2}\right)^4\left[\frac{\sin\pi/2m}{\pi/2m}\right]^4 = \left(\frac{\pi}{2}\right)^4.$$

Example. Consider the lateral surface of a right circular cylinder of altitude H and radius R, so the area of this surface is $2\pi RH$. With n a positive integer divide the lateral area into n horizontal strips each of altitude H/n. With m a positive integer, select m equally spaced points around the top rim, and then on the bottom of the top strip take the m points each under the mid-point of an arc between two of the selected points on the top rim. By joining these points as shown in Fig. 113.1 (with $n = 3$ and $m = 6$), we have $2m$ triangles the sum of whose areas might be considered as approximating the area of the top strip. Doing the same for each strip we obtain $2mn$ congruent triangles. The sum of these triangles is thus $2mn$ times the area of any one of them.

We now find the area T of one such triangle (magnified as illustrated in Fig. 113.2). We have

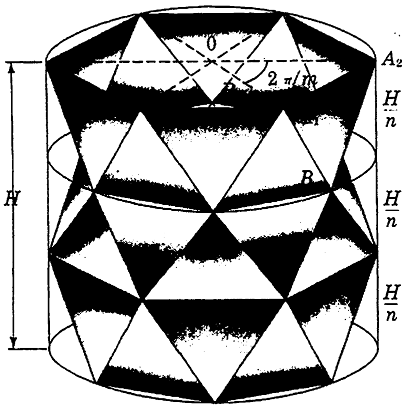


Figure 113.1

$$T = \tfrac{1}{2}(A_1A_2)(CB) = (A_1C)(CB)$$

$$= R\sin\frac{\pi}{m}\sqrt{(DB)^2 + (CD)^2}$$

$$= R\sin\frac{\pi}{m}\sqrt{\left(\frac{H}{n}\right)^2 + (OD - OC)^2}$$

$$= R\sin\frac{\pi}{m}\sqrt{\left(\frac{H}{n}\right)^2 + \left(R - R\cos\frac{\pi}{m}\right)^2}$$

$$= R\sin\frac{\pi}{m}\sqrt{\left(\frac{H}{n}\right)^2 + 4R^2\sin^4\frac{\pi}{2m}}.$$

Thus, the sum $S(m,n)$ of the areas of the $2mn$ triangles is given by

$$S(m,n) = 2mnR\sin\frac{\pi}{m}\sqrt{\frac{H^2}{n^2} + 4R^2\sin^4\frac{\pi}{2m}}$$

$$= 2Rm\sin\frac{\pi}{m}\sqrt{H^2 + 4R^2n^2\sin^4\frac{\pi}{2m}}$$

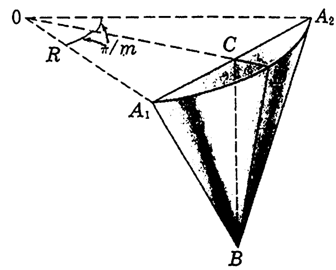


Figure 113.2

and we hope to find "the limit" of this expression as $m \to \infty$ and $n \to \infty$ (and hence all sides of all triangles approach 0).

One way of assuring that both $m \to \infty$ and $n \to \infty$ is to first let $n = m$ and then to let $m \to \infty$. Since (upon setting $n = m$)

$$S(m,m) = 2Rm\sin\frac{\pi}{m}\sqrt{H^2 + 4R^2m^2\sin^4\frac{\pi}{2m}}$$

we have, by using (1) and (2), that

$$\lim_{m\to\infty} S(m,m) = 2R\pi\sqrt{H^2 + 4R^2\cdot 0} = 2\pi RH. \tag{4}$$

We could set $n = m^2$ and then let $m \to \infty$. We thus obtain (using (1) and (3))

$$\lim_{m\to\infty} S(m,m^2) = \lim_{m\to\infty} 2Rm\sin\frac{\pi}{m}\sqrt{H^2 + 4R^2m^4\sin^4\frac{\pi}{2m}} = 2R\pi\sqrt{H^2 + 4R^2\left(\frac{\pi}{2}\right)^4}. \tag{5}$$

Since the results in (4) and (5) are different, it follows that $\lim\limits_{(m,n)\to(\infty,\infty)} S(m,n)$ does not exist.

PROBLEM

Show that $\lim\limits_{m\to\infty} S(m,m^3) = \infty.$

114. Area of a Surface

Let A be a region of the xy-plane and let f be a function whose domain contains A and is such that the partial derived functions f_x and f_y are continuous. Hence, the function $\sqrt{f_x^2 + f_y^2 + 1}$ is continuous so the double integral

$$\iint_A \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}\, dA$$

exists by an extension of Theorem A6.8. Let S be the portion of the graph of

$$z = f(x,y) \tag{1}$$

whose projection on the xy-plane is A. We now define

$$\textbf{(area of } S\textbf{)} = \iint_A \sqrt{f_x^2(x,y) + f_y^2(x,y) + 1}\, dA. \tag{2}$$

Toward motivating this definition we first make four observations:

I. Given a triangle and a line l both in the same plane, the triangle either has a side parallel to l or may be divided into two triangles each having a side parallel to l.

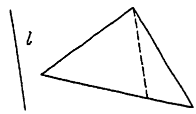


Figure 114.1

II. A rectangle in the same plane may be divided into triangles each having a side parallel to l. (First divide the rectangle into two triangles by a diagonal and then use I.)

III. Let a plane intersect the xy-plane in a line l and at an angle $\gamma \neq 90°$. In the xy-plane take a triangle t with base of length b parallel to l and let T be the triangle in the other plane which projects onto t. Then the base of T is also parallel to l and is also of length b, but the altitudes H of T and h of t are related by $h/H = |\cos \gamma|$ so that $H = h|\sec \gamma|$. Thus†

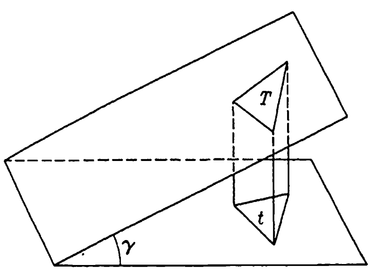


Figure 114.2

$$(\text{area of } T) = (\text{area of } t)\ |\sec \gamma|.$$

IV. If r is a rectangle in the xy-plane and R is the figure in the other plane which projects into r, then

$$(\text{area of } R) = |\sec \gamma|\ (\text{area of } r).$$

In the xy-plane establish a fine net of lines parallel to the axes. The meshes (that is, small rectangles) of this net which lie in the region A we now name $r_1, r_2, \cdots, r_n$. In each r_p, for $p = 1, 2, \cdots, n$, select a point $(x_p, y_p, 0)$, obtain the tangent plane to the graph of (1) at the point $(x_p, y_p, f(x_p, y_p))$,

† For another derivation of this formula see p. 326.

denote the angle this plane makes with xy-plane by γ_p, and let R_p be the figure in this tangent plane which projects into r_p. Hence

$$(3) \qquad \sum_{p=1}^{n} (\text{area of } R_p) = \sum_{p=1}^{n} |\sec \gamma_p| \ (\text{area of } r_p).$$

It seems that (area of R_p) should be, since R_p is in a tangent plane, a reasonable estimate for area of the portion of S that projects onto r_p and thus that (3) is an estimate of area for all of S.

Notice that γ_p is also an angle between the z-axis and the normal to the graph at $(x_p,y_p,f(x_p,y_p))$. Since

$$\vec{n} = \vec{i} f_x(x_p,y_p) + \vec{j} f_y(x_p,y_p) - \vec{k}$$

is such a normal vector, whereas $\vec{k}$ is normal to the xy-plane, we have

$$\cos \gamma_p = \frac{\vec{n} \cdot \vec{k}}{|\vec{n}|\,|\vec{k}|} = \frac{-1}{\sqrt{f_x^2(x_p,y_p) + f_y^2(x_p,y_p) + 1}} \quad \begin{array}{l} \text{since } \vec{i} \cdot \vec{k} = \vec{j} \cdot \vec{k} = 0 \\ \text{and } -\vec{k} \cdot \vec{k} = -1. \end{array}$$

Hence $|\sec \gamma_p| = \sqrt{f_x^2(x_p,y_p) + f_y^2(x_p,y_p) + 1}$ and the estimate (3) for area of S takes the form

$$\sum_{p=1}^{n} \sqrt{f_x^2(x_p,y_p) + f_y^2(x_p,y_p) + 1} \quad (\text{area of } r_p),$$

which is an approximating sum of the integral in (2).

In a specific situation the double integral of (2) is evaluated by means of a twofold iterated integral with limits depending upon the equations of the bounding curves of the region A. The following example illustrates how space rectangular coordinates and plane polar coordinates may be used together.

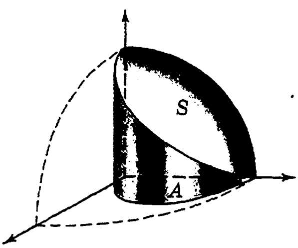


Figure 114.3

Example. On a sphere of radius 3 and center at the origin a portion S lies above the region A enclosed by one loop of the curve in the xy-plane having equation $\rho = 3 \cos 2\theta$. Find the area of S.

Solution. To obtain the form of the integrand, the upper hemisphere is expressed as the graph of $z = \sqrt{9 - x^2 - y^2}$. Hence

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{9 - x^2 - y^2}}, \quad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{9 - x^2 - y^2}}$$

and the double integral for the area of S is

$$\iint_A \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA = \iint_A \sqrt{\frac{x^2 + y^2}{9 - x^2 - y^2} + 1}\, dA$$

$$= \iint_A \frac{3}{\sqrt{9 - x^2 - y^2}}\, dA.$$

But since $A = \{(\rho,\theta) \mid -\pi/4 \le \theta \le \pi/4,\ 0 \le \rho \le 3\cos 2\theta\}$ we transform to polar coordinates for the iterated integral evaluation. From the transformation formulas $x = \rho\cos\theta$, $y = \rho\sin\theta$ we have $\sqrt{9 - x^2 - y^2} = \sqrt{9 - \rho^2}$ and then have (remembering to use $\rho\, d\rho\, d\theta$ and not merely $d\rho\, d\theta$) that

$$(\text{area of } S) = \int_{-\pi/4}^{\pi/4}\int_0^{3\cos 2\theta} \frac{3}{\sqrt{9-\rho^2}}\,\rho\, d\rho\, d\theta = \text{etc.} = 9\left(\frac{\pi}{2} - 1\right) \text{ units}^2.$$

(Note: In evaluating this integral, it is necessary to remember that

$$\sqrt{1 - \cos^2 2\theta} = |\sin 2\theta|).$$

PROBLEMS

1. Use (2) to find the area of a sphere of radius r:
   a. By using only rectangular coordinates.
   b. By using polar coordinates in the xy-plane.
2. Find the area of the portion of the graph of $z = x^2 + y$ which lies above the triangle with vertices (0,0,0), (1,0,0), and (1,1,0).
3. Set up an iterated integral for the area of the portion of the sphere of radius 3 and center at the origin which lies above the triangle of Prob. 2. Now evaluate the inside integral, thus leaving the area expressed as a single integral.
4. Find the area of $S = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2},\ z = xy\}$. (Hint: Use cylindrical coordinates.)
5. Find the area of the portion of the graph of $z^2 = x^2 + y^2$ which lies above the square in the xy-plane with sides of length 4 parallel to the axes and center at the origin.
6. A portion of the plane having equation $Ax + By + Cz + D = 0$ lies inside the cylinder having equation $x^2 + y^2 = a^2$. Find the area of this portion.
7. A thin sheet of material of density δ is spread uniformly on the portion of the graph of $z = 4 - (x^2 + y^2)$ which lies above the xy-plane. Show that the attraction of this matter on a particle of unit mass at the origin is given (after the inside integral is evaluated) by
$$\vec{k}2\pi K\,\delta\int_0^2 \frac{(4-\rho^2) + \sqrt{1 + 4\rho^3}}{\sqrt{(16 - 7\rho^2 + \rho^4)^3}}\,\rho\, d\rho.$$
8. Set up (but do not attempt to evaluate) an integral expressing the vertical component of attraction of the material of density δ spread uniformly over the surface of Prob. 2 on a particle of unit mass at (0,1,0).
9. Find the area of the portion of the cone having equation $z^2 = 2(x^2 + y^2)$ which is above the xy-plane and below the plane having equation $z = 1 - x$.

115. Partial Derivative Systems

Every indefinite integral formula is also a derivative formula. For example, integral table formula 59:

$$\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u} + c \tag{1}$$

tells us, without the necessity of reworking the details, that

$$\frac{d}{du}\left[-\frac{\sqrt{a^2 - u^2}}{a^2 u} + c\right] = \frac{1}{u^2\sqrt{a^2 - u^2}} + 0. \tag{2}$$

Indefinite integral formulas may be used also as "partial integral" formulas. From (1) we may write

$$\int \frac{dx}{x^2\sqrt{y^2 - x^2}} = -\frac{\sqrt{y^2 - x^2}}{y^2 x} + \varphi(y) \tag{3}$$

where the additive function φ has only the variable not used in the integration and thus is "the constant of integration." Formula (3) holds since from (2)

$$\frac{\partial}{\partial x}\left[-\frac{\sqrt{y^2 - x^2}}{y^2 x} + \varphi(y)\right] = \frac{1}{x^2\sqrt{y^2 - x^2}} + \frac{\partial \varphi(y)}{\partial x} = \frac{1}{x^2\sqrt{y^2 - x^2}} + 0.$$

Example 1. Find a function f given that both

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{x^2\sqrt{y^2 - x^2}} \quad \text{and} \quad f(2,y) = 2 + \sin y. \tag{4}$$

Solution. From the first equation of (4), it follows from (3) that

$$f(x,y) = -\frac{\sqrt{y^2 - x^2}}{y^2 x} + \varphi(y). \tag{5}$$

Can $\varphi(y)$ be determined so the second equation of (4) is also satisfied? From (5)

$$f(2,y) = -\frac{\sqrt{y^2 - 4}}{y^2 2} + \varphi(y) \quad \text{which must equal } 2 + \sin y.$$

Thus, both equations of (4) are satisfied by the function f defined by

$$f(x,y) = -\frac{\sqrt{y^2 - x^2}}{y^2 x} + \frac{\sqrt{y^2 - 4}}{2y^2} + 2 + \sin y.$$

The two equations in (4) are typical of one type of partial derivative system; another type is illustrated in the following example. A function which satisfies both equations of such a system is said to be "a solution of the system."

Example 2. Solve the partial derivative system

$$\text{(6)} \qquad \frac{\partial f(x,y)}{\partial x} = y \cos xy + \frac{1}{x} \quad \text{and} \quad \frac{\partial f(x,y)}{\partial y} = x \cos xy + e^{-y}$$

Solution. If f satisfies the first equation above, then

$$f(x,y) = \int \frac{\partial f(x,y)}{\partial x}\, dx = \int \left(y \cos xy + \frac{1}{x}\right) dx = \sin xy + \ln |x| + \varphi(y).$$

The second equation is satisfied by the function f defined by

$$f(x,y) = \int \frac{\partial f(x,y)}{\partial y}\, dy = \int \left(x \cos xy + e^{-y}\right) dy = \sin xy - e^{-y} + \psi(x),$$

where the function ψ is independent of y. By inspection it follows, upon setting $\varphi(y) = -e^{-y}$ and $\psi(x) = \ln |x|$, that

$$f(x,y) = \sin xy + \ln |x| - e^{-y}$$

satisfies both equations of (6) so is a solution of (6).

This function plus any constant is also a solution of (6).

Example 3. Show that there is no simultaneous solution of the equations

$$\text{(7)} \qquad \frac{\partial f(x,y)}{\partial x} = \cos xy + \frac{1}{x} \quad \text{and} \quad \frac{\partial f(x,y)}{\partial y} = \cos xy + e^{-y}.$$

Solution. If f satisfies the first of these equations, then

$$\text{(8)} \qquad f(x,y) = \int \frac{\partial f(x,y)}{\partial x}\, dx = \int \left(\cos xy + \frac{1}{x}\right) dx = \frac{\sin xy}{y} + \ln |x| + \varphi(y),$$

whereas if f satisfies the second equation, then

$$\text{(9)} \qquad f(x,y) = \int \frac{\partial f(x,y)}{\partial y}\, dy = \int (\cos xy + e^{-y})\, dy = \frac{\sin xy}{x} - e^{-y} + \psi(x).$$

Since the terms containing both x and y do not agree, then it is hopeless to select $\varphi(y)$ in y alone and $\psi(x)$ in x alone in such a way as to make (8) and (9) the same. Thus, there is no function f that satisfies both equations of (7).

PROBLEMS

1. Solve each of the following partial derivative systems:
   a. $f_x(x,y) = 3x^2y - 2xy^2$ and $f(1,y) = y + \sin y$.
   b. $f_y(x,y) = \sqrt{xy}$ and $f(x,4) = x^2 + 5\sqrt{x}$.
   c. $\dfrac{\partial f(x,y)}{\partial y} = \dfrac{x}{x^2 + y^2}$ and $f(x,1) = \sqrt{x^2 + 1}$.
   d. $\dfrac{\partial f(x,y)}{\partial x} = \dfrac{x}{x^2 + y^2}$ and $f(0,y) = \ln y^2$.

e. $\dfrac{\partial f(x,y)}{\partial x} = \dfrac{1}{x^2\sqrt{x^2 - y^2}}$ and $f(4,y) = 0$.

f. $f_x(x,y) = \cos(xy + y^2)$ and $f(0,y) = 0$.

g. $\dfrac{\partial f(x,y)}{\partial y} = -\dfrac{2x}{y^2}e^{2x/y}$ and $f(x,2) = e^x$.

h. $f_y(x,y) = \ln(x^2 + y^2)$ and $f(x,3) = \sin x$.

2. Determine a function (if one exists) such that:

a. $\dfrac{\partial f(x,y)}{\partial x} = \dfrac{1}{y}$ and $\dfrac{\partial f(x,y)}{\partial y} = -\dfrac{x}{y^2} + 1$.

b. $f_x(x,y) = y$ and $f_y(x,y) = -x$.

c. $\dfrac{\partial f(x,y)}{\partial x} = \dfrac{y}{x^2 + y^2} + e^x$ and $\dfrac{\partial f(x,y)}{\partial y} = \dfrac{-x}{x^2 + y^2}$.

d. $\dfrac{\partial f(x,y)}{\partial x} = \dfrac{x}{x^2 + y^2}$ and $\dfrac{\partial f(x,y)}{\partial y} = \dfrac{y}{x^2 + y^2} + y$.

e. $f_x(x,y) = y^2$ and $f_y(x,y) = -1/x$.

f. $\dfrac{\partial f(x,y)}{\partial x} = -\dfrac{y}{x^2}e^{2y/x}$ and $f(x,1) = e^x$.

116. Differentiable Functions

The increment and differential notation, as used in connection with functions of one variable (Secs. 41 and 43), is extended to functions of two variables. With x and y denoting values of the independent variables, then Δx and Δy are used for arbitrary numbers called **increments** of x and y, respectively, and the corresponding **increment** of a function f at (x,y) is defined by

$$\text{(1)} \qquad \Delta f(x,y) = f(x + \Delta x, y + \Delta y) - f(x,y).$$

Arbitrary numbers dx and dy as used in (2) are called **differentials** for x and y, respectively, and the corresponding **differential** of f at (x,y) is defined (assuming $f_x(x,y)$ and $f_y(x,y)$ exist) by

$$\text{(2)} \qquad df(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

In some books $df(x,y)$ as defined by (2) is called "a total differential."

Example 1. With f the function defined by $f(x,y) = x^2 - xy$, find the increment and the differential of f at (8,5) corresponding to $\Delta x = dx = 0.02$ and $\Delta y = dy = -0.01$.

Solution.

$$\Delta f(8,5) = f(8 + 0.02, 5 - 0.01) - f(8,5) = [(8.02)^2 - (8.02)(4.99)] - [8^2 - 8 \cdot 5]$$

$$= [64.3204 - 40.0198] - [64 - 40] = 0.3006 \quad \text{and}$$

$$df(8,5) = \left[\frac{\partial(x^2 - xy)}{\partial x}\right]_{(8,5)}(0.02) + \left[\frac{\partial(x^2 - xy)}{\partial y}\right]_{(8,5)}(-0.01)$$

$$= \left[2x - y\right]_{(8,5)}(0.02) + \left[-x\right]_{(8,5)}(-0.01)$$

$$= [11](0.02) + [-8](-0.01) = 0.30.$$

As in Example 1, the arbitrary numbers Δx and dx may be chosen equal if desired, and also we may set $\Delta y = dy$.

The graph of $z = f(x,y)$ and its tangent plane (assumed to exist) at a point (x_0,y_0,z_0) may be used to show (as in Fig. 116.1) a geometric interpretation of $\Delta f(x_0,y_0)$ and $df(x_0,y_0)$ for $\Delta x = dx = h$ and $\Delta y = dy = k$. The fact that the tangent plane at (x_0,y_0,z_0) also contains the point

$$(x_0 + h, y_0 + k, z_0 + df(x_0,y_0))$$

may be seen by substituting these coordinates into the equation of the tangent plane $z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$.

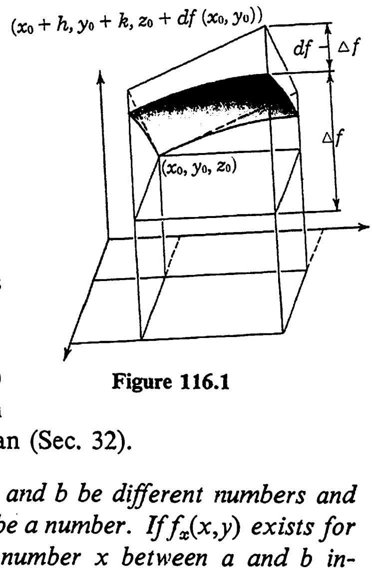


Figure 116.1

An algebraic similarity (given below) between $\Delta f(x,y)$ and $df(x,y)$ depends upon the following extension of the law of the mean (Sec. 32).

Let f be a function of two variables, let a and b be different numbers and let y be a number. If $f_x(x,y)$ exists for each number x between a and b inclusive, then there is a number ξ between a and b such that

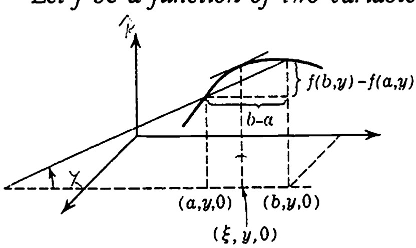


Figure 116.2

$$(3) \qquad \frac{f(b,y) - f(a,y)}{b - a} = f_x(\xi,y).$$

Among the various ways formula (3) may be written are

$$f(b,y) - f(a,y) = f_x(\xi,y)(b - a) \quad \text{and}$$

$$(4) \qquad f(x + \Delta x,y) - f(x,y) = f_x(\xi,y)\,\Delta x, \quad \xi \text{ between } x \text{ and } x + \Delta x.$$

In the same way (assuming existence of f_y) there is a number η between y and $y + \Delta y$ such that

$$f(x,y + \Delta y) - f(x,y) = f_y(x,\eta)\,\Delta y. \tag{5}$$

Notice that in (4) the second variable has the same value throughout, whereas in (5) the first variable has the same value throughout.

Let f_x and f_y exist at all points within a circle having center (x,y). With $(x + \Delta x, y + \Delta y)$ within this circle

$$\begin{aligned} \Delta f(x,y) &= f(x + \Delta x,y + \Delta y) - f(x,y) \\ &= f(x + \Delta x,y + \Delta y) - f(x,y + \Delta y) + f(x,y + \Delta y) - f(x,y) \end{aligned} \tag{6}$$

since the same amount $f(x,y + \Delta y)$ was subtracted and added. In the first two terms of (6) the second variable has the value $y + \Delta y$, whereas in the third and fourth terms the first variable has the value x. Thus, there are numbers ξ and η such that

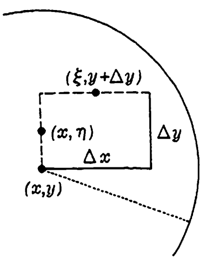


Figure 116.3

$$\Delta f(x,y) = f_x(\xi,y + \Delta y)\,\Delta x + f_y(x,\eta)\,\Delta y, \tag{7}$$

where (see Fig. 116.3) ξ is between x and $x + \Delta x$ if $\Delta x \neq 0$ but $\xi = x$ if $\Delta x = 0$, and η is between y and $y + \Delta y$ if $\Delta y \neq 0$ but $\eta = y$ if $\Delta y = 0$.

Upon setting $\Delta x = dx$ and $\Delta y = dy$ the expression (7) for $\Delta f(x,y)$ and the expression for $df(x,y)$ given in (2) are algebraically quite similar. This similarity, and the above geometric interpretation, suggests investigating (as we do below) conditions under which Δf and df approximate one another in some sense consistent with corresponding notions for functions of one variable.

The increment $\Delta f(x,y)$, as defined by (1), requires that both (x,y) and $(x + \Delta x, y + \Delta y)$ be in the domain of f, whereas $df(x,y)$, as defined by (2), requires that (x,y) be in the domains of both f_x and f_y. Even with (x,y) in the domains of f_x and f_y, so that $df(x,y)$ is defined, we do not say that f is differentiable at (x,y), but give the following definition.

DEFINITION. *If f_x and f_y exist at each point within a circle having center at (x,y) and if (with $\Delta x = dx$ and $\Delta y = dy$)*

$$\lim_{(dx,dy)\to(0,0)} \frac{\Delta f(x,y) - df(x,y)}{|dx| + |dy|} = 0, \tag{8}$$

then f is said to be ***differentiable***† *at (x,y)*

† Recall that if a function f of one variable is differentiable at x, then (see p. 132) $\lim\limits_{\Delta x = dx \to 0} \dfrac{\Delta f(x) - df(x)}{dx} = 0$. This property of differentiable functions of one variable motivated the definition of differentiable functions of two variables.

The following theorem gives sufficient conditions for a function to be differentiable at a point and reveals that many functions met in practice are differentiable over most of their domains.

THEOREM 116. *A function f is differentiable at (x,y) if f_x and f_y are both continuous at (x,y).*

PROOF. Let f_x and f_y be continuous at (x,y), let ϵ be an arbitrary positive number and determine $\delta > 0$ such that if $(u - x)^2 + (v - y)^2 < \delta^2$ then $f_x(u,v)$ and $f_y(u,v)$ are not only defined but also both inequalities

$$|f_x(u,v) - f_x(x,y)| < \epsilon \quad \text{and} \quad |f_y(u,v) - f_y(x,y)| < \epsilon$$

hold. With dx and dy given such that

$$0 < (dx)^2 + (dy)^2 < \delta^2 \tag{9}$$

we choose ξ between x and $x + dx$ if $dx \neq 0$ but $\xi = x$ if $dx = 0$ and η between y and $y + dy$ if $dy \neq 0$ but $\eta = y$ if $dy = 0$ such that (7) holds with $\Delta x = dx$ and $\Delta y = dy$. Now $(\xi, y + dy)$ and (x,η) are within the circle of radius δ and center (x,y) so that

$$|f_x(\xi,y + dy) - f_x(x,y)| < \epsilon \quad \text{and} \quad |f_y(x,\eta) - f_y(x,y)| < \epsilon.$$

It therefore follows for $\Delta x = dx$ and $\Delta y = dy$ that

$$\begin{aligned}
|\Delta f(x,y) - df(x,y)| &= |[f_x(\xi,y + dy)\,dx + f_y(x,\eta)\,dy] \\
&\qquad - [f_x(x,y)\,dx + f_y(x,y)\,dy]| \\
&\le |f_x(\xi,y + dy) - f_x(x,y)|\,|dx| \\
&\qquad + |f_y(x,\eta) - f_y(x,y)|\,|dy| \\
&< \epsilon|dx| + \epsilon|dy| = \epsilon[|dx| + |dy|].
\end{aligned}$$

We have thus shown that whenever (9) is satisfied then

$$\frac{|\Delta f(x,y) - df(x,y)|}{|dx| + |dy|} < \epsilon$$

which means that (8) holds and hence that f is differentiable at (x,y) as we wished to prove.

117. Exact Differentials

Whenever a function f is given, then finding $df(x,y)$ requires only the ability to compute the partial derivatives of f. An inverse problem is illustrated in the following example.

Example. Show that one of the following expressions is the differential of a function, but the other is not:

$$(xy^3 + 3x^2)\,dx + (x^2y^2 + 2y)\,dy \quad \text{and} \quad (2xy^3 + x^2)\,dx + (3x^2y^2 + y)\,dy.$$

Solution. If there is a function f having the first expression as its differential, then both of the equations

$$f_x(x,y) = xy^3 + 3x^2 \quad\text{and}\quad f_y(x,y) = x^2y^2 + 2y$$

must hold and therefore both of the equations

$$f(x,y) = \frac{x^2y^3}{2} + x^3 + \varphi(y) \quad\text{and}\quad f(x,y) = \frac{x^2y^3}{3} + y^2 + \psi(x)$$

must hold for some choice of $\varphi(y)$ and $\psi(x)$. But since the terms involving x and y together do not agree, there is no function whose differential is the first expression.

By the same technique used on the second expression, we seek a function f such that

$$f(x,y) = x^2y^3 + \frac{x^3}{3} + \varphi(y) = x^2y^3 + \frac{y^2}{2} + \psi(x).$$

Thus $f(x,y) = x^2y^3 + x^3/3 + y^2/2$ has the second expression as its differential.

In applied or theoretical work, an expression having the form $M(x,y)\,dx + N(x,y)\,dy$ may arise and whenever it does the further discussion of the situation will be simpler if there is a function f having this expression as its differential.

DEFINITION. *Let M and N be functions of two variables. Then*

$$M(x,y)\,dx + N(x,y)\,dy$$

*is said to be an **exact differential** if there is a function f having*

$$\frac{\partial f(x,y)}{\partial x} = M(x,y) \quad \textit{and} \quad \frac{\partial f(x,y)}{\partial y} = N(x,y)$$

and hence such that

$$df(x,y) = \frac{\partial f(x,y)}{\partial x}\,dx + \frac{\partial f(x,y)}{\partial y}\,dy \equiv M(x,y)\,dx + N(x,y)\,dy.$$

PROBLEMS

1. For the function f as defined, find the increment and the differential of f at (x_0,y_0) corresponding to $\Delta x = dx$ and $\Delta y = dy$ as given. Also, divide the increment by the differential.

 a. $f(x,y) = x^2 - xy + y$; $x_0 = 2$, $y_0 = -3$, $\Delta x = dx = -0.1$, $\Delta y = dy = 0.2$.

 b. $f(x,y) = \dfrac{x+y}{x-y}$; $x_0 = 4$, $y_0 = -3$, $\Delta x = dx = \Delta y = dy = 0.01$.

 c. $f(x,y) = \cos(xy)$; $x_0 = 2$, $y_0 = \dfrac{\pi}{6}$, $\Delta x = dx = 0.05$, $\Delta y = dy = \dfrac{\pi}{90}$.

 d. $f(x,y) = \tan^{-1}\dfrac{x}{y}$; $x_0 = y_0 = 1$, $\Delta x = dx = -0.02$, $\Delta y = dy = -0.01$.

2. Find $df(x,y)$ given that

a. $f(x,y) = 3x^2 + 4xy + y^2$.

b. $f(x,y) = \dfrac{Ax + By}{Cx + Dy}$.

c. $f(x,y) = \tan^{-1}(x \sin y)$.

d. $f(x,y) = (x - y) \ln (x^2 + y^2)$.

e. $f(x,y) = x^{\sin y}$.

f. $f(x,y) = \dfrac{\sqrt{y^2 - x^2}}{y^2 x}$.

3. Show that if

a. $z = \sqrt{x^2 + y^2}$ or $z = -\sqrt{x^2 + y^2}$, then $dz = \dfrac{x\,dx + y\,dy}{z}$.

b. $z = \sqrt{a^2 - (x^2 + y^2)}$ or $z = -\sqrt{a^2 - (x^2 + y^2)}$, then

$$dz = -\frac{x\,dx + y\,dy}{z}.$$

4. Let u and v be functions of two variables.

a. Define the function f by $f(x,y) = u(x,y)v(x,y)$ and check in turn that

$$\begin{aligned} f_x &= uv_x + vu_x \quad \text{and} \quad f_y = uv_y + vu_y, \\ df &= (uv_x + vu_x)\,dx + (uv_y + vu_y)\,dy \\ &= u(v_x\,dx + v_y\,dy) + v(u_x\,dx + u_y\,dy) \\ &= u\,dv + v\,du. \end{aligned}$$

b. Define f by $f(x,y) = u(x,y)/v(x,y)$ and obtain the usual formula for the differential of a quotient.

c. Show that $du^n(x,y) = nu^{n-1}(x,y)\,du(x,y)$.

d. Show that $d \cos u(x,y) = -\sin u(x,y)\,du(x,y)$.

5. For each of the following pairs of expressions, show that one is an exact differential and that the other is not.

a. $y\,dx - x\,dy$, $\dfrac{y\,dx - x\,dy}{y^2}$

b. $y\,dx - x\,dy$, $\dfrac{y\,dx - x\,dy}{x^2}$

c. $y\,dx + x\,dy$, $\dfrac{y\,dx + x\,dy}{x^2}$

d. $x\,dx + y\,dy$, $\dfrac{x\,dx + y\,dy}{y^2}$

e. $(x + x\sqrt{x^2 + y^2})\,dx + y\,dy$, $(x^2 + y^2)^{-1/2}[(x + x\sqrt{x^2 + y^2})\,dx + y\,dy]$.

f. $(y + x^2y^2)\,dx - x\,dy$, $3y^{-2}[(y + x^2y^2)\,dx - x\,dy]$.

6. Let f be the function defined by $f(x,y) = \sqrt{|xy|}$. Show:

a. That $df(0,0) = 0$.

b. That f is not differentiable at $(0,0)$.

118. Implicit Functions

With f a function of two variables, the sets

$$\{(x,y,z) \mid z = f(x,y)\} \quad \text{and} \quad \{(x,y,0) \mid f(x,y) = 0\}$$

are visualized as a surface and a curve; the curve being the profile of the surface in the xy-plane. For a point $(x,y,0)$ to be on this profile there must be a relation between x and y. Thus, with $(x,y,0)$ and $(x + \Delta x, y + \Delta y, 0)$ both on the profile, then not only is $f(x,y) = 0$ but we must have as well $f(x + \Delta x, y + \Delta y) = 0$ so that

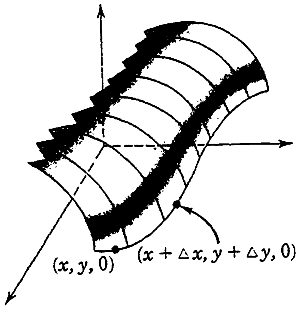


Figure 118

$$(1)\quad \Delta f(x,y) = f(x + \Delta x, y + \Delta y) - f(x,y) = 0 - 0 = 0.$$

Assuming that f_x and f_y exist, there is a number ξ between x and $x + \Delta x$, and a number η between y and $y + \Delta y$ such that

$$\Delta f(x,y) = f_x(\xi, y + \Delta y)\,\Delta x + f_y(x,\eta)\,\Delta y = 0$$

(see (7) of Sec. 116). This equation leads to

$$\frac{\Delta y}{\Delta x} = -\frac{f_x(\xi, y + \Delta y)}{f_y(x,\eta)}, \quad \text{if } f_y(x,\eta) \neq 0.$$

Thus, if f_x and f_y are continuous, if $f_y(x,y) \neq 0$, and if $\lim\limits_{\Delta x\to 0} \Delta y = 0$, then

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = -\lim_{\Delta x\to 0}\frac{f_x(\xi, y + \Delta y)}{f_y(x,\eta)} = -\frac{\lim\limits_{\Delta x\to 0} f_x(\xi, y + \Delta y)}{\lim\limits_{\Delta x\to 0} f_y(x,\eta)} = -\frac{f_x(x,y)}{f_y(x,y)}$$

Any relation between the variables implied by

$$(2)\qquad f(x,y) = 0$$

is said to be an **implicit function.** In a region throughout which (2) determines y uniquely in terms of x and $f_y(x,y) \neq 0$, then

$$(3)\qquad D_x y = -\frac{f_x(x,y)}{f_y(x,y)}.$$

provided f_x and f_y are continuous.

Example 1. Given $x^2y - x\sin y + 4 = 0$, find $D_x y$ and $D_y x$.

Solution.

$$\frac{\partial(x^2y - x\sin y + 4)}{\partial x} = 2xy - \sin y, \qquad \frac{\partial(x^2y - x\sin y + 4)}{\partial y} = x^2 - x\cos y$$

$$D_x y = -\frac{(2xy - \sin y)}{x^2 - x\cos y} \quad \text{if} \quad x^2 - x\cos y \neq 0, \quad \text{and}$$

$$D_y x = -\frac{x^2 - x\cos y}{2xy - \sin y} \quad \text{if} \quad 2xy - \sin y \neq 0.$$

By using differentials $dy = D_x y\, dx$, as defined in connection with functions of a single variable, (3) may be written as

$$\frac{dy}{dx} = -\frac{f_x(x,y)}{f_y(x,y)} \text{ and then as } f_x(x,y)\,dx + f_y(x,y)\,dy = 0$$

which is exactly the form of the total differential $df(x,y)$ of a function of two variables. Thus, a perfectly formal way of finding dy/dx from $f(x,y) = 0$ is to set $df(x,y) = d(0) = 0$, expand:

$$df(x,y) = f_x(x,y)\,dx + f_y(x,y)\,dy = 0,$$

and then solve for dy/dx.

Example 2. Find dy/dx given that $x^2 - 3xy + y^2 = 4$.

Solution. Set $x^2 - 3xy + y^2 - 4 = 0$ and then write

$$d(x^2 - 3xy + y^2 - 4) = (2x - 3y)\,dx + (-3x + 2y)\,dy = d(0) = 0,$$

$$(3x - 2y)\,dy = (2x - 3y)\,dx, \quad \frac{dy}{dx} = \frac{2x - 3y}{3x - 2y} \quad \text{if} \quad 3x \neq 2y.$$

119. Families

The equation $x^2 + y^2 + 2cy = 0$ may be written as $x^2 + (y + c)^2 = c^2$ and thus for each number c represents a circle with center at $(0,-c)$ and radius $|c|$. Also, the equation

$$(1) \qquad x^2 + y^2 + 2cy = 0$$

may be thought of as representing the family of all circles tangent to the x-axis at the origin. In this interpretation, c is called the **parameter** of the family. For each value of the parameter

$$(2) \quad d(x^2 + y^2 + 2cy) = 2x\,dx + 2(y + c)\,dy = d(0) = 0.$$

Upon eliminating c between (1) and (2) the result is the differential equation

$$2x\,dx + 2\left(y - \frac{x^2 + y^2}{2y}\right)dy = 0;$$

$$\text{that is, } 2xy\,dx + (y^2 - x^2)\,dy = 0.$$

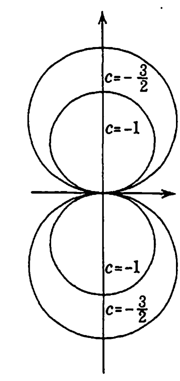


Figure 119

Definition. *Given a family of implicit functions* $f(x,y,c) = 0$, *the result of eliminating the parameter c between*

$$f(x,y,c) = 0 \quad \textit{and} \quad df(x,y,c) = 0$$

is called the ***differential equation of the family*** $f(x,y,c) = 0$ *(or of the family of curves represented by* $f(x,y,c) = 0$*). Also,* $f(x,y,c) = 0$ *is said to be the* ***primitive*** *of the resulting differential equation.*

Thus, $2xy\,dx + (y^2 - x^2)\,dy = 0$ is the differential equation of the family of circles tangent to the x-axis at the origin. Also, $x^2 + y^2 + 2cy = 0$ is the primitive of the differential equation $2xy\,dx + (y^2 - x^2)\,dy = 0$.

PROBLEMS

1. Find D_xy, if it exists, given:

a. $xy + x^2y^3 - 3 = 0$.
b. $xe^y + y \cot x = 3$.
c. $x^2 + y^2 + 1 = 0$.
d. $\sin x + \cos y = 3$.
e. $x^y = 4$.
f. $x^{\ln y} = 2$.

2. Find an equation of the tangent to the graph of the given equation at the points indicated.

a. $xy + \ln(x + y) + 2 = 0$; $(2,-1)$, $(-1,2)$.

b. $2 \sin x \cos y = 1$; $\left(\dfrac{\pi}{6},0\right)$, $\left(\dfrac{\pi}{2},\dfrac{\pi}{3}\right)$.

c. $x^2 + xy - y^2 = 5$; points with abscissa 3.

d. $x^{2/3} + y^{2/3} = 5$; points with ordinate 8.

3. Find the differential equation having the primitive:

a. $xy - cy^2 - 4 = 0$.
b. $x(y^2 - c) + y = 0$.
c. $y = (c - \cos x)e^{-x}$.
d. $y = e^{cx}$.
e. $cy + x - \sin x = 0$.
f. $y + c(x - \sin x) = 0$.

4. Find the differential equation of the family of all:

a. Circles tangent to the x-axis at the point $(1,0)$.

b. Circles tangent to the y-axis at the origin.

c. Central conics with vertices at $(\pm 2,0)$.

d. Parabolas with vertices at the origin and foci on the x-axis.

e. All lines passing through the point $(-1,2)$.

120. Functions of Three Variables

Given a function F of three variables with values denoted by x, y, and z, the partial derived functions F_x, F_y, and F_z are defined by

$$F_x(x,y,z) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x,y,z) - F(x,y,z)}{\Delta x}$$

with analogous definitions for F_y and F_z. For example

$$\frac{\partial}{\partial z}(x^2 - 3xyz + z^3 \sin y - 4) = -3xy + 3z^2 \sin y.$$

The total differential of F is defined by

$$dF(x,y,z) = F_x(x,y,z)\,dx + F_y(x,y,z)\,dy + F_z(x,y,z)\,dz.$$

Functions of one variable were represented geometrically by using two mutually perpendicular axes and functions of two variables by three mutually perpendicular axes. A natural tendency is to expect a geometric representation of

$$\{(x,y,z,w) \mid w = F(x,y,z)\}$$

by means of four mutually perpendicular axes, but these we cannot visualize. This inability does not, however, in any way disqualify us from using functions of three or more variables, partials of such functions, or other concepts involving more than two variables.

We have already visualized implicit functions

$$F(x,y,z) = 0 \tag{1}$$

when we represented such equations as $x^2 + y^2 + z^2 - 4 = 0$ graphically. Considering (1) as expressing z in terms of x and y (at least throughout some range of x, y, and z) as represented by a solid, then previous results about implicit differentiation extend to

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} \quad\text{and}\quad \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} \quad\text{if}\quad F_z(x,y,z) \neq 0. \tag{2}$$

By means of these formulas, the equations of tangent planes and normal lines to surfaces may be put into more symmetric form. For upon expressing some (or all if possible) of the graph of (1) as the graph of

$$z = f(x,y), \quad \text{where } F(x,y,f(x,y)) \equiv 0, \tag{3}$$

then at a point (x_0,y_0,z_0) of this graph the tangent plane has equation

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0), \quad \text{where } z_0 = f(x_0,y_0). \tag{4}$$

From (3) and (2)

$$f_x(x_0,y_0) = \left.\frac{\partial z}{\partial x}\right]_{(x_0,y_0)} = -\frac{F_x(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)} \quad\text{and}\quad f_y(x_0,y_0) = -\frac{F_y(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}.$$

Hence, (4) becomes

$$z - z_0 = -\frac{F_x(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}(x - x_0) - \frac{F_y(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}(y - y_0)$$

which may be put in the more symmetric form

(5) $F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0.$

Hence, $F_x(x_0,y_0,z_0)$, $F_y(x_0,y_0,z_0)$, $F_z(x_0,y_0,z_0)$ are direction numbers of the normal line to the surface at (x_0,y_0,z_0) so this normal has parametric equations

$$x = x_0 + tF_x(x_0,y_0,z_0), \quad y = y_0 + tF_y(x_0,y_0,z_0), \quad z = z_0 + tF_z(x_0,y_0,z_0).$$

Example 1. Find equations of the tangent plane and normal line to the graph of $x^2 + y^2 + (z - 3)^3 = 6$ at the point $(2,-1,4)$.

Solution. Set $F(x,y,z) = x^2 + y^2 + (z - 3)^3 - 6$ so the graph has equation $F(x,y,z) = 0$. Now $F_x(2,-1,4) = 2(2) = 4$, $F_y(2,-1,4) = -2$, and $F_z(2,-1,4) = 3(z - 3)^2\Big]_{(2,-1,4)} = 3$. Thus, the tangent plane has equation

$$4(x - 2) - 2(y + 1) + 3(z - 4) = 0 \quad \text{or} \quad 4x - 2y + 3z = 22,$$

and the normal line has equations

$$x = 2 + 4t, \quad y = -1 - 2t, \quad z = 4 + 3t \quad \text{or} \quad \frac{x-2}{4} = \frac{y+1}{-2} = \frac{z-4}{3}.$$

Since $F_x(x_0,y_0,z_0)$, $F_y(x_0,y_0,z_0)$, $F_z(x_0,y_0,z_0)$ are direction numbers of the normal line to the graph of (1) at the point (x_0,y_0,z_0), then the vector

$$\vec{i}F_x(x_0,y_0,z_0) + \vec{j}F_y(x_0,y_0,z_0) + \vec{k}F_z(x_0,y_0,z_0)$$

with initial end at (x_0,y_0,z_0) is normal to the graph. This vector is called the **gradient** of F at (x_0,y_0,z_0) (abbreviated **grad** F). Also, a special vector symbol $\vec{\nabla}$, read "del," is introduced by

$$\vec{\nabla}F = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}. \tag{6}$$

Example 2. Given $F(x,y,z) = 2x^2y + yz^2 + 3xz$, find $\vec{\nabla}F(-1,2,3)$.

Solution. $\vec{\nabla}F(-1,2,3) = \Big[(4xy + 3z)\vec{i} + (2x^2 + z^2)\vec{j} + (2yz + 3x)\vec{k}\Big]_{(-1,2,3)}$

$$= \vec{i} + 11\vec{j} + 9\vec{k}. \tag{7}$$

In Example 2 note that $F(-1,2,3) = 13$ so the point $(-1,2,3)$ is not on the graph of $F(x,y,z) = 0$, but is on the graph of $F(x,y,z) - 13 = 0$ and at this point (7) is normal to this second surface.

The del $\vec{\nabla}$ is also used in connection with a function f of two variables by setting

$$\vec{\nabla}f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}. \tag{8}$$

PROBLEMS

1. By considering the given equation as defining z in terms of x and y, find $\partial z/\partial x$ and $\partial z/\partial y$.

   a. $xz - x^2y + yz^3 = 4.$ c. $e^{xy} - \ln z - 5 = 0.$

   b. $x + \sin(yz) - 1 = 0.$ d. $\tan^{-1}(x + z) + y = 4.$

2. From the equations of Prob. 1, find $\partial x/\partial y$ and $\partial x/\partial z$.

3. Find equations of the tangent plane and normal line to the surface whose equation is given at the point indicated. In each case find the points where the normal line pierces the coordinate planes.

   a. $x^2 + y^2 + z^2 = 14, \quad (-1,2,3).$ d. $2^{xz} - y = 1, \quad (1,1,1).$

   b. $x^2y + y^2 + z^2 = 7, \quad (1,2,-1).$ e. $x \ln |x^2 + z^2 - 1| = y, \quad (1,0,1).$

   c. $xy + yz = 2xz, \quad (3,-3,-1).$ f. $x \cos(yz) + z + 1 = 0, \quad \left(3, \frac{\pi}{2}, 2\right).$

4. Show that the graphs of $2x^2 + 2y^2 - z^2 = 25$ and $x^2 + y^2 = 5z$ have a common tangent plane at the point $(4,3,5)$.

5. The line having equations $x/2 = y/2 = z$ pierces the surface having equation $2x^2 + 4y^2 + z^2 = 100$ in two points. Find the acute angle between the given line and the normal to the surface at each of these points.

6. Check that the graphs of the given equations contain the given point. Show that at this point the normals to the graphs are perpendicular.

   a. $x^2 - y^2 + z^2 + 2 = 0, \quad x^2 + y^2 + 3z^2 = 8; \quad (-1,2,1).$

   b. $x^2 - 10y^2 + 3z^2 = 21, \quad 4x^2 + 5y^2 - 3z^2 = 24; \quad (-3,0,2).$

7. Determine a, b, and c in such a way that the graphs of the equations $ax^2 - 5y^2 + 2z^2 = 6$ and $bx^2 + y^2 + z^2 = c$ will have the point $(-1,1,2)$ on their intersection and at this point have perpendicular normals.

8. Find $\vec{\nabla} F(x,y,z)$ if:

   a. $F(x,y,z) = x^3y + xyz + 3yz^2.$ b. $F(x,y,z) = xy \cos z + x \sin(xyz).$

9. Let (x_0,y_0,z_0) be a point on the graph of $F(x,y,z) = c$. By using properties of the dot product, show that

$$\vec{\nabla} F(x_0,y_0,z_0) \cdot [\vec{i}(x - x_0) + \vec{j}(y - y_0) + \vec{k}(z - z_0)] = 0$$

   is an equation of the tangent plane to this graph at (x_0,y_0,z_0).
   Also, with $\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$ and $d\vec{r} = \vec{i}dx + \vec{j}dy + \vec{k}dz$ show that

$$dF = \vec{\nabla} F \cdot d\vec{r}.$$

121. Change of Variables

Let f be a function of two variables defined by

$$f(x,y) = x^2 - xy, \tag{1}$$

and let F be the function of one variable defined by

$$F(t) = f(3 \cos t, 2 \sin t).$$

Then $F(t) = 9 \cos^2 t - 6 \cos t \sin t$ and $F'(t)$ is obtained as

$$F'(t) = -18 \cos t \sin t - 6 \cos^2 t + 6 \sin^2 t. \tag{2}$$

In some situations it is not only important to know the derivative of F, but it is also desirable to know how this derivative is related to the partial derivatives of f. As shown below

$$F'(t) = f_x(3 \cos t, 2 \sin t) D_t 3 \cos t + f_y(3 \cos t, \sin t) D_t 2 \sin t \tag{3}$$

which may be checked by noting that

$$f_x(x,y) = 2x - y \quad \text{so} \quad f_x(3 \cos t, 2 \sin t) = 6 \cos t - 2 \sin t,$$
$$f_y(x,y) = -x \quad \text{so} \quad f_y(3 \cos t, 2 \sin t) = -3 \cos t$$

and hence (since $D_t 3 \cos t = -3 \sin t$ and $D_t 2 \sin t = 2 \cos t$) the right side of (3) is

$$(6 \cos t - 2 \sin t)(-3 \sin t) + (-3 \cos t)2 \cos t$$
$$= -18 \cos t \sin t + 6 \sin^2 t - 6 \cos^2 t$$

which agrees with the right side of (2).

For a general development, let f be a function of two variables and let x and y each be a function of a single variable. Select a number t in the domains of both x and y so that $(x(t),y(t))$ is an ordered pair of numbers and if this ordered pair is in the domain of f, then $f(x(t),y(t))$ is a number. Thus, the set of ordered pairs of the form

$$[t, f(x(t),y(t))]$$

is a function which we denote by F. Hence, for t and $t + \Delta t$ both in the domain of F we have

$$F(t + \Delta t) - F(t) = f(x(t + \Delta t), y(t + \Delta t)) - f(x(t),y(t)). \tag{4}$$

For convenience in writing, we drop the t on the right side of (4) after replacing $x(t + \Delta t)$ by $x + \Delta x$ and $y(t + \Delta t)$ by $y + \Delta y$ thus obtaining

$$F(t + \Delta t) - F(t) = f(x + \Delta x, y + \Delta y) - f(x,y).$$

Assuming that the partials f_x and f_y exist, there is a number ξ between x and $x + \Delta x$ and a number η between y and $y + \Delta y$ such that

$$F(t + \Delta t) - F(t) = f_x(\xi,y + \Delta y)\,\Delta x + f_y(x,\eta)\,\Delta y$$

(see (7) of Sec. 116). Now divide both sides by Δt. If f_x and f_y are continuous and if $D_t x$ and $D_t y$ exist, then

$$\begin{aligned} f_x(x,y)\,D_t x + f_y(x,y)\,D_t y &= \lim_{\Delta t \to 0}\left[f_x(\xi,y + \Delta y)\frac{\Delta x}{\Delta t} + f_y(x,\eta)\frac{\Delta y}{\Delta t}\right] \\ &= \lim_{\Delta t \to 0}\frac{F(t + \Delta t) - F(t)}{\Delta t} = F'(t). \end{aligned}$$

Thus, if t is such that $x'(t)$, $y'(t)$, $f_x(x(t),y(t))$ and $f_y(x(t),y(t))$ all exist, and if f_x and f_y are continuous at $(x(t),y(t))$, then we have the formula

$$D_t f(x(t),y(t)) = f_x(x(t),y(t))D_t x(t) + f_y(x(t),y(t))D_t y(t). \tag{5}$$

It is customary to use incomplete notation and from

$$z = f(x,y), \tag{6}$$

in a context where x and y are known to be functions expressible in terms of t alone, to write

$$\frac{dz}{dt} = f_x(x,y)\frac{dx}{dt} + f_y(x,y)\frac{dy}{dt}. \tag{7}$$

In differential notation $dz = f_x(x,y)\,dx + f_y(x,y)\,dy$ so a formal way of remembering (7) is to divide both sides of this differential formula by dt. An even further abstraction of (5) is made by writing

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

By starting with (6), but under the condition that x and y are functions of two variables with values denoted by t and s, then the above argument, applied to each variable separately, yields

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \quad \text{and} \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}. \tag{8}$$

Formulas (7) and (8) are described as derivative or partial derivative formulas under **change of variables**, or as **the chain rule**.

Example. Given $z = f(x,y)$ with $x = \rho\cos\theta$ and $y = \rho\sin\theta$ show that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \rho}\cos\theta - \frac{1}{\rho}\frac{\partial z}{\partial \theta}\sin\theta.$$

Solution. From (8) with ρ in place of t and θ in place of s,

$$\frac{\partial z}{\partial \rho} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \rho} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta, \quad \text{and}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-\rho\sin\theta) + \frac{\partial z}{\partial y}\rho\cos\theta.$$

Multiply each term of the first equation by $\rho\cos\theta$, each term of the second equation by $-\sin\theta$, and then add corresponding terms:

$$\frac{\partial z}{\partial \rho}\rho\cos\theta - \frac{\partial z}{\partial \theta}\sin\theta = \frac{\partial z}{\partial x}(\rho\cos^2\theta + \rho\sin^2\theta) = \rho\frac{\partial z}{\partial x}$$

which, upon division by ρ, is the desired equation.

In a fairly prevalent mode of dependent-independent-variable manner of speaking, we shall indicate why it is sometimes said:

If z is a dependent variable depending upon x and y, then

(9) $$dz = z_x\,dx + z_y\,dy$$

whether x and y are independent variables or are themselves dependent variables.

In the first place, if x and y are independent variables, then (9) is the definition of dz where $z = f(x,y)$ so that $z_x = f_x$ and $z_y = f_y$.

On the other hand, if x and y are dependent variables with (say) s and t the independent variables, then (9) translates into

(10) $$dx = x_s\,ds + x_t\,dt \quad \text{and} \quad dy = y_s\,ds + y_t\,dt.$$

Now, however, z depends intermediately upon x and y but ultimately upon s and t, so that another translation of (9) is

(11) $$dz = z_s\,ds + z_t\,dt$$

wherein, according to (8),

$$z_s = z_x x_s + z_y y_s \quad \text{and} \quad z_t = z_x x_t + z_y y_t.$$

By substituting these expressions for z_s and z_t into (11) we obtain

$$\begin{aligned} dz &= (z_x x_s + z_y y_s)\,ds + (z_x x_t + z_y y_t)\,dt \\ &= z_x(x_s\,ds + x_t\,dt) + z_y(y_s\,ds + y_t\,dt) \end{aligned}$$

and now the substitution using (10) yields $dz = z_x\,dx + z_y\,dy$ which looks exactly like (9).

PROBLEMS

1. Continuing with the above example, show that:

a. $\dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial \rho}\sin\theta + \dfrac{1}{\rho}\dfrac{\partial z}{\partial \theta}\cos\theta.$ b. $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = \left(\dfrac{\partial z}{\partial \rho}\right)^2 + \dfrac{1}{\rho^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2.$

2. In each of the following find $D_t z$ by two methods: (i) by expressing z explicitly in terms of t, and (ii) by using Formula 7.

   a. $z = x^2 - xy$; $x = e^t$, $y = \ln t$.

   b. $z = x^2 + 2xy - y^2$; $x = \cos t$, $y = \sin t$.

   c. $z = \sin 2x \cos y$; $x = e^t$, $y = 2t$.

   d. $z = \sqrt{1 + y^2} + \ln |x^2 - y^2|$; $x = \sec t$, $y = \tan t$.

3. Find $\partial z/\partial t$ and $\partial z/\partial s$ by two methods.

   a. $z = x^2 - xy$; $x = e^t \cos s$, $y = e^t \sin s$.

   b. $z = x^2 + 2xy - y^2$; $x = s \cos t$, $y = s \sin t$.

   c. $z = \sin 2x \cos y$; $x = se^t$, $y = st$.

   d. $z = x^y$; $x = 1 + t^2$, $y = e^s$.

4. Let f be a function of a single variable, let u be a function of two variables, and define a function F of two variables by

$$F(x,y) = f(u(x,y)).$$

Under proper conditions derive the formulas

$$F_x(x,y) = f'(u(x,y))\frac{\partial u(x,y)}{\partial x} \quad\text{and}\quad F_y(x,y) = f'(u(x,y))\frac{\partial u(x,y)}{\partial y}.$$

(Note: This situation is usually described by saying, "If $z = f(u)$ and $u = g(x,y)$, then

$$\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x} \quad\text{and}\quad \frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}.\text{"}\Big)$$

Use these formulas and show that if:

   a. $z = f(x + 2y)$, then $\dfrac{\partial z}{\partial x} = f'(x + 2y)$ and $\dfrac{\partial z}{\partial y} = 2f'(x + 2y)$.

   (Hint: Set $u = x + 2y$.)

   b. $z = f(x - 2y)$, then $2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

   c. $z = f\left(\dfrac{y}{x}\right)$, then $\dfrac{\partial z}{\partial x} = -\dfrac{y}{x^2}f'\left(\dfrac{y}{x}\right)$ and $\dfrac{\partial z}{\partial y} = \dfrac{1}{x}f'\left(\dfrac{y}{x}\right)$.

   d. $z = f(xy)$, then $\dfrac{\partial z}{\partial x} = yf'(xy)$ and $\dfrac{\partial z}{\partial y} = xf'(xy)$.

   e. $z = xf\left(\dfrac{y}{x}\right)$, then $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.

   f. $z = x + f(xy)$, then $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = x$.

122. Second Partials

Given a function f of two variables, then f_x and f_y are functions of two variables and these functions may have partial derivatives denoted by

$$(1) \qquad f_{xx},\ f_{xy},\ f_{yx},\ f_{yy}.$$

The two middle terms are defined by

$$f_{xy}(x,y) = \lim_{\Delta y \to 0} \frac{f_x(x,y+\Delta y) - f_x(x,y)}{\Delta y} \quad \text{and}$$

$$f_{yx}(x,y) = \lim_{\Delta x \to 0} \frac{f_y(x+\Delta x,y) - f_y(x,y)}{\Delta x}.$$

The functions in (1) are called **second partials** of f.

Example 1. Obtain the four second partials of the function f defined by $f(x,y) = x^2y + xy^3$.

Solution. First $f_x(x,y) = 2xy + y^3$ and $f_y(x,y) = x^2 + 3xy^2$. Hence

$$f_{xx}(x,y) = \frac{\partial(2xy + y^3)}{\partial x} = 2y, \qquad f_{xy}(x,y) = \frac{\partial(2xy + y^3)}{\partial y} = 2x + 3y^2,$$

$$f_{yx}(x,y) = \frac{\partial(x^2 + 3xy^2)}{\partial x} = 2x + 3y^2, \qquad f_{yy}(x,y) = \frac{\partial(x^2 + 3xy^2)}{\partial y} = 6xy.$$

The order of taking partials in f_{xy} is described as "x first, y second" whereas the order in f_{yx} is "y first, x second." Even though these orders are reversed, it so happens that in most cases $f_{xy} = f_{yx}$ as stated in the following theorem. First we prove a lemma.

LEMMA. *If f_{xy} and f_{yx} both exist on the rectangle shown, then there are points (x_1,y_1) and (x_2,y_2) in the rectangle such that*

$$f_{xy}(x_1,y_1) = f_{yx}(x_2,y_2).$$

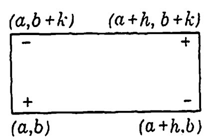


Figure 122

PROOF. We arbitrarily set

$$F(x) = f(x,b+k) - f(x,b),\ a \le x \le a+h,$$

$$G(y) = f(a+h,y) - f(a,y),\ b \le y \le b+k,$$

$$H(h,k) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b).$$

The $+$ and $-$ on the figure indicate the signs assigned to the values of f at these corners in forming $H(h,k)$. By direct substitution we check that $F(a+h) - F(a) = H(h,k)$. By the Law of the Mean, there is an x_1 between a and $a+h$ such that $F(a+h) - F(a) = F'(x_1)h$ so that $H(h,k) = F'(x_1)h$.

We now compute F' and see that

$$\begin{aligned} F'(x_1) &= f_x(x_1,b+k) - f_x(x_1,b) \\ &= f_{xy}(x_1,y_1)k \text{ for some } y_1 \text{ between } b \text{ and } b+k \end{aligned}$$

by another application of the Law of the Mean. Thus

$$H(h,k) = f_{xy}(x_1,y_1)hk.$$

Now use G in a similar way to show that there is a point (x_2,y_2) in the rectangle such that $H(h,k) = f_{yx}(x_2,y_2)kh$. Hence when these two expressions for $H(h,k)$ are set equal to one another, the stated equality follows.

THEOREM 122.1. *If f_{xy} and f_{yx} are continuous in a region and (a,b) is any point in this region, then*

$$f_{xy}(a,b) = f_{yx}(a,b). \tag{2}$$

PROOF. With $\epsilon > 0$ arbitrary, choose $\delta > 0$ such that the square with opposite corners (a,b) and $(a+\delta,b+\delta)$ lies in the region and if (x,y) is in the square, then both

$$|f_{xy}(x,y) - f_{xy}(a,b)| < \frac{\epsilon}{2} \quad \text{and} \quad |f_{yx}(x,y) - f_{yx}(a,b)| < \frac{\epsilon}{2}.$$

By the Lemma, let (x_1,y_1) and (x_2,y_2) be in the square and such that $f_{xy}(x_1,y_1) = f_{yx}(x_2,y_2)$. Hence

$$\begin{aligned} |f_{xy}(a,b) - f_{yx}(a,b)| &= |f_{xy}(a,b) - f_{xy}(x_1,y_1) + f_{yx}(x_2,y_2) - f_{yx}(a,b)| \\ &\le |f_{xy}(a,b) - f_{xy}(x_1,y_1)| + |f_{yx}(x_2,y_2) - f_{yx}(a,b)| < \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, (2) must hold.

The following theorem is an application of Theorem 122.1.

THEOREM 122.2. *Let M, N, M_y, and N_x be continuous in a region. If $M\,dx + N\,dy$ is an exact differential in this region, then for (x,y) in this region*

$$M_y(x,y) = N_x(x,y). \tag{4}$$

PROOF. Let $M\,dx + N\,dy$ be an exact differential, let f be a function having this expression as its differential so that, for (x,y) in the region,

$$df(x,y) = M(x,y)\,dx + N(x,y)\,dy$$

and hence

$$f_x(x,y) = M(x,y) \quad \text{and} \quad f_y(x,y) = N(x,y).$$

Since M_y and N_x exist in the region, then

$$f_{xy}(x,y) = M_y(x,y) \quad \text{and} \quad f_{yx}(x,y) = N_x(x,y). \tag{5}$$

Since M_y and N_x are continuous in the region, so are f_{xy} and f_{yx}. Consequently, $f_{xy}(x,y) = f_{yx}(x,y)$, which, because of (5), states that (4) holds.

The converse of Theorem 122.2 is also true; roughly, *If* (4) *holds, then* $M\,dx + N\,dy$ *is an exact differential.* At this time we are unable to prove this converse, but will give a proof in Sec. 127.

If we are given an expression $M\,dx + N\,dy$, then Theorem 122.2 tells us there is no use looking for a function whose differential is this expression if $M_y \neq N_x$, whereas the converse says that if $M_y = N_x$ then there is a function f such that $df = M\,dx + N\,dy$. For example, given

$$(6) \qquad (xy^3 + x^2)\,dx + (x^2y^2 + y)\,dy$$

take $\dfrac{\partial}{\partial y}(xy^3 + x^2) = 3xy^2$ and $\dfrac{\partial}{\partial x}(x^2y^2 + y) = 2xy^2$, see that the results are not equal and know that (6) is not an exact differential. However, from

$$2xy^3\,dx + 3x^2y^2\,dy \quad \text{take} \quad \frac{\partial(2xy^3)}{\partial y} = 6xy^2, \quad \frac{\partial(3x^2y^2)}{\partial x} = 6xy^2$$

and then with perfect confidence proceed to find a function having the expression $2xy^3\,dx + 3x^2y^2\,dy$ as its differential.

PROBLEMS

1. Find the four second partials of the function indicated and check that $z_{xy} = z_{yx}$.

a. $z = x^3 - 2xy^2$. c. $z = \cos(2x + 3y)$. e. $z = e^{y/x}$.

b. $z = \dfrac{x}{x+y}$. d. $z = \tan^{-1}\dfrac{y}{x}$. f. $z = \cosh\dfrac{y}{x}$.

2. Verify each of the following:

a. If $z = \sqrt{x^2 + y^2}$, then $z_{xx} + z_{yy} = z^{-1}$.

b. If $z = \sqrt{x^2 + y^2}$, then $(z_x)^2 + (z_y)^2 = 1$.

c. If $f(x,y) = x^4 - 3x^2y^2 + y^3$, then $f_{xx}(-1,2) = -12$, $f_{xy}(-1,2) = 24$, $f_{yy}(-1,2) = 6$.

d. If $f(x,y) = x^2\cos y$, then $f_{xx}\left(2,\frac{\pi}{3}\right) = 1$, $f_{xy}\left(2,\frac{\pi}{3}\right) = -2\sqrt{3}$, $f_{yy}\left(2,\frac{\pi}{3}\right) = -2$.

e. If $z = \dfrac{xy}{x+y}$, then $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x\,\partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = 0$.

f. If $z = \cos(x + 2y) + \sin(x - 2y)$, then $4z_{xx} = z_{yy}$.

g. If $z = (x + 3y)^5 + e^{(x-3y)}$, then $9z_{xx} = z_{yy}$.

3. Let f and g each be functions of one variable and be such that f'' and g'' both exist. With c a constant and w the function of two variables defined by

$$w(x,t) = f(x + ct) + g(x - ct) \quad \text{show that} \quad w_{tt} = c^2 w_{xx}.$$

4. A function f of two variables is said to be **harmonic** in a region if

$$f_{xx}(x,y) + f_{yy}(x,y) = 0$$

for each point (x,y) of the region. Show that each of the functions defined below is harmonic in all of the xy-plane except for the points indicated.

a. $f(x,y) = e^x \sin y$.
c. $f(x,y) = \ln (x^2 + y^2), \quad (x,y) \neq (0,0)$.
b. $f(x,y) = x^3 - 3xy^2$.
d. $f(x,y) = e^{(x^2-y^2)} \cos (2xy)$.
e. $f(x,y) = \ln \sqrt{x^2 + y^2} + \tan^{-1} (y/x), \quad (x,y) \neq (0,0), \quad x \neq 0$.

5. The first partials of second partials are called **third partials.** For example

$$\frac{\partial f_{xy}}{\partial x} = f_{xyx}.$$

a. For f a function of two variables, list the eight possible third partials.

b. Find all eight third partials of:

1) $z = x^3 - x^2y^4$.
3) $z = e^x \cos y$.
2) $z = \sin x \cos y$.
4) $z = \ln (x^2 + y^2)$.

c. Use the result symbolized by (2) and show that

$$f_{xxy} = f_{xyx} = f_{yxx} \quad \text{and} \quad f_{xyy} = f_{yxy} = f_{yyx}.$$

123. Directional Derivatives

Select a point (x_0,y_0) within a region where a function f and its partials f_x and f_y are continuous. Also let α be an angle. Make the change of variables given by

(1) $$x = x_0 + s \cos \alpha, \quad y = y_0 + s \sin \alpha$$

thus obtaining a function g of one variable defined by

$$g(s) = f(x_0 + s \cos \alpha, \quad y_0 + s \sin \alpha).$$

Since $D_s(x_0 + s \cos \alpha) = \cos \alpha$ and $D_s(x_0 + s \sin \alpha) = \sin \alpha$, we have

$$g'(s) = f_x(x_0 + s \cos \alpha, y_0 + s \sin \alpha) \cos \alpha + f_y(x_0 + s \cos \alpha, y_0 + s \sin \alpha) \sin \alpha$$

(see (7) of Sec. 121). In particular

(2) $$g'(0) = f_x(x_0,y_0) \cos \alpha + f_y(x_0,y_0) \sin \alpha.$$

To obtain a geometric interpretation of (2):

1) Visualize the line in the xy-plane having (1) as parametric equations. This line has inclination α and passes through (x_0,y_0).
2) Erect the plane perpendicular to the xy-plane and containing this line.
3) Observe the curve in which this plane intersects the surface having equation $z = f(x,y)$.
4) Notice that (2) is the "slope" of this curve at the point $(x_0,y_0, f(x_0,y_0))$.

It is thus natural to call (2) the **directional derivative**, in the direction α, of f at (x_0,y_0).

Given a point (x,y) and an angle α, the directional derivative in the direction α of f at (x,y) is denoted by

(3) $$\mathscr{D}_\alpha f(x,y) = f_x(x,y)\cos\alpha + f_y(x,y)\sin\alpha$$

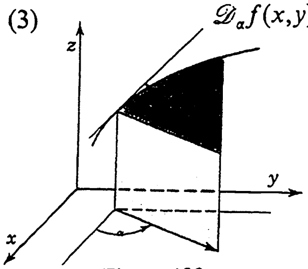


Figure 123

where a script $\mathscr{D}$, rather than D, is used.†

Example 1. Given $f(x,y) = 3x^2y + 4x$, find:

(a) $\mathscr{D}_\alpha f(-1,4)$.
(b) The directional derivative at $(-1,4)$ in the direction toward $(2,8)$.
(c) The maximum and minimum of all directional derivatives of f at $(-1,4)$.

Solution. $f_x(-1,4) = 6xy + 4]_{(-1,4)} = -20$, $f_y(-1,4) = 3x^2]_{(-1,4)} = 3$ so the answer to (a) is

(4) $$\mathscr{D}_\alpha f(-1,4) = -20\cos\alpha + 3\sin\alpha.$$

(b) The vector from $(-1,4)$ to $(2,8)$ is $(2+1)\vec{i} + (8-4)\vec{j} = 3\vec{i} + 4\vec{j}$, and since $\sqrt{3^2+4^2} = 5$ the unit vector from $(-1,4)$ toward $(2,8)$ is $\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$. Thus, the desired angle α_0 has $\cos\alpha_0 = \frac{3}{5}$ and $\sin\alpha_0 = \frac{4}{5}$ so the answer to (b) is

$$\mathscr{D}_{\alpha_0} f(-1,4) = -20(\tfrac{3}{5}) + 3(\tfrac{4}{5}) = -\tfrac{48}{5}.$$

(c) *First Method.* We set the derivative with respect to α of the right side of (4) equal to 0:

$$\frac{d}{d\alpha}(-20\cos\alpha + 3\sin\alpha) = 20\sin\alpha + 3\cos\alpha = 0 \quad \text{so} \quad \tan\alpha = -\tfrac{3}{20}.$$

We need consider only solutions for α with $-180° < \alpha \le 180°$ and, moreover, need not find α itself since from $\tan\alpha = -\frac{3}{20}$ it follows that

$$\cos\alpha = \cos\tan^{-1}\frac{-3}{20} = \frac{1}{\sec\tan^{-1}(-\frac{3}{20})} = \frac{\pm 1}{\sqrt{1+\tan^2\tan^{-1}(-\frac{3}{20})}} = \frac{\pm 1}{\sqrt{1+\frac{9}{400}}}$$

$$= \pm\frac{20}{\sqrt{409}} \quad \text{and} \quad \sin\alpha = \cos\alpha\tan\alpha = \pm\frac{20}{\sqrt{409}}\left(\frac{-3}{20}\right) = \mp\frac{3}{\sqrt{409}}.$$

Hence, upon substituting into (4) we obtain the extreme values

$$-20\left(\pm\frac{20}{\sqrt{409}}\right) + 3\left(\mp\frac{3}{\sqrt{409}}\right) = \mp\frac{400+9}{\sqrt{409}} = \mp\sqrt{409}.$$

† Other notations for directional derivatives are sometimes used; among them f_s, $\dfrac{df}{ds}$, and $D_s f$.

(c) *Second Method.* By multiplying and dividing the right side of (4) by $\sqrt{(-20)^2 + 3^2} = \sqrt{409}$ we obtain

$$\mathscr{D}_\alpha f(-1,4) = \sqrt{409}\left[\frac{-20}{\sqrt{409}} \cos\alpha + \frac{3}{\sqrt{409}} \sin\alpha\right].$$

The segment from the origin to the point $(-20,3)$ has inclination θ where $\cos\theta = -20/\sqrt{409}$ and $\sin\theta = 3/\sqrt{409}$. Thus

$$(5) \qquad \mathscr{D}_\alpha f(-1,4) = \sqrt{409}(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = \sqrt{409}\cos(\theta - \alpha).$$

For all possible values of α the maximum of (5) occurs when $\cos(\theta - \alpha) = 1$ and hence when $\alpha = \theta$ and then the maximum value is $\sqrt{409}$. Also, the minimum occurs when $\cos(\theta - \alpha) = -1$, hence when $\alpha = \theta \pm 180°$ and the minimum value is $-\sqrt{409}$.

PROBLEMS

1. Find the directional derivative of the function f defined by

   a. $f(x,y) = \tan^{-1}(y/x)$ at $(1,2)$ in the direction $\alpha = \pi/4$.

   b. $f(x,y) = \ln(x^2 + y^2)$ at $(3,4)$ in the direction $\alpha = 5\pi/6$.

   c. $f(x,y) = e^{xy}$ at $(1,2)$ in the direction from $(1,2)$ toward $(3,0)$.

   d. $f(x,y) = \sqrt{x^2 + y^2}$ at $(3,4)$ in the direction from $(3,4)$ toward $(0,0)$.

   e. $f(x,y) = xy$ at (a,b) in the direction from (a,b) toward $(0,0)$.

2. Given that the function f is defined by $f(x,y) = ye^{x/2}$ find:

   a. $\mathscr{D}_\alpha f(0,2\sqrt{3})$.

   b. The directions in which the directional derivatives of f at $(0,2\sqrt{3})$ have maximum and minimum values and find these values.

3. Let (x_0,y_0) be such that $f_x^2(x_0,y_0) + f_y^2(x_0,y_0) \neq 0$. Show that

$$\mathscr{D}_\alpha f(x_0, y_0) = \sqrt{f_x^2(x_0,y_0) + f_y^2(x_0,y_0)}\cos(\theta - \alpha)$$

   where θ is such that

$$\cos\theta = \frac{f_x(x_0,y_0)}{\sqrt{f_x^2(x_0,y_0) + f_y^2(x_0,y_0)}} \quad \text{and} \quad \sin\theta = \frac{f_y(x_0,y_0)}{\sqrt{f_x^2(x_0,y_0) + f_y^2(x_0,y_0)}}.$$

4. For each of the following indicated functions and points, find the values of α for which the directional derivative is zero, maximum, and minimum.

   a. $y^2e^{x/2}$, $(2,4)$ and $(2,-4)$.

   b. $\sqrt{x^2 + y^2}$, $(3,4)$ and $(-3,4)$.

   c. $\ln\sqrt{x^2 + y^2}$, $(3,4)$ and $(-3,4)$.

   d. $\tan^{-1}(y/x)$, $(1,-1)$ and $(-1,1)$.

124. Vectors and Directional Derivatives

In the xy-plane a line making an angle α with x-axis makes the angle $\beta = 90° - \alpha$ with the y-axis. Thus, the directional derivative in the direction α of a function f of two variables may be written as

$$\begin{aligned}\mathscr{D}_\alpha f(x,y) &= f_x(x,y)\cos\alpha + f_y(x,y)\sin(90° - \beta)\\ &= f_x(x,y)\cos\alpha + f_y(x,y)\cos\beta.\end{aligned}$$

In three dimensions a single angle does not determine a unique direction, but a set α, β, γ of direction angles does. Thus, with

(1) $$\vec{v} = \vec{i}\cos\alpha + \vec{j}\cos\beta + \vec{k}\cos\gamma$$

and F a function of three variables, the directional derivative in the direction $\vec{v}$ of F at (x,y,z) is

(2) $$\mathscr{D}_{\vec{v}}F(x,y,z) = F_x(x,y,z)\cos\alpha + F_y(x,y,z)\cos\beta + F_z(x,y,z)\cos\gamma.$$

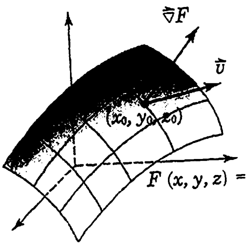


Figure 124

As defined earlier $\vec{\nabla}F = \vec{i}F_x + \vec{j}F_y + \vec{k}F_z$. The dot product of this vector with $\vec{v}$ is

(3) $$\begin{aligned}(\vec{\nabla}F)\cdot\vec{v} &= \\ (\vec{i}F_x + \vec{j}F_y + \vec{k}F_z)&\cdot(\vec{i}\cos\alpha + \vec{j}\cos\beta + \vec{k}\cos\gamma)\\ &= F_x\cos\alpha + F_y\cos\beta + F_z\cos\gamma.\end{aligned}$$

For a geometric interpretation of $\mathscr{D}_{\vec{v}}F$ consider the family of surfaces represented by

$$F(x,y,z) = c$$

with c the parameter of the family. With (x_0,y_0,z_0) in the domain of F, there is a member of the family of surfaces containing the point (x_0,y_0,z_0); in fact, that member having equation

(4) $$F(x,y,z) = c_0 \quad \text{where } c_0 = F(x_0,y_0,z_0).$$

At the point (x_0,y_0,z_0) the normal to this surface has direction numbers $F_x(x_0,y_0,z_0)$, $F_y(x_0,y_0,z_0)$, $F_z(x_0,y_0,z_0)$. Thus, the vector

$$\vec{\nabla}F(x_0,y_0,z_0) = \vec{i}F_x(x_0,y_0,z_0) + \vec{j}F_y(x_0,y_0,z_0) + \vec{k}F_z(x_0,y_0,z_0)$$

drawn with its initial end at (x_0,y_0,z_0) is normal to the graph of (4). The angle θ between this normal vector and the vector $\vec{v}$ is such that, from the definition of the dot product,

$$\begin{aligned}\cos\theta &= \frac{\vec{\nabla}F(x_0,y_0,z_0)\cdot\vec{v}}{|\vec{\nabla}F(x_0,y_0,z_0)|\,|\vec{v}|}\\ &= \frac{\mathscr{D}_{\vec{v}}F(x_0,y_0,z_0)}{\sqrt{F_x^2(x_0,y_0,z_0) + F_y^2(x_0,y_0,z_0) + F_z^2(x_0,y_0,z_0)}}\end{aligned}$$

by (3) and the fact that $|\vec{v}| = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. Thus

$$(5) \quad \mathscr{D}_{\vec{v}} F(x_0,y_0,z_0) = \sqrt{F_x^2(x_0,y_0,z_0) + F_y^2(x_0,y_0,z_0) + F_z^2(x_0,y_0,z_0)} \cos \theta.$$

If $\theta = 0$ then $\vec{v}$ has the same direction as $\vec{\nabla} F(x_0,y_0,z_0)$, but the opposite direction if $\theta = 180°$. Hence, the maximum value of the directional derivative of F at (x_0,y_0,z_0) is

$$(6) \qquad \sqrt{F_x^2(x_0,y_0,z_0) + F_y^2(x_0,y_0,z_0) + F_z^2(x_0,y_0,z_0)}$$

and occurs when the direction and sense is the same as $\vec{\nabla} F(x_0,y_0,z_0)$, whereas the minimum is the negative of (6) and occurs when the sense is reversed.

125. Tangents to Space Curves

Let F and G be functions of three variables and let (x_0,y_0,z_0) be a specific point such that both $F(x_0,y_0,z_0) = 0$ and $G(x_0,y_0,z_0) = 0$. Then the graphs of

$$(1) \qquad F(x,y,z) = 0 \quad \text{and} \quad G(x,y,z) = 0$$

are visualized as surfaces intersecting in a curve passing through the point (x_0,y_0,z_0). If at this point the surfaces have tangent planes which do not coincide, then these tangent planes intersect in a line tangent at (x_0,y_0,z_0) to the curve of intersection of the surfaces. The vectors $\vec{\nabla} F(x_0,y_0,z_0)$ and $\vec{\nabla} G(x_0,y_0,z_0)$, with initial ends (x_0,y_0,z_0), are normal to the respective surfaces. The cross product

$$(2) \qquad \vec{\nabla} F(x_0,y_0,z_0) \times \vec{\nabla} G(x_0,y_0,z_0)$$

is a vector perpendicular to both normals, hence lies in both tangent planes and, therefore, lies along the tangent line to the curve of intersection of the surfaces. We shall obtain equations for this tangent line.

For economy in writing, we omit (x_0,y_0,z_0) but insert a subscript zero as a reminder that functions and their partials are evaluated at (x_0,y_0,z_0). Thus (2) becomes

$$\begin{aligned}
\vec{\nabla} F_0 \times \vec{\nabla} G_0 &= (\vec{i} F_x + \vec{j} F_y + \vec{k} F_z)_0 \times (\vec{i} G_x + \vec{j} G_y + \vec{k} G_z)_0 \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix}_0 \\
(3) \qquad &= \vec{i} \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}_0 + \vec{j} \begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}_0 + \vec{k} \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}_0 .
\end{aligned}$$

The coefficients of $\vec{i}, \vec{j}$, and $\vec{k}$ are thus direction numbers of the desired tangent line and parametric equations of this line are

$$x = x_0 + t\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}_0, \quad y = y_0 + t\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}_0, \quad z = z_0 + t\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}_0.$$

Example. Check that the point $(\frac{1}{2}, -\sqrt{3}/2, 1)$ is on both the sphere and the cone having equations

$$(x-2)^2 + y^2 + z^2 = 4 \quad \text{and} \quad x^2 + y^2 = z^2.$$

At this point find parametric equations of the line tangent to the curve of intersection of this sphere and cone.

Solution. Set $F(x,y,z) = (x-2)^2 + y^2 + z^2 - 4$ and $G(x,y,z) = x^2 + y^2 - z^2$. Thus

$$F_x\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1\right) = 2\left(\frac{1}{2} - 2\right) = -3, \quad F_y\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1\right) = -\sqrt{3},$$

$$F_z\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1\right) = 2;$$

$$G_x\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1\right) = 1, \quad G_y\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1\right) = -\sqrt{3}, \quad G_z\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 1\right) = -2.$$

Hence, parametric equations of the tangent line are

$$x = \tfrac{1}{2} + t\begin{vmatrix} -\sqrt{3} & 2 \\ -\sqrt{3} & -2 \end{vmatrix} = \tfrac{1}{2} + 4\sqrt{3}t, \quad y = -\frac{\sqrt{3}}{2} - 4t, \quad z = 1 + 4\sqrt{3}t.$$

Determinants of the type appearing in (3) are of such frequent occurrence in more advanced mathematics and applications that a special notation is introduced. For example

$$\frac{\partial(F,G)}{\partial(y,z)} \quad \text{is used in place of} \quad \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}$$

and is called the **Jacobian** of F,G with respect to y,z.

PROBLEMS

1. Find the directional derivative of F at the given point and in the direction either given by a vector or described.

a. $F(x,y,z) = xy + y^2z + xyz^2$, $(1,5,-2)$, $2\vec{i} - 3\vec{j} + \vec{k}$.

b. $F(x,y,z) = x^2 - y^2 + xz$, $(0,1,-1)$, $\vec{i} - 2\vec{j} + 2\vec{k}$.

c. $F(x,y,z) = x\sqrt{5y^2 - 4z^3}$, $(0,1,-1)$, $\vec{i}$.

d. $F(x,y,z) = xyz$, $(-2,1,3)$, from this point toward $(2,1,0)$.

e. $F(x,y,z) = x + y + z$, $(-2,1,3)$, from this point toward the origin.

2. Find the maximum directional derivative of F at the given point.

   a. $F(x,y,z) = \ln \sqrt{x^2 + y^2 + z^2}$, $(2,-2,1)$.

   b. $F(x,y,z) = \sin(xyz)$, $\left(2,-1,\frac{\pi}{6}\right)$.

   c. $F(x,y,z) = \tan^{-1}\left(\frac{xy}{z}\right)$, $(1,-2,2)$.

3. Show that the maximum directional derivative of the function F defined by $F(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ is the same at all points except the origin (where the directional derivative is undefined).

4. Find parametric equations of the tangent line to the curve of intersection of the graphs of the given equations at the given point.

   a. $x^2 + y^2 = z^2$, $x + y + z = 12$; $(3,4,5)$.

   b. $x^2 + y^2 + z^2 = 4x$, $x^2 + y^2 = z^2$; $(1,-1,\sqrt{2})$.

   c. $x^2 + y^2 = 8z^2$, $x^2 + y^2 + z^2 = 9$; $(-2,2,1)$.

5. Check that the curve in the example also passes through the point $(\frac{1}{2},\sqrt{3}/2,1)$. Show that the tangent to the curve at this point intersects the tangent obtained in the example by finding the point of intersection.

6. Given $x = \rho\cos\theta$, $y = \rho\sin\theta$ show that $\dfrac{\partial(x,y)}{\partial(\rho,\theta)} = \rho$.

7. Given $z = f(x,y)$ and $w = g(x,y)$ while $x = \varphi(u,v)$ and $y = \psi(u,v)$ show that

$$\frac{\partial(z,w)}{\partial(u,v)} = \frac{\partial(z,w)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}.$$

126. Line Integrals

Let P be the function of two variables defined by $P(x,y) = x\sin y$ and let C be the graph of $y = 2x^2$ for $0 \le x \le 1$. Then C joins the points $(0,0)$ and $(1,2)$ and

$$\int_0^1 P(x,2x^2)\,dx = \int_0^1 x\sin 2x^2\,dx = -\tfrac{1}{4}\cos 2x^2\Big]_0^1 = \tfrac{1}{4}(1-\cos 2).$$

Given two points (a,c) and (b,d), given a curve C joining these points and having equation $y = f(x)$, $a \le x \le b$, with f' continuous, and given a continuous function P of two variables, then

$$\int_a^b P(x,f(x))\,dx$$

is called the **line** (or **curvilinear**) **integral** of $P(x,y)\,dx$ over C from (a,c) to (b,d) and is represented by

$$\int_{C\,(a,c)}^{(b,d)} P(x,y)\,dx.$$

If C is the graph of $x = g(y)$, $c \le y \le d$, and Q is a continuous function of two variables, then

$$\int_c^d Q(g(y),y)\,dy = \int_{C\,(a,c)}^{(b,d)} Q(x,y)\,dy$$

is the line integral of $Q(x,y)\,dy$ over C from (a,c) to (b,d). Also, the line integral of $P(x,y)\,dx + Q(x,y)\,dy$ over C from (a,c) to (b,d) is defined by

$$\int_{C\,(a,c)}^{(b,d)} \{P(x,y)\,dx + Q(x,y)\,dy\} = \int_{C\,(a,c)}^{(b,d)} P(x,y)\,dx + \int_{C\,(a,c)}^{(b,d)} Q(x,y)\,dy.$$

If C has parametric equations $x = x(t)$, $y = y(t)$ with $t_0 \le t \le t_1$ where x' and y' are continuous, then

$$\int_{C\,(a,c)}^{(b,d)} P(x,y)\,dx = \int_{t_0}^{t_1} P(x(t),y(t))x'(t)\,dt$$

with a similar expression for the line integral of $Q(x,y)\,dy$.

Example 1. With C the portion of the graph of $8y = x^2$ joining the points (0,0) and (4,2), find

$$I = \int_C \{(x + y)\,dx + x^2 \cos y\,dy\}.$$

Solution. In the first term replace y by $x^2/8$ and use the x-limits 0 and 4, but in the second term replace x^2 by $8y$ and use y-limits to obtain

$$\begin{aligned} I &= \int_0^4 \left(x + \frac{x^2}{8}\right) dx + \int_0^2 8y \cos y\,dy \\ &= \left[\frac{x^2}{2} + \frac{x^3}{24}\right]_0^4 + 8\left[y \sin y + \cos y\right]_0^2 \\ &= 8 + \tfrac{8}{3} + 8[2 \sin 2 + \cos 2 - 1] = \tfrac{8}{3} + 8(2 \sin 2 + \cos 2). \end{aligned}$$

Example 2. Replace the curve of Example 1 by the graph of the parametric equations $x = 4t$, $y = 2t$ for $0 \le t \le 1$, joining the same points (0,0) and (4,2).

Solution. The integral is now

$$\int_0^1 \{(4t + 2t)\, d(4t) + (4t)^2 \cos 2t\, d(2t)\}$$
$$= \int_0^1 24t\, dt + \int_0^1 32t^2 \cos 2t\, dt$$
$$= 12t^2\Big]_0^1 + 32\left[\frac{t^2}{2}\sin 2t + \frac{t}{2}\cos 2t - \frac{\sin 2t}{4}\right]_0^1$$
$$= 12 + 32\left[\frac{1}{2}\sin 2 + \frac{1}{2}\cos 2 - \frac{1}{4}\sin 2\right] = 12 + 8[\sin 2 + 2\cos 2].$$

If the initial and terminal ends of C have been definitely stated, it is customary to omit (a,c) and (b,d) from the integral sign, and to further abbreviate

$$\int_{C\ (a,c)}^{(b,d)} \{P(x,y)\, dx + Q(x,y)\, dy\} \quad \text{to} \quad \int_C P\, dx + Q\, dy.$$

By a **regular arc** is meant a graph of $y = f(x)$ with f' continuous or $x = g(y)$ with g' continuous, or $x = x(t)$, $y = y(t)$ with x' and y' both continuous. By a **regular curve** C is meant that C is made up of regular arcs $C_1, C_2, \cdots, C_n$ joined successively end to end (see Fig. 126) and, by definition

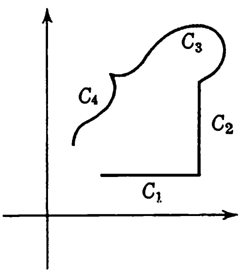


Figure 126

$$\int_C P\, dx + Q\, dy = \sum_{k=1}^{n} \int_{C_k} P\, dx + Q\, dy.$$

Example 3. Replace the curve of Example 1 by the regular curve C consisting of the segment C_1 from (0,0) to (4,0) followed by the segment C_2 from (4,0) to (4,2).

Solution. Since $y = 0$ on C_1 while $x = 4$ on C_2,

$$\int_{C_1} \{(x + y)\, dx + x^2 \cos y\, dy\} = \int_0^4 \{(x + 0)\, dx + x^2 \cos 0\, d0\} = \int_0^4 x\, dx = 8,$$
$$\int_{C_2} \{(x + y)\, dx + x^2 \cos y\, dy\} = \int_0^2 \{(4 + y)\, d4 + 4^2 \cos y\, dy\}$$
$$= \int_0^2 16 \cos y\, dy = 16 \sin 2,$$
$$\int_C \{(x + y)\, dx + x^2 \cos y\, dy\} = 8 + 16 \sin 2.$$

If a curve C is cut in at most one point by each vertical line, then integration along C from left to right is indicated by $\overrightarrow{\int}_C$ whereas the reverse direction of integration is indicated by $\overleftarrow{\int}_C$. If R is a region bounded by a regular closed curve C, then integration over C keeping R to the left is indicated by $\ointctrclockwise$ and is called **integration around** R in the **positive** direction. Integration around R in the reverse (or negative) direction is denoted by $\ointclockwise$. Schematically

$$\overleftarrow{\int} = -\overrightarrow{\int} \quad \text{and} \quad \ointctrclockwise = -\ointclockwise.$$

Example 4. For C the boundary of $R = \{(x,y) \mid 0 \le x \le 2,\ x^2/4 \le y \le x/2\}$, find

(1) $$\ointctrclockwise y\,dx + dy \quad \text{and} \quad \ointctrclockwise y\,dx + x\,dy.$$

Solution. C consists of two regular arcs joining (0,0) and (2,1) where the

lower arc C_1 has equation $y = x^2/4$, but the
upper arc C_2 has equation $y = x/2$.

Hence, for the first integral in (1)

$$\int_{C_1} (y\,dx + dy) = \int_0^2 \frac{x^2}{4}\,dx + \int_0^1 dy = \frac{x^3}{12}\Big]_0^2 + y\Big]_0^1 = \frac{5}{3},$$

$$\int_{C_2} (y\,dx + dy) = \int_2^0 \frac{x}{2}\,dx + \int_1^0 dy = \frac{x^2}{4}\Big]_2^0 + y\Big]_1^0 = -2, \quad \text{and}$$

$$\ointctrclockwise (y\,dx + dy) = \tfrac{5}{3} - 2 = -\tfrac{1}{3}.$$

For the second integral in (1)

$$\int_{C_1} (y\,dx + x\,dy) = \int_0^1 \frac{x^2}{4}\,dx + \int_0^1 \sqrt{4y}\,dy = \frac{x^3}{12}\Big]_0^2 + \frac{4}{3}y^{3/2}\Big]_0^1 = 2,$$

$$\int_{C_2} (y\,dx + x\,dy) = \int_2^0 \frac{x}{2}\,dx + \int_1^0 2y\,dy = \frac{x^2}{4}\Big]_2^0 + y^2\Big]_1^0 = -2, \quad \text{and}$$

$$\ointctrclockwise (y\,dx + x\,dy) = 2 - 2 = 0.$$

Let C be a regular curve with initial and terminal points (a,c) and (b,d). In order along C select points

(2) $$(x_0,y_0),\ (x_1,y_1),\ \cdots,\ (x_{n-1},\ y_{n-1}),\ (x_n,y_n)$$

where $x_0 = a$, $x_n = b$; $y_0 = c$, $y_n = d$. With P and Q continuous functions of two variables form the sum

(3) $$\sum_{k=1}^{n} \{P(x_k,y_k)(x_k - x_{k-1}) + Q(x_k,y_k)(y_k - y_{k-1})\}.$$

As $n \to \infty$ in such a way that $x_k - x_{k-1} \to 0$ and $y_k - y_{k-1} \to 0$, this sum approaches

(4) $$\int_C P\,dx + Q\,dy,$$

but no attempt is made at a proof here.

In case P and Q are components of a force function $\vec{F}$ defined by

$$\vec{F}(x,y) = \vec{i}P(x,y) + \vec{j}Q(x,y)$$

for each point (x,y) of a region including C, then the work W in moving an object over C is defined by

$$W = \int_C P\,dx + Q\,dy.$$

To see why such a definition is made consider the force

$$\vec{F}(x_k,y_k) = \vec{i}P(x_k,y_k) + \vec{j}Q(x_k,y_k)$$

at the kth point of (2). The vector from (x_{k-1},y_{k-1}) to (x_k,y_k) is

$$\vec{i}(x_k - x_{k-1}) + \vec{j}(y_k - y_{k-1}),$$

and an approximation of the work in moving an object over this chord (and thus over the corresponding arc of C) should be†

$$\begin{aligned}[\vec{i}P(x_k,y_k) + \vec{j}Q(x_k,y_k)] \cdot [\vec{i}(x_k - x_{k-1}) + \vec{j}(y_k - y_{k-1})]\\ = P(x_k,y_k)(x_k - x_{k-1}) + Q(x_k,y_k)(y_k - y_{k-1})\end{aligned}$$

which is the kth term in the sum (3) whose limit is (4).

It is, moreover, customary to let $\vec{r} = \vec{i}x + \vec{j}y$ be the vector from the origin to an arbitrary point (x,y) of C, to set $d\vec{r} = \vec{i}\,dx + \vec{j}\,dy$ so that

$$\begin{aligned}\vec{F} \cdot d\vec{r} &= (\vec{i}P + \vec{j}Q) \cdot (\vec{i}\,dx + \vec{j}\,dy)\\ &= P\,dx + Q\,dy,\end{aligned}$$

and then to say: *If a body is moved over a curve C defined by a vector function $\vec{r}$, where C lies in a force field $\vec{F}$, then the work is*

$$W = \int_C \vec{F} \cdot d\vec{r}.$$

PROBLEMS

1. Find $\int_C (x + y)\,dx - \sqrt{x}\,dy$ where C is the graph of:

   a. $y = 2x;\quad 0 \le x \le 1.$

   b. $x = t,\quad y = 2t;\quad 0 \le t \le 1.$

   c. $x = t^2,\quad y = 2t^2;\quad 0 \le t \le 1.$

   d. $y = 2x^2;\quad 0 \le x \le 1.$

   e. $x = \sin t,\quad y = 1 - \cos 2t;\quad 0 \le t \le \pi/2.$

† Recall that if a constant force $\vec{f}$ moves a body from the initial to the terminal end of a vector $\vec{u}$ then the work is the scalar product $\vec{f} \cdot \vec{u}$.

2. Find $\int_C (x^2 + 2xy)\,dx + (x^2 - y)\,dy$ where C has initial point $(-1,0)$, terminal point $(1,2)$ and is a portion of the graph of:

a. $y = x + 1$.
c. $x = -1$, then $y = 2$.
b. $2y = (x + 1)^2$.
d. $x = -1 + t$, $y = t$.
e. $x = 1 + 2\cos t$, $y = 2\sin t$; $-\pi \le t \le \pi/2$.

3. Let C consist of the line segments from $(0,0)$ to $(2,1)$ and then from $(2,1)$ to $(2,3)$. Find $\int_C P\,dx + Q\,dy$ if:

a. $P(x,y) = x\sqrt{y}$, $Q(x,y) = x^3 + y$.
b. $P(x,y) = x^3 + y$, $Q(x,y) = x\sqrt{y}$.
c. $P(x,y) = x\sin(\pi y/2)$, $Q(x,y) = ye^x$.
d. $P(x,y) = ye^x$, $Q(x,y) = x\sin(\pi y/2)$.

4. Find $\oint_C y^2\,dx + x\,dy$ where C is:

a. The square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$.
b. The square with vertices $(\pm 1, \pm 1)$.
c. The circle with center $(0,0)$ and radius 2.
d. The graph of $x = 2\cos^3 t$, $y = 2\sin^3 t$; $0 \le t \le 2\pi$.

127. Green's Theorem

The relation between a line integral and a double integral, as expressed below in (1), has important physical applications. The following theorem would be Green's theorem if the curve C were as general as we later show is possible.

THEOREM 126.1. *Let R be a region bounded by a regular curve C which has the property that each horizontal and each vertical line cuts C in at most two points. Then*

$$(1) \quad \oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \{Q_x(x,y) - P_y(x,y)\}\,dR$$

provided P_y and Q_x are continuous at each point of R and C.

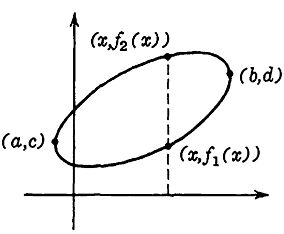


Figure 127.1

PROOF. Let (a,c) be the left-most and (b,d) the right-most point of C. Let f_1 and f_2 be functions such that

$$R = \{(x,y) \mid a \le x \le b,\ f_1(x) \le y \le f_2(x)\}.$$

Designate the graph of $y = f_1(x)$, $a \le x \le b$ by C_1 and the graph of $y = f_2(x)$, $a \le x \le b$ by C_2. By the equality of double and twofold iterated integrals

$$\begin{aligned}
\iint_R P_y(x,y)\,dR &= \int_a^b \int_{f_1(x)}^{f_2(x)} P_y(x,y)\,dy\,dx \\
&= \int_a^b P(x,y)\Big]_{y=f_1(x)}^{y=f_2(x)} dx \qquad \text{[by the Fundamental Theorem of Calculus]} \\
&= \int_a^b \{P(x,f_2(x)) - P(x,f_1(x))\}\,dx \\
&= \int_{C_2} P(x,y)\,dx - \int_{C_1} P(x,y)\,dx \qquad \text{[by the definition of a line integral]} \\
&= -\int_{C_1} P(x,y)\,dx - \int_{C_2} P(x,y)\,dx \\
&\qquad\qquad \left[\text{since } \int_{C_2} = -\int_{C_2}\right] \\
&= -\oint_C P(x,y)\,dx, \quad \text{so that}
\end{aligned}$$

(2) $$\oint_C P(x,y)\,dx = -\iint_R P_y(x,y)\,dR.$$

By starting with the double integral over R of Q_x show that

(3) $$\oint_C Q(x,y)\,dy = \iint_R Q_x(x,y)\,dR.$$

The equations (2) and (3) then yield (1) and Theorem 126.1 is proved.

A slight extension would show that (1) also holds if portions of the regular curve C are horizontal or vertical line segments.

If R is the region shown in Fig. 127.2, then insert the "cross cut" C_3. Let R_1 be the portion of R below C_3 and R_2 the portion of R above C_3. Then by applying Theorem 126.1 to R_1 and R_2 separately:

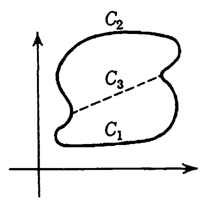


Figure 127.2

$$\begin{aligned}
\iint_R (Q_x - P_y)\,dR &= \iint_{R_1} (Q_x - P_y)\,dR + \iint_{R_2} (Q_x - P_y)\,dR \\
&= \left\{\int_{C_1} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy\right\} \\
&\qquad + \left\{\int_{C_3} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy\right\} \\
&= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy \\
&\qquad\qquad \left[\text{since } \int_{C_3} + \int_{C_3} = 0\right] \\
&= \oint_C P\,dx + Q\,dy.
\end{aligned}$$

By this crosscut method, (1) holds for any region (such as in Fig. 127.3) which may be divided into a finite number of subregions by regular arcs (some of which may be line segments) such that (1) holds for each of these subregions.

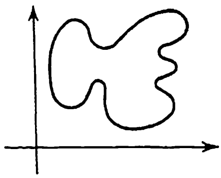


Figure 127.3

Green's theorem simplifies evaluation of some line integrals.

Example. Find $\oint_C (x^2 + y^2)\, dx + \sin y\, dy$ where C is the boundary of the region $\{(x,y) \mid 0 \le x \le 1, x^2 \le y \le x\}$.

Solution. Here $P(x,y) = x^2 + y^2$, $Q(x,y) = \sin y$, $P_y(x,y) = 2y$, $Q_x(x,y) = 0$ and by Green's theorem

$$\oint_C (x^2 + y^2)\, dx + \sin y\, dy = \int_0^1 \int_{x^2}^{x} (0 - 2y)\, dy\, dx = -\int_0^1 y^2 \Big]_{x^2}^{x} dx = \text{etc.} = -\frac{2}{15}.$$

A region is said to be **simply connected** if every simple closed curve in the region surrounds only points of the region. For example, the region between two concentric circles is not simply connected.

THEOREM 127.2. *Let G be a simply connected region within which P_y and Q_x are continuous and such that at each (x,y) in G*

$$P_y(x,y) = Q_x(x,y). \tag{4}$$

I. *If C is any regular simple closed curve in G, then*

$$\int_C P\, dx + Q\, dy = 0 \textit{ for either order along } C. \tag{5}$$

II. *If (a,b) and (x,y) are any two points in G and if C_1 and C_2 are any two regular curves in G joining (a,b) to (x,y), then*

$$\int_{C_1} P\, dx + Q\, dy = \int_{C_2} P\, dx + Q\, dy. \tag{6}$$

III. *The expression $P\, dx + Q\, dy$ is an exact differential in G; that is, there is a function F such that for each point (x,y) in G*

$$dF(x,y) = P(x,y)\, dx + Q(x,y)\, dy.$$

Note 1: The property stated in II is sometimes described as "*Curvilinear integration of $P\, dx + Q\, dy$ between two points is independent of the path in a simply connected region throughout which $P_y = Q_x$.*" More advanced work in mathematics and physics is necessary to appreciate the importance of this fact.

Note 2: Theorem 122.2 is in essence: *If $M\, dx + N\, dy$ is an exact differential, then $M_y = N_x$.* In Sec. 122 the converse (*If $M_y = N_x$, then $M\, dx + N\, dy$* is an exact differential) was stated and a proof promised for later. Part III is this converse with, however, M replaced by P and N by Q to conform with the notation of the present section.

PROOF of I. Let C be any simple closed curve in G and let R be the region C surrounds. Hence, (1) holds for this C and R. In addition, from (4), $Q_x(x,y) - P_y(x,y) = 0$ for (x,y) in R. Thus, for this R the double integral in (1) has value 0 and hence so does the line integral; that is, (5) holds.

PROOF of II. With (a,b), (x,y), C_1 and C_2 as given, assume first that C_1 lies entirely below C_2 except for the points (a,b) and (x,y). Let C be the curve from (a,b) to (x,y) along C_1 and then back to (a,b) along C_2. Thus, from Part I

$$0 = \oint_C P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$$
$$= \int_{C_1} P\,dx + Q\,dy - \int_{C_2} P\,dx + Q\,dy$$

and hence (6) holds. If C_1 and C_2 cross each other, take a third curve C_3 connecting (a,b) and (x,y) completely above both C_1 and C_2 except for (a,b) and (x,y). Then by what has just been proved

$$\int_{C_3} P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy \quad \text{and}$$
$$\int_{C_3} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy$$

so again (6) holds.

PROOF of III. A translation of Part II is: The curvilinear integral of $P\,dx + Q\,dy$ from (a,b) to (x,y) is, under hypothesis (4), independent of the (regular) curve joining these points and hence depends only upon the points themselves. We may thus consider (a,b) as fixed and define a function F by setting, for each (x,y) in G,

$$F(x,y) = \int_{(a,b)}^{(x,y)} P\,dx + Q\,dy$$

wherein this new notation for the curvilinear integral (which does not specify the path but only the end points) may be used only when (4) holds, as it does here. Let Δx be such that the line segment joining (x,y) to $(x + \Delta x,y)$ is also in G. Then

$$F(x + \Delta x,y) = \int_{(a,b)}^{(x+\Delta x,y)} P\,dx + Q\,dy.$$

Now consider a path from (a,b) to $(x + \Delta x,y)$ which goes from (a,b) to (x,y) and then from (x,y) along the line segment to $(x + \Delta x,y)$ so that

$$F(x + \Delta x,y) = \int_{(a,b)}^{(x,y)} P\,dx + Q\,dy + \int_{(x,y)}^{(x+\Delta x,y)} P\,dx + Q\,dy$$
$$= F(x,y) + \int_{(x,y)}^{(x+\Delta x,y)} P\,dx + Q\,dy.$$

Since the segment from (x,y) to $(x + \Delta x,y)$ is parallel to the x-axis, we reduce this integral (by considering $dy = 0$) to

$$\int_x^{x+\Delta x} P\,dx.$$

To this integral we apply the Law of the Mean for integrals (p. 186) and determine a number ξ between x and $x + \Delta x$ for which

$$\int_x^{x+\Delta x} P\,dx = P(\xi,y)\,\Delta x.$$

It therefore follows that

$$F(x + \Delta x,y) - F(x,y) = P(\xi,y)\ \Delta x \quad \text{or} \quad \frac{F(x + \Delta x,y) - F(x,y)}{\Delta x} = P(\xi,y).$$

Since P is continuous at (x,y) we now let $\Delta x \to 0$ and obtain

$$F_x(x,y) = P(x,y).$$

In a similar way it follows that $F_y(x,y) = Q(x,y)$. Hence

$$dF(x,y) = F_x(x,y)\,dx + F_y(x,y)\,dy = P(x,y)\,dx + Q(x,y)\,dy$$

which shows, as we wished to prove, that if (4) holds in G then $P\,dx + Q\,dy$ is an exact differential in G.

The converse of Theorem 127.2II is:

THEOREM 127.3. *Let G be a simply connected region in which P_y and Q_x are continuous. If for every regular simple closed curve C in G*

$$(7) \qquad \int_C P\,dx + Q\,dy = 0,$$

then $P_y(x,y) = Q_x(x,y)$ for each point (x,y) in G.

PROOF. Assume (x_0,y_0) is inside G and $P_y(x_0,y_0) \neq Q_x(x_0,y_0)$. From the continuity of P_y and Q_x, there is a circular region R in G with center (x_0,y_0) such that at each point (x,y) of R, $Q_x(x,y) - P_y(x,y)$ is different from zero and has the same sign as $Q_x(x_0,y_0) - P_y(x_0,y_0)$. The boundary C of R is certainly regular, and from (1)

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dR \neq 0$$

contrary to the fact that (7) holds for every regular curve in G.

PROBLEMS

1. Work Prob. 4, Sec. 126 (p. 410) by using Green's theorem.

2. For C a regular simple closed curve, check that

   a. $\int_C (xy\cos x + \sin y)\,dx + (x\cos y + x\sin x + \cos x)\,dy = 0.$

   b. $\int_C (2xy^3 + 5)\,dx + (3x^2y^2 - 4)\,dy = 0.$

   c. $\int_C \frac{2x}{\sqrt{1+y^2}}\,dx - \frac{x^2y}{(1+y^2)^{3/2}}\,dy = 0.$

   d. $\int_C e^{xy}(1 + xy)\,dx + x^2e^{xy}\,dy = 0.$

   e. $\int_C F_x(x,y)\,dx + F_y(x,y)\,dy = 0$ if the second partials of F are continuous at each point within and on C.

3. Let $P(x,y) = x$ and $Q(x,y) = xy$. Show that

$$\int_C P\,dx + Q\,dy = 0$$

for every circle with center at the origin. Notice that $Q_x(x,y) \neq P_y(x,y)$. Does this contradict Theorem 127.3?

4. Let F be a function of two variables such that F_{xx}, F_{xy}, and F_{yy} are everywhere continuous. Let (a,c) and (b,d) be any two points. Prove that if C is any regular curve from (a,c) to (b,d), then

$$\int_C dF(x,y) = F(b,d) - F(a,c).$$

(Hint: Show that the equation holds for the curve C_1 consisting of the line segment from (a,c) to (b,c) followed by the segment from (b,c) to (b,d). Then use Theorem 127.2 II.)

CHAPTER 12

Approximations

Applications of mathematics usually involve the occupational hazard of obtaining numerical results. At each stage of a computation there is a positive probability of a mistake whose effect may be large, small, or anything in between with no a priori method of control. In contrast to mistakes, which are blunders occurring without intent, most computations involve errors made intentionally for practical or expedient reasons, but kept within a preassigned range of uncertainty suitable to the problem at hand. In this chapter a few approximation methods are given *with error estimates*. The approximation methods mentioned previously in this book (for example, Newton's Method) were given as background material and have been accompanied with apologies for not including criteria for judging accuracy.

Approximation theory (which develops methods for computing to within any desired tolerance of an exact, but usually unattainable, ideal) has long been a respectable branch of analysis and now, with the advent of high-speed electronic computers, is indispensable. In fact, the indefinite phrase "Close enough for all practical purposes" is a cliché of pious hope seldom heard around a modern computing center where "how close?" is an ever-present question which should be answered before expensive time is scheduled on a machine so reliable that mistakes occur (if at all) less frequently than once in a human lifetime of computing.

128. Taylor's Theorem

After reading Taylor's theorem, it should be seen that the Law of the Mean (Sec. 32) is the special case of Taylor's theorem in which $n = 1$.

THEOREM 128 (Taylor's Theorem). *Let a and b be numbers, $a \neq b$, let n be a positive integer, and let f be a function whose nth derivative $f^{(n)}(x)$ exists for each number x between a and b inclusive. Then there is a number ξ_n between a and b such that*

$$(1)\quad f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \frac{f'''(a)}{3!}(b-a)^3 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + \frac{f^{(n)}(\xi_n)}{n!}(b-a)^n.$$

The following proof of Taylor's theorem for $n = 4$ illustrates a procedure by which the theorem may be proved for any positive integer n.

PROOF for $a < b$ and $n = 4$. Let S_4 be the number such that

$$(2) \quad f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \frac{f'''(a)}{3!}(b - a)^3 + S_4 \cdot (b - a)^4.$$

Let φ be the function defined for each number x such that $a \leq x \leq b$ by

$$(3) \quad \varphi(x) = -f(b) + f(x) + f'(x)(b - x) + \frac{f''(x)}{2!}(b - x)^2 + \frac{f'''(x)}{3!}(b - x)^3 + S_4 \cdot (b - x)^4.$$

Notice that $\varphi'(x)$ exists for $a \leq x \leq b$ since each term on the right of (3) has a derivative. Also, by taking the derivative, then

$$\varphi'(x) = 0 + f'(x) + \{-f'(x) + f''(x)(b - x)\} + \left\{-\frac{f''(x)}{2!}2(b - x) + \frac{f'''(x)}{2!}(b - x)^2\right\}$$
$$+ \left\{-\frac{f'''(x)}{3!}3(b - x)^2 + \frac{f^{(4)}(x)}{3!}(b - x)^3\right\} - 4S_4 \cdot (b - x)^3 \quad \text{so that}$$

$$(4) \quad \varphi'(x) = \frac{f^{(4)}(x)}{3!}(b - x)^3 - 4S_4 \cdot (b - x)^3.$$

In (3) substitute $x = a$:

$$\varphi(a) = -f(b) + f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \frac{f'''(a)}{3!}(b - a)^3 + S_4 \cdot (b - a)^4 = 0$$

where the $= 0$ follows from (2). Also, in (3) substitute $x = b$ and see that

$$\varphi(b) = -f(b) + f(b) + f'(b)(b - b) + \frac{f''(b)}{2!}(b - b)^2 + \frac{f'''(b)}{3!}(b - b)^3 + S_4 \cdot (b - b)^4 = 0.$$

Hence, $\varphi(a) = 0$, $\varphi(b) = 0$, and $\varphi'(x)$ exists for $a < x < b$ and thus by Rolle's theorem (see Sec. 32) there is a number ξ_4 such that $a < \xi_4 < b$ and $\varphi'(\xi_4) = 0$. Hence, from (4)

$$0 = \varphi'(\xi_4) = \frac{f^{(4)}(\xi_4)}{3!}(b - \xi_4)^3 - 4 \cdot S_4 \cdot (b - \xi_4)^3.$$

Consequently, upon solving this equation for S_4:

$$S_4 = \frac{f^{(4)}(\xi_4)}{4!}.$$

This expression for S_4 substituted into (2) yields (1) with $n = 4$. A similar proof may be made if $b < a$.

Example. Substitute $f = \sin$, $a = \pi/3$, $b = 13\pi/36$, and $n = 4$ into (1).

Solution. First $f(a) = \sin \pi/3 = \sqrt{3}/2$ and $f(b) = \sin(13\pi/36)$. Also

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad \text{and} \quad f^{(4)}(x) = \sin x.$$

Hence $f'(a) = \frac{1}{2}, f''(a) = -\sqrt{3}/2, f'''(a) = -\frac{1}{2}, f^{(4)}(\xi_4) = \sin \xi_4$, and

$$\sin \frac{13\pi}{36} = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{13\pi}{36} - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!}\left(\frac{13\pi}{36} - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!}\left(\frac{13\pi}{36} - \frac{\pi}{3}\right)^3 + \frac{\sin \xi_4}{4!}\left(\frac{13\pi}{36} - \frac{\pi}{3}\right)^4,$$

$$(5)\quad \sin 65° = \frac{\sqrt{3}}{2} + \frac{1}{2}\frac{\pi}{36} - \frac{\sqrt{3}}{2 \cdot 2!}\left(\frac{\pi}{36}\right)^2 - \frac{1}{2 \cdot 3!}\left(\frac{\pi}{36}\right)^3 + \frac{\sin \xi_4}{4!}\left(\frac{\pi}{36}\right)^4, \quad \frac{\pi}{3} < \xi_4 < \frac{13\pi}{36}.$$

On the right of (5), the first four terms may be approximated to the number of decimal places known for $\sqrt{3}$ and π, but the fifth term is unknown since ξ_4 is only limited by inequalities. However, as a generous estimate, $0 < \sin \xi_4 < 1$ and thus the last term is positive and certainly less than

$$\frac{1}{4!}\left(\frac{\pi}{36}\right)^4 = \frac{1}{24}\left(\frac{3.14159\cdots}{36}\right)^4 < \frac{1}{24}\left(\frac{3.6}{36}\right)^4$$
$$= \tfrac{1}{24}(0.1)^4 = \tfrac{1}{24}(0.000\,1) < 0.000\,005.$$

Thus, upon computing each of the first four terms to 6 decimal places, their sum will agree with sin 65° to at least 5 decimal places.

One use of Taylor's theorem is to set up such finite arithmetic sums for computations with error estimates.

PROBLEMS

Write the formula of Taylor's theorem for:

1. $f(x) = e^x; \quad a = 0, \quad b = 2, \quad n = 5.$

2. $f(x) = \cos x; \quad a = \frac{\pi}{4}, \quad b = \frac{\pi}{2}, \quad n = 4.$

3. $f(x) = \ln x; \quad a = 1, \quad b = 1.2, \quad n = 4.$

4. $f(x) = \sqrt{x}; \quad a = 1, \quad b = 1.2, \quad n = 4.$

5. $f(x) = \tan^{-1} x; \quad a = 0, \quad b = 1, \quad n = 3.$

6. $f(x) = x^{-1/3}; \quad a = 8, \quad b = 8.5, \quad n = 3.$

7. $f(x) = \dfrac{1}{1 + x}; \quad a = 0, \quad b = \frac{1}{2}, \quad n = 4.$

8. $f(x) = e^{\sin x}; \quad a = \frac{\pi}{2}, \quad b = \frac{3\pi}{4}, \quad n = 3.$

9. $f(x) = x^4; \quad a = 2, \quad b = 2.01, \quad n = 5.$

10. $f(x) = \tan x; \quad a = \frac{\pi}{4}, \quad b = 1, \quad n = 3.$

129. The Remainder Term

The expression (1) of Sec. 128 is referred to as "**Taylor's formula** for the expansion of $f(b)$ around the point a." Because of the connection with series established later (and to emphasize the difference in the nature of the last term and those preceding it), this formula is written as

$$(1) \quad f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + R_n \quad \text{where the remainder}$$

$$(2) \quad R_n = \frac{f^{(n)}(\xi_n)}{n!}(b-a)^n \quad \text{for some number } \xi_n \text{ between } a \text{ and } b.$$

In (1), $f^{(k)}(a)$ is the kth derivative of f evaluated at a, and the zero-derivative $f^0(a)$ is interpreted as $f(a)$.

Example. Find how many terms of Taylor's formula with $f(x) = \ln x$, $a = 1$, and $b = 1.2$ are necessary to compute $\ln(1.2)$ with error less than 5×10^{-4}.

Solution. First $f'(x) = x^{-1}, f''(x) = -x^{-2}, f'''(x) = 2x^{-3}, f^{(4)}(x) = -3!x^{-4}$, etc. Therefore, by considering only the formula for the remainder, there are numbers $\xi_1, \xi_2, \cdots$ all between 1 and 1.2 such that

$$|R_1| = |f'(\xi_1)|\,(1.2 - 1) = \frac{1}{\xi_1}(0.2) < 0.2 \quad \left(\text{since } 0 < \frac{1}{\xi_1} < 1\right),$$

$$|R_2| = \frac{|f''(\xi_2)|}{2!}(1.2-1)^2 = \left(\frac{1}{\xi_2}\right)^2 \frac{(0.2)^2}{2} < \frac{0.04}{2} < 0.02,$$

$$|R_3| = \frac{|f'''(\xi_3)|}{3!}(1.2-1)^3 = 2\left(\frac{1}{\xi_3}\right)^3 \frac{(0.2)^3}{6} < \frac{0.008}{3} < 0.003,$$

$$|R_4| = \frac{|f^{(4)}(\xi_4)|}{4!}(1.2-1)^4 = 3!\left(\frac{1}{\xi_4}\right)^4 \frac{(0.2)^4}{4!} < \frac{(0.2)^4}{4} = \frac{0.0016}{4} < 0.0005.$$

Thus, R_4 is the first remainder we are sure is in absolute value less than 5×10^{-4}. The computation of $\ln(1.2)$ from the first four terms of Taylor's formula with $a = 1$ and $b = 1.2$ will, therefore, be in error less than 5×10^{-4}. This computation was not asked for, but note that

$$f(1.2) = \ln(1.2), \quad f(1) = \ln 1 = 0, \quad f'(1) = 1, \quad f''(1) = -1, \quad f'''(1) = 2,$$

and hence, from the first four terms of Taylor's formula, and the preceding information about R_4,

$$\ln(1.2) = 0 + 1(0.2) - \frac{1}{2!}(0.2)^2 + \frac{2}{3!}(0.2)^3 \pm 5 \times 10^{-4}$$

$$= 0.1827 \pm 5 \times 10^{-4}.$$

PROBLEMS

1. Use Taylor's formula for $f(x) = \sqrt{1+x}$ with $a = 0$, $b = 0.2$, and $n = 4$ to compute $\sqrt{1.2}$ approximately. Also estimate the error.
2. Compute approximately $\cos 61°$ by using Taylor's formula with $f(x) = \cos x$, $a = \pi/3$, $b = 61\pi/180$, and $n = 4$. Estimate the error.
3. Find how many terms of Taylor's formula with $f(x) = e^x$, $a = 0$, and $b = 1$ are required to compute e with error not exceeding 5×10^{-5}.
4. How many terms of Taylor's formula with $f(x) = \sin x$, $a = 0$, and $b = \pi/180$ are required to compute $\sin 1°$ with error not exceeding 5×10^{-6}.
5. Compute $\sqrt[3]{9}$ with error less than 5×10^{-3}.
6. Compute $\sqrt[3]{4 + (1.9)^2}$ with error less than 5×10^{-3}.

130. Alternative Notation

By a change of notation, Taylor's formula for the expansion of $f(a + h)$ about the point a is

$$(1)\quad f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}h^{n-1} + R_n$$

where $R_n = \dfrac{f^{(n)}(a + \theta_n h)}{n!}h^n$ for some number θ_n such that $0 < \theta_n < 1$.

To obtain this formula set $b = a + h$ in (1) of Sec. 128. Hence, $b - a = h$ and any number between a and $a + h$ may be represented as $a + \theta h$ by a proper choice of θ with $0 < \theta < 1$.

In particular the formula for expansion of $f(h)$ around $a = 0$ appears as

$$(2)\quad f(h) = f(0) + f'(0)h + \frac{f''(0)}{2!}h^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}h^{n-1} + R_n$$

where $R_n = \dfrac{f^{(n)}(\theta_n h)}{n!}h^n$ for some θ_n such that $0 < \theta_n < 1$. Expansion around $a = 0$ is usually called **Maclaurin's** expansion.

Example 1. Use Maclaurin's expansion to express $\sin 40°$ to within 5×10^{-5}.

Solution. With $f(x) = \sin x$, then $f(0) = \sin 0 = 0$,

$$f'(0) = \cos 0 = 1,\quad f''(0) = -\sin 0 = 0,$$
$$f'''(0) = -\cos 0 = -1,\quad f^{(4)}(0) = \sin 0 = 0,\quad \text{etc.}$$

The remainder is either expressed in terms of the sine or cosine and

$$|R_n| = \frac{1}{n!}\left|\pm \sin\left(\theta_n \frac{40\pi}{180}\right)\right|\left(\frac{40\pi}{180}\right)^n < \frac{1}{n!}\left(\frac{2\pi}{9}\right)^n \quad \text{if } n \text{ is even and}$$

$$|R_n| = \frac{1}{n!}\left|\pm \cos\left(\theta_n \frac{40\pi}{180}\right)\right|\left(\frac{40\pi}{180}\right)^n < \frac{1}{n!}\left(\frac{2\pi}{9}\right)^n \quad \text{if } n \text{ is odd.}$$

A check will show that $2\pi/9 < 0.7$. Therefore

$$|R_2| < \frac{1}{2}(0.7)^2 = \frac{1}{2}(0.49) < 0.25,$$

$$|R_3| < \frac{0.7}{3}(0.25) = \frac{1}{3}(0.175) < 0.06,$$

$$|R_4| < \frac{0.7}{4}(0.06) = \frac{1}{4}(0.042) < 0.011,$$

$$|R_5| < \frac{0.7}{5}(0.011) = \frac{1}{5}(0.007\,7) < 0.001\,6,$$

$$|R_6| < \frac{0.7}{6}(0.001\,6) = \frac{1}{6}(0.001\,12) < 0.000\,2,$$

$$|R_7| < \frac{0.7}{7}(0.000\,2) < 0.000\,02 < 5 \times 10^{-5}.$$

Thus seven terms are sufficient, but every other term has value zero so that

$$\text{(3)} \qquad \sin 40° = 0 + \frac{2\pi}{9} + 0 - \frac{1}{3!}\left(\frac{2\pi}{9}\right)^3 + 0 + \frac{1}{5!}\left(\frac{2\pi}{9}\right)^5 + 0 \pm 5 \times 10^{-5}$$

$$= \frac{2\pi}{9}\left[1 - \frac{1}{3!}\left(\frac{2\pi}{9}\right)^2 + \frac{1}{5!}\left(\frac{2\pi}{9}\right)^4\right] \pm 5 \times 10^{-5}.$$

Example 2. Express $\sin 40°$ to within 5×10^{-5} by using Taylor's expansion around $a = \pi/4$.

Solution. Since $\frac{\pi}{4} - \frac{40\pi}{180} = \frac{\pi}{36}$ the remainder is such that

$$|R_n| < \frac{1}{n!}\left(\frac{\pi}{36}\right)^n < \frac{1}{n!}(0.1)^n$$

whether n is odd or even. Hence

$$|R_2| < \frac{(0.1)^2}{2} = \frac{0.01}{2} = 0.005 > 5 \times 10^{-5},$$

$$|R_3| < \frac{0.1}{3}(0.005) < 0.000\,2 > 5 \times 10^{-5},$$

$$|R_4| < \frac{0.1}{4}(0.000\,2) < 0.000\,005 < 5 \times 10^{-5},$$

$$\sin 40° = \sin\frac{\pi}{4} + \left(\cos\frac{\pi}{4}\right)\left(-\frac{\pi}{36}\right) - \frac{1}{2!}\left(\sin\frac{\pi}{4}\right)\left(-\frac{\pi}{36}\right)^2 - \frac{1}{3!}\left(\cos\frac{\pi}{4}\right)\left(-\frac{\pi}{36}\right)^3 \pm 5 \times 10^{-5}$$

$$\text{(4)} \qquad = \frac{\sqrt{2}}{2}\left[1 - \frac{\pi}{36} - \frac{1}{2}\left(\frac{\pi}{36}\right)^2 + \frac{1}{6}\left(\frac{\pi}{36}\right)^3\right] \pm 5 \times 10^{-5}.$$

Notice that there is one more term in (4) than in (3).

131. Remainder in Other Forms

The following more general form of Taylor's theorem is useful in some connections.

THEOREM 131. *Let a and b be numbers, $a < b$, let p be a positive number, let n be a positive integer, and let f be a function for which the nth derivative $f^{(n)}(x)$ exists for each number x such that $a \le x \le b$. There is then a number ξ_n such that $a < \xi_n < b$ and*

$$(1)\quad f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)(b-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(b-a)^{n-1}}{(n-1)!} + R_n$$

where
$$R_n = \frac{f^{(n)}(\xi_n)(b-\xi_n)^{n-p}(b-a)^p}{p(n-1)!}.$$

PROOF. Let S_n be the number such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)(b-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(b-a)^{n-1}}{(n-1)!} + S_n \cdot (b-a)^p$$

and let ϕ be the function defined for each number x such that $a \le x \le b$ by

$$\phi(x) = -f(b) + f(x) + f'(x)(b-x) + \frac{f''(x)(b-x)^2}{2!} + \cdots + \frac{f^{(n-1)}(x)(b-x)^{n-1}}{(n-1)!} + S_n \cdot (b-x)^p.$$

Thus $\phi(a) = 0$ by the definition of S_n, $\phi(b) = 0$ since $p > 0$, and

$$\begin{aligned}\phi'(x) &= f'(x) + \{-f'(x) + f''(x)(b-x)\} + \left\{-\frac{f''(x)2(b-x)}{2!} + \frac{f'''(x)(b-x)^2}{2!}\right\} \\ &\quad + \left\{-\frac{f'''(x)3(b-x)^2}{3!} + \frac{f^{(4)}(x)(b-x)^3}{3!}\right\} + \cdots \\ &\quad + \left\{-\frac{f^{(n-1)}(x)\cdot(n-1)(b-x)^{n-2}}{(n-1)!} + \frac{f^{(n)}(x)(b-x)^{n-1}}{(n-1)!}\right\} - S_n p(b-x)^{p-1} \\ &= \frac{f^{(n)}(x)\cdot(b-x)^{n-1}}{(n-1)!} - S_n p(b-x)^{p-1}, \qquad a \le x \le b.\end{aligned}$$

The function ϕ is thus seen to satisfy the conditions of Rolle's theorem (Sec. 32) and we accordingly let ξ_n be a number such that $a < \xi_n < b$ and

$$0 = \phi'(\xi_n) = \frac{f^{(n)}(\xi_n)(b-\xi_n)^{n-1}}{(n-1)!} - S_n p(b-\xi_n)^{p-1}.$$

Therefore
$$S_n = \frac{f^{(n)}(\xi_n)(b-\xi_n)^{n-1}}{(n-1)!\,p(b-\xi_n)^{p-1}} = \frac{f^{(n)}(\xi_n)(b-\xi_n)^{n-p}}{p(n-1)!}.$$

We thus set $R_n = S_n \cdot (b-a)^p$, note that

$$(2)\qquad R_n = \frac{f^{(n)}(\xi_n)(b-\xi_n)^{n-p}(b-a)^p}{p(n-1)!}$$

and hence see that the theorem is proved.

The remainder R_n given in (2) is said to be the **Schlömilch** form of the remainder.

Upon setting $p = n$ in (2) we obtain

$$R_n = \frac{f^{(n)}(\xi_n)(b-a)^n}{n!} \tag{3}$$

which is called the **Lagrange** form of the remainder. Taylor's formula is therefore seen to have remainder in the Lagrange form.

Upon setting $p = 1$ in (2) we obtain

$$R_n = \frac{f^{(n)}(\xi_n)(b-\xi_n)^{n-1}(b-a)}{(n-1)!} \tag{4}$$

which is called the **Cauchy** form of the remainder. Also see page 447.

PROBLEMS

1. Obtain an expression for computing $\cos 40°$ to within 5×10^{-5} by using:
   a. Maclaurin's expansion.
   b. Taylor's expansion about $\pi/6$.

2. Apply Taylor's expansion for a suitable f, a, and h and approximate the indicated number to within the given error.
   a. $\sqrt{17}$; 5×10^{-3}.
   b. $\dfrac{1}{\sqrt[3]{63}}$; 5×10^{-3}.
   c. $\ln(0.9)$; 5×10^{-4}.
   d. $\tan 46°$; 5×10^{-3}.
   e. $\dfrac{1}{(21)^2}$; 5×10^{-4}.
   f. $e^{\sin 1°}$; 5×10^{-3}.

3. By using Maclaurin's expansion prove:
   a. $1 + \dfrac{h}{2} - \dfrac{h^2}{8} < \sqrt{1+h} < 1 + \dfrac{h}{2}$ for $h > 0$.
   b. $e^h - 1 - h - \dfrac{h^2}{2} < \dfrac{h^3e^h}{6}$ for $h > 0$.
   c. $|\sin h - h| \le \dfrac{|h|^3}{6}$.

4. Estimate the error in replacing the length of a circular arc by the length of its chord if the central angle of the arc does not exceed 10°.

132. Polynomial Approximations

Taylor's formula for the expansion of $f(x)$ around a is

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n \quad \text{where} \tag{1}$$

$$R_n = \frac{f^{(n)}(\xi_n)}{n!}(x-a)^n \quad \text{for some } \xi_n \text{ between } a \text{ and } x.$$

This formula is used to obtain polynomial approximations of a function on a given range of the independent variable.

Example 1. Show that $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} \pm 0.000\,3$ for $0 \leq x \leq \frac{1}{2}$.

Solution. Let $f(x) = e^{-x}$ and $a = 0$. First

$$f'(x) = -e^{-x}, \quad f''(x) = e^{-x}, \quad f'''(x) = -e^{-x},$$

$$f^{(4)}(x) = e^{-x}, \quad \text{and} \quad f^{(5)}(x) = -e^{-x}.$$

Consequently $f(0) = f''(0) = f^{(4)}(0) = e^0 = 1$ and $f'(0) = f'''(0) = -e^0 = -1$. From these values and formula (1) there is a number ξ_5 between 0 and x such that

$$(2) \qquad e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + R_5 \quad \text{where} \quad R_5 = -\frac{e^{-\xi_5}}{5!}x^5.$$

With

$$0 \leq x \leq \tfrac{1}{2}, \quad \text{then} \quad 0 < \xi_5 < \tfrac{1}{2}, \quad 0 < e^{-\xi_5} < 1, \quad 0 < x^5 \leq (\tfrac{1}{2})^5 = \tfrac{1}{32} \quad \text{and}$$

$$|R_5| < \frac{1}{5!}\frac{1}{32} < 0.000\,3 \quad \text{as we wished to show.}$$

Example 2. Given that $0 \leq x \leq \frac{1}{2}$ show that

$$(3) \qquad e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} \pm 0.000\,009.$$

Solution. Replace x in (2) by u so that

$$e^{-u} = 1 - u + \frac{u^2}{2} - \frac{u^3}{3!} + \frac{u^4}{4!} + R_5 \quad \text{where} \quad R_5 = -\frac{e^{-\xi_5}}{5!}u^5, \quad 0 < \xi_5 < u.$$

Now in this relation set $u = x^2$ and obtain

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} + R_5 \quad \text{where} \quad R_5 = -\frac{e^{-\xi_5}}{5!}(x^2)^5, \quad 0 < \xi_5 < x^2.$$

If $0 < x \leq \frac{1}{2}$, then $0 < \xi_5 < (\frac{1}{2})^2$, $0 < e^{-\xi_5} < 1$, $0 < (x^2)^5 < 1/2^{10}$ and

$$|R_5| < \frac{1}{5!}\frac{1}{2^{10}} < 0.000\,009.$$

Hence, if $0 \leq x \leq \frac{1}{2}$, then (3) holds.

A polynomial approximation of a function, with an error estimate valid throughout an interval, may be used to approximate the definite integral with error estimate.

Example 3. Show that $\int_0^{1/2} e^{-x^2}\,dx = 0.461\,280 \pm 5 \times 10^{-6}$.

Solution. Since (3) holds for $0 \le x \le \frac{1}{2}$, then

$$\int_0^{1/2} e^{-x^2}\,dx = \int_0^{1/2}\left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} \pm 0.000\,009\right)dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{3!7} + \frac{x^9}{4!9} \pm 0.000\,009x\Big]_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{3(2^3)} + \frac{1}{10(2^5)} - \frac{1}{3!7(2^7)} + \frac{1}{4!9(2^9)} \pm \frac{0.000\,009}{2}$$

$$= 0.461\,280 \pm 0.000\,004\,5 = 0.461\,280 \pm 5 \times 10^{-6}.$$

PROBLEMS

1. a. Obtain $\sin x = x - \frac{x^3}{3!} \pm \frac{1}{5!2^5}$ for $0 \le x \le \frac{1}{2}$.

   b. Use this result to approximate $\int_0^{1/\sqrt{2}} \sin t^2\,dt$ with error estimate.

2. a. Obtain $\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} \pm \frac{1}{2(8^4)}$ for $0 \le x \le \frac{1}{8}$.

   b. Use this result to approximate $\int_0^{1/2} \sqrt{1 + t^3}\,dt$ with error estimate.

3. a. Obtain $\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 \pm \frac{1}{4^7}$ for $0 \le x \le \frac{1}{4}$.

   b. Use this result to approximate $\tan^{-1}\frac{1}{2}$ with error estimate.

4. a. Show that $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \pm \frac{1}{7!}$ for $0 < x \le 1$.

   b. Use this result to approximate, with error estimate, the improper integral

$$\int_0^1 \frac{\sin x}{x}\,dx.$$

5. Approximate the following integrals with errors not exceeding 0.001.

   a. $\int_0^{1/2} \frac{dx}{1 + x^4}$. c. $\int_0^1 e^{x^2}\,dx$. e. $\int_0^1 \frac{e^x - 1}{x}\,dx$.

   b. $\int_0^{1/2} \frac{dx}{\sqrt{1 - x^3}}$. d. $\int_0^1 xe^x\,dx$. f. $\int_0^{0.5} \sin x^2\,dx$.

6. A tower of unknown height x stands on the equator. Find x if a wire 10 ft longer than the equator and passing through the top of the tower fits around the

earth tightly and then over the top of the tower without sag. (Hint: Let R = (4000)(5280) ft and let θ be the angle at the center of the earth between the radii to foot of the tower and one of the points of tangency of the wire and the equator. Show that $R(\tan\theta - \theta) = 5$, express x in terms of $\sec\theta$, and (since θ is small) use the first two non-zero terms of the Maclaurin expansions to approximate $\tan\theta$ and $\sec\theta$.)

7.† Let a, b, c denote the longer leg, the shorter leg, and the hypotenuse, respectively, of a right triangle. Show that the value in degrees of the smaller acute angle is given approximately by

$$B = \frac{b}{2c + a}\,172°.$$

$\left(\text{Hint: Write } \frac{b}{2c+a} = \frac{\sin B}{2 + \cos B} = f(B) \text{ and use the first term of Maclaurin's expansion of } f(B).\right)$

133. Simpson's Rule

A method of approximating a definite integral is based on the following fact:

MIDPOINT RULE. *If $\phi(x)$ is a polynomial of third or lower degree, then*

(1) $$\int_a^b \phi(x)\,dx = \frac{b-a}{6}\left[\phi(a) + \phi\left(\frac{a+b}{2}\right) + \phi(b)\right].$$

PROOF. First let $p(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3$. Then

(2) $$\int_{-1}^{1} p(s)\,ds = 2c_0 + \frac{2}{3}c_2 \quad \text{(independent of } c_1 \text{ and } c_3\text{)}.$$

Since
$$\begin{aligned} p(-1) &= c_0 - c_1 + c_2 - c_3 \\ p(0) &= c_0 \\ p(1) &= c_0 + c_1 + c_2 + c_3, \end{aligned}$$
then
$$c_0 = p(0),\ 2c_0 + 2c_2 = p(-1) + p(1) \text{ so that}$$
$$2c_2 = p(-1) + p(1) - 2p(0).$$

Consequently, from (2),

(3) $$\int_{-1}^{1} p(s)\,ds = 2p(0) + \frac{1}{3}[p(-1) + p(1) - 2p(0)]$$

† For more details of this formula see the article by R. A. Johnson, "Determination of an Angle of a Right Triangle, Without Tables," *American Mathematical Monthly*, Vol. XXVII, 1920, pp. 368–369; also editorial comment, pp. 365, 366.

$$= \frac{1}{3}[p(-1) + 4p(0) + p(1)].$$

This result (3), for any polynomial up to third degree, will be used to obtain (1). Make the transformation

$$(4) \qquad x = \frac{a+b}{2} + \frac{b-a}{2}s \quad \text{so that} \quad dx = \frac{b-a}{2}ds,$$

$s = -1$ when $x = a$, $s = 0$ when $x = \dfrac{a+b}{2}$, and $s = 1$ when $x = b$. Set

$$p(s) = \phi\left[\frac{a+b}{2} + \frac{b-a}{2}s\right].$$

This polynomial $p(s)$ is the same degree in s that $\phi(x)$ is in x. Then

$$\int_a^b \phi(x)\,dx = \int_{-1}^1 p(s)\frac{b-a}{2}\,ds = \frac{b-a}{2}\frac{1}{3}[p(-1) + 4p(0) + p(1)]$$
$$= \frac{b-a}{6}\left[\phi(a) + 4\phi\left(\frac{a+b}{2}\right) + \phi(b)\right].$$

This mid-point rule is used to obtain an approximation of a definite integral of any function f.

SIMPSON'S RULE. *With n an even number, with $x_k = a + k\dfrac{b-a}{n}$ and with $y_k = f(x_k)$ for $k = 0, 1, 2, \cdots, n$, then*

$$(5) \quad \int_a^b f(x)\,dx \sim \frac{b-a}{3n}\{y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\}.$$

DERIVATION. Pass a parabolic (or third-degree) arc through the three points (a,y_0), (x_1,y_1), and (x_2,y_2). The function defining this arc has the same values as f at a, x_1, and x_2 and its integral from a to x_2 will, presumably, approximate the integral of f from a to x_2. Without even knowing the function defining this arc, its integral (from the mid-point rule) is $\dfrac{x_2 - a}{6}(y_0 + 4y_1 + y_2)$. Since $x_2 - a = 2(b-a)/n$, it thus follows that

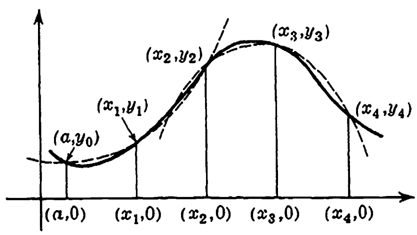


Figure 133

$$\int_a^{x_2} f(x)\,dx \sim \frac{b-a}{3n}(y_0 + 4y_1 + y_2).$$

By the same reasoning applied to $x_2 \leq x \leq x_4$, then to $x_4 \leq x \leq x_6$, etc. to $x_{n-2} \leq x \leq x_n$ it follows that

$$\int_{x_2}^{x_4} f(x)\,dx \sim \frac{b-a}{3n}(y_2 + 4y_3 + y_4), \quad \text{etc.}$$

$$\int_{x_{n-2}}^{x_n} f(x)\,dx \sim \frac{b-a}{3n}(y_{n-2} + 4y_{n-1} + y_n).$$

By adding these approximations, the formula (5) is obtained.

Example 1. By using Simpson's rule with $n = 4$, find an approximation of

$$\int_{-1}^{3} \sqrt{4 + x^3}\,dx.$$

Solution. Since $a = -1$, $b = 3$, and $n = 4$, then $(b - a)/n = 1$,

$$a = -1, \quad x_1 = 0, \quad x_2 = 1, \quad x_3 = 2, \quad \text{and} \quad x_4 = 3.$$

Therefore with $y = \sqrt{4 + x^3}$ the computation is:

$$\begin{array}{llll||rl}
y_0 = \sqrt{4 + (-1)^3} & = \sqrt{3} & = 1.732 & & y_0 = & 1.732 \\
y_1 = \sqrt{4 + 0^3} & = \sqrt{4} & = 2.000 & & 4y_1 = & 8.000 \\
y_2 = \sqrt{4 + 1^3} & = \sqrt{5} & = 2.236 & & 2y_2 = & 4.472 \\
y_3 = \sqrt{4 + 2^3} & = \sqrt{12} & = 3.464 & & 4y_3 = & 13.856 \\
y_4 = \sqrt{4 + 3^3} & = \sqrt{31} & = 5.568 & & y_4 = & 5.568 \\
\hline
& & & & s = & 33.628
\end{array}$$

Hence $$\int_{-1}^{3} \sqrt{4 + x^3}\,dx \sim \frac{3 - (-1)}{3(4)}\,33.628 = 11.209.$$

Data on a physical experiment, with equally spaced observations, represents isolated values of a function whose analytic expression is not known, but even so an approximation of the integral of this function over the range of observations may be desired. In such a case Simpson's rule may be used.

Example 2. Given

x	0	0.5	1	1.5	2
y	1.000	1.649	2.718	4.482	7.389

, then

$$\int_0^2 y\,dx \sim \frac{2}{3 \cdot 4}[1.000 + 4(1.649) + 2(2.718) + 4(4.482) + 7.389]$$

$$= \tfrac{1}{6}[38.349] = 6.3915.$$

Actually in the table $y = e^x$ so the integral $= e^2 - 1 = 6.389$.

Simpson's rule is one of the most widely used methods for approximating definite integrals. The following error estimate is useful to know. The proof is given in the next section.

ERROR FOR SIMPSON'S RULE. *In the notation of the statement of Simpson's rule, the error is given as:*

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{3n}\{y_0 + 4y_1 + 2y_2 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\}\right|$$

$$\leq \frac{(b-a)^5 M}{180n^4} \tag{6}$$

where M is any constant such that $M \geq |f^{(4)}(x)|$ for $a \leq x \leq b$.

Example 3. Find a number n which in Simpson's rule will yield

$$\int_0^{0.5} \sin e^x\,dx \quad \text{accurate to three decimal places.}$$

Solution. Set $f(x) = \sin e^x$ and find the fourth derivative

$$f^{(4)}(x) = e^x \cos e^x - 7e^{2x} \sin e^x - 6e^{3x} \cos e^x + e^{4x} \sin e^x.$$

To be sure to find a number M such that $M \geq |f^{(4)}(x)|$ for $0 \leq x \leq 0.5$ take the absolute value of each term, use $|\sin e^x| \leq 1$, $|\cos e^x| \leq 1$ so that surely

$$|f^{(4)}(x)| \leq e^x + 7e^{2x} + 6e^{3x} + e^{4x} \leq e^{0.5} + 7e + 6e^{1.5} + e^2, \quad 0 \leq x \leq 0.5.$$

From a table of powers of e a perfectly safe value to use for M is

$$1.649 + 7(2.718) + 6(4.482) + 7.389 = 44.952$$

or any larger number. With $M = 45$, then from (6) the result of applying Simpson's rule will be within 5×10^{-4} of being correct if n is an even integer satisfying

$$\frac{(0.5)^5 45}{180n^4} < 5 \times 10^{-4} \quad \text{so that} \quad \frac{(0.5)^5}{20} 10^4 < n^4,$$

and hence $n > (0.5)10\sqrt[4]{(0.5)/20} = 5\sqrt[4]{0.025} = 1.99$ by logarithmic computation. Thus, $n = 2$ may be used with confidence that the result will be within the stated accuracy; that is, the mid-point rule

$$\int_0^{0.5} \sin e^x\,dx = \frac{0.5}{6}(\sin 1 + 4 \sin e^{0.25} + \sin e^{0.5})$$

with computation carried to four decimal places, will round accurate to three decimal places. By using a table of powers of e and a radian-trig. table, the result is 0.473.

134. Error for Simpson's Rule

Let h and M be positive numbers and let g be a function whose fourth derivative exists and is such that

$$|g^{(4)}(x)| \leq M \quad \text{for } -h \leq x \leq h. \tag{1}$$

We shall first obtain an upper bound for the error in using the mid-point rule to approximate the integral of g from $-h$ to h. To do so let φ be the function defined by

$$\varphi(t) = \int_{-t}^{t} g(x)\,dx - \frac{t}{3}[g(-t) + 4g(0) + g(t)] \quad \text{for } 0 \le t \le h. \tag{2}$$

Let G be a function such that $G'(x) = g(x)$ for $-h \le x \le h$ so that

$$\int_{-t}^{t} g(x)\,dx = G(t) - G(-t) \quad \text{and} \quad \frac{d}{dt}\int_{-t}^{t} g(x)\,dx = G'(t) + G'(-t) = g(t) + g(-t).$$

We now compute three successive derivatives of φ:

$$\begin{aligned}
\varphi'(t) &= g(t) + g(-t) - \tfrac{1}{3}[g(-t) + 4g(0) + g(t)] - \frac{t}{3}[-g'(-t) + g'(t)] \\
&= \tfrac{2}{3}[g(t) + g(-t)] - \tfrac{4}{3}g(0) - \frac{t}{3}[g'(t) - g'(-t)], \\
\varphi''(t) &= \tfrac{2}{3}[g'(t) - g'(-t)] - 0 - \tfrac{1}{3}[g'(t) - g'(-t)] - \frac{t}{3}[g''(t) + g''(-t)] \\
&= \tfrac{1}{3}[g'(t) - g'(-t)] - \frac{t}{3}[g''(t) + g''(-t)],
\end{aligned}$$

$$\begin{aligned}
\varphi'''(t) &= \tfrac{1}{3}[g''(t) + g''(-t)] - \tfrac{1}{3}[g''(t) + g''(-t)] - \frac{t}{3}[g'''(t) - g'''(-t)] \\
&= -\frac{t}{3}[g'''(t) - g'''(-t)].
\end{aligned} \tag{3}$$

By substituting $t = 0$ notice, for later use, that

$$\varphi(0) = 0, \quad \varphi'(0) = 0, \quad \text{and} \quad \varphi''(0) = 0. \tag{4}$$

With t a number such that $0 < t \le h$ we now apply the Law of the Mean to g''' for the closed interval $I[-t,t]$ and determine a number ξ_t such that

$$g'''(t) - g'''(-t) = g^{(4)}(\xi_t)[t - (-t)] = g^{(4)}(\xi_t) \cdot 2t.$$

Thus, from (3), $\varphi'''(t) = -\tfrac{2}{3}t^2 g^{(4)}(\xi_t)$ and then, from (1), $|\varphi'''(t)| \le \tfrac{2}{3}t^2 M$ or

$$-\tfrac{2}{3}Mt^2 \le \varphi'''(t) \le \tfrac{2}{3}Mt^2 \quad \text{for } 0 \le t \le h.$$

Hence, for $0 \le s \le t \le h$ it follows that

$$\begin{gathered}
-\tfrac{2}{3}M\int_0^s t^2\,dt \le \int_0^s \varphi'''(t)\,dt \le \tfrac{2}{3}M\int_0^s t^2\,dt, \\
-\tfrac{2}{9}Mt^3\Big]_0^s \le \varphi''(t)\Big]_0^s \le \tfrac{2}{9}Mt^3\Big]_0^s, \\
-\tfrac{2}{9}Ms^2 \le \varphi''(s) - \varphi''(0) \le \tfrac{2}{9}Ms^3.
\end{gathered}$$

Since $\varphi''(0) = 0$ and these inequalities hold for $0 \le s \le t \le h$, then

$$-\tfrac{2}{9}Mt^3 \le \varphi''(t) \le \tfrac{2}{9}Mt^3 \quad \text{for } 0 \le t \le h.$$

By two more similar integrations it follows that

$$-\frac{2M}{9 \cdot 4 \cdot 5}t^5 \le \varphi(t) \le \frac{2M}{9 \cdot 4 \cdot 5}t^5 \quad \text{for } 0 \le t \le h.$$

In particular for $t = h$ we have $|\varphi(h)| \leq \frac{M}{90} h^5$; that is,

$$\left| \int_{-h}^{h} g(x)\,dx - \frac{h}{3}[g(-h) + 4g(0) + g(h)] \right| \leq \frac{M}{90} h^5.$$

For a function f such that $|f^{(4)}(x)| \leq M$ for $c - h \leq x \leq c + h$, then (by making a translation of axes with the new origin at the point $(c,0)$) the above result yields

$$\text{(4)} \qquad \left| \int_{c-h}^{c+h} f(x)\,dx - \frac{h}{3}[f(c - h) + 4f(c) + f(c + h)] \right| \leq \frac{M}{90} h^5.$$

We are now ready to obtain the expression as given on p. 429 for the error in using Simpson's rule for approximating

$$\int_a^b f(x)\,dx \quad \text{where} \quad |f^{(4)}(x)| \leq M \quad \text{wherever} \quad a \leq x \leq b.$$

Let n be an even number, let $h = (b - a)/n$, let $x_k = a + kh$, and let

$$y_k = f(x_k) \quad \text{for } k = 0, 1, 2, \cdots, n.$$

Then by considering intervals centered at $x_1, x_3, x_5, \cdots, x_{n-1}$ and each of length $2h$, it follows from (4) that

$$\left| \int_a^{x_2} f(x)\,dx - \frac{b - a}{3n}(y_0 + 4y_1 + y_2) \right| \leq \frac{M}{90}\left(\frac{b - a}{n}\right)^5,$$

$$\left| \int_{x_2}^{x_4} f(x)\,dx - \frac{b - a}{3n}(y_2 + 4y_3 + y_4) \right| \leq \frac{M}{90}\left(\frac{b - a}{n}\right)^5$$

etc.

$$\left| \int_{x_{n-2}}^{b} f(x)\,dx - \frac{b - a}{3n}(y_{n-2} + 4y_{n-1} + y_n) \right| \leq \frac{M}{90}\left(\frac{b - a}{n}\right)^5.$$

There are $n/2$ of these inequalities and together they yield

$$\left| \int_a^b f(x)\,dx - \frac{b - a}{3n}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \right|$$
$$\leq \frac{n}{2}\frac{M}{90}\left(\frac{b - a}{n}\right)^5 = \frac{(b - a)^5 M}{180n^4}.$$

PROBLEMS

1. a. Approximate $\int_1^2 x^{-1}\,dx$ by using Simpson's rule with $n = 10$ and thus obtain an approximation of ln 2.

   b. Show that the error of Part a is $< 5 \times 10^{-5}$.

2. a. Express the length of the graph of $4y = \sqrt{25 - x^2}$ from $(0,\frac{5}{4})$ to $(3,1)$ as an integral.

   b. Do the same for the graph of $x = \sqrt{25 - 16y^2}$ from $(5,0)$ to $(3,1)$.

c. Approximate these integrals by using Simpson's rule with $n = 2$ in Part a and $n = 4$ in Part b.

d. Use Part c to approximate the circumference of the ellipse having equation $x^2 + 16y^2 = 25$.

3. Approximate the length of the graph of $y = \sin x$ between $(0,0)$ and $(\pi/2,1)$ by using Simpson's rule with $n = 4$.

4. By using Simpson's rule with $n = 4$ approximate the "elliptic integral"

$$\int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5 \sin^2 x}}.$$

5. Find a number n such that the error in computing

$$\int_0^1 \sin e^x \, dx$$

by Simpson's rule will be less than 5×10^{-4}.

6. Use Simpson's rule to approximate the integral of the function whose tabular values are given:

a.

x	0	1	2	3	4
y	25.2	17.5	40.3	35.0	30.7

b.

x	0.5	1.0	1.5	2.0	2.5	3.0	3.5
y	20.2	9.79	6.56	4.92	3.91	3.24	2.75

135. L'Hospital's Rules

We first prove: THE EXTENDED LAW OF THE MEAN. *Let f and g be functions satisfying the three conditions:*

(a) *f and g are both continuous on a closed interval $I[a,b]$.*

(b) *$f'(x)$ and $g'(x)$ both exist and $g'(x) \neq 0$ for $a < x < b$.*

Then there is a number X such that

$$a < X < b \quad and \quad \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(X)}{g'(X)}. \tag{1}$$

PROOF. Notice first that $g(b) - g(a) \neq 0$ since (by the Law of the Mean) there is a number ξ such that $g(b) - g(a) = g'(\xi)(b - a)$, which is $\neq 0$ since $g'(\xi) \neq 0$ because $a < \xi < b$.

Next, let φ be the function defined for $a \leq x \leq b$ by

$$\varphi(x) = [f(b) - f(a)][g(x) - g(a)] - [f(x) - f(a)][g(b) - g(a)]$$

so that $\varphi(a) = 0$ and $\varphi(b) = 0$. Also, for $a < x < b$

$$\varphi'(x) = [f(b) - f(a)]g'(x) - f'(x)[g(b) - g(a)].$$

Thus, by Rolle's theorem (Sec. 32) let X be such that $a < X < b$ and

$$0 = \varphi'(X) = [f(b) - f(a)]g'(X) - f'(X)[g(b) - g(a)].$$

Since $g'(X) \neq 0$ and $g(b) - g(a) \neq 0$, the equation in (1) follows.

The following theorem (l'Hospital's Rule I) may be used for determining limits of some quotients in which both numerator and denominator approach zero.

RULE I. *Let f and g be functions whose domains include the open intervals $I(a,c)$ and $I(c,b)$ and are such that:*

(a) $\lim_{x\to c} f(x) = 0$ *and* $\lim_{x\to c} g(x) = 0$.

(b) $f'(x)$ *and* $g'(x)$ *exist and* $g'(x) \neq 0$ *for* $a < x < c$ *or* $c < x < b$.

(c) $\lim_{x\to c} \dfrac{f'(x)}{g'(x)}$ *exists, or is* $+\infty$, *or is* $-\infty$.

Then also

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}. \tag{2}$$

PROOF. If c is not in the domain of f or g (or if $f(c)$ or $g(c)$ is different from zero) then define $f(c) = g(c) = 0$. Now f and g are continuous on the open interval $I(a,b)$ and the Extended Law of the Mean may be applied to any closed sub-interval of $I(a,b)$. For x such that $a < x < c$ or $c < x < b$ let X be between x and c and such that

$$\frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(X)}{g'(X)} \quad \text{and hence} \quad \frac{f(x)}{g(x)} = \frac{f'(X)}{g'(X)}.$$

As $x \to c$ then $X \to c$ and thus (2) follows.

Example 1. Find $\lim_{x\to 2} \dfrac{e^{x-2} - e^{2-x}}{\sin(x - 2)}$.

Solution. Upon setting $f(x) = e^{x-2} - e^{2-x}$ and $g(x) = \sin(x - 2)$, then $f'(x) = e^{x-2} + e^{2-x}$, $g'(x) = \cos(x - 2)$ and

$$\lim_{x\to 2} \frac{f'(x)}{g'(x)} = \lim_{x\to 2} \frac{e^{x-2} + e^{2-x}}{\cos(x - 2)} = \frac{e^0 + e^0}{\cos 0} = \frac{2}{1} = 2.$$

Consequently, by l'Hospital's Rule I also

$$\lim_{x\to 2} \frac{f(x)}{g(x)} = 2; \quad \text{that is,} \quad \lim_{x\to 2} \frac{e^{x-2} - e^{2-x}}{\sin(x - 2)} = 2.$$

For some functions f and g such that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ it happens that also $\lim_{x\to c} f'(x) = 0$ and $\lim_{x\to c} g'(x) = 0$. In such cases apply Rule I again to the quotient $f'(x)/g'(x)$.

Example 2.

$$\lim_{x\to 3}\frac{e^{x-3}+e^{3-x}-2}{1-\cos(x-3)}=\lim_{x\to 3}\frac{D_x(e^{x-3}+e^{3-x}-2)}{D_x[1-\cos(x-3)]}$$

$$=\lim_{x\to 3}\frac{e^{x-3}-e^{3-x}}{\sin(x-3)} \qquad \text{(form “0/0” so try Rule I again)}$$

$$=\lim_{x\to 3}\frac{D_x(e^{x-3}-e^{3-x})}{D_x\sin(x-3)}=\lim_{x\to 3}\frac{e^{x-3}+e^{3-x}}{\cos(x-3)}=2.$$

Two mistakes are so common that we warn against them.

FIRST. *Do not differentiate the fraction* $f(x)/g(x)$; the new quotient is obtained by differentiating the numerator for a new numerator and differentiating the denominator for a new denominator.

SECOND. *Do not apply Rule* I *unless both numerator and denominator approach* 0. Thus

$$\lim_{x\to 2}\frac{x^2-4}{x-3}\neq\lim_{x\to 2}\frac{D_x(x^2-4)}{D_x(x-3)}=\lim_{x\to 2}\frac{2x}{1}=4,$$

and the reason for not applying Rule I is because the denominator does not approach 0. By Theorem 17

$$\lim_{x\to 2}\frac{x^2-4}{x-3}=\frac{\lim\limits_{x\to 2}(x^2-4)}{\lim\limits_{x\to 2}(x-3)}=\frac{2^2-4}{2-3}=\frac{0}{-1}=0.$$

If $f(x)$ and/or $g(x)$ are defined only on one side of c, then l'Hospital's Rule I may be used with $x\to c+$ or $x\to c-$ as the case may be.

Example 3.

$$\lim_{x\to 0+}\frac{1-\cos\sqrt{x}}{x}=\lim_{x\to 0+}\frac{D_x(1-\cos\sqrt{x})}{D_x x}=\lim_{x\to 0+}\frac{\sin\sqrt{x}}{2\sqrt{x}}=\frac{1}{2}\cdot 1=\frac{1}{2}.$$

RULE II. *In Rule* I *replace the conditions* (*a*) *by*

$$(\mathrm{a_{II}}) \qquad \lim_{x\to c}|g(x)|=\infty.$$

The conclusion is then the same as in Rule I.

A proof is given at the end of this section.

Example 4. Find $\displaystyle\lim_{x\to(\pi/2)-}\frac{\sec x}{\ln\sec x}$.

Solution. The denominator becomes infinite as $x \to (\pi/2) -$ so earlier limit theorems are not applicable, but Rule II is:

$$\lim_{x\to(\pi/2)-} \frac{\sec x}{\ln \sec x} = \lim_{x\to(\pi/2)-} \frac{D_x \sec x}{D_x \ln \sec x}$$

$$= \lim_{x\to(\pi/2)-} \frac{\sec x \tan x}{\left(\dfrac{\sec x \tan x}{\sec x}\right)} = \lim_{x\to(\pi/2)-} \sec x = \infty.$$

RULE III. *The conclusions of Rule* I *and* II *remain the same if $x \to c$ is replaced either by $x \to \infty$ or by $x \to -\infty$.*

PROOF for $x \to \infty$. Make the transformation $t = \dfrac{1}{x}$ and set

$$F(t) = f(x) \quad \text{and} \quad G(t) = g(x).$$

Notice that $t \to 0+$ as $x \to \infty$, that $D_xF(t) = D_xf(x) = f'(x)$ but also

$$D_xF(t) = D_tF(t)D_xt = F'(t)\left(-\frac{1}{x^2}\right) = F'(t)(-t^2).$$

Similar relations hold for G and g and hence

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{t\to 0+} \frac{F'(t)(-t^2)}{G'(t)(-t^2)} = \lim_{t\to 0+} \frac{F'(t)}{G'(t)} = \lim_{t\to 0+} \frac{F(t)}{G(t)} \quad \begin{Bmatrix}\text{by Rule I or II as} \\ \text{the case may be}\end{Bmatrix}$$

$$= \lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

Example 5.

$$\lim_{x\to\infty} \frac{x^3}{e^x} = \lim_{x\to\infty} \frac{D_x x^3}{D_x e^x} = \lim_{x\to\infty} \frac{3x^2}{e^x} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{6x}{e^x} \qquad \left(\text{again } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{6}{e^x} = 0.$$

Example 6. Find $\lim\limits_{x\to-\infty} \dfrac{x^3}{e^x}$.

Solution. Since the numerator $\to -\infty$ whereas the denominator $\to 0$ as $x \to -\infty$, none of the l'Hospital Rules apply. Since, however, $e^x > 0$ for any number x, then

$$\lim_{x\to-\infty} \frac{x^3}{e^x} = -\infty.$$

PROOF of l'Hospital's Rule II. With f and g satisfying the condition

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L, \quad \text{where} \quad |L| \neq \infty.$$

Let $\epsilon > 0$ be an arbitrary number. Let $\delta_1 > 0$ be such that

$$\text{(3)} \qquad \text{if} \quad 0 < |x - c| < \delta_1, \quad \text{then} \quad g(x) \neq 0 \quad \text{and} \quad L - \frac{\epsilon}{2} < \frac{f'(x)}{g'(x)} < L + \frac{\epsilon}{2}.$$

Choose a definite number x_0 such that $0 < |x_0 - c| < \delta_1$ and then let x be any number between x_0 and c. By the Extended Law of the Mean, let X be a number between x_0 and x such that

$$\frac{f'(X)}{g'(X)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)}.$$

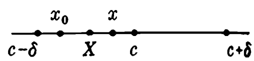


Figure 135

Hence also $0 < |X - c| < \delta_1$, and thus from (2)

$$L - \frac{\epsilon}{2} < \frac{f'(X)}{g'(X)} < L + \frac{\epsilon}{2}; \quad \text{that is} \quad L - \frac{\epsilon}{2} < \frac{f(x) - f(x_0)}{g(x) - g(x_0)} < L + \frac{\epsilon}{2}.$$

By dividing both numerator and denominator of this last inequality by $g(x) \neq 0$, we have

$$\text{(4)} \qquad L - \frac{\epsilon}{2} < \frac{f(x)/g(x) - f(x_0)/g(x)}{1 - g(x_0)/g(x)} < L + \frac{\epsilon}{2}.$$

Since $\lim_{x \to c} |g(x)| = \infty$, while x_0 is a definite number, we see that both

$$\lim_{x \to c} \frac{f(x_0)}{g(x)} = 0 \quad \text{and} \quad \lim_{x \to c} \frac{g(x_0)}{g(x)} = 0$$

which, together with (4), means that there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta, \quad \text{then} \quad L - \epsilon < \frac{f(x)}{g(x)} < L + \epsilon.$$

Since ϵ was arbitrary we have, from the definition of a limit, that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L.$$

A similar proof can be made if L is replaced either by $+\infty$ or by $-\infty$.

PROBLEMS

1. Establish each of the following limits:

a. $\displaystyle\lim_{x \to 0} \frac{x^2 - 2x}{\sin x \ln (3 - x)} = -\frac{2}{\ln 3}.$

b. $\displaystyle\lim_{x \to 2} \frac{x^2 - 2x}{\sin x \ln (3 - x)} = -\frac{2}{\sin 2}.$

c. $\displaystyle\lim_{x \to 0} \frac{x - \dfrac{x^3}{3!} - \sin x}{x^5} = -\frac{1}{5!}.$

d. $\displaystyle\lim_{x \to \infty} \frac{(1.01)^x}{x^5} = \infty.$

2. Find each of the following:

a. $\displaystyle\lim_{x\to 2} \frac{x^3 - 5x + 2}{x^4 + 6x^2 - 40}$.

b. $\displaystyle\lim_{x\to 0} \frac{x^3 - 5x + 2}{x^4 + 6x^2 - 40}$.

c. $\displaystyle\lim_{x\to \infty} \frac{x^3 - 5x + 2}{x^4 + 6x^2 - 40}$.

d. $\displaystyle\lim_{x\to 0} \frac{\tan x}{x}$.

e. $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{x^2}$.

f. $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{\sin^2 x}$.

g. $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{3!}}{x^4}$.

h. $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x - x^2 - x^3}{x^4}$.

i. $\displaystyle\lim_{x\to \infty} \frac{\ln(1 + x^{-1})}{x^{-1}}$.

j. $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^2}$.

k. $\displaystyle\lim_{x\to 0} \frac{x^3}{x - \sin x}$.

l. $\displaystyle\lim_{x\to 0} \frac{x^3}{\tan x - \sin x}$.

m. $\displaystyle\lim_{x\to \pi/2} \frac{\sin 2x}{\pi - 2x}$.

n. $\displaystyle\lim_{x\to 0} \frac{2x}{\sin(\pi - 2x)}$.

o. $\displaystyle\lim_{x\to \pi/2} \frac{1 + \cos 2x}{1 - \sin x}$.

p. $\displaystyle\lim_{x\to \infty} \frac{\ln(e^{3x} + x)}{x}$.

3. In the figure, the circle has radius 3 and is tangent to the y-axis at the origin. For $0 < h < 3$ the point A is $(0,h)$ and B is h units from the origin 0.

The line AB cuts the x-axis at the point E.

a. Express the distance OE in terms of h.

b. Find $\lim_{h\to 0} OE$.

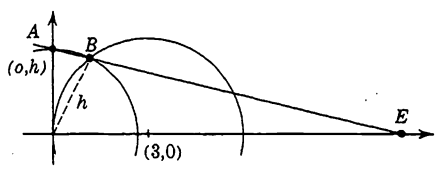


Figure Prob. 3

4. Solve Prob. 3 in case B is chosen so that arc $OB = h$ (instead of chord $OB = h$).

5. Let P be the point (h,h^2) on the graph of $y = x^2$. Connect P with the origin 0. Find

$$\lim_{h\to 0} \frac{\text{chord } OP}{\text{arc } OP}.$$

6. For $g(h)$ an nth degree polynomial in h, and for f a function, the notationa: contrivance "$f(h) \sim g(h)$ for $|h|$ small" means that both

$$\lim_{h\to 0} \frac{f(h)}{g(h)} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{f(h) - g(h)}{h^{n+1}} = \text{constant}.$$

In physics and engineering the following approximations for $|h|$ small are frequently used. Justify each of them.

a. $e^h \sim 1 + h$.

b. $\dfrac{e^h - 1}{h} \sim 1 + \dfrac{h}{2}$.

c. $\dfrac{1}{\sqrt{1-h}} \sim 1 + \dfrac{h}{2}$.

d. $\cosh h \sim 1 + \dfrac{h^2}{2}$.

e. $\sin h \sim h$.

f. $\ln(1+h) \sim h - \dfrac{h^2}{2}$.

136. Other Limit Forms

The following examples show methods to try if:

$\lim f(x) \cdot g(x)$ takes the form $0 \cdot \infty$,

$\lim [f(x) - g(x)]$ takes the form $\infty - \infty$,

$\lim f(x)^{g(x)}$ takes any of the forms 0^0, 1^∞, ∞^0.

In each case changes are made so that one of l'Hospital's rules applies.

Example 1. Find $\lim\limits_{x\to 0} x \ln|x|$ which takes the form $0 \cdot \infty$.

Solution. Upon writing $x \ln|x| = \dfrac{\ln|x|}{x^{-1}}$, then the absolute value of the denominator becomes infinite as $x \to 0$ so l'Hospital's Rule II applies:

$$\lim_{x\to 0} x \ln|x| = \lim_{x\to 0} \frac{\ln|x|}{x^{-1}} = \lim_{x\to 0} \frac{D_x \ln|x|}{D_x x^{-1}} = \lim_{x\to 0} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0}(-x) = 0.$$

Example 2. Find $\lim\limits_{x\to\pi/2} (\sec x - \tan x)$ which takes the form $\pm(\infty - \infty)$.

Solution. First use trigonometric identities and obtain a form to which one of l'Hospital's rules applies:

$$\lim_{x\to\pi/2}(\sec x - \tan x) = \lim_{x\to\pi/2}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to\pi/2}\frac{1 - \sin x}{\cos x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\pi/2}\frac{D_x(1 - \sin x)}{D_x \cos x} = \lim_{x\to\pi/2}\frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0.$$

The two logarithmic relations

$$(1) \qquad b = e^{\ln b} \quad \text{and} \quad \ln b^p = p \ln b$$

are useful in evaluating limits of functions having exponents which approach zero or become infinite.

Example 3. Find $\lim_{x\to 0} |x|^x$ which takes the form 0^0.

Solution. From (1), $|x|^x = e^{\ln |x|^x} = e^{x \ln |x|}$.
The result of Example 1 may be used on the exponent:

$$\lim_{x\to 0} |x|^x = \lim_{x\to 0} e^{x \ln |x|} = e^{\lim_{x\to 0} x \ln |x|} = e^0 = 1.$$

Example 4. Show that $\lim_{x\to 0} (x + 2^x)^{1/x} = 2e$.

Solution. The graph of $y = x + 2^x$ intersects the y-axis at $(0,1)$ and hence, by continuity, there are negative values of x for which $(x + 2^x)^{1/x}$ is defined. Thus, it is permissible to let $x \to 0$ through either positive or negative values. Again, from the two relations of (1):

$$\lim_{x\to 0} (x + 2^x)^{1/x} = \lim_{x\to 0} e^{\ln (x+2^x)^{1/x}} = \lim_{x\to 0} e^{\frac{\ln (x+2^x)}{x}}$$

$$= e^{\lim_{x\to 0} \frac{\ln (x+2^x)}{x}} = e^{\lim_{x\to 0} \frac{D_x \ln (x+2^x)}{D_x x}} = e^{\lim_{x\to 0} \frac{1+2^x \ln 2}{x+2^x}}$$

$$= e^{(1+\ln 2)} = e\, e^{\ln 2} = e\, 2 = 2e.$$

PROBLEMS

1. Establish each of the following:

a. $\lim_{x\to 0} x^3 \ln |x| = 0.$

b. $\lim_{x\to 0} \left[2 \csc^2 x - \frac{1}{1 - \cos x}\right] = \frac{1}{2}.$

c. $\lim_{x\to \infty} \left(1 + \frac{3}{x}\right)^x = e^3.$

d. $\lim_{x\to \infty} (x + a^2)^{1/x^2} = 1.$

e. $\lim_{x\to 0} \left[\frac{1}{\sin x} - \frac{1}{x}\right] = 0.$

f. $\lim_{x\to 0} \left[\frac{1.001}{\sin x} - \frac{1}{x}\right] = \infty.$

g. $\lim_{x\to 0} \left[\frac{1 + x}{\sin x} - \frac{1}{x}\right] = 1.$

h. $\lim_{x\to 0} \left[\frac{1 + ax}{\sin x} - \frac{1}{x}\right] = a.$

i. $\lim_{x\to 0} \left[\frac{1}{\sin^2 x} - \frac{1}{x^2}\right] = \frac{1}{3}.$

j. $\lim_{x\to 0} \left[\frac{1}{\sin^3 x} - \frac{1}{x^3}\right] = \infty.$

2. Obtain the following:

a. $\lim_{x\to 0} (x + e^x)^{1/x} = e^2.$

b. $\lim_{x\to \infty} \frac{e^x + 1}{e^x + x} = 1.$

c. $\lim_{x\to \infty} (x + e^x)^{1/x} = e.$

d. $\lim_{x\to 0+} (1 + x)^{1/x^2} = \infty.$

e. $\lim_{x\to 0-} (1 + x)^{1/x^2} = 0.$

f. $\lim_{x\to 0} (1 + x + x^2)^{1/x} = e.$

3. Examine each of the following limits:

a. $\lim_{x\to 0} x^3 \cot x$.

b. $\lim_{x\to 0} \dfrac{\ln(1+\sin x)}{\sin x}$.

c. $\lim_{x\to 0} \dfrac{\ln(e^{3x}+x)}{x}$.

d. $\lim_{x\to \pi/2} \left(\dfrac{\pi}{2}-x\right)^3 \tan x$.

e. $\lim_{x\to 0} |x|^{(|x|^{-1})}$.

f. $\lim_{x\to 0} |x|^{1-\cos x}$.

g. $\lim_{x\to \infty} x^{(x^{-1})}$.

h. $\lim_{x\to 0} (\csc x - \cot x)$.

i. $\lim_{x\to \infty} \{\ln(e^x+1) - x\}$.

j. $\lim_{x\to 0} \left[\dfrac{\sin x}{x}\right]^{1/x^2}$.

k. $\lim_{x\to \infty} \left[\cos \dfrac{a}{\sqrt{x}}\right]^x$, $a \neq 0$.

l. $\lim_{x\to 0} (e^x - x)^{1/x}$.

4. Show that the graph of each of the following equations has a horizontal asymptote. Find an equation of the asymptote.

a. $y = x \sin \dfrac{a}{x}$.

b. $y = \dfrac{x^3}{e^{x^3}}$.

c. $y = (x^2+2)^{1/x^2}$.

d. $y = \dfrac{x+\ln x}{x \ln x}$.

e. $y = [\cos(1/x)]^{x^2}$.

f. $y = \dfrac{e^{-x}}{\ln(1+x^{-1})}$.

5. With p a positive number and a any number, show that:

a. $\lim_{h\to\infty} he^{-ph} = 0$.

b. $\lim_{h\to\infty} \int_0^h e^{-px}\,dx = \dfrac{1}{p}$.

c. $\lim_{h\to\infty} \int_0^h xe^{-px}\,dx = \dfrac{1}{p^2}$.

d. $\lim_{h\to\infty} \int_0^h x^2e^{-px}\,dx = \dfrac{2}{p^3}$.

e. $\lim_{h\to\infty} \int_0^h x^3e^{-px}\,dx = \dfrac{3!}{p^4}$.

f. $\lim_{h\to\infty} \int_0^h e^{-px}\sin ax\,dx = \dfrac{a}{p^2+a^2}$.

g. $\lim_{h\to\infty} \int_0^h e^{-px}\cos ax\,dx = \dfrac{p}{p^2+a^2}$.

h. $\lim_{h\to\infty} \int_0^h e^{-px}\,x\sin ax\,dx = \dfrac{2ap}{(p^2+a^2)^2}$.

6. Evaluate each of the following improper integrals:

a. $\int_0^1 \ln x\,dx$.

b. $\int_0^1 x\ln x\,dx$.

c. $\int_0^1 \ln^2 x\,dx$.

d. $\int_{-\infty}^0 xe^x\,dx$.

e. $\int_0^\infty x^4e^{-x}\,dx$.

f. $\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$.

137. Taylor's Theorem in Two Variables

Let F be a function of two variables, where F has continuous partial derivatives of as high order as we wish to use. With dx and dy any designated numbers (constants) used for differentials of the first and second variables, the total differential $dF(x,y)$ was defined by

$$dF(x,y) = \frac{\partial F(x,y)}{\partial x}\,dx + \frac{\partial F(x,y)}{\partial y}\,dy.$$

To save space in writing, we omit (x,y) and note that the second differential $d^2F \equiv d(dF)$ is such that

$$\begin{aligned}
d^2F &= d\left[\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy\right] \\
&= \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy\right]dx + \frac{\partial}{\partial y}\left[\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy\right]dy \\
&= \left[\frac{\partial^2 F}{\partial x^2}\,dx + \frac{\partial^2 F}{\partial x\,\partial y}\,dy\right]dx + \left[\frac{\partial^2 F}{\partial y\,\partial x}\,dx + \frac{\partial^2 F}{dy^2}\,dy\right]dy \\
&= \frac{\partial^2 F}{\partial x^2}(dx)^2 + 2\frac{\partial^2 F}{\partial x\,\partial y}\,dx\,dy + \frac{\partial^2 F}{\partial y^2}(dy)^2 \quad \text{since} \quad \frac{\partial^2 F}{\partial y\,\partial x} = \frac{\partial^2 F}{\partial x\,\partial y}.
\end{aligned}$$

Now $d^3F \equiv d(d^2F)$ and by working out the details it will be found that

$$d^3F = \frac{\partial^3 F}{\partial x^3}(dx)^3 + 3\frac{\partial^3 F}{\partial x^2\,\partial y}(dx)^2\,dy + 3\frac{\partial^3 F}{\partial x\,\partial y^2}(dx)(dy)^2 + \frac{\partial^3 F}{\partial y^3}(dy)^3.$$

The similarity of the expressions for d^2F and d^3F with the ordinary binomial expansions

$$(a+b)^2 = a^2 + 2ab + b^2, \quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

should be noted. Such similarities continue to higher differentials so that for any positive integer k:

$$\begin{aligned}
d^kF = \frac{\partial^k F}{\partial x^k}(dx)^k &+ k\frac{\partial^k F}{\partial x^{k-1}\,\partial y}(dx)^{k-1}\,dy \\
&+ \frac{k(k-1)}{2!}\frac{\partial^k F}{\partial x^{k-2}\,\partial y^2}(dx)^{k-2}(dy)^2 + \cdots + \frac{\partial^k F}{\partial y^k}(dy)^k.
\end{aligned}$$

This fact makes it possible to write Taylor's formula for a function of two variables in compact form. The formula to be obtained is:

$$\text{(1)} \quad F(x+dx, y+dy) = F(x,y) + dF(x,y) + \frac{d^2F(x,y)}{2!} + \cdots + \frac{d^{n-1}F(x,y)}{(n-1)!} + R_n$$

where for some numbers ξ and η, with ξ between x and $x + dx$ but η between y and $y + dy$, R_n has the form

$$R_n = \frac{d^n F(\xi,\eta)}{n!}.$$

This formula is obtained from a form of Taylor's formula for a function of one variable by letting f be defined by

$$(2) \qquad f(t) = F(x + t\,dx, y + t\,dy), \quad 0 \leq t \leq 1.$$

For t given and $0 < t$, there is a number τ_n such that $0 < \tau_n < t$ and

$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}t^{n-1} + R_n$$

where
$$R_n = \frac{1}{n!}f^{(n)}(\tau_n)t^n.$$

In particular for $t = 1$

$$(3) \qquad f(1) = f(0) + f'(0) + \frac{f''(0)}{2!} + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!} + R_n$$

where
$$R_n = \frac{f^{(n)}(\tau_n)}{n!}.$$

From (2) we have $f(1) = F(x + dx, y + dy)$ and $f(0) = F(x,y)$. Notice also from (2) that

$$\begin{aligned}
f'(t) &= F_x(x + t\,dx, y + t\,dy)D_t(t\,dx) + F_y(x + t\,dx, y + t\,dy)D_t(t\,dy) \\
&= F_x(x + t\,dx, y + t\,dy)\,dx + F_y(x + t\,dx, y + t\,dy)\,dy, \\
f'(0) &= F_x(x,y)\,dx + F_y(x,y)\,dy = dF(x,y), \\
f''(t) &= F_{xx}(x + t\,dx, y + t\,dy)\,dx\,D_t(t\,dx) \\
&\quad + F_{xy}(x + t\,dx, y + t\,dy)\,dx\,D_t(t\,dy) \\
&\quad + F_{yx}(x + t\,dx, y + t\,dy)\,dy\,D_t(t\,dx) \\
&\quad + F_{yy}(x + t\,dx, y + t\,dy)\,dy\,D_t(t\,dy), \\
f''(0) &= F_{xx}(x,y)(dx)^2 + 2F_{xy}(x,y)\,dx\,dy + F_{yy}(x,y)(dy)^2 \\
&= d^2F(x,y)
\end{aligned}$$

and by continuing in this manner

$$f^{(k)}(0) = d^kF(x,y).$$

Since $0 < \tau_n < 1$ in (3), then upon setting $\xi = x + \tau_n\,dx$ and $\eta = y + \tau_n\,dy$ it follows that ξ is between x and $x + dx$ whereas η is between y and $y + dy$. All these facts show that (1) is merely (3) under the substitution (2).

Example 1. Use three terms of a Taylor expansion of an appropriate function to find an approximation of

$$3.2 \sin 59°.$$

Solution. Set $F(x,y) = x \sin y$. Hence

$$F\left(3.2, \frac{59\pi}{180}\right) = 3.2 \sin 59°, \quad F\left(3, \frac{\pi}{3}\right) = 3 \sin 60° = 3\left(\frac{\sqrt{3}}{2}\right) = 2.598\,077.$$

Thus, use $dx = 0.2$, $dy = -\pi/180$ so that from (1)

$$F\left(3 + 0.2, \frac{\pi}{3} - \frac{\pi}{180}\right) \sim F\left(3, \frac{\pi}{3}\right) + dF\left(3, \frac{\pi}{3}\right) + \frac{1}{2!} d^2F\left(3, \frac{\pi}{3}\right).$$

Since $\qquad dF(x,y) = \sin y\, dx + x \cos y\, dy \quad$ and

$d^2F(x,y) = 0 \cdot (dx)^2 + 2 \cos y\, dx\, dy - x \sin y (dy)^2$, then

$$dF\left(3, \frac{\pi}{3}\right) = \left(\sin \frac{\pi}{3}\right)(0.2) + \left(3 \cos \frac{\pi}{3}\right)\left(-\frac{\pi}{180}\right) = 0.147\,025$$

$$d^2F\left(3, \frac{\pi}{3}\right) = 0 + 2\left(\cos \frac{\pi}{3}\right)(0.2)\left(-\frac{\pi}{180}\right) - 3\left(\sin \frac{\pi}{3}\right)\left(-\frac{\pi}{180}\right)^2 = -0.004\,282.$$

Hence, by keeping 6 decimal places then rounding to 5, we have

$$3.2 \sin 59° = 2.598\,077 + 0.147\,025 + (0.5)(-0.004\,282) = 2.742\,96.$$

Example 2. Expand $e^{x+dx} \sin (y + dy)$ through second-degree terms in dx and dy. Use this expansion to approximate $e^{0.5} \sin 5°$.

Solution. Set $f(x,y) = e^x \sin y$ and find

$$f_x(x,y) = e^x \sin y, \quad f_{xx}(x,y) = e^x \sin y, \quad f_{xy}(x,y) = e^x \cos y,$$

$$f_y(x,y) = e^x \cos y, \quad \text{and} \quad f_{yy}(x,y) = -e^x \sin y. \quad \text{Thus}$$

$$e^{x+dx} \sin (y + dy) \sim e^x \sin y + [e^x \sin y\, dx + e^x \cos y\, dy] + \tfrac{1}{2}[e^x \sin y (dx)^2 + 2e^x \cos y\, dx\, dy - e^x \sin y (dy)^2].$$

Upon setting

$$x = y = 0,\ dx = 0.5 \text{ and } dy = 5\pi/180 = \pi/36, \quad \text{then}$$

$$e^{0.5} \sin 5° \sim 0 + \left[0(0.5) + \frac{\pi}{36}\right] + \frac{1}{2}\left[0(0.5)^2 + 2(0.5)\frac{\pi}{36} - 0\left(\frac{\pi}{36}\right)^2\right]$$

$$= \frac{\pi}{36} + 0.5\frac{\pi}{36} = \frac{\pi}{24}.$$

PROBLEMS

1. a. Verify the Taylor expansion as far as given:

$$\sin (x + dx) \cos (y + dy) \sim \sin x \cos y + [\cos x \cos y\, dx - \sin x \sin y\, dy] + \tfrac{1}{2}[-\sin x \cos y (dx)^2 - 2 \cos x \sin y\, dx\, dy - \sin x \cos y (dy)^2].$$

b. Write down the corresponding expansion of $\cos(x + dx)\sin(y + dy)$.

c. Find the same number of terms for Taylor's expansion of

$$\cos(x + dx)\cos(y + dy).$$

d. Write down the corresponding expansion of $\sin(x + dx)\sin(y + dy)$.

e. Extend the result of Part a through the term involving $(dy)^3$, then set $x = y = 0$ and verify

$$\sin dx \cos dy \sim dx - \tfrac{1}{6}(dx)^3 - \tfrac{1}{2}\, dx(dy)^2.$$

f. By using Taylor's expansion for functions of one variable, write expansions of $\sin(x + dx)$ through $(dx)^2$ and $\cos(y + dy)$ through $(dy)^2$. Multiply the results to obtain an approximation formula for $\sin(x + dx)\cos(y + dy)$.

2. a. Expand $e^x \sin y$ through terms involving $(dy)^2$, then set $x = y = 0$ and obtain

$$e^{dx}\sin dy \sim dy + dx\, dy.$$

b. By the same method show that also

$$e^{dx}\ln(1 + dy) \sim dy + dx\, dy - \tfrac{1}{2}(dy)^2.$$

c. In Parts a and b extend the results through terms involving $(dy)^3$ and find the corresponding approximations of $e^{dx}\sin dy$ and $e^{dx}\ln(1 + dy)$.

138. Maxima and Minima, Two Variables

A function F of two variables is said to have $F(x_0,y_0)$ as a
relative maximum if $F(x,y) \le F(x_0,y_0)$, *but a*
relative minimum if $F(x,y) \ge F(x_0,y_0)$
for all (x,y) *in some neighborhood of* (x_0,y_0).

Given that (x_0,y_0) and a neighborhood of it are in the domains of F, F_x, and F_y and that $F(x_0,y_0)$ is either a rel. max. or a rel. min. of F, then each of the profiles

$$\{(x,y_0,z) \mid z = F(x,y_0)\} \quad \text{and} \quad \{(x_0,y,z) \mid z = F(x_0,y)\}$$

has a tangent line at $(x_0,y_0,F(x_0,y_0))$ parallel to the xy-plane and thus it follows (is necessary) that

$$F_x(x_0,y_0) = 0 \quad \text{and} \quad F_y(x_0,y_0) = 0. \tag{1}$$

Equations (1) are, however, not sufficient for either a rel. max. or a rel. min. of F at (x_0,y_0). For example

$$\frac{\partial(x^2 - y^2)}{\partial x} = 2x \quad \text{and} \quad \frac{\partial(x^2 - y^2)}{\partial y} = -2y$$

are both 0 at (0,0), but F defined by $F(x,y) = x^2 - y^2$ does not have $F(0,0) = 0$ either a rel. max. or a rel. min. since, in particular, the profiles

$$\{(x,0,z) \mid z = F(x,0) = x^2\} \quad \text{and} \quad \{(0,y,z) \mid z = F(0,y) = -y^2\}$$

are parabolas with vertices at the origin, the first with vertex down but the second with vertex up.

The following theorem (a proof is at the end of the section) gives sufficient conditions for relative maxima and relative minima.

THEOREM 138. *Let F be a function, let (x_0,y_0) and a neighborhood of it be in the domain of F, let*

$$F_x(x_0,y_0) = 0, \quad F_y(x_0,y_0) = 0, \quad \textit{and} \tag{1}$$

$$F_{xx}(x_0,y_0)F_{yy}(x_0,y_0) - [F_{xy}(x_0,y_0)]^2 > 0, \tag{2}$$

and let F_{xx}, F_{yy}, and F_{xy} be continuous at (x_0,y_0). Then $F(x_0,y_0)$ is

$$\textit{a rel. max. of } F \textit{ if } F_{xx}(x_0,y_0) < 0, \textit{ but is a rel. min. of } F \textit{ if } F_{xx}(x_0,y_0) > 0. \tag{3}$$

Example 1. Find the dimensions of the box of minimum cost if the volume is V ft^3, the base costs 15¢/ft^2, the lid 10¢/ft^2, and the sides 5¢/ft^2.

Solution. With z ft the depth and the base x ft by y ft, then $V = xyz$ and the cost is

$$C = 15xy + 10xy + 2(5xz + 5yz) = 25xy + 10V\left(\frac{1}{y} + \frac{1}{x}\right).$$

Thus $C_x = 25y - \dfrac{10V}{x^2}$, $C_y = 25x - \dfrac{10V}{y^2}$ so that $C_x = 0$ and $C_y = 0$ has solutions

$$x_0 = y_0 = \sqrt[3]{2V/5} = \sqrt[3]{0.4V}.$$

Since $C_{xx} = \dfrac{20V}{x^3}$, $C_{yy} = \dfrac{20V}{y^3}$, and $C_{xy} = 25$, relation (2) is

$$\frac{20V}{0.4V}\cdot\frac{20V}{0.4V} - (25)^2 = (50)^2 - (25)^2 > 0.$$

Also C_{xx} at (x_0,y_0) is $50 > 0$ so (x_0,y_0) determines a minimum. The box therefore has square base $\sqrt[3]{0.4V}$ on a side and the altitude is $\sqrt[3]{V}(0.4)^{-2/3}$.

Example 2. Find the minimum distance from the origin to the surface S having equation

$$z^2 = xy - 4. \tag{4}$$

Solution. Let F be the function defined by

$$F(x,y) = x^2 + y^2 + xy - 4 \quad \text{with domain} \quad \{(x,y) \mid xy - 4 \ge 0\}. \tag{5}$$

Hence, if (x,y,z) is a point on S (so $z^2 = xy - 4$), then $F(x,y)$ is the square of the distance from the origin to this point. It is necessary to specify the domain in (5) since if (x,y,z) is a point of S, then $z^2 = xy - 4 \ge 0$.

Now $F_x = 2x + y$, $F_y = 2y + x$ and these could both be zero if and only if $x = 0$ and $y = 0$. Since $(0,0)$ is not a point of the domain of F, then F_x and F_y cannot both be zero at any point in the domain of F. Consequently, if F has a minimum, this minimum will occur at a boundary point of the domain of F, i.e., at a point (x,y) such that $xy - 4 = 0$ so $y = 4/x$. For such a point, the square of the distance to the origin is

$$f(x) = F\left(x, \frac{4}{x}\right) = x^2 + \left(\frac{4}{x}\right)^2 + 0 = x^2 + 16x^{-2}.$$

Since $f'(x) = 2x - 32x^{-3}$ it follows that $f'(x) = 0$ if

$$2x - \frac{32}{x^3} = 0 \quad \text{and thus if} \quad x = \pm 2.$$

Also $f''(x) = 2 + 96x^{-4} > 0$. Thus, the desired minimum is

$$\sqrt{f(2)} = \sqrt{2^2 + (4/2)^2} = 2\sqrt{2} \quad \text{or} \quad \sqrt{f(-2)} = 2\sqrt{2}$$

and occurs at the points (2,2,0) and $(-2,-2,0)$ of S.

PROOF of Theorem 138. Let $\delta > 0$ be such that if $|h| < \delta$ and $|k| < \delta$, then $(x_0 + h, y_0 + k)$ is in the domains of the continuity of F_{xx}, F_{yy}, and F_{xy}; such that $F_{xx}(x_0 + h, y_0 + k)$ has the same sign as $F_{xx}(x_0,y_0)$ and such that

$$[F_{xx}F_{yy} - F_{xy}^2]_{(x_0+h,y_0+k)} > 0. \tag{6}$$

Choose any numbers dx and dy such that $|dx| < \delta$ and $|dy| < \delta$. Let ξ be between x_0 and $x_0 + dx$, and η be between y_0 and $y_0 + dy$ such that (from Taylor's formula)

$$F(x_0 + dx, y_0 + dy) = F(x_0,y_0) + [F_x(x_0,y_0)\,dx + F_y(x_0,y_0)\,dy] + \frac{1}{2!}[F_{xx}(dx)^2 + 2F_{xy}\,dx\,dy + F_{yy}(dy)^2]_{(\xi,\eta)}.$$

Since $F_x(x_0,y_0) = F_y(x_0,y_0) = 0$ from (1), it follows that

$$F(x_0 + dx, y_0 + dy) - F(x_0,y_0) = \tfrac{1}{2}[F_{xx}(dx)^2 + 2F_{xy}\,dx\,dy + F_{yy}\,(dy)^2]_{(\xi,\eta)}$$

$$= \frac{F_{xx}(\xi,\eta)}{2}\left\{(dx)^2 + 2\frac{F_{xy}}{F_{xx}}\,dx\,dy + \frac{F_{yy}}{F_{xx}}(dy)^2\right\}_{(\xi,\eta)}$$

$$= \frac{F_{xx}(\xi,\eta)}{2}\left\{(dx)^2 + 2\frac{F_{xy}}{F_{xx}}\,dx\,dy + \left(\frac{F_{xy}}{F_{xx}}\,dy\right)^2 + \frac{F_{yy}}{F_{xx}}(dy)^2 - \left(\frac{F_{xy}}{F_{xx}}\,dy\right)^2\right\}_{(\xi,\eta)}$$

$$= \frac{F_{xx}(\xi,\eta)}{2}\left\{\left[dx + \frac{F_{xy}}{F_{xx}}\,dy\right]^2 + \frac{F_{xx}F_{yy} - (F_{xy})^2}{(F_{xx})^2}(dy)^2\right\}_{(\xi,\eta)}.$$

The first term in { } is ≥ 0 and the second is also ≥ 0 by (6). Thus $F(x_0 + dx, y_0 + dy) - F(x_0,y_0)$ has the same sign as $F_{xx}(\xi,\eta)$ which is the same as the sign of $F_{xx}(x_0,y_0)$; that is,

$$F(x_0 + dx, y_0 + dy) \leq F(x_0,y_0) \text{ if } F_{xx}(x_0,y_0) < 0, \quad \text{but}$$

$$F(x_0 + dx, y_0 + dy) \geq F(x_0,y_0) \text{ if } F_{xx}(x_0,y_0) > 0.$$

PROBLEMS

1. A bin of volume 48 ft^3 is to be put in a basement corner by using the floor and two walls. Find the dimensions for minimum cost if one side costs 5¢/ft^2, the other side 10¢/ft^2, and the lid 15¢/ft^2.

2. Find the dimensions and volume of the rectangular parallelepiped of relative maximum volume which has three faces in the coordinate planes and the vertex opposite the origin in the plane having equation:

   a. $6x + 4y + 3z = 36$. b. $6x - 3y + 2z = 6$.

3. Find the minimum distance from the origin to the surface having equation

   a. $z^2 = xy + 4$. c. $z^2 = x^2y - 8$. e. $2x + 3y + z = 7$.
   b. $z^2 = x^2y + 8$. d. $x^2 + y^2 = 4$. f. $3y + z = 7$.

4. Find the minimum distance between the lines L_1 and L_2 whose parametric equations are:

   a. L_1: $x = 5 + 2s$, $y = 8 + 2s$, $z = 6 - 2s$,
   L_2: $x = -1 + t$, $y = 6 - 2t$, $z = t$.

   b. L_1: $x = -1$, $y = 5 + 3s$, $z = -6 - s$,
   L_2: $x = 2 - 4t$, $y = 6$, $z = 7 + t$.

5. Given that (2) holds, prove:

   a. $f_{xx}(x_0,y_0) \neq 0$. b. $f_{yy}(x_0,y_0)$ and $f_{xx}(x_0,y_0)$ have the same sign.

Let f be a function whose $(n + 1)$st derived function $f^{(n+1)}$ is continuous on the closed interval $I[a,b]$. Define I_k by

$$I_k = \int_a^b \frac{(b - x)^{k-1}}{(k - 1)!} f^{(k)}(x)\, dx, \quad \text{for } k = 1,2, \cdots, n + 1.$$

a. Show that $I_1 = f(b) - f(a)$.

b. By using integration by parts, show that

$$I_k = \frac{f^{(k)}(a)}{k!}(b - a)^k + I_{k+1}, \quad \text{for } k = 1,2, \cdots, n.$$

c. Show that $\sum_{k=1}^{n} I_k$ leads to the expression

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + R_n$$

where $$R_n = \int_a^b \frac{(b - x)^n}{n!} f^{(n+1)}(x)\, dx.$$

(Note: This is called Taylor's formula with remainder in integral form.)

Start with the remainder in integral form, use the Law of the Mean for integrals (page 180), and obtain both the Lagrange and the Cauchy forms of the remainder (page 423).

CHAPTER 13

Series

"Achilles running to overtake a crawling tortoise ahead of him can never overtake it, because he must first reach the place from which the tortoise started; when Achilles reaches that place the tortoise has departed and so is still ahead. Repeating the argument we easily see that the tortoise will always be ahead."† This reasoning of Zeno's (fifth century B.C.) was one of the perplexing paradoxes of antiquity and awaited a satisfactory answer until Friedrich Gauss (1777–1855), "The Prince of Mathematicians" according to E. T. Bell,† started the trend of rigorous work on infinite series.

139. Sequences

A sequence is a function whose domain is the set of all integers greater than or equal to a given integer. For example, the function f defined for each integer $n \geq 1$ by $f(n) = 1/n$ is a sequence which is usually displayed as

$$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots.$$

Problem 1 below shows there are other sequences having 1, $\frac{1}{2}$, $\frac{1}{3}$ as their first three terms. The purpose of this problem is to show that it is necessary to know the definition of the function which is the sequence.

Since $0! = 1$, the function g defined by $g(n) = 1/n!$ for each integer $n \geq 0$ is the sequence

$$1, 1, \frac{1}{2}, \frac{1}{6}, \cdots, \frac{1}{n!}, \cdots.$$

Subscript notation, rather than the usual functional notation, is used in connection with sequences. Thus, the sequence defined by

$$(1) \qquad s_n = \frac{3n}{n-1}$$

† *Men of Mathematics* (New York, Simon and Schuster, Inc., 1937). This is good recreational reading; a blend of biography, history, and philosophy. And don't skip the Introduction.

is understood to have its first (or initial) term corresponding to $n = 2$ and the sequence is displayed as

$$6, \frac{9}{2}, 4, \cdots, \frac{3n}{n-1}, \cdots.$$

The defining relation of a sequence is also called the **general term** of the sequence. If s_n is the general term of a sequence, then the sequence is denoted by $\{s_n\}$ which really means $\{(n,y) \mid y = s_n\}$.

DEFINITION 139.1. *A sequence $\{s_n\}$ is said to be* **convergent**, *and to converge to s, where s is a number, if*

$$\lim_{n\to\infty} s_n = s;$$

that is, if corresponding to each positive number ϵ there is an integer N such that whenever $n \geq N$, then $|s_n - s| < \epsilon$. If a sequence is not convergent, then the sequence is said to be **divergent**.

Thus, the sequence defined by (1) is convergent, and converges to 3, since

$$\lim_{n\to\infty} \frac{3n}{n-1} = \lim_{n\to\infty} \frac{3}{1 - 1/n} = 3.$$

The sequence $1, -1, 1, -1, \cdots, (-1)^n, \cdots$ is divergent and is typical of sequences which are said to **oscillate**.

DEFINITION 139.2. *A sequence $\{s_n\}$ is said to* **diverge to** $+\infty$ *if corresponding to each positive number G there is an integer N such that whenever $n \geq N$, then $s_n > G$ and the limit notation*

$$\lim_{n\to\infty} s_n = +\infty$$

is used even though this means that the limit does not exist. A similar definition is given for divergence of a sequence to $-\infty$.

For example, $1, 2, 4, \cdots, 2^n, \cdots$, diverges to $+\infty$. However,

$$1, -2, 4, \cdots, (-2)^n, \cdots$$

diverges, but neither to $+\infty$ nor to $-\infty$.

Let r be a number. We shall show that the sequence

(2) $$|r|, |r|^2, \cdots, |r|^n, \cdots$$

diverges to $+\infty$ if $|r| > 1$, but converges to 0 if $|r| < 1$; that is,

$$\lim_{n\to\infty} |r|^n = \begin{cases} +\infty & \text{if } |r| > 1 \\ 0 & \text{if } |r| < 1. \end{cases}$$

First, consider $|r| > 1$ and let c be the positive number such that $|r| = 1 + c$. Hence

$$|r|^n = (1 + c)^n \geq 1 + nc,$$

as may be seen by using the binomial expansion of $(1 + c)^n$. Hence, for G a positive number, let N be an integer such that $N > (G - 1)/c$. Then for $n \geq N$ it follows that

$$|r|^n \geq 1 + Nc > 1 + (G - 1) = G \quad \text{so that} \quad \lim_{n\to\infty} |r|^n = +\infty.$$

Hence, the sequence (2) diverges to $+\infty$ if $|r| > 1$.

Next consider $|r| < 1$. If $|r| = 0$, then $|r|^n = 0$ and $\lim_{n\to\infty} |r|^n = 0$. Hence, take $0 < |r| < 1$ and now $1/|r| > 1$. Choose any number $\epsilon > 0$ and then an integer N such that (by the first part)

$$\text{whenever } n > N \quad \text{then } \left[\frac{1}{|r|}\right]^n > \frac{1}{\epsilon}; \quad \text{that is, } |r|^n < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\lim_{n\to\infty} |r|^n = 0$ and hence that the sequence (2) converges to 0 if $|r| < 1$.

Laws of limits for functions hold when the functions are sequences as stated in the following theorem using sequence notation.

THEOREM 139.1. *If $\{s_n\}$ and $\{t_n\}$ are convergent sequences and c is a number, then*

$$\lim_{n\to\infty} cs_n = c \lim_{n\to\infty} s_n,$$

$$\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n,$$

$$\lim_{n\to\infty} (s_n t_n) = \lim_{n\to\infty} s_n \times \lim_{n\to\infty} t_n,$$

and if $\lim_{n\to\infty} t_n \neq 0$, *then* $t_n \neq 0$ *for n sufficiently large and*

$$\lim_{n\to\infty} \frac{s_n}{t_n} = \lim_{n\to\infty} s_n \Big/ \lim_{n\to\infty} t_n.$$

Example 1. Let a be a non-zero number, let r be a number, and let $\{s_n\}$ be the sequence

$$\begin{aligned}
s_0 &= a,\\
s_1 &= a + ar,\\
s_2 &= a + ar + ar^2,\\
&\cdots\cdots\\
s_n &= a + ar + ar^2 + \cdots + ar^n,\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

Show that this sequence converges if and only if $|r| < 1$.

Solution. CASE 1. $|r| \neq 1$ so in particular $1 - r \neq 0$. Notice that

$$\begin{aligned} rs_n &= ar + ar^2 + ar^3 + \cdots + ar^{n+1} \\ &= (a + ar + ar^2 + \cdots + ar^n) + ar^{n+1} - a \\ &= s_n + ar^{n+1} - a, \end{aligned}$$

so that $s_n(r - 1) = ar^{n+1} - a$ and $s_n = (ar^{n+1} - a)/(r - 1)$. Hence, if $|r| < 1$, then $\lim_{n\to\infty} r^{n+1} = 0$,

$$\lim_{n\to\infty} s_n = \frac{-a}{r-1} = \frac{a}{1-r},$$

and the sequence $\{s_n\}$ converges, but if $|r| > 1$ then $\lim_{n\to\infty} |r|^{n+1} = +\infty$ and the sequence $\{s_n\}$ diverges.

CASE 2. $|r| = 1$. If $r = 1$ the sequence is

$$a, 2a, 3a, \cdots, (n + 1)a, \cdots$$

which diverges (to $+\infty$ if $a > 0$, but to $-\infty$ if $a < 0$). If $r = -1$, the sequence is the divergent (oscillating) sequence

$$a, 0, a, 0, \cdots, \frac{a + (-1)^n a}{2}, \cdots.$$

Hence, the sequence converges if and only if $|r| < 1$.

The following theorem will be used repeatedly.

THEOREM 139.2. *If k is a number and $\{u_n\}$ is a sequence such that both*

$$u_1 \le u_2 \le u_3 \le \cdots \le u_n \le \cdots \quad \textit{and} \quad u_n \le k,$$

then the sequence $\{u_n\}$ is convergent and $\lim_{n\to\infty} u_n \le k$.

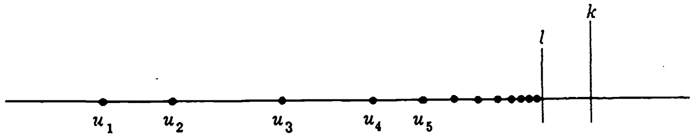


Figure 139.1

PROOF. Let $S = \{x \mid x = u_n \text{ for some integer } n\}$. Then S is bounded above by k and we denote the least upper bound of S by l (which exists by the axiom of Sec. 4). Hence

$$u_n \le l \quad \text{for} \quad n = 1, 2, 3, \cdots$$

since l is an upper bound of S. Let $\epsilon > 0$ be arbitrary. Then $l - \epsilon < l$ so $l - \epsilon$ is not an upper bound of S and we let N be such that $l - \epsilon < u_N$. Consequently, whenever $n > N$ then $l - \epsilon < u_N \leq u_n$. Hence

$$\text{whenever } n > N \quad \text{then} \quad |u_n - l| < \epsilon$$

which states that $\lim\limits_{n\to\infty} u_n$ exists (and is l).

The theorem above is sometimes stated: *If a sequence is monotonically increasing and is bounded above, then the sequence is convergent.*

It follows that *if a sequence is monotonically decreasing and bounded below, then the sequence is convergent.* For if

$$b_1 \geq b_2 \geq b_3 \geq \cdots \geq b_n \geq \cdots \geq \lambda,$$

then $-b_1 \leq -b_2 \leq -b_3 \leq \cdots \leq -b_n \leq \cdots \leq -\lambda$, so that $\lim\limits_{n\to\infty}(-b_n)$ exists. But $-\lim\limits_{n\to\infty}(-b_n) = -(-1)\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b_n$.

Example 2. Show that the sequence $\{s_n\}$ defined by

$$s_0 = \frac{1}{0!}$$

$$s_1 = \frac{1}{0!} + \frac{1}{1!}$$

$$s_2 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}$$

$$\cdots\cdots\cdots\cdots$$

$$s_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

$$\cdots\cdots\cdots\cdots$$

is a convergent sequence.

Solution. Since each term is the preceding term increased by a positive number, we have $s_0 < s_1 < s_2 < \cdots < s_n < \cdots$. Note that $k! = 1 \cdot 2 \cdot 3 \cdots k \geq 2^{k-1}$ and thus $1/k! \leq 1/2^{k-1}$. Consequently

$$s_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 2 + \left(1 - \frac{1}{2^{n-1}}\right) < 3.$$

Thus we have a monotonically increasing sequence bounded above. Hence, by the theorem above, $\lim_{n\to\infty} s_n$ exists; i.e., the sequence is convergent.

Example 3. Let p be a positive number and let $\{s_n\}$ be the sequence

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{2^p} \\ s_3 &= 1 + \frac{1}{2^p} + \frac{1}{3^p} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ s_n &= 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}$$

Show that $\{s_n\}$ converges if and only if $p > 1$.

Solution. Sketch the graph of $y = 1/x^p$, $x > 0$.

CASE 1. $p > 1$ so $p - 1 > 0$. Construct the rectangles shown in Fig. 139.2, note that s_n = sum of areas of first n rectangles, and

$$\begin{aligned} s_n &\le 1 + \int_1^n \frac{1}{x^p}\,dx \\ &= 1 + \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right) \\ &< 1 + \frac{1}{p-1}, \quad p - 1 > 0. \end{aligned}$$

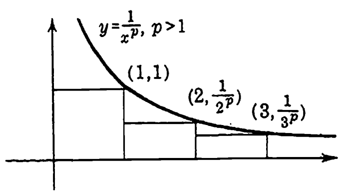


Figure 139.2

Thus, $\{s_n\}$ is a monotonically increasing sequence bounded above and hence is convergent.

CASE 2. $0 < p \le 1$ so $1 - p \ge 0$. Construct the rectangles shown in Fig. 139.3. Again s_n = sum of areas of first n rectangles, but

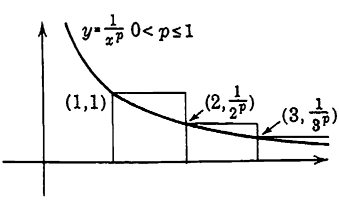


Figure 139.3

$$s_n \ge \int_1^n \frac{1}{x^p}\,dx = \begin{cases} \ln n & \text{if } p = 1 \\ \dfrac{1}{1-p}(n^{1-p} - 1) & \text{if } p < 1. \end{cases}$$

In either situation ($p = 1$ or $p < 1$) we see that merely by choosing n sufficiently large we may make s_n be as large as we please, i.e., $\{s_n\}$ diverges to $+\infty$.

PROBLEMS

1. For the sequence whose nth term u_n is given, show that the first three terms are $1, \frac{1}{2}, \frac{1}{3}$. Also find the fourth term.

a. $u_n = \dfrac{1}{n}$.

b. $u_n = \dfrac{1}{n^3 - 6n^2 + 12n - 6}$.

c. $u_n = \dfrac{n^2 - 3n + 11}{3(n^2 + 2)}$.

d. $u_n = n^2 - 6n + 11 - \dfrac{5}{n}$.

e. $u_n = \dfrac{1}{n\,2^{(n-1)(n-2)(n-3)}}$.

f. $u_n = \dfrac{n + (-1)^n(n - 2)}{2n}$.

2. Find $\lim\limits_{n\to\infty} u_n$ if

a. $u_n = \dfrac{5n}{2n - 1}$.

b. $u_n = \dfrac{5n^2}{2n^2 - 1}$.

c. $u_n = \dfrac{5n^{3/2}}{2n^{3/2} - 1}$.

d. $u_n = \dfrac{5n^{3/2}}{\sqrt{n}(2n - 1)}$.

e. $u_n = \dfrac{(5 + n)^{100}}{5^{n+1}} \cdot \dfrac{5^n}{(4 + n)^{100}}$.

f. $u_n = \dfrac{n\sqrt{n} - 1}{(n + 1)\sqrt{n + 1} - 1}$.

g. $u_n = \dfrac{\sqrt{n + 25}}{\sqrt{n}}$.

h. $u_n = \dfrac{e^{n+1}}{1 + 3^{n+1}} \dfrac{1 + 3^n}{e^n}$.

i. $u_n = \dfrac{\sin(1/2n)}{1/n}$.

j. $u_n = \dfrac{n}{2^n}$.

k. $u_n = \dfrac{n + 2^n}{(n + 1) + 2^{n+1}}$.

l. $u_n = \dfrac{4n^5 + 3n^4 - 5n^3 + 2n^2 - 58n + 6}{3n^5 + 6n^4 - 500n^3 - n + 25}$.

3. For the given definition of u_n, find $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n}$.

a. $u_n = \dfrac{e^n}{1 + 3^n}$.

b. $u_n = \dfrac{e^{n-1}}{1 + 3^{n-1}}$.

c. $u_n = \dfrac{e^n}{1 + 2^n}$.

d. $u_n = \dfrac{n^{100}}{(1.001)^n}$.

e. $u_n = \dfrac{1}{n^{100}(0.999)^n}$.

f. $u_n = \dfrac{10^n}{n!}$.

g. $u_n = \dfrac{n!}{10^n}$.

h. $u_n = \dfrac{n}{5^n}$.

i. $u_n = \dfrac{n}{\sqrt{2n + 3}}$.

4. Let $\{s_n\}$ be the sequence such that

$s_1 = 1,\quad s_2 = 0,\quad s_3 = 1,\quad s_4 = 0,\quad \cdots,\quad s_n = \frac{1}{2}\{1 + (-1)^{n+1}\},\quad \cdots.$

Let $\{a_n\}$ be the sequence defined (taking arithmetic means) by

$$a_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

Show that $\lim\limits_{n\to\infty} a_n = \frac{1}{2}$.

140. Series of Numbers

If $\{u_n\}$ is a sequence of numbers, then

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

is called a **series** (or an **infinite series**) and is also represented by

$$\sum_{n=1}^{\infty} u_n \quad \text{or} \quad \sum_{k=1}^{\infty} u_k.$$

It is sometimes convenient to have the first term of a series associated with zero and to write

$$\sum_{n=0}^{\infty} u_n \quad \text{or} \quad \sum_{k=0}^{\infty} u_k.$$

Thus, if $u_n = 1/n!$ for $n \geq 0$, then we write

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{1}{n!}.$$

If we do not want to be specific as to the first term we write merely

$$\sum u_n.$$

With some series a number is associated, but with other series no number is associated in the following definition.

DEFINITION. *Given a series*

(1) $$u_1 + u_2 + u_3 + \cdots + u_n + \cdots,$$

form the sequence $\{s_n\}$ *(called the sequence of* ***partial sums****) by writing*

$$\begin{aligned} s_1 &= u_1 \\ s_2 &= u_1 + u_2 \\ s_3 &= u_1 + u_2 + u_3 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ s_n &= u_1 + u_2 + u_3 + \cdots + u_n = \sum_{k=1}^{n} u_k \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

If this sequence $\{s_n\}$ *is convergent, then we say the series* (1) *is* ***convergent.*** *Also, if this sequence* $\{s_n\}$ ***converges to l****, we set*

$$l = u_1 + u_2 + u_3 + \cdots + u_n + \cdots = \sum_{n=1}^{\infty} u_n$$

and say that the ***sum*** *of the series is l. If a series is not convergent, we say the series is* ***divergent****, and attach no number to the series.*

Thus from Examples 1 and 3 of Sec. 139, we obtain the following two examples, respectively.

Example 1. For $a \neq 0$, the series

$$a + ar + ar^2 + \cdots + ar^n + \cdots \tag{2}$$

is convergent if $|r| < 1$ $\left(\text{and then converges to } \dfrac{a}{1 - r}\right)$, but is divergent if $|r| \geq 1$.

Example 2. The series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \tag{3}$$

converges if $p > 1$ (although we do not know what it converges to), but diverges if $p \leq 1$.

We now establish the result:

If a series $\Sigma\, u_n$ *converges, then* $\lim\limits_{n\to\infty} u_n = 0$. For if the series converges, then $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} u_k$ and $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n-1} u_k$ both exist and are the same so that

$$0 = \lim_{n\to\infty} \sum_{k=1}^{n} u_k - \lim_{n\to\infty} \sum_{k=1}^{n-1} u_k = \lim_{n\to\infty} \left(\sum_{k=1}^{n} u_k - \sum_{k=1}^{n-1} u_k \right) = \lim_{n\to\infty} u_n.$$

We shall give several results which may be used to test whether a given series converges or diverges. As a consequence of the above result, we obtain the first of these tests.

TEST 1. *If as* $n \to \infty$ *the nth term of a series does not have a limit or else has a limit different from* 0, *then the series diverges.*

As examples

$$1 + 0 - 1 + 0 + \cdots + \sin n\frac{\pi}{2} + \cdots$$

diverges since $\lim\limits_{n\to\infty} \sin n\dfrac{\pi}{2}$ does not exist, whereas

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{n}{n+1} + \cdots \text{ diverges since } \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0.$$

Notice that Test 1 is a test for divergence (and not convergence); that is, even though the nth term approaches zero, the series may still diverge. For example, the divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

[which is (3) with $p = 1$] has $\lim\limits_{n\to\infty} (1/n) = 0$.

TEST 2. *If* $a_n > 0$ *for* $n = 1, 2, 3, \cdots$, *if* $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots$, *and if* $\lim_{n\to\infty} a_n = 0$, *then*

$$a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}a_n + \cdots$$

is a convergent series.

PROOF. The odd partial sums written as

$$\begin{aligned} s_1 &= a_1 \\ s_3 &= (a_1 - a_2) + a_3 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ s_{2n-1} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-3} - a_{2n-2}) + a_{2n-1} > 0 \end{aligned}$$

shows that the odd partial sums are bounded below by zero. Also

$$s_{2n+1} = s_{2n-1} - a_{2n} + a_{2n+1} = s_{2n-1} - (a_{2n} - a_{2n+1}) \leq s_{2n-1}$$

shows that the odd partial sums form a monotonically decreasing sequence. Hence, by Theorem 139.1 we know that $\lim_{n\to\infty} s_{2n-1}$ exists. Since $\lim_{n\to\infty} a_{2n} = 0$ and $s_{2n} = s_{2n-1} - a_{2n}$ it then follows that

$$\lim_{n\to\infty} s_{2n-1} = \lim_{n\to\infty} s_{2n-1} - \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} (s_{2n-1} - a_{2n}) = \lim_{n\to\infty} s_{2n}.$$

Thus, the sequence of partial sums has a limit, i.e., the series converges.

Example 3. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1}\frac{1}{n} + \cdots$ converges.

TEST 3. *If* $\Sigma\, u_n$ *is a series of positive terms and* N *is a positive integer such that*

$$\frac{u_{n+1}}{u_n} \geq 1 \quad \textit{whenever } n \geq N,$$

then $\Sigma\, u_n$ *diverges.*

PROOF. For if the conditions are satisfied, then

$$\frac{u_{N+1}}{u_N} \geq 1 \quad \text{so that} \quad u_{N+1} \geq u_N > 0,$$

$$\frac{u_{N+2}}{u_{N+1}} \geq 1 \quad \text{so that} \quad u_{N+2} \geq u_{N+1} \geq u_N > 0,$$

and it should be seen that whenever $n \geq N$, then $u_n \geq u_N > 0$. Thus u_n does not approach 0 as $n \to \infty$ so the series diverges by Test 1.

Example 4. Show that $\frac{1}{10} + \frac{2}{100} + \frac{6}{1000} + \cdots + \frac{n!}{10^n} + \cdots$ diverges.

Solution. Upon letting $u_n = n!/10^n$ we have $\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \frac{n+1}{10}$, which is greater than 1 whenever $n > 10$. Thus the series diverges by Test 3.

Notice for the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$, that

$$\frac{u_{n+1}}{u_n} = \frac{n}{n+1} \quad < 1 \text{ for every } n,$$

but nevertheless the series diverges (see Example 2 with $p = 1$).

TEST 4. (*The* **Ratio Test**). *Let $\Sigma\, u_n$ be a series of positive terms such that*

$$\rho = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} \tag{4}$$

exists. Then

$$\text{if } \begin{cases} \rho < 1 & \textit{the series converges.} \\ \rho > 1 & \textit{the series diverges.} \\ \rho = 1 & \textit{there is no conclusion.} \end{cases}$$

PROOF. CASE 1. $\rho < 1$. Let r be a number such that $\rho < r < 1$. Since r is greater than ρ and (4) holds, let N be such that

whenever $n \geq N$, then $\frac{u_{n+1}}{u_n} < r = \frac{r^{n+1}}{r^n}$ and thus $\frac{u_{n+1}}{r^{n+1}} < \frac{u_n}{r^n}$.

The last inequality holds between u's with two successive subscripts so long as both these subscripts are $\geq N$. Thus

$$\text{whenever } n > N, \quad \text{then} \quad \frac{u_{n+1}}{r^{n+1}} < \frac{u_n}{r^n} < \frac{u_{n-1}}{r^{n-1}} < \cdots < \frac{u_N}{r^N}.$$

Out of this we separate the fact that

$$\text{whenever } n > N, \quad \text{then} \quad u_n < \frac{u_N}{r^N}\, r^n.$$

Consequently for any integer $m > N$, then

$$\sum_{n=N}^{m} u_n < \frac{u_N}{r^N} \sum_{n=N}^{m} r^n < \frac{u_N}{r^N} \sum_{n=0}^{\infty} r^n = \frac{u_N}{r^N}\, \frac{1}{1-r}$$

where the equality follows from Example 1 and the fact that $0 < r < 1$. Now let C be the constant defined by

$$C = u_1 + u_2 + \cdots + u_{N-1} + \frac{u_N}{r^N} \frac{1}{1-r}.$$

Hence for the positive integer n, we have that

$$\sum_{k=1}^{n} u_k \leq C.$$

Thus, the sequence of partial sums is increasing (since $u_n > 0$) and is bounded above (by C) so is convergent; i.e. the series $\Sigma\, u_n$ is convergent.

CASE 2. $\rho > 1$. There is then an integer N such that

$$\text{whenever } n \geq N \quad \text{then} \quad \frac{u_{n+1}}{u_n} > 1$$

(since the limit is greater than 1) and thus the series diverges by Test 3.

CASE 3. $\rho = 1$. To show that no conclusion can be drawn as to convergence or divergence in this case, we need only exhibit one convergent series with $\rho = 1$ and one divergent series with $\rho = 1$. To do so we note that for p any number, the series having $u_n = 1/n^p$ is such that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^p = 1^p = 1.$$

However (see Example 2), the series $\Sigma \dfrac{1}{n^p}$ converges if $p > 1$, but diverges if $p \leq 1$. Thus, some convergent series have $\rho = 1$, but also some divergent series have $\rho = 1$.

Example 5. Show that the following series converges:

$$100 + 5000 + \frac{100^3}{3!} + \cdots + \frac{100^n}{n!} + \cdots.$$

Solution. Since $u_n = \dfrac{100^n}{n!}$ and $u_{n+1} = \dfrac{100^{n+1}}{(n+1)!}$ we see that

$$\frac{u_{n+1}}{u_n} = \frac{100^{n+1}}{100^n} \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots n(n+1)} = \frac{100}{n+1}.$$

Consequently $\displaystyle\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{100}{n+1} = 0 < 1$, and thus the series converges.

Example 6. Establish the divergence of the series

$$2 + \frac{1}{2} + \frac{8}{27} + \frac{1}{4} + \cdots + \frac{2^n}{n^3} + \cdots.$$

Solution. We compute $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^3} \cdot \frac{n^3}{2^n} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^3 = 2 > 1$, and thus see that the series is divergent.

Example 7. Is the following series convergent or divergent?

$$\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \cdots + \frac{n}{2n+1} + \cdots$$

Solution. The ratio test yields

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n+1}{2n+3} \cdot \frac{2n+1}{n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{2n+1}{2n+3}$$
$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\left(\frac{2+1/n}{2+3/n}\right) = 1 \cdot \frac{2}{2} = 1$$

and from this ($\rho = 1$) there is no conclusion as to the convergence or divergence of the series. Notice, however, that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2},$$

which is not 0, and thus (by Test 1) the series is divergent.

Example 8. For $n = 1, 2, 3, \cdots$, let f_n be the function defined by

$$f_n(x) = \frac{x^n}{n3^n}.$$

Show that $\sum_{n=1}^{\infty} f_n(2)$ converges, but that $\sum_{n=1}^{\infty} f_n(3)$ diverges.

Solution. Upon letting u_n be the nth term of the first series we have

$$u_n = \frac{1}{n}\left(\frac{2}{3}\right)^n, \quad u_{n+1} = \frac{1}{n+1}\left(\frac{2}{3}\right)^{n+1},$$

$$\frac{u_{n+1}}{u_n} = \frac{n}{n+1}\frac{2}{3}, \quad \text{and} \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{2}{3} < 1.$$

Therefore (by Test 4) the first series converges. Since $f_n(3) = \frac{1}{n}$, the second series is $\Sigma \frac{1}{n}$ and therefore diverges (from Example 2 with $p = 1$).

COROLLARY (to Test 4). *If for each integer n, $u_n \neq 0$ and if*

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1, \tag{4}$$

then $\lim_{n\to\infty} u_n = 0$.

PROOF. If (4) holds, then the series $\Sigma|u_n|$ converges, thus $\lim_{n\to\infty} |u_n| = 0$ and hence $\lim_{n\to\infty} u_n = 0$.

Example 9. Show that $\lim_{n\to\infty} \dfrac{n^{100}}{(1.0001)^n} = 0.$

Solution. $\lim_{n\to\infty} \dfrac{(n+1)^{100}}{(1.0001)^{n+1}} \cdot \dfrac{(1.0001)^n}{n^{100}} = \lim_{n\to\infty} \left(\dfrac{n+1}{n}\right)^{100} \dfrac{1}{1.0001} = \dfrac{1}{1.0001} < 1,$

therefore the stated result follows from the corollary.

PROBLEMS

1. Use the ratio test to establish the convergence or divergence of each of the following series:

a. $1000 + \dfrac{1\,000\,000}{2} + \dfrac{1\,000\,000\,000}{6} + \cdots + \dfrac{1000^n}{n!} + \cdots.$

b. $\sum \dfrac{1}{n5^n}$. e. $\sum \dfrac{n^{100}}{(1.001)^n}$. h. $\sum \tan^n 60°$.

c. $\sum \dfrac{n}{5^n}$. f. $\sum \dfrac{1}{n^{100}(0.999)^n}$. i. $\sum \dfrac{\pi^n}{1 + (3.1416)^n}$.

d. $\sum \dfrac{n^{100}}{5^n}$. g. $\sum \sin^n 60°$. j. $\sum \dfrac{\pi^n}{1 + (3.14159)^n}$.

2. Establish the convergence or divergence of each of the following series:

a. $\sum \dfrac{n}{\sqrt{2n+3}}$. d. $\sum \dfrac{n^3}{n!}$. g. $\sum \dfrac{n}{3+4n}$.

b. $\sum \dfrac{1}{n!}$. e. $\sum \dfrac{n^n}{n!}$. h. $\sum \dfrac{(n!)^2}{(2n)!}$.

c. $\sum \dfrac{3^n}{n!}$. f. $\sum \dfrac{1}{n\sqrt{n}}$. i. $\sum \dfrac{(n!)^2}{2(n!)}$.

3. For each positive integer n let f_n be the function defined by the given equation:

a. $f_n(x) = nx^n$. Show that $\Sigma f_n(\frac{1}{2})$ converges and $\Sigma f_n(1)$ diverges.

b. $f_n(x) = \dfrac{x^n}{n(n+1)}$. Show that $\Sigma f_n(\frac{9}{10})$ converges and $\Sigma f_n(\frac{10}{9})$ diverges.

c. $f_n(x) = \dfrac{2n-1}{2^n x^n}$, $x \neq 0$. Show that $\Sigma f_n(\frac{1}{2})$ diverges and $\Sigma f_n(1)$ converges.

d. $f_n(x) = \dfrac{x^n}{n2^n}$. Show that $\Sigma f_n(2)$ diverges, $\Sigma f_n(-2)$ converges, $\Sigma f_n(1.9)$ converges, $\Sigma f_n(2.1)$ diverges.

4. Let $\Sigma\, u_n$ be a series of positive terms.

   a. Show that if $\Sigma\, u_n$ converges, then $\Sigma \dfrac{1}{u_n}$ diverges.

   b. Let $u_n = \dfrac{n}{n+1}$ and show that $\Sigma\, u_n$ and $\Sigma \dfrac{1}{u_n}$ both diverge.

   c. Show that if $\lim\limits_{n\to\infty} u_n$ exists and is not 0, then both $\Sigma\, u_n$ and $\Sigma \dfrac{1}{u_n}$ diverge.

   d. Show that if $\lim\limits_{n\to\infty} u_n$ exists, then $\Sigma \dfrac{1}{u_n}$ diverges.

   e. Give an example of a series $\Sigma\, u_n$ which diverges, but such that $\Sigma \dfrac{1}{u_n}$ converges.

5. What is wrong with each of the following?

   a. $$\begin{aligned} \text{Let } s &= 1 - 1 + 1 - 1 + \cdots \\ &= 1 - (1 - 1 + 1 - 1 + \cdots) \\ &= 1 - s, \quad \therefore 2s = 1 \text{ and } s = \tfrac{1}{2}. \end{aligned}$$

   b. $$\begin{aligned} \text{Let } s &= 1 + 1 + 1 + \cdots \\ &= 1 + (1 + 1 + 1 + \cdots) \\ &= 1 + s, \quad \therefore s - s = 1 \text{ so that } 0 = 1. \end{aligned}$$

   c. $$\begin{aligned} \text{Let } s &= 1 + 2 + 2^2 + 2^3 + \cdots \\ &= 1 + (2 + 2^2 + 2^3 + \cdots) \\ &= 1 + 2(1 + 2 + 2^2 + \cdots) \\ &= 1 + 2s, \quad \therefore -s = 1 \text{ and } s = -1. \end{aligned}$$

6. Let x be a number such that $x < \frac{1}{2}$. Use the corollary and show that
$$\lim_{n\to\infty} \frac{(2n)!}{(n!)^2 2^{2n}} \left(\frac{x}{1-x}\right)^n (1-x)^{1/2} = 0.$$

141. Comparison Tests

It may be possible to establish the convergence or divergence of a given series by comparing the terms of the given series with the terms of another series whose convergence or divergence is known.

TEST 5. *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two series of positive terms. If there is a positive number c such that*
$$u_n \le cv_n$$
for all sufficiently large values of n and:

a. *If the v-series converges, then the u-series converges, but*
b. *If the u-series diverges, then the v-series diverges.*

PROOF. Let c be a positive number and let N be such that
$$\text{whenever } n \ge N, \quad \text{then } u_n \le cv_n.$$

To prove Part a, let the v-series be convergent and converge to l. Hence, for $n \geq N$

$$\sum_{k=N}^{n} u_k \leq c \sum_{k=N}^{n} v_k \leq c \sum_{k=1}^{n} v_k \leq cl.$$

Let $\sum_{k=1}^{N-1} u_k = A$. Hence for any integer n

$$\sum_{k=1}^{n} u_k \leq A + cl$$

and this last expression does not depend upon n. Thus, the sequence of partial sums of the u-series is bounded above and is an increasing sequence (since each term is positive) and the u-series converges.

To prove Part b, let the u-series be divergent. The v-series is therefore also divergent (for if it were convergent, then the u-series would be convergent by Part a).

Example 1. We have already shown that the series

$$(1) \qquad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1}\frac{1}{n} + \cdots$$

is convergent. Show, however, that the series of alternate terms

$$(2) \qquad 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} + \cdots \quad \text{and}$$

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots - \frac{1}{2n} - \cdots$$

are both divergent.

Solution. Even though each term of the series (1) is less than or equal to the corresponding term of the divergent series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/n + \cdots$ nevertheless $\frac{1}{n} \leq 2\frac{1}{2n-1}$. Thus knowing that $\Sigma \frac{1}{n}$ diverges, we use Test 5 (with $c = 2$) to see that (1) diverges.

It should be seen that the series (2) diverges if and only if

$$(3) \qquad \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n} + \cdots$$

diverges and that (3) does diverge since $\frac{1}{n} \leq 2\frac{1}{2n}$.

TEST 6. *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be series of positive terms.*

a. *If $\Sigma\, v_n$ converges and $\lim_{n\to\infty} \frac{u_n}{v_n}$ exists, then $\Sigma\, u_n$ converges.*

b. *If $\Sigma\, v_n$ diverges and $\lim_{n\to\infty} \frac{u_n}{v_n}$ is $+\infty$ or exists and is not zero, then $\Sigma\, u_n$ diverges.*

PROOF. a. If $\Sigma\, v_n$ converges and $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lambda$, then for sufficiently large values of n we have $u_n < (\lambda + 1)v_n$ and thus (by Test 5) $\Sigma\, u_n$ converges.

b. If $\Sigma\, v_n$ diverges and $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \infty$, then given any number $G > 0$ we have, for sufficiently large values of n, that $u_n > Gv_n$ and thus $\Sigma\, u_n$ diverges. Also, if $\Sigma\, v_n$ diverges and $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lambda > 0$, then for sufficiently large values of n, $u_n > \frac{1}{2}\lambda v_n$ and thus $\Sigma\, u_n$ diverges.

Example 2. Show that $\sum\limits_{n=2}^{\infty} \dfrac{1}{\sqrt{n}\ln n}$ diverges.

Solution. Consider $\lim\limits_{n\to\infty} \left(\dfrac{1}{\sqrt{n}\ln n}\right)\Big/\dfrac{1}{n} = \lim\limits_{n\to\infty} \dfrac{\sqrt{n}}{\ln n}$. In order to find this limit note

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x} = \lim_{n\to\infty} \frac{D_x\sqrt{x}}{D_x \ln x} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{x^{-1}} = \lim_{x\to\infty} \frac{1}{2}\sqrt{x} = \infty.$$

Thus $\lim\limits_{n\to\infty} \left(\dfrac{1}{\sqrt{n}\ln n}\right)\Big/\dfrac{1}{n} = \infty$, and since $\Sigma\, \dfrac{1}{n}$ diverges we see by Test 6b that the given series also diverges.

COROLLARY. *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two series of positive terms such that*

$$\lim_{n\to\infty} \frac{u_n}{v_n} \textit{ exists and is } \neq 0, \tag{4}$$

then either both series converge or else both series diverge.

Example 3. Does $\sum\limits_{n=1}^{\infty} \sin \dfrac{\pi}{2n - 1}$ converge or diverge?

Solution. Remembering $\lim\limits_{h\to 0} \dfrac{\sin h}{h} = 1$, we have $\lim\limits_{n\to\infty} \dfrac{\sin(\pi/(2n-1))}{\pi/(2n-1)} = 1 \neq 0.$

Thus, the given series and $\Sigma\, \dfrac{\pi}{2n-1}$ either both converge or else both diverge. Since

$$\lim_{n\to\infty} \frac{\pi/(2n-1)}{1/n} = \frac{\pi}{2} \neq 0$$

and $\Sigma\, \dfrac{1}{n}$ diverges, then $\Sigma\, \dfrac{\pi}{2n-1}$ diverges. Hence the given series diverges.

TEST 7. (Integral test). *Let f be a continuous positive decreasing function defined for $x \geq n_0$ (n_0 an integer) and let $a_n = f(n)$. Then for*

$$\sum_{k=n_0}^{\infty} a_k \quad \textit{and} \quad \int_{n_0}^{\infty} f(x)\,dx \tag{5}$$

the series converges if and only if the improper integral exists.

PROOF. For $k \geq n_0$ and x such that $k \leq x \leq k+1$ we have

$$a_k = f(k) \geq f(x) \geq f(k+1) = a_{k+1} > 0 \quad \text{and}$$

$$a_k = \int_k^{k+1} a_k\,dx \geq \int_k^{k+1} f(x)\,dx \geq \int_k^{k+1} a_{k+1}\,dx = a_{k+1} > 0$$

Consequently, for n an integer such that $n \geq n_0$, then

$$\sum_{k=n_0}^{n} a_k \geq \sum_{k=n_0}^{n} \int_k^{k+1} f(x)\,dx = \int_{n_0}^{n+1} f(x)\,dx \geq \sum_{k=n_0}^{n} a_{k+1} > 0. \tag{6}$$

If the series converges, then (from the left-hand inequality of (6)) the integral in (5) exists, but if the series diverges, then (from the right-hand inequality) the integral does not exist.

Example 4. Show the series $\dfrac{1}{2 \ln 2} + \dfrac{1}{3 \ln 3} + \cdots + \dfrac{1}{n \ln n} + \cdots$ diverges.

Solution. Let $f(x) = \dfrac{1}{x \ln x}$ for $x \geq 2$. Since

$$\lim_{b\to\infty} \int_2^b \frac{dx}{x \ln x} = \lim_{b\to\infty} \int_2^b \frac{d \ln x}{\ln x} = \lim_{b\to\infty} \ln(\ln x)]_2^b = \infty$$

the integral $\displaystyle\int_2^{\infty} \frac{dx}{x \ln x}$ does not exist and (by Test 7) the series diverges.

142. Sums and Differences

The following theorem shows how series may be added or subtracted term by term.

THEOREM 142. *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two given series and for each integer n let*

$$w_n = u_n + v_n \quad \textit{and} \quad z_n = u_n - v_n.$$

Also, let $\Sigma\, u_n$ converge to L_1 and let $\Sigma\, v_n$ converge to L_2. Then $\Sigma\, w_n$ and $\Sigma\, z_n$ are both convergent and converge to $L_1 + L_2$ and $L_1 - L_2$, respectively.

PROOF. We need merely note the existence of the limits

$$\lim_{n\to\infty} \sum_{k=1}^{n} u_k = L_1, \quad \lim_{n\to\infty} \sum_{k=1}^{n} v_k = L_2, \quad \text{and}$$

$$\begin{aligned} L_1 + L_2 &= \lim_{n\to\infty} \sum_{k=1}^{n} u_k + \lim_{n\to\infty} \sum_{k=1}^{n} v_k \\ &= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} u_k + \sum_{k=1}^{n} v_k \right\} \qquad \text{by Theorem 139.1} \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} (u_k + v_k) = \lim_{n\to\infty} \sum_{k=1}^{n} w_k \end{aligned}$$

which proves that $\Sigma\, w_n$ is convergent and converges to $L_1 + L_2$. In the same way $\Sigma\, z_n$ is convergent and converges to $L_1 - L_2$.

In the notation of Theorem 142, it should be noted that $\Sigma\, w_n$ may converge but both $\Sigma\, u_n$ and $\Sigma\, v_n$ diverge, e.g., $\Sigma \dfrac{n-1}{n^2}$ and $\Sigma \dfrac{-1}{n}$ both diverge whereas $\Sigma \left\{ \dfrac{n-1}{n^2} + \dfrac{-1}{n} \right\} = \Sigma \dfrac{-1}{n^2}$ converges. As a simpler example $1 + 1 + 1 + \cdots$ and $-1 - 1 - 1 \cdots$ both diverge but $(1 - 1) + (1 - 1) + (1 - 1) + \cdots$ converges.

PROBLEMS

1. First establish the convergence of $\Sigma\, (e/3)^n$ and then use this series as a comparison to establish the convergence of

$$\sum \frac{e^n}{10 + 3^n}, \quad \sum \frac{10e^n}{5 + 3^n} \quad \text{and} \quad \sum \frac{10 + e^n}{n + 3^n 4}.$$

2. Use a comparison test to establish the convergence or divergence of:

a. $\displaystyle\sum \frac{1}{5 + 3n^2}$.

b. $\displaystyle\sum \frac{2}{5 + 3n^2}$.

c. $\displaystyle\sum \frac{2}{50n + n^2}$.

d. $\displaystyle\sum \frac{2}{50n + 100^2}$.

e. $\displaystyle\sum \frac{2}{25 + \sin n + n^2}$.

f. $\displaystyle\sum \frac{2 + \sin n}{n^2}$.

g. $\displaystyle\sum \frac{n}{2(n + 1)(n + 2)}$.

h. $\displaystyle\sum \frac{2n + 1}{(n + 1)(n + 2)(n + 3)}$.

i. $\displaystyle\sum \frac{1}{\ln n}$.

j. $\displaystyle\sum \sin \frac{\pi}{2n^2 - 1}$.

k. $\displaystyle\sum \cos \frac{\pi}{2n^2 - 1}$.

l. $\displaystyle\sum \tan \frac{\pi}{2n - 1}$.

3. Let f be a function such that $f(n) > 0$ for each positive integer n and such that for some positive number p

$$\lim_{n \to \infty} n^p f(n) \text{ exists and is } \neq 0.$$

Show that $\Sigma f(n)$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

4. Let Σu_n be a series such that $u_n \geq 0$ for all sufficiently large values of n. Show:

a. If $\sqrt[n]{u_n} \geq 1$ for all sufficiently large values of n, then Σu_n diverges.

b. If there is a number r, $0 < r < 1$, such that for all sufficiently large n

$$\sqrt[n]{u_n} < r,$$

then Σu_n converges. (This is known as Cauchy's Root Test.)

c. If $0 < r < 1$, then $\Sigma r^n |\sin n\alpha|$ converges.

5. Show that both of the following series diverge.

a. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{6} + \frac{1}{5} - \frac{1}{10} + \cdots + u_n + \cdots,$

$$\text{where } u_n = \begin{cases} \dfrac{1}{n} \text{ if } n \text{ is odd} \\[2ex] -\dfrac{1}{2(n-1)} \text{ if } n \text{ is even.} \end{cases}$$

b. $1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} + \cdots + u_n + \cdots,$

$$\text{where } u_n = \begin{cases} \dfrac{2}{n+1} \text{ if } n \text{ is odd} \\[2ex] -\dfrac{1}{n} \text{ if } n \text{ is even.} \end{cases}$$

143. Absolute Convergence

The first of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots (-1)^{n+1}\frac{1}{n} + \cdots \quad \text{and} \quad 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

converges, but the second diverges. Also, each term of the second series is the absolute value of the corresponding term of the first series. According to the following definition the convergence of the first series is further qualified as "conditional convergence."

DEFINITION. *A series Σu_n is said to be* ***absolutely convergent*** *if the series $\Sigma |u_n|$ is convergent. A series Σu_n is said to be* ***conditionally convergent*** *if the series itself is convergent, but the series $\Sigma |u_n|$ is divergent.*

Sometimes it is easier to establish the convergence of $\Sigma |u_n|$ than it is the convergence of Σu_n itself. Hence the following test may be useful.

TEST 8. *If $\Sigma\, |u_n|$ is convergent, then $\Sigma\, u_n$ is convergent; that is, if a series is absolutely convergent, then the series is convergent.*

PROOF. Let $u_1, u_2, u_3, \cdots$ be such that $\Sigma\, |u_n|$ is convergent and has sum L. First define

$$v_k = u_k + |u_k| \quad \text{for} \quad k = 1, 2, 3, \cdots.$$

Notice that if $u_k < 0$ then $v_k = 0$ but if $u_k \geq 0$ then $v_k = 2u_k = 2|u_k|$ and hence in either case $0 \leq v_k \leq 2|u_k|$. Consequently

$$0 \leq \sum_{k=1}^{n} v_k \leq 2 \sum_{k=1}^{n} |u_k| \leq 2L.$$

Thus the partial sums of the v-series is a monotonically increasing sequence bounded above by $2L$. Hence

$$\lim_{n\to\infty} \sum_{k=1}^{n} v_k \text{ exists.}$$

Therefore
$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n} v_k - \lim_{n\to\infty} \sum_{k=1}^{n} |u_k| &= \lim_{n\to\infty} \sum_{k=1}^{n} (v_k - |u_k|) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} u_k\end{aligned}$$

and the existence of this limit means (by definition) that $\Sigma\, u_n$ converges.

Example 1. Show that $\Sigma \dfrac{\cos n}{n^2}$ is convergent.

Solution. Since $\left|\dfrac{\cos n}{n^2}\right| \leq \dfrac{1}{n^2}$ and $\Sigma \dfrac{1}{n^2}$ converges, the given series is absolutely convergent and hence the series itself is convergent by Test 8.

From Tests 3, 4, and 7 (and their proofs) we have:

TEST 9. *If for all sufficiently large values of* n

$$u_n \neq 0 \quad \textit{and} \quad \left|\frac{u_{n+1}}{u_n}\right| \geq 1,$$

then the nth *term cannot approach* 0 *as* $n \to \infty$ *and* $\Sigma\, u_n$ *diverges. Also, whenever*

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| \textit{ exists and } = \rho,$$

if $\rho < 1$, then $\Sigma\, u_n$ converges absolutely (hence $\Sigma\, u_n$ converges), but if $\rho > 1$, then $\Sigma\, u_n$ diverges.

Example 2. Determine the convergence or divergence of each of the series

$$\sum f_n(-\tfrac{1}{2}), \quad f_n(-1.1), \quad \text{and} \quad \sum f_n(-1), \quad \text{given } f_n(x) = (-1)^{n+1} \frac{(x+2)^n}{\sqrt{n}}.$$

Solution. Let x be a number such that $x \neq -2$ and note that

$$\lim_{n\to\infty}\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = \lim_{n\to\infty}\frac{|x+2|^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{|x+2|^n} = |x+2|.$$

Since $|-\frac{1}{2}+2| = \frac{3}{2} > 1$, we see that the first series diverges.

Since $|-1.1+2| = 0.9 < 1$, the second series converges.

Since $|-1+2| = 1$, the ratio test is inconclusive for the third series. The third series is, however,

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \cdots + (-1)^{n+1}\frac{1}{\sqrt{n}} + \cdots$$

whose terms alternate in sign, the numerical values of the terms decrease and approach zero, and hence (see Test 2) the series converges.

For further properties of absolutely convergent series and conditionally convergent series, see Appendix A 9.

144. Series of Functions

Let x be a number and consider the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n2^n}. \tag{1}$$

In particular if $x = 0$, then each term is zero, so the series converges to 0.

Let x be different from zero and set $u_n(x) = (-1)^{n+1}x^n/(n2^n)$. Thus

$$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\frac{|x^{n+1}|}{(n+1)2^{n+1}}\cdot\frac{n2^n}{|x^n|} = \lim_{n\to\infty}\frac{n}{n+1}\frac{|x|}{2} = \frac{|x|}{2}.$$

Hence, if $0 < |x| < 2$, then the series converges. If, however, the number x is such that $|x| > 2$, then the series diverges. If $|x| = 2$, then this ratio test does not tell whether the series converges or diverges. Notice, however, that if $x = 2$, then the series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n2^n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n+1}\frac{1}{n} + \cdots$$

which converges, but if $x = -2$, then the series is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-2)^n}{n2^n} = -1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n} - \cdots$$

which diverges. Thus, we see that the series (1) converges if and only if

$$-2 < x \leq 2.$$

DEFINITION. *For $n = 1, 2, 3, \cdots$, let f_n be a function. For x a number the series*

$$\sum_{n=1}^{\infty} f_n(x)$$

either converges or diverges. The collection of all numbers x for which the series converges is called the **region of convergence.**

Thus, for the series (1) the region of convergence is

$$\{x \mid -2 < x \leq 2\}.$$

DEFINITION. *Given a sequence of numbers $a_1, a_2, \cdots, a_n, \cdots$, the series $\Sigma\, a_n x^n$ is said to be a* ***power series in x****, and $\Sigma\, a_n(x - c)^n$ is said to be a* ***power series in $(x - c)$****.*

PROBLEMS

1. Show that each of the following series is absolutely convergent:

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{e^n}$.

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1}$.

c. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

d. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$.

e. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{2^n}$.

f. $\displaystyle\sum_{n=0}^{\infty} \frac{\sin nx}{2^n}$.

2. Show that each of the following series is conditionally convergent:

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2n + 1}}$.

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 + 1}$.

c. $\displaystyle\sum_{n=1}^{\infty} \frac{n \cos n\pi}{(n + 1)(n + 2)}$.

d. $\displaystyle\sum_{n=1}^{\infty} \frac{\tan\left(\frac{\pi}{4} + n\frac{\pi}{2}\right)}{n}$.

e. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n\frac{\pi}{4}}{n}$.

f. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n(n + 1)}}$.

3. State as much as possible about the convergence of each of the following series:

a. $\displaystyle\sum \frac{n^2}{n^3 + 1}$.

b. $\displaystyle\sum (-1)^n \frac{n^2}{n^3 + 1}$.

c. $\displaystyle\sum (-1)^n \frac{n}{3^{n-1}}$.

d. $\displaystyle\sum (-1)^n \frac{n^2}{3^{n-1}}$.

e. $\displaystyle\sum (-1)^n \frac{n^3}{3^{n-1}}$.

f. $\displaystyle\sum (-1)^n \ln\left(\frac{1}{n^2}\right)$.

g. $\displaystyle\sum \frac{(-1)^n}{\ln n^2}$.

h. $\displaystyle\sum (-1)^n \frac{n^2 + 1}{n^3 + 1000}$.

i. $\displaystyle\sum (-1)^n \frac{n^2 + 1}{n^3 + 2^n}$.

j. $\displaystyle\sum (-1)^n \frac{n^3 + 1}{n^2 + 2^n}$.

k. $\displaystyle\sum \frac{4 \cdot 7 \cdots (3n - 2)}{(3n)!} (-1)^{3n}$.

4. Find the region of convergence for:

a. $\sum \frac{x^n}{2^n}$.

b. $\sum \frac{(-x)^n}{2^n}$.

c. $\sum \frac{x^n}{n^2}$.

d. $\sum \frac{x^{2n}}{n}$.

e. $\sum \frac{x^{3n}}{n}$.

f. $\sum \frac{x^n}{n2^n}$.

g. $\sum \frac{x^n}{n + 2^n}$.

h. $\sum \frac{x^n}{n!}$.

i. $\sum \frac{2^n}{x^n}$, $x \neq 0$.

j. $\sum \frac{1}{nx^n}$, $x \neq 0$.

k. $\sum \frac{1}{n(-x)^n}$, $x \neq 0$.

l. $\sum \frac{n!}{x^n}$, $x \neq 0$.

m. $\sum \frac{1}{n!x^n}$, $x \neq 0$.

n. $\sum \frac{(x-2)^n}{3^n}$.

o. $\sum \frac{(2-x)^n}{3^n}$.

p. $\sum \frac{(x+2)^n}{3^n}$.

q. $\sum \frac{(x+2)^n}{n3^n}$.

r. $\sum \frac{(x+2)^n}{\sqrt{n}}$.

s. $\sum \frac{(2x-1)^n}{n(n+1)(n+2)}$.

t. $\sum \frac{x^{n+5}}{10^n}$.

u. $\sum 10^n x^{n+5}$.

v. $\sum \frac{4 \cdot 7 \cdots (3n-2)}{(3n)!} x^{3n}$.

w. $\sum \frac{2 \cdot 5 \cdots (3n-1)}{(3n+1)!} x^{3n+1}$.

5. Show that the regions of convergence of the three series differ at most by two points:

a. $\sum_{n=1}^{\infty} x^n$, $\sum_{n=1}^{\infty} D_x x^n$, $\sum_{n=1}^{\infty} \int_0^x t^n\, dt$.

b. $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$, $\sum_{n=1}^{\infty} D_x \frac{x^n}{n3^n}$, $\sum_{n=1}^{\infty} \int_0^x \frac{t^n}{n3^n}\, dt$.

c. $\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n^2+1}$, $\sum_{n=1}^{\infty} D_x \frac{(x-4)^n}{2n^2+1}$, $\sum_{n=1}^{\infty} \int_4^x \frac{(t-4)^n}{2n^2+1}\, dt$.

d. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$, $\sum_{n=1}^{\infty} D_x \frac{(x+2)^n}{\sqrt{n}}$, $\sum_{n=1}^{\infty} \int_{-2}^x \frac{(t+2)^n}{\sqrt{n}}\, dt$.

145. Functions Represented by Power Series

From Taylor's theorem [see (3) and (4) of Sec. 131 with $a = 0$ and $b = x$] if a function f has its nth derived function $f^{(n)}$ continuous throughout an interval containing the origin, then throughout this interval

$$(1) \quad \left| f(x) - \left\{ f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} \right\} \right| = |R_n(x)|$$

where R_n may be written in two ways, namely, for some number ξ_n between 0 and x

$$(2) \qquad R_n(x) = \frac{f^{(n)}(\xi_n)}{n!} x^n \quad \text{(Lagrange form)}$$

or (possibly with a different number for ξ_n)

$$(3) \qquad R_n(x) = \frac{f^{(n)}(\xi_n)}{(n-1)!} (x - \xi_n)^{n-1} x \quad \text{(Cauchy form).}$$

If for a specific function having all derivatives continuous in an interval containing the origin and for a specific number x in this interval we can show that

$$\lim_{n\to\infty} R_n(x) = 0,$$

then we have that $f(x)$ is represented by its Maclauren series:

$$(4) \qquad f(x) \equiv f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots$$

in the sense that the series converges and converges to $f(x)$.

Example 1. Show that

$$(5) \qquad \sin x \equiv x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots.$$

Solution. Upon setting $f(x) = \sin x$ we see that $f(0) = 0$, $f'(0) = \cos 0 = 1$, $f''(0) = -\sin 0 = 0$, $f'''(0) = -\cos 0 = -1$, etc. Moreover $|f^{(n)}(x)|$ is either $|\sin x|$ or $|\cos x|$ both of which are ≤ 1. Thus, for this function [from (2)]

$$|R_n(x)| \leq \frac{1}{n!} |x|^n.$$

However, regardless of the value of x (see corollary p. 460),

$$\lim_{n\to\infty} \frac{1}{n!} |x|^n = 0.$$

Thus $\lim_{n\to\infty} R_n(x) = 0$ and hence (5) holds for any number x.

Example 2. Show that

$$(6) \ \ln(1 + x) \equiv x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots, \quad \text{for } -1 < x \leq 1.$$

Solution. Upon setting $f(x) = \ln(1 + x)$ for $x > -1$, the nth derivative is

$$(7) \qquad f^{(n)}(x) = (-1)^{n+1}(n-1)!(1+x)^{-n}, \quad x > -1.$$

Thus $f(0) = 0$, whereas $\dfrac{f^n(0)}{n!} = \dfrac{(-1)^{n+1}(n-1)!}{n!} = \dfrac{(-1)^{n+1}}{n}$, and we obtain the series in (6). We need, however, to prove that if $-1 < x \le 1$, then $R_n(x) \to 0$ as $n \to \infty$. This we do by considering two cases, noting first that if $x = 0$, then both sides of (6) are zero.

CASE 1. $-1 < x < 0$ so that $-1 < x < \xi_n < 0$. In this case, use the remainder in the Cauchy form (3) which may now be written as

$$|R_n(x)| = \left|\frac{(n-1)!(1+\xi_n)^{-n}}{(n-1)!}(x-\xi_n)^{n-1}x\right| = \frac{|x-\xi_n|^{n-1}|x|}{(1+\xi_n)^n}$$

$$= \frac{1}{1+\xi_n}\left(\frac{\xi_n - x}{1+\xi_n}\right)^{n-1}|x| = \frac{1}{1+\xi_n}\left(\frac{\xi_n + |x|}{1+\xi_n}\right)^{n-1}|x|.$$

From $-1 < x < \xi_n$, we have $0 < 1 + x < 1 + \xi_n$ and thus

$$0 < \frac{1}{1+\xi_n} < \frac{1}{1+x}.$$

Also $\dfrac{\xi_n + |x|}{1+\xi_n} < |x|$, for if we thought that $\dfrac{\xi_n + |x|}{1+\xi_n} \ge |x|$, then we would have $\xi_n + |x| \ge |x| + \xi_n|x|$, $\xi_n \ge \xi_n|x|$ and hence (since $\xi_n < 0$), $1 \le |x|$ which is not so. Thus in this case

$$|R_n(x)| < \frac{1}{1+x}|x|^n$$

and hence $R_n(x) \to 0$ as $n \to \infty$ since $|x| < 1$.

CASE 2. $0 < x \le 1$. Use the Lagrange form and show that $|R_n(x)| < 1/n$. Thus if $-1 < x \le 1$, then $R_n(x) \to 0$ as $n \to \infty$ and hence (6) holds.

Since (6) is valid for $-1 < x \le 1$ (and for x in this range $0 < 1 + x \le 2$) then (6) may be used to compute the natural logarithm of any positive number ≤ 2 but is seldom used because of other formulas such as (8) below.

Example 3. Use (6) to show that if $|x| < 1$, then

$$\text{(8)} \qquad \ln\left(\frac{1-x}{1+x}\right) = -2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots\right).$$

Solution. Replace x in (6) by $-x$ to obtain

$$\text{(9)} \qquad \ln(1-x) = -\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{1}{n}x_n + \cdots\right) \quad \text{if } -1 \le x < 1.$$

For x in the domain of validity of both (6) and (9), then $|x| < 1$ and now term by term subtraction of (6) from (9) yields (8).

Notice now that if y is any positive number, and we set

$$\frac{1-x}{1+x} = y, \quad \text{then} \quad x = \frac{1-y}{1+y} \quad \text{and} \quad |x| < 1.$$

For example, if $y = 3$ then $x = -0.5$ and from (8)

$$\ln 3 = 2[0.5 + \tfrac{1}{3}(0.5)^3 + \tfrac{1}{5}(0.5)^5 + \cdots].$$

PROBLEMS

1. Sketch the graph of $y = \dfrac{1-x}{1+x}$ for $|x| < 1$. Use derivatives to show that the graph is descending to the right.

2. Show that if (x,y) is a point with $\frac{1}{3} \leq y \leq 3$ on the graph of Prob. 1, then $-\frac{1}{2} \leq x \leq \frac{1}{2}$. (Note: This shows that (8) may be used to compute the natural logarithms of a number between $\frac{1}{3}$ and 3 by a rapidly converging series.)

3. Prove that for all real x

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots$$

4. In (5) replace x by x^2 to obtain

$$\sin (x^2) = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \frac{1}{7!}x^{14} + \cdots$$

Next let $f(x) = \sin (x^2)$, find $f'(x)$, $f''(x)$, etc., to see how much more trouble it is to obtain the above expansion directly.

5. Obtain the formal expansion as far as given

$$\ln \cos x = -\frac{1}{2}x^2 - \frac{2}{4!}x^4 - \frac{10}{6!}x^6 - \cdots$$

which certainly does not hold if $\cos x$ is zero or negative. Representation is too hard to establish since the form of the n-th derivative is too complicated.

6. Show that if $-\frac{1}{2}\pi \leq x \leq \frac{3}{2}\pi$, then

$$\sqrt{1 + \sin x} = 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} - \frac{x^3}{2^3 \cdot 3!} + \frac{x^4}{2^4 \cdot 4!} + \frac{x^5}{2^5 \cdot 5!} - \cdots$$

but if $\frac{3}{2}\pi < x < \frac{7}{2}\pi$ the negative of the series must be used.
(Hint: $\sqrt{1 + \sin x} = |\sin \frac{1}{2}x + \cos \frac{1}{2}x|$.)

7. We know that $\cos t \le 1$. With $0 < x$ check in sequence

$$\int_0^x \cos t\,dt \le \int_0^x dt, \quad \sin x \le x, \quad \int_0^x \sin t\,dt \le \int_0^x t\,dt,$$

$$-\cos x + 1 \le \frac{1}{2}x^2, \quad 1 - \frac{1}{2}t^2 \le \cos t, \quad \int_0^x \left(1 - \frac{1}{2}t^2\right) dt \le \int_0^x \cos t\,dt, \text{ etc.}$$

Continue obtaining inequality, integral, inequality, etc. to see that partial sums of sine and cosine series emerge.

146. Calculus of Power Series

From previous work (including examples and problems) it follows that if a power series $\Sigma\, a_n x^n$ is such that

$$\text{(1)} \qquad \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \text{ exists and } = r > 0,$$

then $\Sigma\, |a_n x^n|$ and $\Sigma\, a_n x^n$ converge whenever $-r < x < r$ and $\Sigma\, a_n x^n$ diverges whenever $x < -r$ or $x > r$, but it requires other tests to determine the convergence or divergence of $\Sigma\, a_n(-r)^n$ and $\Sigma\, a_n r^n$.

Any power series $\Sigma\, a_n x^n$ converges at $x = 0$ and $\Sigma\, n!x^n$ converges only if $x = 0$. On the other hand $\Sigma\, (n!)^{-1}x^n$ converges for all values of x.

The following theorem does not depend upon the limit in (1).

THEOREM 146.1. *Let $\Sigma\, a_n x^n$ be a power series which converges for some number $x \ne 0$, but does not converge for all x. Then there is a number $r > 0$ (called the **radius of convergence** of the series) such that*

$$\text{(2)} \qquad \sum |a_n x^n| \textit{ converges whenever } |x| < r \textit{ but}$$

$$\text{(3)} \qquad \sum a_n x^n \quad \textit{diverges whenever } r < |x|.$$

PROOF. Let $x_1 \ne 0$ be a number such that $\Sigma\, a_n x_1^n$ converges and let x_2 be a number such that $\Sigma\, a_n x_2^n$ diverges. We first prove that

$$\begin{matrix}\text{(4)}\\ \text{(5)}\end{matrix} \qquad \sum |a_n x^n| \begin{cases}\text{converges whenever } |x| < |x_1|, \text{ but} \\ \text{diverges whenever } |x_2| < |x|.\end{cases}$$

To prove (4) note, from the convergence of $\Sigma\, a_n x_1^n$, that

$$\lim_{n\to\infty} a_n x_1^n = 0 \quad \text{so} \quad \lim_{n\to\infty} |a_n x_1^n| = 0.$$

Let N be such that whenever $n > N$, then $|a_n x_1^n| < 1$ and let c be the largest of the $N + 1$ numbers $|a_1 x_1|, |a_2 x_1^2|, \cdots, |a_N x_1^N|$, and 1. Thus c is $\ge$ all the numbers

$$|a_1 x_1|, |a_2 x_1^2|, |a_3 x_1^3|, \cdots, |a_n x_1^n|, \cdots.$$

Now let x be a number such that $|x| < |x_1|$, note for this x that

$$|a_n x^n| = \left| a_n x_1^n \left(\frac{x}{x_1}\right)^n \right| = |a_n x_1^n| \left|\frac{x}{x_1}\right|^n \leq c\left|\frac{x}{x_1}\right|^n$$

and hence (by comparison Test 5) $\Sigma\, |a_n x^n|$ converges since the geometric series $\Sigma\, c|x/x_1|^n$ converges because $|x/x_1| < 1$. Thus (4) is established.

To see (5), let x be a number such that $|x_2| < |x|$. Then for this x, $\Sigma\, |a_n x^n|$ diverges (for if it converged then $\Sigma\, a_n x^n$ would converge and then $\Sigma\, |a_n x_2^n|$ would converge by (4) with x replaced by x_2 and x_1 by x). Thus (5) is established.

With (4) and (5) established, let S be the set defined by

$$S = \{x \mid \textstyle\sum |a_n x^n| \text{ converges}\}.$$

Then S is not empty since $x = 0$ is in S but even more any number x such that $0 < |x| < |x_1|$ is in S by (4). Also, any number b such that $|x_2| < b$ is an upper bound of S by (5). Thus S has a least upper bound (by the axiom of Sec. 4) which we call r. Then $r > 0$ since in particular $0 < |x_1| \leq r$.

We now show that (2) and (3) hold for this number r.

First let x be a number such that $|x| < r$ and choose a number $\underline{x}$ such that $|x| < |\underline{x}| < r$. Then $\underline{x}$ is in S so $\Sigma\, |a_n \underline{x}^n|$ converges so $\Sigma\, a_n \underline{x}^n$ converges and now $\Sigma\, |a_n x^n|$ converges by (4) with x_1 replaced by $\underline{x}$. This establishes (2).

Next, let x be any number such that $r < |x|$ and choose a number $\bar{x}$ such that $r < |\bar{x}| < |x|$. Then $|\bar{x}|$ is not in S so $\Sigma\, |a_n \bar{x}^n|$ diverges. Hence $\Sigma\, a_n x^n$ diverges (for if it converged then $\Sigma\, |a_n \bar{x}^n|$ would converge by (4) with x replaced by $\bar{x}$ and x_1 replaced by x). This establishes (3), and hence the theorem.

The notion of radius of convergence is extended by saying that a power series $\Sigma\, a_n x^n$ which converges only for $x = 0$ has radius of convergence $r = 0$, but if it converges for all x its radius of convergence is $r = \infty$.

THEOREM 146.2. *Let $\Sigma\, a_n x^n$ have radius of convergence r (finite or infinite), but $r > 0$. Then each of the two series*

$$(6) \qquad \sum n a_n x^{n-1} \quad \text{and} \quad \sum \frac{a_n}{n+1} x^{n+1}$$

has the same radius of convergence r.

PROOF. Let x be a number such that $0 < |x| < r$. We first show that $\Sigma\, |n a_n x^{n-1}|$ converges. Let x_1 be a number such that $|x| < |x_1| < r$. Hence $\Sigma\, a_n x_1^n$ converges so that $\lim\limits_{x\to\infty} |a_n x_1^n| = 0$. As in the proof of Theorem 146.1, let c be greater than or equal to all the numbers $|a_1 x_1|, |a_2 x_1^2|, \cdots, |a_n x_1^n|, \cdots$. Hence

$$(7) \qquad |n a_n x^{n-1}| = n\left|\frac{1}{x}\, a_n x_1^n \left(\frac{x}{x_1}\right)^n\right| = \frac{n}{|x|}\, |a_n x_1^n| \left|\frac{x}{x_1}\right|^n \leq \frac{n}{|x|}\, c \left|\frac{x}{x_1}\right|^n.$$

Upon setting $u_n = \dfrac{n}{|x|}\, c \left|\dfrac{x}{x_1}\right|^n$, it follows that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n+1}{n}\left|\frac{x}{x_1}\right| = \left|\frac{x}{x_1}\right| < 1.$$

Thus $\Sigma\, u_n$ converges and then from (7) $\Sigma\, |na_nx^{n-1}|$ converges.

In a similar way $\Sigma \left|\dfrac{a_n}{n+1}\, x^{n+1}\right|$ converges whenever $|x| < r$. Thus, both series (6) converge if $|x| < r$. We need yet to show that both of these series diverge if $r < |x|$ whenever r is finite.

With r finite, let x be a number such that $r < |x|$ and then let x_2 be a number such that $r < |x_2| < |x|$ and note that

$$\sum a_n x_2^n \text{ diverges.}$$

We now state that $\Sigma\, na_nx^{n-1}$ diverges; for if it converged then (by the first part of this proof with a_n replaced by na_n and the exponent n replaced by $n - 1$) the series

$$\sum \left| na_n \frac{x_2^{(n-1)+1}}{(n-1)+1} \right| = \sum |a_n x_2^n|$$

would converge so $\Sigma\, a_nx_2^n$ would converge contrary to the above stated fact that this series $\Sigma\, a_nx_2^n$ diverges.

In the same way the second series in (6) diverges if $r < |x|$.

The following theorem is sometimes stated: *A power series may be differentiated and integrated term by term within its region of convergence.*

THEOREM 146.3. *Let $\Sigma\, a_nx^n$ have radius of convergence $r > 0$. Then*

(a) $$D_x\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=0}^{\infty} D_x a_k x^k \qquad \textit{for} \quad -r < x < r$$

(b) $$\int_a^b \left(\sum_{k=0}^{\infty} a_k x^k\right) dx = \sum_{k=0}^{\infty}\left(\int_a^b a_k x^k\, dx\right) \quad \textit{for} \quad \begin{cases} -r < a < r \\ -r < b < r. \end{cases}$$

PROOF of (a). Since $D_x a_k x^k = k a_k x^{k-1}$, the series on the right in (a) has (by Theorem 146.2) the same radius of convergence r as $\Sigma\, a_kx^k$. By letting f be the function defined by

(8) $$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{for} \quad -r < x < r,$$

it remains, in proving Part (a), to show that

$$f'(x) = \sum_{k=1}^{\infty} ka_k x^{k-1} \quad \text{for} \quad -r < x < r, \tag{9}$$

where summation is started with $k = 1$ instead of $k = 0$ since $D_x a_0 = 0$.

Toward proving (9), let $\underline{x}$ be a specific number such that $-r < \underline{x} < r$. Hence $|\underline{x}| < r$ and we choose a number c such that

$$|\underline{x}| < c < r. \tag{10}$$

The radius of convergence of $\Sigma\, k(k - 1)a_k x^{k-2}$ is also r (by applying Theorem 146.2 with $\Sigma\, ka_k x^{k-1}$ as the first series). In particular (since $0 < c < r$) the series $\Sigma\, k(k - 1)a_k c^{k-2}$ converges absolutely; and we let d be the number

$$d = \sum_{k=2}^{\infty} k(k - 1)|a_k|c^{k-2}. \tag{11}$$

With these preliminaries ready for use, let $\Delta x \neq 0$ be such that also $|\underline{x} + \Delta x| < c$. Hence, both $f(\underline{x})$ and $f(\underline{x} + \Delta x)$ may be computed by means of (1) so that

$$\begin{aligned} f(\underline{x} + \Delta x) - f(\underline{x}) &= \sum_{k=0}^{\infty} a_k(\underline{x} + \Delta x)^k - \sum_{k=0}^{\infty} a_k \underline{x}^k \\ &= \sum_{k=0}^{\infty} a_k[(\underline{x} + \Delta x)^k - \underline{x}^k] \\ &= a_0 \cdot 0 + a_1\,\Delta x + \sum_{k=2}^{\infty} a_k[(\underline{x} + \Delta x)^k - \underline{x}^k]. \end{aligned}$$

Now use Taylor's formula (with remainder after two terms) and select x_k for $k = 2, 3, 4, \cdots$ such that x_k is between $\underline{x}$ and $\underline{x} + \Delta x$ and

$$(\underline{x} + \Delta x)^k = \underline{x}^k + k\underline{x}^{k-1}\,\Delta x + \frac{k(k - 1)}{2!}\, x_k^{k-2}(\Delta x)^2 \quad \text{so that}$$

$$a_k[(\underline{x} + \Delta x)^k - \underline{x}^k] = ka_k\underline{x}^{k-1}\,\Delta x + \frac{k(k - 1)}{2!}\, a_k x_k^{k-2}(\Delta x)^2.$$

Notice that $|x_k| < c$ so that $|a_k x_k^{k-2}| \leq |a_k|c^{k-2}$. Since the series below on the right converges (see (11)), then by comparison Test 5, the series on the left also converges and the inequality holds:

$$\sum_{k=2}^{\infty} \frac{k(k - 1)}{2!}\, |a_k x_k^{k-2}| \leq \sum_{k=2}^{\infty} \frac{k(k - 1)}{2}\, |a_k|c^{k-2} = \frac{d}{2}. \tag{12}$$

Hence $\Sigma\, k(k-1)a_k x_k^{k-2}$ itself converges so that

$$f(\underline{x}+\Delta x)-f(\underline{x}) = a_1\,\Delta x + \sum_{k=2}^{\infty}\left[ka_k\underline{x}^{k-1}\,\Delta x + \frac{k(k-1)}{2}\,a_k x_k^{k-2}(\Delta x)^2\right]$$

$$= a_1\,\Delta x + \Delta x\sum_{k=2}^{\infty} ka_k\underline{x}^{k-1} + \frac{(\Delta x)^2}{2}\sum_{k=2}^{\infty} k(k-1)a_k x_k^{k-2},$$

$$\frac{f(\underline{x}+\Delta x)-f(\underline{x})}{\Delta x} - a_1 - \sum_{k=2}^{\infty} ka_k\underline{x}^{k-1} = \frac{\Delta x}{2}\sum_{k=2}^{\infty} k(k-1)a_k x_k^{k-2},$$

$$\left|\frac{f(\underline{x}+\Delta x)-f(\underline{x})}{\Delta x} - \sum_{k=1}^{\infty} ka_k\underline{x}^{k-1}\right| = \frac{|\Delta x|}{2}\left|\sum_{k=2}^{\infty} k(k-1)a_k x_k^{k-2}\right|$$

$$\le \frac{|\Delta x|}{2}\sum_{k=2}^{\infty} k(k-1)|a_k x_k^{k-2}|$$

so that $$\left|\frac{f(\underline{x}+\Delta x)-f(\underline{x})}{\Delta x} - \sum_{k=1}^{\infty} ka_k\underline{x}^{k-1}\right| \le \frac{|\Delta x|}{2}\,d \qquad \text{by (5).}$$

Notice that d is a constant (certainly independent of $\underline{x}$ and Δx). Hence, as $\Delta x \to 0$ the right side approaches 0 so also the left side approaches 0; that is

$$f'(\underline{x}) = \lim_{\Delta x\to 0}\frac{f(\underline{x}+\Delta x)-f(\underline{x})}{\Delta x} = \sum_{k=1}^{\infty} ka_k\underline{x}^{k-1}.$$

Since $\underline{x}$ was any number such that $-r < \underline{x} < r$, it follows that (9) holds and this completes the proof of Part (a).

PROOF of (b). Let a and b be any two numbers such that both

$$-r < a < r \quad\text{and}\quad -r < b < r.$$

Also, let f and g be the functions defined by

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad\text{and}\quad g(x) = \sum_{k=0}^{\infty}\frac{a_k}{k+1}\,x^{k+1} \quad\text{for}\quad -r < x < r,$$

this being permissible since the series defining g also has radius of convergence r by Theorem 146.2. First note that

$$(13)\qquad g(x)\Big]_a^b = g(b) - g(a) = \sum_{k=0}^{\infty}\frac{a_k}{k+1}\,b^{k+1} - \sum_{k=0}^{\infty}\frac{a_k}{k+1}\,a^{k+1}$$

$$= \sum_{k=0}^{\infty}\frac{a_k}{k+1}\,(b^{k+1}-a^{k+1}) = \sum_{k=0}^{\infty}\left(\int_a^b a_k x^k\,dx\right).$$

By applying Part (a) to g (instead of to f) we have that

$$g'(x) = \sum_{k=0}^{\infty} D_x\frac{a_k}{k+1}\,x^{k+1} = \sum_{k=0}^{\infty} a_k x^k = f(x)$$

so, by the Fundamental Theorem of Calculus,

$$\int_a^b f(x)\,dx = g(x)\Big]_a^b.$$

Hence, from the definition of $f(x)$ and the relation (13) we have

$$\int_a^b \left(\sum_{k=0}^{\infty} a_k x^k\right) dx = \sum_{k=0}^{\infty} \left(\int_a^b a_k x^k\,dx\right)$$

which is the equation of Part (b) and thus Theorem 146.3 is proved.

PROBLEMS

1. Obtain the power series in x for $\cos x$ from the power series for $\sin x$ by:

   a. Differentiation. b. Integration.

2. a. Obtain the power series for $\dfrac{1}{1+x}$, $-1 < x < 1$ by differentiating the power series for $\ln(1 + x)$.

   b. Using the result of Part a find a series expansion for $(1 + x)^{-2}$, $-1 < x < 1$.

   c. Check that the series in Part b is the formal binomial expansion of $(1 + x)^{-2}$.

3. a. For $a > 0$ use (6) of Sec. 145 to show that if $-a < x \leq a$ then

$$\ln(a + x) = \ln a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \cdots + (-1)^{n+1}\frac{x^n}{na^n} + \cdots.$$

   b. Proceeding as in Prob. 2, find the formal binomial expansion of $(a + x)^{-2}$ and note the validity for $-a < x < a$.

4. Show, for any number x, that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

5. Using Prob. 4, find the series expansion in powers of x for $\cosh x$ and $\sinh x$.

6. a. Let $f(x) = (1 - x)^{-1/2}$ for $x < 1$. Show that

$$f^{(n)}(x) = \frac{3 \cdot 5 \cdots (2n-1)}{2^n} \frac{1}{(1-x)^n\sqrt{1-x}}.$$

   b. Use the Cauchy form of $R_n(x)$ to prove that if $-1 < x < 1$, then

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{2^2 2!}x^2 + \cdots + \frac{3 \cdot 5 \cdots (2n-1)}{2^n n!}x^n + \cdots,$$

   c. Use the result of Part b to see that if $-1 < x < 1$, then

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{2^2 2!}x^4 + \cdots + \frac{3 \cdot 5 \cdots (2n-1)}{2^n n!}x^{2n} + \cdots,$$

d. Use the result of Part c to see that for $-1 < x < 1$

$$\arcsin x = x + \frac{1}{2 \cdot 3} x^3 + \frac{3}{2^2 2! 5} x^5 + \cdots + \frac{3 \cdot 5 \cdots (2n-1)}{2^n n! (2n+1)} x^{2n+1} + \cdots.$$

7. With $x \neq 0$, λ any number and n a positive integer such that $2n > \lambda$, replace x in the formula of Prob. 4 by $1/x^2$ and see that

$$|x|^\lambda e^{1/x^2} = |x|^\lambda \left(1 + \frac{1}{x^2} + \frac{1}{2x^4} + \cdots + \frac{1}{n!} \frac{1}{x^{2n}} + \cdots\right) > \frac{1}{n!} \frac{1}{|x|^{2n-\lambda}}.$$

The last term $\to \infty$ as $x \to 0$. Hence we first see that

$$\lim_{x \to 0} \frac{1}{x^\lambda e^{1/x^2}} = 0 \quad \text{and then that} \quad \lim_{x \to 0} \frac{P(x) e^{-1/x^2}}{x^m} = 0$$

for any polynomial $P(x)$ and any integer m. This limit may be used in Part b below.

Let f be the function defined by

$$f(x) = e^{-1/x^2} \quad \text{for } x \neq 0, \text{ but } f(0) = 0. \tag{14}$$

a. Prove that

$$f'(x) = 2e^{-1/x^2}/x^3 \quad \text{for } x \neq 0, \text{ and that } f'(0) = 0.$$

b. Prove there is a polynomial $P_n(x)$ such that

$$f^{(n)}(x) = \frac{P_n(x)}{x^{3n}} e^{-1/x^2} \quad \text{for } x \neq 0, \text{ and that } f^{(n)}(0) = 0.$$

(Note: This shows that every term of the Maclauren series for this $f(x)$ is zero, so the series converges for all x, but not to $f(x)$ for $x \neq 0$.)

8. Naïvely replace x in the formula of Prob. 4 by hD_x and write

$$e^{hD_x} = 1 + hD_x + \frac{h^2}{2!} D_x^2 + \frac{h^3}{3!} D_x^3 + \cdots.$$

Then argue either for or against interpreting

$$f(x + h) = e^{hD_x} f(x)$$

as a shorthand for the formal Taylor's series

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots.$$

CHAPTER 14

Differential Equations

This chapter is largely a bag of tricks. If you follow the suggested procedures you can fill reams of paper with technique-improving manipulations and obtain an excellent review of indefinite integration, but if in the end you do not know exactly what is going on do not blame yourself entirely. The recognized discipline known as "Differential Equations" has a splendid history, an exciting present, and a promising future; its theoretical aspects intrigue pure mathematicians and its tasty flavor of applications assures its respect in those of a practical turn. One short chapter cannot possibly do justice to so venerable a subject, but may give some hint of why calculus is such a basic course. Please reserve judgment as to whether you "like" Differential Equations until you are exposed to more than its tricky caprices and at least see an unequivocal definition of a solution of a differential equation.

147. An Example

In a vertical plane take points P and Q with P to the left of Q. Fasten the ends of a uniform flexible cable (weight w lb/ft) at P and Q, the cable being long enough for the lowest point of the sag to be below the level of both P and Q. Through this lowest point pass a vertical line for the y-axis of a coordinate system (unit 1 ft) and let $(0,a)$ be the coordinates of this lowest point. The problem is to find an equation $y = f(x)$ of the curve formed by the cable.

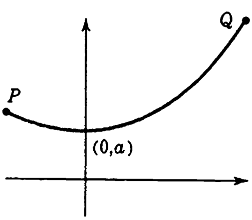


Figure 147.1

Cut the cable at $(0,a)$ and attach a force with proper direction and magnitude to maintain the shape of the cable from $(0,a)$ to Q. This force has no vertical component and is directed to the left. Represent the force by the vector $-\vec{i}H$, where H is a positive number.

Between $(0,a)$ and Q select a point (x,y) on the cable (so $y = f(x)$), cut the cable at this point and attach a single force to maintain the shape of the cable between $(0,a)$ and (x,y). The vector $\vec{u}$ representing this force has horizontal and vertical components $\vec{i}h$ and $\vec{j}v$. It is natural to assume $\vec{u}$ to be tangent to the curve at (x,y) so that

(1) $$f'(x) = \frac{v}{h}.$$

A physical principle to be relied upon is: *For all forces acting on a static body, the sum of horizontal components is zero and the sum of vertical components is zero.*

For the horizontal components acting on the cable from $(0,a)$ to (x,y):

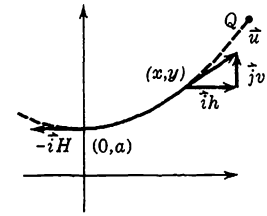


Figure 147.2

(2) $$-\vec{i}H + \vec{i}h = \vec{0} \quad \text{so} \quad h = H.$$

The length of the cable from $(0,a)$ to (x,y) is $\int_0^x \sqrt{1 + f'^2(t)}\, dt$. The downward force due to the weight of the cable added to the vertical components at the ends must sum to $\vec{0}$; but the vertical component at the end $(0,a)$ is zero and hence

$$-\vec{j}w\int_0^x \sqrt{1 + f'^2(t)}\, dt + \vec{j}v = \vec{0} \quad \text{so} \quad v = w\int_0^x \sqrt{1 + f'^2(t)}\, dt.$$

This equation for v together with (1) and (2) yield

$$f'(x) = \frac{w}{H}\int_0^x \sqrt{1 + f'^2(t)}\, dt.$$

By the Fundamental Theorem of Calculus $\frac{d}{dx}\int_0^x \sqrt{1 + f'^2(t)}\, dt = \sqrt{1 + f'^2(x)}$. Since w and H do not depend upon x:

(3) $$f''(x) = \frac{w}{H}\sqrt{1 + f'^2(x)}.$$

This equation together with the known values

(4) $$f'(0) = 0 \quad \text{and} \quad f(0) = a$$

are sufficient to find $f(x)$ by methods developed in this chapter. Later it will be shown, using (3) and (4), that

(5) $$y = f(x) = \frac{H}{2w}(e^{wx/H} + e^{-wx/H}) + a - \frac{H}{w}.$$

This equation is simplified by choosing the x-axis so that $a = H/w$. Hence, (5) becomes

$$y = \frac{a}{2}(e^{x/a} + e^{-x/a})$$

which is the usual form for the equation of a **catenary**.

Equation (3) is an example of a **derivative equation** and associated equations (4) are called **initial conditions**. In differential notation (3) and (4) appear as

$$(6)\quad \frac{d^2y}{dx^2} = \frac{w}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \text{ where } \frac{dy}{dx} = 0 \text{ and } y = a \text{ when } x = 0,$$

and is called a **differential equation** with initial conditions. Also, (5) is said to be the solution of this system.

The study of some physical, electrical, chemical, or other phenomena may lead to differential equations with initial conditions. The setting up of a differential system usually involves principles or assumptions not primarily mathematical. After such a system is obtained the second step, a purely mathematical one, is to relate x and y without derivatives (such as (5)). The third step is to interpret the relation between x and y in the original setting. The current chapter presents methods pertinent to the second step in this sequence, leaving the first and third steps to disciplines in which mathematics is applied.

148. Definitions

A differential equation is an equation in two or more variables and differentials (or derivatives) of these variables. It is usually known which of the variables are to be considered as functions, with the others as independent variables. A differential equation involving one or more functions and their ordinary derivatives with respect to a single independent variable is an ordinary differential equation.† Examples are:

$$(1)\qquad dy = \sin x\, dx$$

$$(2)\qquad \frac{d^2y}{dx^2} + y = 0 \qquad (\text{or } y'' + y = 0)$$

$$(3)\qquad x\left(\frac{dy}{dx}\right)^2 = 1 \qquad (\text{or } xy'^2 = 1).$$

The order of the derivative (or differential) of highest order in an equation is called the **order** of the differential equation. Thus (1) is of order 1, (2) is of order 2, and (3) is of order 1. If a differential equation is a polynomial in the derivatives involved, the degree of the highest order derivative is called the **degree** of the differential equation. Thus (1) and (2) are of degree 1, (3) is of degree 2, and (6) of Sec. 147 is of degree 2. In case all derivatives

† A partial differential equation contains partial derivatives and will not be considered in this book.

of y and y itself occur to the first degree, the differential equation is said to be **linear**. Thus

$$x^2 \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} + y \ln |x| = 0 \quad \text{is linear,}$$

but the first degree equation $y'' + (y')^2 = \sin x$ is not linear.

A one parameter family of equations $f(x,y,c) = 0$ in two variables (without their derivatives) is the primitive (see Sec. 119) of a first order differential equation obtained by eliminating the parameter c between the equations

$$f(x,y,c) = 0 \quad \text{and} \quad df(x,y,c) = 0. \tag{4}$$

Example 1. Find a differential equation having primitive $x(c + y) = c^2$.

Solution. Set $f(x,y,c) = x(c + y) - c^2 = 0$ so that $df(x,y,c) = 0$ is

$$(c + y)\,dx + x\,dy = 0 \quad \text{and hence} \quad c = -\frac{x\,dy + y\,dx}{dx} = -\left(x\frac{dy}{dx} + y\right).$$

This expression for c substituted into $x(c + y) - c^2 = 0$ yields

$$x\left(-x\frac{dy}{dx}\right) - \left(x\frac{dy}{dx} + y\right)^2 = 0 \quad \text{or} \quad x^2\left(\frac{dy}{dx}\right)^2 + (2xy + x^2)\frac{dy}{dx} + y^2 = 0.$$

The reverse problem of starting with a first-order differential equation and then seeking its primitive was considered in Sec. 119 and will now be studied in more detail. In this setting the primitive will be referred to as the **general solution** of the differential equation; a **particular solution** is obtained by assigning a specific value to the parameter.

Example 2. Find the general solution of the differential equation

$$xy\,dx - (1 + x^2)\,dy = 0. \tag{5}$$

Solution. Divide each term by $y(1 + x^2)$ to obtain

$$\frac{x}{1 + x^2}\,dx - \frac{dy}{y} = 0. \tag{6}$$

Since $\dfrac{d}{dx}\dfrac{1}{2}\ln(1 + x^2) = \dfrac{x}{1 + x^2}$ and $\dfrac{d}{dx}\ln|y| = \dfrac{1}{y}\dfrac{dy}{dx}$, (6) may be written as

$$\frac{d}{dx}\left[\frac{1}{2}\ln(1 + x^2) - \ln|y|\right] = 0 \quad \text{so that}$$

$$\tfrac{1}{2}\ln(1 + x^2) - \ln|y| = c \tag{7}$$

is the general solution. Equivalent forms of this general solution are

$$\ln\sqrt{1+x^2} - \ln|y| = c, \quad \ln\frac{|y|}{\sqrt{1+x^2}} = -c,$$

$$\frac{|y|}{\sqrt{1+x^2}} = e^{-c}, \quad |y| = e^{-c}\sqrt{1+x^2}.$$

Since c is arbitrary then e^{-c} is positive but otherwise is arbitrary. Since y occurs only in $|y|$, the last of the above equations may be written in the more usable form

$$y = C\sqrt{1+x^2}, \quad C \text{ arbitrary}.$$

After changing (5) into (6) a quite formal procedure is to "indefinitely integrate" each term:

$$\int\frac{x}{1+x^2}\,dx - \int\frac{dy}{y} = 0,$$

apply formulas of indefinite integrals, and add an arbitrary constant to obtain (7). This method is applicable whenever the differential equation may be algebraically manipulated into a form in which each term involves only one of the variables times the differential of that variable; this is the **variables separable** case. There are other cases to be considered later.

In solving a differential system the procedure is to find the general solution of the differential equation and then obtain the particular solution which also satisfies the supplementary conditions.

Example 3. Solve the system

(8) $$y^2(1+x)\,dx - x^3\,dy = 0; \quad y = 2 \quad \text{when} \quad x = 1.$$

Solution. The variables are separable:

$$\frac{1+x}{x^3}\,dx - \frac{1}{y^2}\,dy = 0, \quad \int(x^{-3}+x^{-2})\,dx - \int y^{-2}\,dy = 0$$

$$-\frac{1}{2x^2} - \frac{1}{x} + \frac{1}{y} = c; \quad \text{general solution.}$$

Set $x = 1$ and $y = 2$ to determine c and find $c = -1$. Thus (8) has solution

$$-\frac{1}{2x^2} - \frac{1}{x} + \frac{1}{y} = -1 \quad \text{or} \quad y = \frac{2x^2}{1+2x-2x^2}.$$

There is no accepted "simplest form" of a general (or even a particular) solution, but whenever one variable may conveniently be expressed explicitly in terms of the other this should be done.

PROBLEMS

1. Find the differential equation having primitive:

a. $y = cx - c^3$. d. $y = ce^{-x}$. g. $y = x + c\sqrt{1 + x^2}$.
b. $xy = c(y + 1)$. e. $y = e^{-x} + ce^{-2x}$. h. $y^2 = c^2e^{-x^2} - 1$.
c. $xy = c(x + 1)$. f. $2y = x + cx^{-1}$. i. $2cy = x^2 - c^2$.

2. Find the general solution of:

a. $(1 + y)\,dx + (2 - x)\,dy = 0$.
b. $xy\,dx + (1 - x)\,dy = 0$.
c. $(3 - y)\,dx + xy\,dy = 0$.
d. $e^{x+y}\,dx + dy = 0$.
e. $2^{x+y}\,dx + 3^{x+y}\,dy = 0$.
f. $(1 - y^2)\,dx + dy = 0$.
g. $(1 + y^2)\,dx + x^2\,dy = 0$.
h. $\sec y\,dx + y\,dy = 0$.
i. $x^2(1 - \cos 2y)\,dx + (x + 1)\,dy = 0$.
j. $x^2y\,dy - dx = x^2\,dx$.

3. Solve the derivative system:

a. $\dfrac{dy}{dx} = \dfrac{x - 3}{y + 2}$; $y = 3$ when $x = 2$.

b. $\dfrac{dy}{dx} = \dfrac{y + 2}{x - 3}$; $y = 3$ when $x = 2$.

c. $ye^{2x}\,dx + 2(e^x + 1)\,dy = 0$; $y = 3$ when $x = \ln 8$.

d. $\sqrt{1 - x^2}\,dy = \sqrt{y}\,dx$; $y = 1$ when $x = 0$.

e. $2\sqrt{y}\,dx + \dfrac{dy}{y + 1} = 0$; $y = 3$ when $x = 0$.

f. $2\sqrt{y + 1}\,dx + \dfrac{dy}{y} = 0$; $y = 3$ when $x = 0$.

149. Substitutions

The variables are not separable in the differential equation

$$(x^2 + xy + y^2)\,dx - xy\,dy = 0. \tag{1}$$

By factoring out x^2 (assuming $x \neq 0$) this equation may be written as

$$x^2\left\{\left[1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2\right] dx - \frac{y}{x}\,dy\right\} = 0 \quad\text{or}\quad \left[1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2\right] dx - \frac{y}{x}\,dy = 0.$$

Set $y/x = v$ so $y = vx$ and $dy = v\,dx + x\,dv$. The differential equation becomes $(1 + v + v^2)\,dx - v(v\,dx + x\,dv) = 0$ or upon collecting terms

$$(1 + v)\,dx - vx\,dv = 0$$

and the variables x and v are separable. Thus, as in Sec. 148

$$\frac{dx}{x} - \frac{v}{1+v}\,dv = 0, \quad \frac{dx}{x} - \left(1 - \frac{1}{1+v}\right)dv = 0,$$

$$\ln|x| - v + \ln|1+v| = c$$

so a form of the general solution of (1) is (since $v = y/x$)

$$\ln|x| - \frac{y}{x} + \ln\left|1 + \frac{y}{x}\right| = c.$$

The principle here is: *If $M(x,y)\,dx + N(x,y)\,dy = 0$ is such that after factoring out x to some power, the variables appear only in the combination y/x, then set $y/x = v$ so*

$$\text{(2)} \qquad y = vx \quad \text{and} \quad dy = v\,dx + x\,dv.$$

In the resulting equation the variables x and v are separable. For if x and y appear only as y/x, then after the substitutions (2) the equation appears in the form

$$f(v)\,dx + g(v)(v\,dx + x\,dv) = 0$$

in which the variables x and v are separable into

$$\frac{dx}{x} + \frac{g(v)}{f(v) + vg(v)}\,dv = 0.$$

Example 1. Solve the differential system

$$(\sqrt{x^2 + y^2} + y)\,dx - x\,dy = 0; \quad y = -3 \quad \text{when} \quad x = 4.$$

Solution. Considering $x > 0$ so $\sqrt{x^2} = x$, factor out x:

$$x\left[\left(\sqrt{1 + \left(\frac{y}{x}\right)^2} + \frac{y}{x}\right)dx - dy\right] = 0.$$

The substitution (2) yields $(\sqrt{1 + v^2} + v)\,dx - (v\,dx + x\,dv) = 0$ so that

$$\sqrt{1 + v^2}\,dx - x\,dv = 0, \quad \frac{dx}{x} - \frac{dv}{\sqrt{1 + v^2}} = 0,$$

$$\ln x - \ln(v + \sqrt{1 + v^2}) = c, \quad \ln x - \ln\left(\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}\right) = c, \quad \text{or}$$

$$\ln\frac{x^2}{y + \sqrt{x^2 + y^2}} = c.$$

The side conditions $y = -3$ when $x = 4$ shows that $c = \ln 8$. Thus, a form of the desired particular solution is

$$\frac{x^2}{y + \sqrt{x^2 + y^2}} = 8 \quad \text{for} \quad x > 0.$$

One might not suspect that the graph of this equation is half of a parabola, but by rationalizing the denominator, etc., we obtain

$$x^2 = 16y + 64 \quad \text{for} \quad x > 0.$$

The substitution illustrated in the next example is merely a translation of axes.

Example 2. Solve $(3x - 2y - 9)\,dx + (2x + y + 1)\,dy = 0$.

Solution. Substitute $x = X + h$, $y = Y + k$ with h and k constants:

$$(3X - 2Y + 3h - 2k - 9)\,dX + (2X + Y + 2h + k + 1)\,dY = 0.$$

This equation is simplified by choosing h and k such that both

$$3h - 2k - 9 = 0 \quad \text{and} \quad 2h + k + 1 = 0.$$

The simultaneous solution is $h = 1$, $k = -3$ so that

$$(3X - 2Y)\,dX + (2X + Y)\,dY = 0 \quad \text{where} \quad x = X + 1,\ y = Y - 3.$$

The differential equation is of the form for setting $Y = VX$:

$$(3 - 2V)\,dX + (2 + V)(V\,dX + X\,dV) = 0,$$

$$(3 + V^2)\,dX + (2 + V)X\,dV = 0, \quad \frac{dX}{X} + \frac{2 + V}{3 + V^2}\,dV = 0,$$

$$\ln|X| + \frac{2}{\sqrt{3}}\tan^{-1}\frac{V}{\sqrt{3}} + \frac{1}{2}\ln(3 + V^2) = c.$$

Now $X = x - 1$, $Y = y + 3$, $V = Y/X = (y + 3)/(x - 1)$ so the general solution is

$$\ln|x - 1| + \frac{2}{\sqrt{3}}\tan^{-1}\frac{y + 3}{\sqrt{3}(x - 1)} + \frac{1}{2}\ln\left[3 + \frac{(y + 3)^2}{(x - 1)^2}\right] = c.$$

Example 3. Solve $(2x - y + 1)\,dx + (4x - 2y - 5)\,dy = 0$.

Solution. The substitution $x = X + h$, $y = Y + k$ as in Example 2 leads to the equations

$$2h - k + 1 = 0, \quad 4h - 2k - 5 = 0$$

which have no simultaneous solutions. Notice, however, that the combination $2x - y$ is present in both terms of the given equation which suggests setting

$$2x - y = v \quad \text{so that} \quad y = 2x - v \quad \text{and} \quad dy = 2\,dx - dv.$$

The equation becomes $(v + 1)\,dx + (2v - 5)(2\,dx - dv) = 0$ which is

$$(v + 1 + 4v - 10)\,dx - (2v - 5)\,dv = 0.$$

The variables x and v are separable so previous methods may be applied, then resubstitution made, to yield the general solution

$$x - \tfrac{2}{5}(2x - y) + \tfrac{7}{25}\ln|5(2x - y) - 9| = c.$$

PROBLEMS

1. Each of the following may be solved by the method illustrated in Example 1.

   a. $(3x - y)\,dx + (x + y)\,dy = 0.$
   b. $y\,dx - (x + y)\,dy = 0.$
   c. $(xy + y^2)\,dx = (x^2 + xy + y^2)\,dy.$
   d. $y^2\,dx = (x^2 + xy)\,dy.$
   e. $(x^2 + xy - y^2)\,dx + xy\,dy = 0.$
   f. $(x^3 - x^2y + xy^2)(dx + dy) = y^3\,dx.$

2. Work Probs. 1c and 1d by first making the substitution $x = uy$.

3. Use the method of Example 2 or Example 3 to solve:

   a. $(3x - y - 5)\,dx + (x + y + 1)\,dy = 0.$
   b. $(2x + y + 1)\,dx + (4x + 2y - 5)\,dy = 0.$
   c. $(2x - 4y + 1)\,dx - (3x - 6y + 2)\,dy = 0.$
   d. $(y + 1)\,dx + (x + y)\,dy = 0.$

150. Linear Equation of First Order

With p and q functions of one variable, the differential equation

$$\frac{dy}{dx} + p(x)y = q(x) \tag{1}$$

is linear and of first order. Toward obtaining a method for solving differential equations of this form, let P be any function such that

$$\frac{dP(x)}{dx} = p(x) \quad \text{that is, set} \quad P(x) = \int p(x)\,dx. \tag{2}$$

Multiply both sides of (1) by $e^{P(x)}$:

$$e^{P(x)}\frac{dy}{dx} + e^{P(x)}p(x)y = e^{P(x)}q(x). \tag{3}$$

The left side of (3) is a perfect derivative; in fact (3) may be written as

$$\frac{d}{dx}[e^{P(x)}y] = e^{P(x)}q(x) \tag{4}$$

since upon expanding the left side of (4) we obtain

$$e^{P(x)}\frac{dy}{dx} + y\frac{d}{dx}e^{P(x)} = e^{P(x)}q(x),$$

$$e^{P(x)}\frac{dy}{dx} + ye^{P(x)}\frac{dP(x)}{dx} = e^{P(x)}q(x)$$

which is the same as (3) because $\dfrac{dP(x)}{dx} = p(x)$ by (2).

From (4) it follows that

$$e^{P(x)}y = \int e^{P(x)}q(x)\,dx \tag{5}$$

and this may be used to solve equations of the form (1).

Example 1. Solve the differential equation $x\,dy + 2y\,dx = 10x^3\,dx$.

Solution. Upon dividing both sides by $x\,dx$ the result is

$$\frac{dy}{dx} + \frac{2}{x}y = 10x^2 \quad (\text{provided } x \neq 0)$$

which is in the form (1) with $p(x) = 2/x$ and $q(x) = 10x^2$. Thus, from (2)

$$P(x) = \int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2 \quad \text{and} \quad e^{P(x)} = e^{\ln x^2} = x^2. \tag{6}$$

Hence, for this particular problem, equation (5) becomes

$$\begin{aligned} x^2y &= \int x^2(10x^2)\,dx \\ &= 2x^5 + c. \end{aligned} \tag{7}$$

Upon dividing both sides by x^2 we obtain

$$y = 2x^3 + cx^{-2}$$

which, as a check will show, is the solution of the given equation.

Since P is any function such that $P'(x) = p(x)$ we choose the simplest such function by **not** adding an arbitrary constant to the integral in (6). Since, however, we want the most general function y satisfying the differential equation, we **do add** an arbitrary constant in (7).

A differential equation which can be put in the form (1) should be recognized even if the letters are different, as illustrated in the next two examples. The equation in Example 2 appears in the study of electric circuits.

Example 2. With A, L, R and ω constants, solve the system

$$L\frac{di}{dt} + Ri = A\sin\omega t, \quad i = 0 \text{ when } t = 0. \tag{8}$$

Solution. The implication is that t is the independent variable and i the dependent variable. Upon dividing by L, the equation

$$\frac{di}{dt} + \frac{R}{L}i = \frac{A}{L}\sin\omega t$$

is in the form (1) with $p(t)$ the constant R/L and $q(t) = (A/L)\sin\omega t$. Hence, corresponding to (2),

$$P(t) = \int \frac{R}{L}\,dt = \frac{R}{L}t$$

(where no constant of integration is added) and (5) takes the form

$$e^{(R/L)t}i = \int e^{(R/L)t}\frac{A}{L}\sin \omega t\, dt$$

$$= \frac{A/L}{(R/L)^2 + \omega^2}e^{(R/L)t}\left[\frac{R}{L}\sin \omega t - \omega \cos \omega t\right] + c$$

by Table Formula **156.** Since $i = 0$ when $t = 0$, it follows that

$$c = \frac{(A/L)\omega}{(R/L)^2 + \omega^2}.$$

Consequently, the solution of the differential system (8) is

$$\text{(9)} \qquad i = \frac{A/L}{(R/L)^2 + \omega^2}\left[\frac{R}{L}\sin \omega t - \omega \cos \omega t + \omega e^{-(R/L)t}\right]$$

$$= \frac{A}{R^2 + L^2\omega^2}(R \sin \omega t - L\omega \cos \omega t + L\omega e^{-(R/L)t}).$$

Example 3. Solve the differential equation

$$\text{(10)} \qquad (x - xy + 2)\, dy + y\, dx = 0.$$

Solution. In attempting to put this equation in the form (1) we write

$$\frac{dy}{dx} + \frac{1}{x - xy + 2}y = 0$$

which might appear to be in the form (1) with $g(x) = 0$. Notice, however, that the coefficient of y, namely $\dfrac{1}{x - xy + 2}$, cannot be set equal to $p(x)$ since it involves y as well as x. Incidentally, none of the previous methods (variables separable, $y/x = v$, translation, etc.) are applicable either.

Returning to (10), write it as $y\, dx + (x - xy + 2)\, dy = 0$,

$$\frac{dx}{dy} + \frac{x - xy + 2}{y} = 0, \quad \frac{dx}{dy} + \frac{1 - y}{y}x = -\frac{2}{y}$$

which is in the form (1) with x and y interchanged throughout. Therefore, set

$$p(y) = \frac{1 - y}{y}, \quad q(y) = -\frac{2}{y},$$

$$P(y) = \int\left(\frac{1}{y} - 1\right) dy = \ln|y| - y, \quad \text{and} \quad e^{P(y)} = e^{\ln|y| - y} = |y|e^{-y}$$

so the equation corresponding to (5) is

$$[|y|e^{-y}]x = \int |y|e^{-y}\left(\frac{-2}{y}\right) dy.$$

Since $|y| = y$ if $y > 0$ but $|y| = -y$ if $y < 0$ we obtain in either case

$$ye^{-y}x = -2\int e^{-y}\,dy = 2e^{-y} + c, \quad y \neq 0.$$

Hence, the solution of (10), giving x in terms of y, is

$$x = \frac{2}{y} + \frac{c}{y}e^{y}.$$

PROBLEMS

1. Solve the linear differential equation of first order:

a. $(2x^3y - 1)\,dx + x^4\,dy = 0.$

b. $\dfrac{dy}{dx} = 1 - y\cot x.$

c. $x\,dy + y\,dx = \sin x\,dx.$

d. $4xy\,dx + (x^2 + x - 2)\,dy = 0.$

e. $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{\ln x}{x^2}.$

f. $e^{3x}[dy + (3y + 6)\,dx] = 2x\,dx.$

2. Each of the following is a linear differential equation of first order (i.e., may be put in the form (1) except for different letters).

a. $d\rho + \rho\cos\theta\,d\theta = \sin 2\theta\,d\theta.$

b. $\sin\theta\,d\rho + (2\rho\cos\theta + \sin 2\theta)\,d\theta = 0.$

c. $t\,ds = 2(t^4 + s)\,dt.$

d. $du = 2(4v - u)\,dv.$

e. $dy - dx = x\cot y\,dy.$

f. $e^{y}(x\,dy - dx) = 2\,dy.$

3. Find the particular solution satisfying the differential equation and the side condition.

a. $x^2\,dy + (2xy - 1)\,dx = 0;\quad y = 2$ when $x = 1.$

b. $x^2\,dy + (x - xy - y)\,dx = 0;\quad y = 2$ when $x = 1.$

c. $dy + (y - \sin x)\cos x\,dx = 0;\quad y = 0$ when $x = \pi.$

d. $y' = x + y;\quad y = 0$ when $x = -1.$

151. The Bernoulli Equation

A differential equation of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^{a}; \quad a \neq 0, \quad a \neq 1 \tag{1}$$

is known as a **Bernoulli equation.** Upon multiplying each term of (1) by $(1 - a)y^{-a}$ the result is

$$(1 - a)y^{-a}\frac{dy}{dx} + (1 - a)p(x)y^{1-a} = (1 - a)q(x)$$

and the first term may be rewritten to give

$$\frac{d}{dx}y^{1-a} + (1 - a)p(x)y^{1-a} = (1 - a)q(x).$$

This equation may now be written as

$$\frac{du}{dx} + (1 - a)p(x)u = (1 - a)q(x) \quad \text{where} \quad u = y^{1-a} \tag{2}$$

which is a linear differential equation of first order to which the method of Sec. 150 may be applied.

Example. Solve the equation $xy' + y = x^5y^4$.

Solution. Upon dividing by x this equation becomes

$$\frac{dy}{dx} + \frac{1}{x}y = x^4y^4$$

which is a Bernoulli equation with $a = 4$. Hence by (2), since $1 - a = -3$,

$$\frac{du}{dx} + (-3)\frac{1}{x}u = -3x^4 \quad \text{where} \quad u = y^{-3}.$$

By the method of Sec. 150, the solution for u in terms of x is

$$u = cx^3 - \tfrac{3}{2}x^5.$$

Consequently, the original equation has solution

$$y = (cx^3 - \tfrac{3}{2}x^5)^{-1/3}.$$

PROBLEMS

1. Solve each of the Bernoulli equations:

a. $x\,dy = y(y^2 + 1)\,dx$.

b. $x\dfrac{dy}{dx} = y - x^3y^3$.

c. $y\,dy + (2 + x^2 - y^2)\,dx = 0$.

d. $\dfrac{dx}{dt} - 2x = 4x^{3/2}\sin t$.

2. The following miscellaneous set of problems reviews the methods for solving differential equations as given so far in this chapter and in Sec. 117.

a. $(4x^3y^3 + 3)\,dx + 3x^4y^2\,dy = 0$.

b. $(2xy + \cos x)\,dx + (x^2 - 1)\,dy = 0$.

c. $(3x - y + 5)\,dx + (6x - 2y + 1)\,dy = 0$.

d. $(x + 2y - 4)\,dx - (2x - 4y)\,dy = 0$.

e. $y\,dx + x\,dy = 2\,dx + 3\,dy$.

f. $y(1 - x)\,dx + x^2(1 - y^2)\,dy = 0$.

g. $(x + 1)y' = y + (x + 1)e^{2x}y^3$.

h. $ds + (2s - s^2)\,dt = 0$.

i. $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$.

j. $(xe^{y/x} - y)\,dx + x\,dy = 0$.

k. $dx = (\sec t - x\tan t)\,dt$.

l. $dr = (r\cot\theta + \tan\theta)\,d\theta$.

m. $(u + v + 2)\,du = (u - v - 4)\,dv$.

n. $\dfrac{dy}{dx} = \dfrac{x + y - 1}{x - y + 1}$.

152. Second Order, Linear, Constant Coefficients

A differential equation of the form

(1) $$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x); \quad a_1 \text{ and } a_2 \text{ constants,}$$

is of second order (since the second derivative is the highest order derivative present), is linear (since y and its derivatives occur to the first power), and has constant coefficients 1, a_1, and a_2. Two equations are associated with (1); the **homogeneous** differential equation

(2) $$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$$

and the algebraic equation, called the **characteristic equation,**

(3) $$r^2 + a_1r + a_2 = 0.$$

In this section we state how the roots of the characteristic equation (3) are related to the solutions of the homogeneous equation (2). In the next section it will be shown how, in some circumstances, the solutions of (2) may be used to find the solutions of (1).

In a more advanced course in differential equations it is proved that if $s_1(x)$ and $s_2(x)$ are both solutions of (2), and if neither $s_1(x)$ nor $s_2(x)$ is a constant times the other, then any solution of (2) may be written as $c_1s_1(x) + c_2s_2(x)$ for suitably chosen constants c_1 and c_2. Under these circumstances $s_1(x)$ and $s_2(x)$ are said to be **linearly independent** (that is, neither is a constant times the other) and are called **specific solutions** of (2) and $c_1s_1(x) + c_2s_2(x)$ is said to be the **general solution** of (2).

For the ordinary quadratic equation (3), the roots are either real and unequal, real and equal, or conjugate complex numbers $a + ib$, $a - ib$ with $b \neq 0$. Upon letting r_1 and r_2 be the roots of (3), the following are facts about the solutions of the homogeneous differential equation (2):

Case	Conditions	Specific Solutions	General Solution
1.	$r_1 \neq r_2$ real	e^{r_1x}, e^{r_2x}	$c_1e^{r_1x} + c_2e^{r_2x}$
2.	$r_1 = r_2$	e^{r_1x}, xe^{r_1x}	$(c_1 + c_2x)e^{r_1x}$
3.	$\begin{cases} r_1 = a + ib \\ r_2 = a - ib \end{cases}$	$e^{ax}\cos bx$, $e^{ax}\sin bx$	$e^{ax}(c_1\cos bx + c_2\sin bx)$.

To indicate the truth in Case 1, let r_1 and r_2 be real with $r_1 \neq r_2$, note that (3) is $(r - r_1)(r - r_2) = r^2 - (r_1 + r_2)r + r_1r_2 = 0$ and the corresponding form of (2) is

(2) $$\frac{d^2y}{dx^2} - (r_1 + r_2)\frac{dy}{dx} + r_1r_2y = 0.$$

Into the left side of (2) substitute $y = e^{r_1x}$:

$$r_1^2e^{r_1x} - (r_1 + r_2)r_1e^{r_1x} + r_1r_2e^{r_1x} = e^{r_1x}[r_1^2 - (r_1 + r_2)r_1 + r_1r_2] \equiv 0.$$

Thus $y = e^{r_1x}$ is a specific solution of (2) in Case 1, and in the same way it may be checked that $y = e^{r_2x}$ is also a solution.

Cases 2 and 3 may be checked in the same way.

Example 1. Solve the homogeneous differential equation

$$4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 0.$$

Solution. This equation and its characteristic equation are

$$\frac{d^2y}{dx^2} - \frac{3dy}{dx} + \tfrac{9}{4}y = 0 \quad \text{and} \quad r^2 - 3r + \tfrac{9}{4} = (r - \tfrac{3}{2})^2.$$

Since the characteristic equation has both roots equal to $\frac{3}{2}$, the homogeneous differential equation has general solution (see Case 2)

$$y = (c_1 + c_2x)e^{(3/2)x}.$$

Example 2. Solve $\dfrac{d^2y}{dx^2} + 4y = 0.$

Solution. The characteristic equation is $r^2 + 0 \cdot r + 4 = r^2 + 4 = 0$ whose roots are $r = \pm 2i = 0 \pm 2i$. The desired solution is

$$y = e^{0 \cdot x}(c_1 \cos 2x + c_2 \sin 2x) = c_1 \cos 2x + c_2 \sin 2x.$$

Example 3. Solve the derivative system

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0; \quad y = 0 \quad \text{and} \quad \frac{dy}{dx} = 6 \quad \text{when} \quad x = 0.$$

Solution. The characteristic equation is $r^2 + 2r - 8 = (r + 4)(r - 2) = 0$ and the general solution of the differential equation is

$$y = c_1e^{-4x} + c_2e^{2x}.$$

Hence $\dfrac{dy}{dx} = -4c_1e^{-4x} + 2c_2e^{2x}$ and, from the associated conditions,

$$0 = c_1e^{-4\cdot 0} + c_2e^{2\cdot 0} = c_1 + c_2 \quad \text{and} \quad 6 = -4c_1 + 2c_2.$$

The simultaneous solution of these two equations is $c_1 = -1, c_2 = 1$ so the solution of the given differential system is

$$y = -e^{-4x} + e^{2x}.$$

PROBLEMS

1. Solve each of the second order, homogeneous differential equations with constant coefficients:

a. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0.$

b. $2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0.$

c. $4\dfrac{d^2y}{dx^2} - 20\dfrac{dy}{dx} + 25y = 0.$

d. $\dfrac{d^2y}{dx^2} - 4y = 0.$

e. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 13y = 0.$

f. $\dfrac{d^2y}{dx^2} + 9y = 0.$

2. Solve each of the following differential equations:

a. $\dfrac{d^2s}{dt^2} - 2\dfrac{ds}{dt} + 5s = 0.$

b. $\dfrac{d^2s}{dt^2} - 4\dfrac{ds}{dt} + 5s = 0.$

c. $\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} = 0.$

d. $\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} - x = 0.$

e. $\dfrac{d^2u}{dv^2} + \dfrac{du}{dv} - u = 0.$

f. $\dfrac{d^2x}{dy^2} + 2\dfrac{dx}{dy} - 4x = 0.$

3. Solve each of the differential systems:

a. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 5y = 0;\quad y = 1$ and $\dfrac{dy}{dx} = -1$ when $x = 0.$

b. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 5y = 0;\quad y = 5$ and $\dfrac{dy}{dx} = 3$ when $x = 0.$

c. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 5y = 0;\quad y = 4$ and $\dfrac{dy}{dx} = 2$ when $x = 0.$

d. $4\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 9y = 0;\quad y = 2$ and $\dfrac{dy}{dx} = 2$ when $x = 0.$

4. A particle moves on a line in such a way that its acceleration function a and position function s are related by

$$a = -4s.$$

The particle "starts" when $t = 0$ at $s = 2$ with velocity 6. Show that the motion is simple harmonic (see Sec. 38) with amplitude $\sqrt{13}$, period π, and phase 0.49 (in radians).

5. Find the law of motion under the following conditions with the usual interpretations in terms of acceleration and velocity.
   a. $a = -9s$; $s = 0$ and $v = 4$ when $t = 0$.
   b. $2a = -s$; $s = 0$ and $v = 1$ when $t = 0$.
   c. $4a = -9s$; $s = \sqrt{3}$ and $v = \frac{3}{2}$ when $t = 0$.
   d. $a + 4s = 0$; $s = -1$ and $v = 2\sqrt{3}$ when $t = 0$.

153. Undetermined Coefficients

With $f(x) \not\equiv 0$, then the second-order linear differential equation with constant coefficients

$$(1) \qquad \frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x); \quad a_1 \text{ and } a_2 \text{ constants,}$$

is **nonhomogeneous** and its associated homogeneous equation is

$$(2) \qquad \frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0.$$

The previous section contained methods for solving (2) and we now denote the general solution of (2) by y_H. After introducing some terminology, we show how to use y_H to find, at least in a wide variety of cases, a solution y_P of (1) where y_P is free of arbitrary constants and is called the **particular solution** of (1). Then

$$(3) \qquad y = y_P + y_H$$

is also a solution of (1) and is called the **complete solution** of (1).

The method of "Undetermined Coefficients" is applicable whenever $f(x)$ is the sum of terms containing x^m (with m a positive integer) $\sin px$, $\cos px$, e^{qx}, and a constant together with products of such terms. Corresponding to each of these terms is associated a **company** as listed below:

Term	Company
x^m	$\{x^m, x^{m-1}, \cdots, x, 1\}$
$\sin px$	$\{\sin px, \cos px\}$
$\cos px$	$\{\sin px, \cos px\}$
e^{qx}	$\{e^{qx}\}$
c, a constant	$\{1\}$

Two companies are combined to form a third company consisting of all products of members of the two companies. Thus, if $f(x)$ contains the terms $x^2 \sin 2x$ then x^2 has the company $\{x^2, x, 1\}$, $\sin 2x$ has the company $\{\sin 2x, \cos 2x\}$, and the term $x^2 \sin 2x$ has the company

$$\{x^2 \sin 2x, x^2 \cos 2x, x \sin 2x, x \cos 2x, \sin 2x, \cos 2x\}.$$

The method of undetermined coefficients for finding the particular solution y_P of (1) is outlined as follows:

1. Find the general solution y_H of the homogeneous equation (2).
2. Construct the company of each term of $f(x)$.
3. If any company contains another company discard the smaller company.
4. If any company has a member which is a term in y_H, then replace each member of that company by the member multiplied by the lowest integral power of x for which no member of the new company is a term of y_H.
5. Assume y_P is a linear combination of all members of all those companies obtained.
6. Substitute y_P into the left side of (1), set the result identically equal to $f(x)$, and from the result determine the (constant) coefficients of y_P.

Example 1. Solve the non-homogeneous differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^3 + 2x + 3 + e^{2x}. \tag{4}$$

Solution. 1. The homogeneous equation, characteristic equation, and y_H are

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0, \quad (r^2 - 4r + 4) = (r - 2)^2 = 0, \quad \text{and}$$

$$y_H = (c_1 + c_2x)e^{2x} = c_1e^{2x} + c_2xe^{2x}.$$

2. Since $f(x) = x^3 + 2x + 3 + e^{2x}$ the companies to consider are

$$\{x^3,x^2,x,1\}, \quad \{x,1\}, \quad \{1\}, \quad \{e^{2x}\}.$$

3. The elements of the second and third companies are in the first so retain only

$$\{x^3,x^2,x,1\}, \quad \{e^{2x}\}.$$

4. Since e^{2x} occurs in y_H replace $\{e^{2x}\}$ by $\{xe^{2x}\}$ but since $\{xe^{2x}\}$ also occurs in y_H the companies of 3 are replaced by

$$\{x^3,x^2,x,1\}, \quad \{x^2e^{2x}\}.$$

5. Assume $y_P = Ax^3 + Bx^2 + Cx + D + Ex^2e^{2x}$.

6. $\dfrac{dy_P}{dx} = 3Ax^2 + 2Bx + C + 2Exe^{2x} + 2Ex^2e^{2x}$ and

$$\begin{aligned}\frac{d^2y_P}{dx^2} &= 6Ax + 2B + 2E(e^{2x} + 2xe^{2x}) + 2E(2xe^{2x} + 2x^2e^{2x})\\ &= 6Ax + 2B + 2Ee^{2x} + 8Exe^{2x} + 4Ex^2e^{2x}.\end{aligned}$$

Hence, by substituting y_P into (4), we have

$$\begin{aligned}&6Ax + 2B + 2Ee^{2x} + 8Exe^{2x} + 4Ex^2e^{2x}\\ &\qquad - 4(3Ax^2 + 2Bx + C + 2Exe^{2x} + 2Ex^2e^{2x})\\ &\qquad + 4(Ax^3 + Bx^2 + Cx + D + Ex^2e^{2x}) \equiv x^3 + 2x + 3 + e^{2x},\end{aligned}$$

and hence upon collecting like terms:

$$4Ax^3 + (4B - 12A)x^2 + (4C - 8B + 6A)x + (4D - 4C + 2B) + (4E - 8E + 4E)x^2e^{2x} + (-8E + 8E)xe^{2x} + 2Ee^{2x} \equiv x^3 + 0 \cdot x^2 + 2x + 3 + e^{2x}.$$

Now by equating coefficients

$$\begin{aligned} 4A &= 1 & &\text{so} \quad A = \tfrac{1}{4}, \\ 4B - 12A &= 0 & &\text{so} \quad B = \tfrac{3}{4}, \\ 4C - 8B + 6A &= 2 & &\text{so} \quad C = \tfrac{13}{8}, \\ 4D - 4C + 2B &= 3 & &\text{so} \quad D = 2, \\ 2E &= 1 & &\text{so} \quad E = \tfrac{1}{2}. \end{aligned}$$

Thus $y_P = \frac{1}{4}x^3 + \frac{3}{4}x^2 + \frac{13}{8}x + 2 + \frac{1}{2}x^2e^{2x}$ is the particular solution of (4) and the complete solution of (4) is

$$y = (\tfrac{1}{4}x^3 + \tfrac{3}{4}x^2 + \tfrac{13}{8}x + 2 + \tfrac{1}{2}x^2e^{2x}) + (c_1 + c_2x)e^{2x}.$$

Example 2. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = \sin x.$

Solution. 1. From the characteristic equation $r^2 - r = r(r - 1) = 0$, then

$$y_H = c_1e^{0 \cdot x} + c_2e^x = c_1 + c_2e^x.$$

2. The only company is $\{\sin x, \cos x\}$
3. and 4. Nothing to be done.
5. Assume $y_P = A \sin x + B \cos x$.
6. $(-A \sin x - B \cos x) - (A \cos x - B \sin x) \equiv \sin x$,

$$\left.\begin{aligned} -A + B &= 1 \\ -A - B &= 0 \end{aligned}\right\}, \quad A = -\tfrac{1}{2}, \quad B = \tfrac{1}{2}, \quad y_P = -\tfrac{1}{2}\sin x + \tfrac{1}{2}\cos x.$$

Thus, the complete solution is $y = \frac{1}{2}(-\sin x + \cos x) + c_1 + c_2e^x$.

PROBLEMS

1. Solve each of the nonhomogeneous differential equations:

a. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = x + 1 + e^x.$

b. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x + 1 + e^x.$

c. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = \cos x.$

d. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = e^x \cos x.$

e. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = \cos x.$

f. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^x \cos x.$

g. $\dfrac{d^2y}{dx^2} + 4y = \sin^2 x.$

h. $\dfrac{d^2y}{dx^2} - y = \sin^2 x \sin 2x.$

2. Solve each of the derivative systems.

a. $\dfrac{d^2y}{dx^2} + 4y = 4e^x;\quad y = 0$ and $\dfrac{dy}{dx} = 1$ when $x = 0$.

b. $\dfrac{d^2y}{dx^2} - y = \cos 2x;\quad y = 1$ and $\dfrac{dy}{dx} = 0$ when $x = 0$.

c. $\dfrac{d^2s}{dt^2} + 4\dfrac{ds}{dt} + 5s = 8 \sin t;\quad s = 2$ and $\dfrac{ds}{dt} = -1$ when $t = 0$.

d. $\dfrac{d^2s}{dt^2} + 5\dfrac{ds}{dt} + 4s = 8 \sin t;\quad s = 2$ and $\dfrac{ds}{dt} = -1$ when $t = 0$.

154. Linear, Constant Coefficients

With $f(x) \not\equiv 0$ and n a positive integer, the differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x) \tag{1}$$

is nonhomogeneous, is linear, is of nth order, has constant coefficients $1, a_1, \cdots, a_{n-1}, a_n$, has associated homogeneous differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0 \tag{2}$$

and characteristic equation

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n = 0. \tag{3}$$

By the Fundamental Theorem of Algebra, the algebraic equation (3) may be written as the product of linear factors of the form $(r - r_k)$, with r_k real, and quadratic factors of the form $(r^2 - 2ar + a^2 + b^2)$ where, since the coefficients are real, such quadratic factors produce conjugate complex roots of (3) of the form $a \pm ib$. The factors of (3) and the solutions of (2) are related as follows:

Factors of (3)	Specific solutions of (2)
$(r - r_k)^p$, r_k real	$e^{r_k x}, xe^{r_k x}, x^2e^{r_k x}, \cdots, x^{p-1}e^{r_k x}$
$(r^2 - 2ar + a^2 + b^2)^p$	$e^{ax}\cos bx, e^{ax}\sin bx, xe^{ax}\cos bx, xe^{ax}\sin bx, \cdots, x^{p-1}e^{ax}\cos bx, x^{p-1}e^{ax}\sin bx$

The general solution of (2) is then the linear combination (with arbitrary constant coefficients) of *all* specific solutions of (2).

Example 1. Solve the homogeneous fifth order equation

$$\frac{d^5y}{dx^5} - 6\frac{d^4y}{dx^4} + 13\frac{d^3y}{dx^3} - 14\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = 0. \tag{4}$$

Solution. The characteristic equation

$$r^5 - 6r^4 + 13r^3 - 14r^2 + 12r - 8 = 0 \tag{5}$$

has integer coefficients (with 1 as the first coefficient) so the possible rational roots are ± 1, ± 2, ± 4, ± 8; that is, the factors of the last coefficient -8. By trial, using synthetic division, 2 is a root:

2|	1	−6	13	−14	12	−8	
		2	−8	10	−8	8	
2|	1	−4	5	−4	4	0,	zero, so try 2 again
		2	−4	2	−4		
2|	1	−2	1	−2	0	,	zero, so try 2 again
		2	0	2			
	1	0	1	0			

Hence, (5) may be written as $(r - 2)^3(r^2 + 0 \cdot r + 1) = (r - 2)^3(r^2 + 1) = 0$. The triple root 2 and the complex roots $\pm i = 0 \pm i$ show that (4) has general solution

$$\begin{aligned} y &= c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x} + c_4e^{0\cdot x}\cos x + c_5e^{0\cdot x}\sin x \\ &= (c_1 + c_2x + c_3x^2)e^{2x} + c_4\cos x + c_5\sin x. \end{aligned}$$

A repeated quadratic factor, such as $(r^2 - 2ar + a^2 + b^2)^p$ with $p = 2$ or more, occurs so seldom that we do not even give an example.

The method of undetermined coefficients (Sec. 153) applies (when the coefficients are constants) to linear nonhomogeneous differential equations of order greater than 2 as well as when the order is 2.

Example 2. Solve the nonhomogeneous equation

$$\frac{d^5y}{dx^5} - 6\frac{d^4y}{dx^4} + 13\frac{d^3y}{dx^3} - 14\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = 250\sin x. \tag{6}$$

Solution. 1. Solve the homogeneous equation (see Example 1) and set

$$y_H = (c_1 + c_2x + c_3x^2)e^{2x} + c_4\cos x + c_5\sin x.$$

2. The only company of $f(x) = 250\sin x$ is $\{\sin x, \cos x\}$.
3. Nothing to do.
4. Since $\sin x$ (and/or $\cos x$) occurs in y_H replace the company by $\{x\sin x, x\cos x\}$.

$$\begin{array}{lrcl|l}
5. & \text{Assume } y_P & = & Ax \sin x + Bx \cos x. & -8 \\
6. & y_P' & = & Ax \cos x - Bx \sin x + A \sin x + B \cos x & 12 \\
 & y_P'' & = & -Ax \sin x - Bx \cos x + 2A \cos x - 2B \sin x & -14 \\
 & y_P''' & = & -Ax \cos x + Bx \sin x - 3A \sin x - 3B \cos x & 13 \\
 & y_P^{(4)} & = & Ax \sin x + Bx \cos x - 4A \cos x + 4B \sin x & -6 \\
 & y_P^{(5)} & = & Ax \cos x - Bx \sin x + 5A \sin x + 5B \cos x & 1
\end{array}$$

$$(-8A-12B+14A+13B-6A-B)x\sin x+(-8B+12A+14B-13A-6B+A)x\cos x$$

$$+(12A+28B-39A-24B+5A)\sin x+(12B-28A-39B+24A+5B)\cos x=250\sin x,$$

$$(-22A + 4B) \sin x + (-4A - 22B) \cos x = 250 \sin x,$$

$$\left.\begin{array}{l} -22A + 4B = 250 \\ -4A - 22B = 0 \end{array}\right\} \quad A = -11, \quad B = 2.$$

Therefore, the complete solution of (6) is

$$y = -11x \sin x + 2x \cos x + (c_1 + c_2x + c_3x^2)e^{2x} + c_4 \cos x + c_5 \sin x.$$

PROBLEMS

1. Solve the differential equation $\dfrac{d^5y}{dx^5} + \dfrac{d^3y}{dx^3} = f(x)$ given that:

a. $f(x) = 2.$
b. $f(x) = x.$
c. $f(x) = x^2.$
d. $f(x) = \sin x.$
e. $f(x) = 2 \sin x + 3 \cos x.$
f. $f(x) = e^x.$

2. Replace the differential equation of Prob. 2 by

$$\frac{d^3y}{dx^2} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = f(x).$$

3. Solve each of the differential equations

a. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 4 \sin x.$

b. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + y = 4e^{-x}.$

c. $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} + 8y = 8x^2.$

d. $4\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} - 15\dfrac{dy}{dx} + 18y = 49e^{-2x}.$

e. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} = 6x + 2xe^x.$

f. $\dfrac{d^3y}{dx^3} = 6x + 2xe^x.$

g. $\dfrac{d^3y}{dx^3} - y = 9e^{-0.5x} + 91 \cos \dfrac{\sqrt{3}}{2}x.$

h. $\dfrac{d^4y}{dx^4} - y = e^x(1 + \cos x).$

155. Variation of Parameters

Consider again a nonhomogeneous, linear differential equation with constant coefficients:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = f(x).$$

In case $f(x)$ or any term of $f(x)$ is not x^m, e^{ax}, $\sin bx$, $\cos bx$, or a product of one or more of these, then the method of undetermined coefficients (Sec. 153) is not applicable. Another method, called **variation of parameters**, is illustrated in the following example.

Example. Solve the differential equation

(a) $$y'' + y = \sec x \tan x.$$

Solution. Step 1. Solve the homogeneous equation $y'' + y = 0$ and call y_H the solution so

(b) $$y_H = c_1 \cos x + c_2 \sin x.$$

Step 2. Assume there is a particular solution of the form

(c) $$y_P = A(x)\cos x + B(x)\sin x$$

(obtained from y_H by replacing the constants c_1 and c_2 by functions).

Step 3. Find $y_P' = -A(x)\sin x + B(x)\cos x + A'(x)\cos x + B'(x)\sin x$ and simplify this by setting

(d) $$A'(x)\cos x + B'(x)\sin x = 0 \quad \text{so that}$$

(e) $$y_P' = -A(x)\sin x + B(x)\cos x.$$

Find y_P'' from this expression for y_P':

(f) $$y_P'' = -A(x)\cos x - B(x)\sin x - A'(x)\sin x + B'(x)\cos x.$$

[Note: If the equation were of higher than second order we would simplify y_P'' by setting the terms involving $A'(x)$, etc., equal to zero.]

Step 4. Substitute y_P and its derivatives for y and its derivatives in given differential equation. In this case substitute (f), (e), and (c) into (a):

$$[-A(x)\cos x - B(x)\sin x - A'(x)\sin x + B'(x)\cos x] + 0\cdot[-A(x)\sin x + B(x)\cos x] + [A(x)\cos x + B(x)\sin x] = \sec x \tan x$$

and simplify [the terms in $A(x)$ and $B(x)$ always cancel]:

(g) $$-A'(x)\sin x + B'(x)\cos x = \sec x \tan x.$$

Step 5. From (g) and (d) [we copy (d) over for convenience]

(d) $$A'(x)\cos x + B'(x)\sin x = 0$$

solve for $A'(x)$ and $B'(x)$:

$$A'(x)(\sin^2 x + \cos^2 x) = -\sin x \sec x \tan x, \quad A'(x) = -\tan^2 x$$
$$B'(x)(\cos^2 x + \sin^2 x) = \cos x \sec x \tan x, \quad B'(x) = \tan x.$$

Step 6. Integrate to find the simplest expressions for $A(x)$ and $B(x)$:

$$\left.\begin{aligned} A(x) &= \int(-\tan^2 x)\,dx = x - \tan x \\ B(x) &= \int \tan x\,dx = -\ln|\cos x| \end{aligned}\right\} \text{No constant of integration is added.}$$

Step 7. Substitute these expressions for $A(x)$ and $B(x)$ into the assumed form [namely (c)] of y_P:

$$\begin{aligned} y_P &= (x - \tan x)\cos x + (-\ln|\cos x|)\sin x \\ &= x\cos x - \sin x - \sin x \ln|\cos x|. \end{aligned}$$

The complete solution is then

$$y = y_P + y_H = x\cos x - \sin x - \sin x \ln|\cos x| + c_1 \cos x + c_2 \sin x.$$

In cases where the method of undetermined coefficients is applicable, the method of variation of parameters may also be used, but the integrations in Step 6 may be difficult.

PROBLEMS

1. Solve the differential equation $y'' + y = f(x)$ after substituting:
   a. $f(x) = \sec x$. b. $f(x) = \sec^2 x$. c. $f(x) = \tan^2 x$. d. $f(x) = \sec^2 x \csc x$.
2. Replace the differential equation of Prob. 1 by $y'' + 4y = f(x)$.
3. Solve
   a. $y'' + 4y' + 4y = x^{-2}e^{-2x}$.
   b. $y'' - 2y' + 2y = e^x \sec x \csc x$.
   c. $y'' - 3y' + 2y = e^x(1 + e^x)^{-1}$.
   d. $y''' + y' = \sec x$.
4. As a comparison, solve each of the following by the method of variation of parameters and also by the method of undetermined coefficients.
   a. $y'' - 5y' + 6y = 10 \sin x$.
   b. $y'' - 2y' + 5y = 5x^2 + 6x + 3$.

156. Missing Variables

In this section two types of differential equations are considered whose solutions may be made to depend upon the solutions of lower order differential equations.

A. DEPENDENT VARIABLE MISSING. In the equations of Examples 1 and 2, only derivatives of y occur and not y itself.

Example 1. Solve $xy'' - y' = (y')^3$.

Solution. Substitute $y' = u$, and $y'' = u'$, to obtain the equation $xu' - u = u^3$ whose order is one less than the order of the given equation. The variables u and x are separable:

$$\frac{du}{u^3 + u} = \frac{dx}{x}.$$

By using partial fractions (see Sec. 76) this equation is

$$\left(\frac{1}{u} - \frac{u}{u^2+1}\right) du = \frac{dx}{x} \quad \text{so} \quad \ln|u| - \frac{1}{2}\ln(u^2+1) = \ln|c_1x|$$

$$\frac{u^2}{u^2+1} = (c_1x)^2 \quad \text{and thus} \quad u^2 = \frac{(c_1x)^2}{1-(c_1x)^2}.$$

But $u = dy/dx$ and therefore

$$\frac{dy}{dx} = \frac{c_1x}{\sqrt{1-(c_1x)^2}}$$ (Note: $\pm$ not necessary since c_1 is arbitrary.)

which, by an ordinary indefinite integration, gives the desired solution

$$y = -\frac{1}{c_1}\sqrt{1-(c_1x)^2} + c_2.$$

Example 2. With w and H constants, solve the derivative system

(2) $$\frac{d^2y}{dx^2} = \frac{w}{H}\sqrt{1+\left(\frac{dy}{dx}\right)^2} \quad \text{where} \quad \frac{dy}{dx} = 0 \quad \text{and} \quad y = a \quad \text{when} \quad x = 0.$$

Solution. Set $\dfrac{dy}{dx} = v$ so $\dfrac{d^2y}{dx^2} = \dfrac{dv}{dx}$ and the differential equation in (2) becomes $\dfrac{dv}{dx} = \dfrac{w}{H}\sqrt{1+v^2}$ in which the variables v and x are separable:

$$\frac{dv}{\sqrt{1+v^2}} = \frac{w}{H}dx \quad \text{so that} \quad \ln(v+\sqrt{1+v^2}) = \frac{w}{H}x + c_1.$$

Since $v = y'$ we have $v = 0$ when $x = 0$ so $\ln(0+\sqrt{1+0^2}) = 0 + c_1$. Thus $c_1 = 0$, hence $\ln(v+\sqrt{1+v^2}) = wx/H$ and thus

$$v+\sqrt{1+v^2} = e^{wx/H}, \quad \sqrt{1+v^2} = e^{wx/H} - v, \quad 1+v^2 = e^{2wx/H} - 2ve^{wx/H} + v^2,$$

$$v = \frac{1}{2}\frac{e^{2wx/H}-1}{e^{wx/H}} = \frac{1}{2}[e^{wx/H} - e^{-wx/H}].$$

Since $v = dy/dx$, a simple indefinite integration gives

$$y = \frac{1}{2}\frac{H}{w}(e^{wx/H} + e^{-wx/H}) + c_2.$$

But $y = a$ when $x = 0$ so $a = \dfrac{1}{2}\dfrac{H}{w}(e^0 + e^0) + c_2$ and thus

$$y = \frac{1}{2}\frac{H}{w}(e^{wx/H} + e^{-wx/H}) + a - \frac{H}{w}.$$

[Note: Example 2 fulfills a promise made earlier (see the sentence including (5) of Sec. 147).]

B. INDEPENDENT VARIABLE MISSING. If an equation involves $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, but x appears only in derivatives, the method is to set

$$\frac{dy}{dx} = u \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{du}{dx}.$$

An additional step, **which is easily forgotten,** *is to write the equation as if y (instead of x) were the independent variable* by using the formula

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} \quad \text{so that} \quad \frac{du}{dx} = \frac{du}{dy}u.$$

Thus, in this situation the recommended substitutions are

$$\frac{dy}{dx} = u \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{du}{dy}u. \tag{3}$$

Example 3. $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 2y\dfrac{dy}{dx}.$

Solution. The substitutions (3) change this equation into

$$y\frac{du}{dy}u + u^2 = 2yu \quad \text{so} \quad u\left[\frac{du}{dy} + \frac{1}{y}u - 2\right] = 0 \quad \text{and} \tag{4}$$

$$u = 0 \quad \text{or} \quad \frac{du}{dy} + \frac{1}{y}u = 2.$$

This is a linear equation (see Sec. 151) wherein we set

$$p(y) = \frac{1}{y}, \quad P(y) = \int p(y)\,dy = \ln|y| \quad \text{and know that}$$

$$e^{\ln|y|}u = \int e^{\ln|y|}\,2\,dy, \quad |y|u = \int 2|y|\,dy, \quad yu = \int 2y\,dy = y^2 + c_1.$$

Thus $$u = \frac{y^2 + c_1}{y}, \quad \frac{dy}{dx} = \frac{y^2 + c_1}{y}, \quad \frac{y}{y^2 + c_1}dy = dx,$$

$$x = \tfrac{1}{2}\ln|y^2 + c_1| + c_2. \tag{5}$$

The equation $u = 0$; that is, $\dfrac{dy}{dx} = 0$ has solution $y = c$. Notice that (4) has u as one factor so in addition to (5) the given equation has the **trivial** solution $y = c$.

PROBLEMS

1. Solve

a. $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} = \left(\dfrac{dy}{dx}\right)^2.$

b. $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = \left(\dfrac{dy}{dx}\right)^2.$

c. $x\dfrac{d^2y}{dx^2} + 1 = \left(\dfrac{dy}{dx}\right)^2.$

d. $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = \dfrac{dy}{dx}.$

e. $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0.$

f. $y\dfrac{d^2y}{dx^2} + 4y^2 = \dfrac{1}{2}\left(\dfrac{dy}{dx}\right)^2.$

2. Continue from equations (3) and show that

$$\frac{d^3y}{dx^3} = \frac{d^2u}{dy^2}u^2 + \left(\frac{du}{dy}\right)^2 u \quad \text{and}$$

$$\frac{d^4y}{dx^4} = \frac{d^3u}{dy^3}u^3 + 4\frac{d^2u}{dy^2}\frac{du}{dy}u^2 + \left(\frac{du}{dy}\right)^3 u.$$

157. Integrating Factors

So far in this chapter (and in Sec. 117) some common types of differential equations have been given, together with definite procedures to follow in each case. A complete listing of types for which methods have been developed would be a ponderous tome and, even so, would not include all differential equations met in practice.

This introduction to finding solutions of differential equations in closed form is concluded by illustrating dependence upon experience and insight.

Example 1. $(x^3 - y)\,dx + x\,dy = 0.$

Solution. Write $x^3\,dx + x\,dy - y\,dx = 0$ and recognize $x\,dy - y\,dx$ as the numerator of $d\left(\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2}$. The given equation may be written, by dividing by x^2, as

$$x\,dx + \frac{x\,dy - y\,dx}{x^2} = 0, \quad d\left(\frac{x^2}{2}\right) + d\left(\frac{y}{x}\right) = 0, \quad \text{so as} \quad d\left(\frac{x^2}{2} - \frac{y}{x}\right) = 0.$$

Hence, the solution is $\dfrac{x^2}{2} - \dfrac{y}{x} = c$ or $y = cx + \dfrac{x^3}{2}$.

The given equation could have been written as

$$\frac{dy}{dx} - \frac{1}{x}y = -x^2$$

which is a linear equation but for this problem the above procedure (if recognized) is easier than the method of Sec. 150.

Notice the numerators of each of the following:

$$d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}, \qquad d\left(-\frac{y}{x}\right) = \frac{y\,dx - x\,dy}{x^2},$$

$$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}, \qquad d\left(-\frac{x}{y}\right) = \frac{x\,dy - y\,dx}{y^2},$$

$$d\ln(x^2 + y^2) = \frac{2(x\,dx + y\,dy)}{x^2 + y^2}, \quad d\ln\sqrt{x^2 + y^2} = \frac{x\,dx + y\,dy}{x^2 + y^2},$$

$$d\tan^{-1}\frac{y}{x} = \frac{d(y/x)}{1 + (y/x)^2} = \frac{(x\,dy - y\,dx)/x^2}{(x^2 + y^2)/x^2} = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

and also $d(xy) = x\,dy + y\,dx$, and $d(x^n + y^n) = n(x^{n-1}\,dx + y^{n-1}\,dy)$. Should part of a differential equation be $x\,dy - y\,dx$, $x\,dx + y\,dy$, or $y\,dx + x\,dy$, it may be possible to multiply through the whole equation by an expression so the resulting equation is recognizable as a differential equal to zero. Such a multiplier is called an **integrating factor.**

Example 2. $x(1 - xy)\,dy + y\,dx = 0$.

Solution. $x\,dy + y\,dx - x^2y\,dy = 0$, $d(xy) - x^2y\,dy = 0$,

$$\frac{d(xy)}{x^2y^2} - \frac{dy}{y} = 0, \quad -d(xy)^{-1} - d\ln|y| = 0, \quad d[(xy)^{-1} + \ln|y|] = 0,$$

and the solution is $(xy)^{-1} + \ln|y| = c$.

At one stage the integrating factor $1/(xy)^2$ was used.

Example 3. $(x^3 + y)\,dx + (x^2y - x)\,dy = 0$.

Solution. $(y\,dx - x\,dy) + (x^3\,dx + x^2y)\,dy = 0$. The first parenthesis is the numerator of $d(x/y)$ so try the integrating factor $1/y^2$:

$$\frac{y\,dx - x\,dy}{y^2} + \frac{x^3\,dx + x^2y\,dy}{y^2} = 0, \quad d\left(\frac{x}{y}\right) + \left[\frac{x^3}{y^2}\,dx + \frac{x^2}{y}\,dy\right] = 0$$

wherein the bracket is not a perfect differential. But $y\,dx - x\,dy$ is also the numerator of $d(-y/x)$ so try the integrating factor $1/x^2$:

$$\frac{y\,dx - x\,dy}{x^2} + \frac{x^3\,dx + x^2y\,dy}{x^2} = 0, \quad d\left(-\frac{y}{x}\right) + [x\,dx + y\,dy] = 0,$$

$$d\left(-\frac{y}{x}\right) + \frac{1}{2}\,d(x^2 + y^2) = 0, \quad d\left[-\frac{y}{x} + \frac{x^2 + y^2}{2}\right] = 0$$

so the solution is $\dfrac{x^2 + y^2}{2} - \dfrac{y}{x} = c$.

PROBLEMS

1. In each of the following an integrating factor may be used, although in some cases another method will also work.

a. $(3 - x^2 - y)\,dx + x\,dy = 0.$

b. $y\,dx + x(xy^2 + 1)\,dy = 0.$

c. $x(xy^2 - 1)\dfrac{dy}{dx} + x^4 + y = 0.$

d. $xy' + y(1 + x^4y) = 0.$

e. $x\,dy - y\,dx = (x^2 + y^2)\,dx.$

f. $x\,dy - y\,dx = (x^2 - y^2)\,dx.$

g. $\dfrac{dy}{dx} + \dfrac{y}{x} = \sin x.$

h. $s\,dt - t\,ds = (s^2 + 1)\,ds.$

2. The following miscellaneous collection will provide a review of all methods given so far in this introduction to differential equations.

a. $\dfrac{y - x}{xy}\,dx + \dfrac{x}{y^2}\,dy = 0.$

b. $2\dfrac{d^2s}{dt^2} - 5\dfrac{ds}{dt} + 3s = 0.$

c. $ye^{3x}\,dx - (1 + e^x)\,dy = 0.$

d. $(x + 2)y' - x + 2y = 0.$

e. $(x + 3y)\,dx + (y - 3x)\,dy = 0.$

f. $(x + x\sin t)x' = (1 + 2x)^2.$

g. $(y^2 - y)\,dx = x^2\,dy.$

h. $x^2\,dy + (4xy + 2)\,dx = 0.$

i. $y' + ay = e^{-ax}.$

j. $y' - xy = xy^{-1}.$

k. $y''' = y.$

l. $\cos y\dfrac{dy}{dx} + 3\sin y = 2x.$

m. $2\dfrac{d^3y}{dx^3} + 5\dfrac{d^2y}{dx^2} - 22\dfrac{dy}{dx} + 15y = 0.$

n. $(x\sqrt{x^2 + y^2} - y)\,dx + (y\sqrt{x^2 + y^2} - x)\,dy = 0.$

o. $(xy - 2x^2)\,dy = (x^2 + y^2 + 3xy)\,dx.$

p. $y''' + 9y'' + 27y' + 27y = e^{-3x}.$

q. $y'' = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$

r. $(x - 3y + 3)\,dx = (2x - 6y + 1)\,dy.$

s. $x\sin(y/x)\,dy = (y\sin y/x + x)\,dx.$

t. $(4x - y - 2)\,dx + (x + y + 2)\,dy = 0.$

u. $(3x^2 - 6xy)\,dx - (3x^2 + 2y)\,dy = 0.$

v. $y'' = 15y - 2y' + 130\sin x.$

158. Power-Series Method

If a power series converges to zero:

$$0 = b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1} + b_nx^n + \cdots, \tag{1}$$

for **all** x such that $|x| < a$, then all coefficients are zero; that is

$$b_0 = 0, \quad b_1 = 0, \quad b_2 = 0, \cdots, b_n = 0, \cdots. \tag{2}$$

Let (1) hold for all x such that $|x| < a$. First set $x = 0$ so that $0 = b_0 + b_1 \cdot 0 + b_2 \cdot 0 + \cdots$ and hence $b_0 = 0$. Since (1) holds, then term-by-term derivatives (see Sec. 146) are permissible:

$$D_x 0 = D_x b_0 + D_x b_1 x + D_x b_2 x^2 + \cdots + D_x b_n x^n + \cdots,$$

$$0 = 0 + b_1 + 2b_2 x + 3b_3 x^2 + \cdots + nb_n x^{n-1} + \cdots \quad \text{for all } |x| < a. \tag{3}$$

In this equation set $x = 0$ and see that $b_1 = 0$. The derivative of both sides of (3) yields

$$0 = 2b_2 + 3 \cdot 2b_3 x + 4 \cdot 3b_4 x^2 + \cdots + n(n-1)b_n x^{n-2} + \cdots \quad \text{for all } |x| < a$$

and hence $b_2 = 0$. By continuing in this way, (2) holds.

Conversely, if (2) holds, then (1) does also.

The fact that (1) holds if and only if (2) holds will be used to obtain solutions in terms of power series for some differential equations. As will be seen below, the power-series method entails considerable manipulation and hence, in practice, is not turned to until other methods have been tried unsuccessfully. We shall, however, illustrate the method by a simple example which could be solved more easily by other methods.

Example 1. Determine constants $a_0, a_1, a_2, \cdots, a_n, \cdots$ such that

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n + \cdots \tag{4}$$

will be the general solution of the differential equation

$$2\frac{dy}{dx} - y = 0. \tag{5}$$

Solution. It is assumed that the power series in (4) converges in some interval with center at the origin and hence (see Sec. 146) that its derivative may be obtained by taking derivatives term-by-term:

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n-1)a_{n-1}x^{n-2} + na_n x^{n-1} + \cdots \tag{6}$$

with the result converging in the same interval. Subtract the terms of (4) from two times the terms of (6) and collect like powers of x to obtain

$$0 = 2\frac{dy}{dx} - y = (2a_1 - a_0) + (2 \cdot 2a_2 - a_1)x + (2 \cdot 3a_3 - a_2)x^2 + \cdots + (2na_n - a_{n-1})x^{n-1} + \cdots. \tag{7}$$

From the above discussion all coefficients must be zero:

(8)

$$2a_1 - a_0 = 0, \quad 2 \cdot 2a_2 - a_1 = 0, \quad 2 \cdot 3a_3 - a_2 = 0, \cdots, 2na_n - a_{n-1} = 0, \cdots.$$

Hence $a_1 = \frac{1}{2}a_0$, $a_2 = \frac{1}{4}a_1 = \frac{1}{8}a_0$, $a_3 = \frac{1}{6}a_2 = \frac{1}{6}(\frac{1}{8}a_0) = \frac{1}{48}a_0$, etc. The first four terms of the series in (4) are thus

$$y = a_0 + \tfrac{1}{2}a_0x + \tfrac{1}{8}a_0x^2 + \tfrac{1}{48}a_0x^3 + \cdots.$$

From these few terms, however, we cannot determine whether the series converges or not. We thus write the first n equations of (8) as

$$\begin{aligned} 2a_1 &= a_0 \\ 2 \cdot 2a_2 &= a_1 \\ 2 \cdot 3a_3 &= a_2 \\ &\cdots\cdots \\ 2na_n &= a_{n-1}. \end{aligned}$$

The product of the n terms on the left is equal to the product of the terms on the right:

$$2^n n!\, a_1a_2a_3 \cdots a_n = a_0a_1a_2 \cdots a_{n-1}.$$

By cancelling factors, the result is $2^n n!\, a_n = a_0$ so that

$$a_n = \frac{a_0}{2^n n!}$$

and thus (4) becomes

$$y = a_0 + \frac{a_0}{2}x + \frac{a_0}{2^2 2!}x^2 + \frac{a_0}{2^2 3!}x^3 + \cdots + \frac{a_0}{2^n n!}x^n + \cdots,$$

$$(9) \qquad y = a_0\left[1 + \frac{x}{2} + \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \cdots + \frac{1}{n!}\left(\frac{x}{2}\right)^n + \cdots\right].$$

At this point, as a matter of logic, we have not proved that (9) is the general solution of (5). We have merely shown: *If (5) has a solution in the form (4), then (9) must be this solution.* It remains first to determine whether (9) converges and if it does then to check whether (9) satisfies (5).

The series in (9) converges for $x = 0$ and for $x \neq 0$ the ratio test may be used:

$$\lim_{n\to\infty}\left|\frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+1} \div \frac{1}{n!}\left(\frac{x}{2}\right)^n\right| = \lim_{n\to\infty}\frac{|x|}{2(n+1)} = 0 < 1.$$

Thus (see p. 469), the series in (9) converges for all x and hence its derivative may be obtained by taking term-by-term derivatives with the resulting series converging for all x (see p. 477). A check may now be made to show that (9) satisfies the given equation $2y' - y = 0$.

Notice that a_0 is not determined and indeed is arbitrary; it is the arbitrary constant of the general solution† of the first-order differential equation (5).

The summation index n on either side of

$$(10) \qquad \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

† By other methods, the solution of (5) is $y = ce^{x/2}$. Notice also that the series in (9) is equal to $a_0e^{x/2}$.

is a "dummy index" in the sense that either side of (10) written out is $2 \cdot 1a_2x^0 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + \cdots$ and does not contain n.

In the next example we use (10), and it is well to have a method of starting with the left side of (10) and obtaining the right side. On the left of (10) set $m = n - 2$. Hence, if $n = 2$ then $m = 0$, if $n = 3$ then $m = 1$, etc. Also, $n = m + 2$ and hence

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m.$$

Now, on the right (and on the right only) change the dummy index m to n and the result is (10).

Example 2. Obtain the power-series solution of

$$(1 - x^2)y'' - 6xy' - 4y = 0. \tag{11}$$

Solution. We first assume (11) has a solution in the form (4) (but write (4) using summation notation), obtain the corresponding series for y' and y'' (on the right of the double line below), then multiply by the respective coefficients given on the left of the double line, and add the results in the form (12):

$$\begin{array}{r||ll}
-4 & y = \sum_{n=0}^{\infty} a_n x^n & \\
-6x & y' = \sum_{n=1}^{\infty} na_n x^{n-1} & \text{(summation starts with } n = 1) \\
(1 - x^2) & y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} & \text{(starts with } n = 2)
\end{array}$$

$$\left[\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n\right] - \sum_{n=1}^{\infty} 6na_n x^n - \sum_{n=0}^{\infty} 4a_n x^n = 0. \tag{12}$$

The first summation is now changed to agree with the others in having x^n instead of x^{n-2} (see (10)):

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 6na_n x^n - \sum_{n=0}^{\infty} 4a_n x^n = 0.$$

The second summation starts with $n = 2$, the third with $n = 1$, and we thus write out the terms for $n = 0$ and $n = 1$, then collect all summations from $n = 2$ under one summation sign:

$$2 \cdot 1a_2x^0 + 3 \cdot 2a_3x - 6a_1x - 4a_0x^0 - 4a_1x$$
$$+ \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} - n(n-1)a_n - 6na_n - 4a_n]x^n = 0.$$

Upon collecting terms we thus have

$$(2a_2 - 4a_0) + (6a_3 - 10a_1)x + \sum_{n=2}^{\infty} [(n+2)(n+1)a_{n+2} - (n+4)(n+1)a_n]x^n = 0.$$

Since all terms must be zero for any x in the eventual interval of convergence, then

(13) $$a_2 = 2a_0, \quad a_3 = \tfrac{5}{3}a, \quad \text{and}$$

(14) $$a_{n+2} = \frac{n+4}{n+2}a_n \quad \text{for} \quad n = 2, 3, 4, \cdots.$$

Notice that the subscripts n and $n + 2$ in (14) differ by 2. It is thus natural to group the coefficients with even subscripts by replacing n in (14) by $2k$:

(15) $$a_{2k+2} = \frac{2k+4}{2k+2}a_{2k} = \frac{k+2}{k+1}a_{2k}. \quad \text{for} \quad k = 1, 2, 3, \cdots$$

and for the coefficients with odd subscripts to replace n in (14) by $2k - 1$:

(16) $$a_{2k+1} = \frac{(2k-1)+4}{(2k-1)+2}a_{2k-1} = \frac{2k+3}{2k+1}a_{2k-1} \quad \text{for} \quad k = 2, 3, 4, \cdots.$$

We now arrange two lines according to evenness or oddness of subscripts (using (13) for the first two entries and (15) or (16) for later ones):

$$\begin{array}{ll} a_2 = 2a_0 & a_3 = \tfrac{5}{3}a_1 \\ a_4 = \tfrac{3}{2}a_2 & a_5 = \tfrac{7}{5}a_3 \\ a_6 = \tfrac{4}{3}a_4 & a_7 = \tfrac{9}{7}a_5 \\ \cdots\cdots & \cdots\cdots \\ a_{2k+2} = \dfrac{k+2}{k+1}a_{2k} & a_{2k+1} = \dfrac{2k+3}{2k+1}a_{2k-1}. \end{array}$$

From the first column (by setting the product of all terms on the left equal to the product of terms on the right)

$$a_{2k+2} = (k+2)a_0 \quad \text{for} \quad k = 0, 1, 2, \cdots$$

and from the second column

$$a_{2k+1} = \frac{2k+3}{3}a_1 \quad \text{for} \quad k = 0, 1, 2, \cdots.$$

Hence $$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{k=0}^{\infty} a_{2k+2}x^{2k+2} + \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1}$$

$$= a_0 + \sum_{k=0}^{\infty}(k+2)a_0 x^{2k+2} + \sum_{k=0}^{\infty}\frac{2k+3}{3}a_1 x^{2k+1},$$

(17) $$y = a_0\left[1 + \sum_{k=0}^{\infty}(k+2)x^{2k+2}\right] + \frac{a_1}{3}\sum_{k=0}^{\infty}(2k+3)x^{2k+1}.$$

Upon applying the ratio test to the first series:

$$\lim_{k\to\infty}\left|\frac{(k+3)x^{2k+4}}{(k+2)x^{2k+2}}\right| = |x|^2 \lim_{k\to\infty}\frac{k+3}{k+2} = |x|^2,$$

the first series is seen to converge if $|x|^2 < 1$; that is, if $-1 < x < 1$. The ratio test also shows that the second series in (17) converges for $-1 < x < 1$.

A check will now show that (17) is a solution of the differential equation (11), *but only for* $-1 < x < 1$.

Notice that a_0 and a_1 are arbitrary and that the general solution of the second-order differential equation (11) must have two arbitrary constants.

PROBLEMS

1. Even though some other method may be easier, find the series solution of:
   a. $y' = 2xy$.
   b. $y' = y - x$.
   c. $y'' + y = 0$.
   d. $y'' + y' = 0$.
   e. $(x^2 + 1)y'' + 6xy' + 6y = 0$.
   f. $(x^2 - 1)y'' - 6y = 0$.
   g. $y'' - xy' + 2y = 0$.
   h. $(x^2 + 1)y'' - 4xy' + 6y = 0$.
2. Find the particular solution of the differential equation which also satisfies the initial condition.
   a. $y'' + xy' - 2y = 0$; $y = 1$ and $y' = -2$ when $x = 0$.
   b. $y'' + xy' + 3y = x^2$; $y = 2$ and $y' = 1$ when $x = 0$.

159. Indicial Equation

As in the previous section, the differential equation

$$2xy'' + y' - y = 0 \tag{1}$$

is assumed to have its general solution in the form

$$y = \sum_{n=0}^{\infty} a_n x^n. \tag{2}$$

This assumption leads to

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 0,$$

$$a_1 x^0 - a_0 x^0 + \sum_{n=2}^{\infty} [2n(n-1)a_n + na_n - a_{n-1}]x^{n-1} = 0.$$

Consequently $a_1 = a_0$ and

$$a_n = \frac{1}{n(2n-1)} a_{n-1}, \quad n = 2, 3, 4, \cdots.$$

By the usual scheme $a_n = \dfrac{1}{n!\,3 \cdot 5 \cdot \cdots \cdot (2n-1)} a_0$ for $n = 2, 3, 4, \cdots$ and thus (2) becomes

$$y = a_0\left[1 + x + \sum_{n=2}^{\infty} \frac{1}{n!\,3 \cdot 5 \cdots (2n-1)} x^n\right]. \tag{3}$$

A check will show that (3) is a solution of (1). Unfortunately, however, (3) contains only one arbitrary constant so is not the general solution of the second-order differential equation (1), since the general solution of (1) requires two arbitrary constants.

Fiom experience it has been found that a differential equation whose general solution is not expressible as an ordinary power series may have one solution $y_1(x)$ in the form

$$y = x^r\left[1 + \sum_{n=1}^{\infty} a_n x^n\right] = x^r + \sum_{n=1}^{\infty} a_n x^{n+r} \tag{4}$$

for one value of r and another solution $y_2(x)$ in the same form (4) but with a different value of r. Whenever this happens then

$$y = c_1 y_1(x) + c_2 y_2(x),$$

with c_1 and c_2 arbitrary constants, is the general solution of the given differential equation. The following example is an illustration of the technique.

Example. Show there are two values of r, and corresponding values of a_1, a_2, a_3, $\cdots$, for each of which (4) is a solution of (1).

Solution. We rewrite (4), its first and second derivatives, and multiply by the coefficients in (1):

$$\begin{array}{r||l} -1 & y = x^r + \sum_{n=1}^{\infty} a_n x^{n+r} \\ 1 & y' = rx^{r-1} + \sum_{n=1}^{\infty} (n + r)a_n x^{n+r-1} \\ 2x & y'' = r(r-1)x^{r-2} + \sum_{n=1}^{\infty} (n + r)(n + r - 1)a_n x^{n+r-2}, \end{array}$$

$$2r(r-1)x^{r-1} + rx^{r-1} - x^r + \sum_{n=1}^{\infty} [2(n + r)(n + r - 1) + (n + r)]a_n x^{n+r-1} - \sum_{n=1}^{\infty} a_n x^{n+r} = 0,$$

$$r(2r-1)x^{r-1} - x^r + \sum_{n=1}^{\infty} (n + r)(2n + 2r - 1)a_n x^{n+r-1} - \sum_{n=2}^{\infty} a_{n-1} x^{n+r-1} = 0,$$

$$r(2r-1)x^{r-1} - x^r + (1 + r)(2 + 2r - 1)a_1 x^r + \sum_{n=2}^{\infty} [(n + r)(2n + 2r - 1)a_n - a_{n-1}]x^{n+r-1} = 0.$$

Upon equating coefficients of powers of x to zero, we have

$$r(2r-1) = 0, \tag{5}$$

$$a_1 = \frac{1}{(1 + r)(2r + 1)}, \tag{6}$$

$$a_n = \frac{1}{(n + r)(2n + 2r - 1)} a_{n-1} \quad \text{for} \quad n = 2, 3, 4, \cdots. \tag{7}$$

The equation (5) is called the **indicial equation** for (1). This indicial equation has roots $r = 0$ and $r = \frac{1}{2}$. We now show that (1) has a solution in the form (4) for each of these roots.

CASE 1. $r = 0$. Then (6) and (7) become

$$a_1 = 1 \quad \text{and} \quad a_n = \frac{1}{n(2n-1)}\,a_{n-1} \quad \text{for} \quad n = 2, 3, 4, \cdots.$$

Hence $a_n = \dfrac{1}{n!\,3\cdot 5\cdots(2n-1)}$ for $n = 2, 3, \cdots$ and (4) becomes

$$y_1(x) = 1 + x + \sum_{n=2}^{\infty} \frac{1}{n!\,3\cdot 5\cdots(2n-1)}\,x^n, \quad \text{for all } x.$$

CASE 2. $r = \frac{1}{2}$. Then (6) and (7) become

$$a_1 = \tfrac{1}{3} \quad \text{and} \quad a_n = \frac{1}{(2n+1)n}\,a_{n-1} \quad \text{for} \quad n = 2, 3, 4, \cdots.$$

Thus $a_n = \dfrac{1}{5\cdot 7\cdots(2n+1)\,n!}\,\dfrac{1}{3} = \dfrac{1}{3\cdot 5\cdot 7\cdots(2n+1)\,n!}$ for $n = 1, 2, 3, \cdots$ and (4) becomes

$$y_2(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{1}{3\cdot 5\cdots(2n+1)\,n!}\,x^n\right] \quad \text{for all } x > 0.$$

A check will show that both $y_1(x)$ and $y_2(x)$ satisfy (1) in their respective ranges and thus

$$y = c_1y_1(x) + c_2y_2(x)$$

is the general solution of (1) throughout the range $x > 0$.

If the indicial equation has equal roots, or the difference of the roots is an integer, then further analysis is necessary and we leave this to a later course.

PROBLEMS

1. Find two solutions $y_1(x)$ and $y_2(x)$ of the differential equation

a. $2xy'' + (1 + 2x)y' + 4y = 0$.

b. $3xy'' + (2 - x)y' + 2y = 0$.

c. $2xy'' + (2x + 1)y' + 2y = 0$.

d. $4xy'' + 2y' + y = 0$.

2. Extend the method of Sec. 158 to find the general solution of:

a. $y'' - (x - \frac{1}{2})y = 0$ in the form $y = \Sigma\, a_n(x - \frac{1}{2})^n$.

b. $y'' - 2(x + 1)y' - 3y = 0$ in the form $y = \Sigma\, a_n(x + 1)^n$.

160. Taylor Series Solutions

Let y be a function all of whose derivatives exist at $x = 0$. Then the Taylor expansion about $x = 0$:

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots \tag{1}$$

is valid (that is, the series converges to $y(x)$) provided the remainder $R_n(x)$ after n terms approaches 0 as $n \to \infty$ (see Secs. 129–131). We shall show how (1) may be used to obtain approximations of solutions of some differential equations with initial conditions.

Example. Find the first six terms of (1) if

$$y'' + xy' - 2y = \sin x; \quad y = 1 \quad \text{and} \quad y' = \tfrac{1}{2} \quad \text{when} \quad x = 0. \tag{2}$$

Solution. The initial conditions may be written as

$$y(0) = 1 \quad \text{and} \quad y'(0) = \tfrac{1}{2} \tag{3}$$

thus furnishing the first two terms of (1). The differential equation itself may be written as

$$y'' = -xy' + 2y + \sin x \quad \text{so} \quad y''(0) = -0(\tfrac{1}{2}) + 2(1) + \sin 0 = 2 \tag{4}$$

and we have the coefficient of x^2 in (1). The derivative of both sides of the first equation of (4) yields

$$\begin{aligned} y''' &= -(xy'' + y') + 2y' + \cos x \\ &= -xy'' + y' + \cos x \quad \text{and} \quad y'''(0) = -0(2) + \tfrac{1}{2} + \cos 0 = \tfrac{3}{2}. \end{aligned}$$

Then $y^{(4)} = -xy''' - y'' + y'' - \sin x = -xy''' - \sin x, \quad y^{(4)}(0) = 0.$ Also

$$y^{(5)} = -xy^{(4)} - y''' - \cos x \quad \text{and} \quad y^{(5)}(0) = -\tfrac{3}{2} - 1 = -\tfrac{5}{2}.$$

Hence, the first six terms of (1), where y is presumably the solution of (2), is

$$y = 1 + \frac{1}{2}x + \frac{2}{2!}x^2 + \frac{3}{2 \cdot 3!}x^3 + \frac{0}{4!}x^4 - \frac{5}{2 \cdot 5!}x^5 + \cdots,$$

$$y = 1 + \frac{1}{2}x + x^2 + \frac{1}{4}x^3 - \frac{1}{48}x^5 + \cdots. \tag{5}$$

By continuing the above process it should be seen how any desired number of coefficients may be computed. As yet, however, we have no assurance that the resulting series will converge, and it may be difficult to determine the nth coefficient in terms of n. It is thus well to know the following theorem whose proof we cannot give here.†

THEOREM. *If $P(x)$, $Q(x)$, and $f(x)$ have Taylor expansions about $x = 0$ which are valid for $|x| < a$, then the solution of*

$$y'' + P(x)y' + Q(x)y = f(x); \quad y(0) = a_0,\ y'(0) = a_1$$

(notice the coefficient of y'' is 1) also has a Taylor expansion about $x = 0$ which is valid for $|x| < a$.

† See page 363 of *Ordinary Differential Equations* by Wilfred Kaplan (Boston: Addison-Wesley Publishing Company, Inc., 1958).

In (2), $P(x) = x = 0 + x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots,$

$Q(x) = -2 + 0 \cdot x + 0 \cdot x^2 + \cdots,$ and

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

are valid for all x so the series of which (5) gives the first six terms is a solution of (2) valid for all x.

The Taylor expansion of the solution of

$$(1 - x)y'' + xy' - 2y = \sin x; \quad y(0) = 1, y'(0) = \tfrac{1}{2}$$

is valid only for $|x| < 1$ since, in order to apply the above theorem, the equation should be written

$$y'' + \frac{x}{1-x}y' - \frac{2}{1-x}y = \frac{\sin x}{1-x}$$

for which $P(x) = \dfrac{x}{1-x} = x + x^2 + x^3 + \cdots$ for $|x| < 1$ with the same interval of convergence for the power series expansions of

$$Q(x) = -\frac{2}{1-x} \quad \text{and} \quad f(x) = \frac{\sin x}{1-x}.$$

PROBLEMS

1. Find the first six terms of the Taylor expansion of the solution of:

a. $y'' + xy' - y = \sin x; \quad y = 1, \quad y' = \frac{1}{2}$ if $x = 0$.

b. $y'' + xy' - y = \sin x; \quad y = 0, \quad y' = 0$ if $x = 0$.

c. $(1 + x)y'' + xy' - y = \sin x; \quad y = 1, \quad y' = \frac{1}{2}$ if $x = 0$.

d. $y'' + e^x y' - y = 1 + x - \sin x; \quad y = 1, \quad y' = 0$ if $x = 0$.

2. If the supplementary conditions are given at $x = x_0$ (instead of at $x = 0$) then the Taylor expansion

$$y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \cdots$$

may be used. Find five terms of the Taylor expansion of y if:

a. $xy'' + x^2y' - 2y = \ln x; \quad y = 0$ and $y' = \frac{1}{2}$ if $x = 1$.

b. $x^2y'' - 2xy' + (x - 2)y = 0; \quad y = \frac{1}{2}$ and $y' = 0$ if $x = 2$.

While working on a Ph.D. thesis in Physics, the candidate obtained a function f of a single variable satisfying

(1) $$f(x_1)f(x_2 - x_3) + f(x_2)f(x_3 - x_1) + f(x_3)f(x_1 - x_2) = 0$$

for all real x_1, x_2, x_3. Further work was delayed until a simple expression for $f(x)$ could be obtained. As far as the physicist was concerned, $f(x)$ could be assumed to have a Taylor expansion converging to it for all x.

A graduate student in Mathematics helped out as follows.

First set $x_1 = x_2 = x_3 = 0$ to obtain $3f^2(0) = 0$ so that

(2) $$f(0) = 0.$$

If $f(x) = 0$ for all x, then (1) is satisfied, but this trivial solution has no physical significance. Thus consider that there is an x_1 such that

(3) $$f(x_1) \neq 0.$$

Take $x_2 = x_1$. Then from (1) and (3)

$$f(x_1 - x_3) + f(x_3 - x_1) = 0.$$

Upon setting $x = x_1 - x_3$, then

(4) $$f(x) = -f(-x)$$

so that f is an odd function.

Next set $x_2 - x_3 = x_1$ in (1):

$$f^2(x_1) + f(x_1 + x_3)f(x_3 - x_1) + f(x_3)f(-x_3) = 0.$$

Set $x_1 = x$, $x_3 = y$, and use (4) to obtain

(5) $$f^2(x) - f^2(y) = f(x + y)f(x - y).$$

As an undergraduate, the graduate student in Mathematics had solved the following problem:

> Find every twice-differentiable real-valued function f with domain all real numbers and satisfying the functional equation (5) for all real numbers x and y.

which was given at the William Lowell Putnam Mathematical Competition of 1963. He took the derivative of both sides of (5) first with respect to x and then with respect to y and proceeded from there. If this is not a sufficient hint, a solution may be found in *The American Mathematical Monthly*, Vol. 71 (June-July 1964), 640.

Appendix

A1. Proofs of Limit Theorems. Some of the theorems of Sections 17, 19, and 20 were not proved and these proofs will now be provided. The theorems are restated for easy reference.

THEOREM 17. *Let f and g be functions whose limits exist at c:*

$$\lim_{x\to c} f(x) = L_1 \quad \text{and} \quad \lim_{x\to c} g(x) = L_2,$$

and for every number $\delta > 0$ the domain of f, the domain of g and $\{x \mid 0 < |x - c| < \delta\}$ have numbers in common. Then

I. $$\lim_{x\to c} [f(x) + g(x)] = L_1 + L_2,$$

II. $$\lim_{x\to c} [f(x)g(x)] = L_1L_2.$$

For $L_2 \neq 0$

III. $$\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{L_2} \quad \text{and} \quad \lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

Also, for $L_1 > 0$ and n a positive integer

IV. $$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L_1}.$$

Recall that I is the only part which has been proved. (See p. 52.) Let ϵ be an arbitrary number. We now use this number in the proofs of the remaining parts.

PROOF of II. We shall first prove the auxiliary results:

a. $$\lim_{x\to c} L_2[f(x) - L_1] = 0, \quad \lim_{x\to c} L_1[g(x) - L_2] = 0, \quad \text{and}$$

b. $$\lim_{x\to c} [f(x) - L_1][g(x) - L_2] = 0.$$

Let δ be such that if $0 < |x - c| < \delta$ then $|f(x) - L_1| < \dfrac{\epsilon}{|L_2| + 1}$. Hence, if $0 < |x - c| < \delta$, then

$$|L_2[f(x) - L_1]| = |L_2|\,|f(x) - L_1| \leq |L_2| \frac{\epsilon}{|L_2| + 1} \leq \epsilon$$

which establishes the first part of a. The second part of a. follows in a similar manner. Next, let $\delta_1 > 0$ and $\delta_2 > 0$ be such that

$$\text{if} \quad 0 < |x - c| < \delta_1, \quad \text{then} \quad |f(x) - L_1| < \sqrt{\epsilon} \quad \text{and}$$

$$\text{if} \quad 0 < |x - c| < \delta_2, \quad \text{then} \quad |g(x) - L_2| < \sqrt{\epsilon}.$$

δ be the smaller of δ_1 and δ_2. Thus

$$\text{if}\quad 0 < |x - c| < \delta, \quad \text{then}\quad |[f(x) - L_1][g(x) - L_2]| = |f(x) - L_1|\,|g(x) - L_2| < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon$$

which says that b. holds.

Now notice that

$$f(x)g(x) = [f(x) - L_1][g(x) - L_2] + L_2[f(x) - L_1] + L_1[g(x) - L_2] + L_1L_2.$$

As $x \to c$ the first three terms on the right approach 0 while the fourth term is the constant L_1L_2. By part I the sum of the limits as $x \to c$ of the terms on the right is the limit as $x \to c$ of the sum and thus the limit as $x \to c$ of the left term exists and is equal to L_1L_2, thus establishing II.

Proof of III. First choose $\delta_1 > 0$ such that

$$\text{if}\quad 0 < |x - c| < \delta_1, \quad \text{then}\quad |g(x) - L_2| < \tfrac{1}{2}|L_2|$$

which is possible since $|L_2| > 0$ because $L_2 \neq 0$ in this part. Hence, if $0 < |x - c| < \delta_1$, then

$$|L_2| = |L_2 - g(x) + g(x)| \le |L_2 - g(x)| + |g(x)| < \tfrac{1}{2}|L_2| + |g(x)| \quad \text{and}$$

$$|L_2| - \tfrac{1}{2}|L_2| < |g(x)|; \quad \text{that is,}$$

$$\text{if}\quad 0 < |x - c| < \delta_1, \quad \text{then}\quad |g(x)| > \tfrac{1}{2}|L_2| \quad\text{and}\quad \frac{1}{|g(x)|} < \frac{2}{|L_2|}.$$

Next choose δ_2 such that if $0 < |x - c| < \delta_2$, then $|g(x) - L_2| < \frac{1}{2}|L_2|^2\epsilon$. Let δ be the smaller of δ_1 and δ_2. Hence, if $0 < |x - c| < \delta$, then

$$\left|\frac{1}{g(x)} - \frac{1}{L_2}\right| = \frac{|L_2 - g(x)|}{|g(x)L_2|} = \frac{|g(x) - L_2|}{|L_2|}\cdot\frac{1}{|g(x)|} \le \frac{|g(x) - L_2|}{|L_2|}\cdot\frac{2}{|L_2|}$$

$$= |g(x) - L_2|\frac{2}{|L_2|^2} < \tfrac{1}{2}|L_2|^2\epsilon\cdot\frac{2}{|L_2|^2} = \epsilon,$$

and this says that

(1) $$\lim_{x\to c}\frac{1}{g(x)} = \frac{1}{L_2} = \frac{1}{\lim\limits_{x\to c} g(x)}$$

which is the first part of III. By omitting $x \to c$ to save space

$$\frac{L_1}{L_2} = \frac{\lim f(x)}{\lim g(x)} = \lim f(x)\cdot\frac{1}{\lim g(x)} = \lim f(x)\cdot\lim\frac{1}{g(x)} \text{ by (1)}$$

$$= \lim\left[f(x)\frac{1}{g(x)}\right] \text{ by II}$$

$$= \lim\frac{f(x)}{g(x)}$$

which is the second part of III and thus the proof of III is complete.

Proof of IV. An auxiliary result we shall need is:

(2) $$\text{If } t > \frac{s}{2} > 0, \quad \text{then}\quad |t - s| < \frac{s}{s^n}|t^n - s^n|$$

where n is a positive integer.

The following sequence of steps shows that (2) holds:

$$t^n - s^n = (t - s)(t^{n-1} + t^{n-2}s + t^{n-3}s^2 + \cdots + ts^{n-2} + s^{n-1}),$$

$$|t^n - s^n| > |t - s|\left(\frac{s^{n-1}}{2^{n-1}} + \frac{s^{n-2}}{2^{n-2}}s + \frac{s^{n-3}}{2^{n-3}}s^2 + \cdots + \frac{s}{2}s^{n-2} + s^{n-1}\right)$$

$$= |t - s|\, s^{n-1}\left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \cdots + \frac{1}{2} + 1\right)$$

$$= |t - s|\, s^{n-1}\left(2 - \frac{1}{2^n}\right) \quad \text{since } 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^n}$$

$$> |t - s|\frac{s^n}{s} \quad \text{since } 2 - \frac{1}{2^n} > 1,$$

which upon multiplying by s/s^n yields the second inequality of (2).

In this part IV it is given that $L_1 > 0$. Thus $L_1/2^n$ is less than L_1 and since $\lim_{x\to c} f(x) = L_1$ then for all x sufficiently close to c it follows that $f(x)$ will be greater than $L_1/2^n$. Let $\delta_1 > 0$ be such that if $0 < |x - c| < \delta_1$, then $f(x) > L_1/2^n$. Hence

(3) $$\text{if } 0 < |x - c| < \delta, \text{ then } \sqrt[n]{f(x)} > \frac{\sqrt[n]{L_1}}{2}.$$

Now from (2), with $t = \sqrt[n]{f(x)}$ and $s = \sqrt[n]{L_1}$, it follows that

(4) $$\text{if } 0 < |x - c| < \delta_1, \text{ then } \sqrt[n]{f(x)} > \frac{\sqrt[n]{L_1}}{2} \text{ and}$$

$$|\sqrt[n]{f(x)} - \sqrt[n]{L_1}| < \frac{\sqrt[n]{L_1}}{L_1}|f(x) - L_1|.$$

Next choose $\delta_2 > 0$ such that

(5) $$\text{if } 0 < |x - c| < \delta_2, \text{ then } |f(x) - L_1| < \frac{L_1}{\sqrt[n]{L_1}}\epsilon.$$

Finally, with δ the smaller of δ_1 and δ_2 it follows that

$$\text{if } 0 < |x - c| < \delta, \text{ then } |\sqrt[n]{f(x)} - \sqrt[n]{L_1}| < \frac{\sqrt[n]{L_1}}{L_1}|f(x) - L_1| \quad \text{by (4)}$$

$$< \frac{\sqrt[n]{L_1}}{L_1}\cdot\frac{L_1}{\sqrt[n]{L_1}}\epsilon \quad \text{by (5)}$$

which by definition means that $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L_1}$.

All parts of Theorem 17 have now been proved.

THEOREM 19. *With c, a, and L numbers, let f be a function such that*

$$\lim_{t\to a} f(t) = L,$$

let u be o function whose range, except possibly u(c), is in the domain of f and is such that

(6) $$u(x) \neq a \quad \text{if} \quad x \neq c, \quad \text{but}$$

$$\lim_{x\to c} u(x) = a.$$

Then the composition function of f upon u also has limit L at c:

(7) $$\lim_{x\to c} f[u(x)] = L.$$

PROOF. Given an arbitrary number $\epsilon > 0$, let e be a positive number such that (since f has limit L at a)

$$\text{whenever} \quad 0 < |t - a| < e, \quad \text{then} \quad |f(t) - L| < \epsilon. \tag{8}$$

Corresponding to this number $e > 0$ let $\delta > 0$ be such that (since u has limit a at c)

$$\text{whenever} \quad 0 < |x - c| < \delta, \quad \text{then} \quad |u(x) - a| < e,$$

or even more $0 < |u(x) - a| < e$ from (6). Thus, whenever x is such that $0 < |x - c| < \delta$, then $0 < |u(x) - a| < e$ and then in turn, by using (8), $|f[u(x)] - L| < \epsilon$. Hence

$$\text{whenever} \quad 0 < |x - c| < \delta, \quad \text{then} \quad |f[u(x)] - L| < \epsilon$$

and this, from the definition of a limit means that (7) holds.

In case the outside function f is continuous, then slightly different conditions for interchanging the limit and f are given in:

THEOREM 20.1. *For c and a numbers and for f and u functions such that*

$$\lim_{x \to c} u(x) = a,$$

and such that f is continuous at a, then

$$\begin{aligned}\lim_{x \to c} f[u(x)] &= f(a)\\ &= f[\lim_{x \to c} u(x)].\end{aligned}$$

PROOF. In the proof of Theorem 19 replace L throughout by $f(a)$. Now repeat the proof of Theorem 19 with the minor changes of replacing $0 < |t - a| < e$ by merely $|t - a| < e$ and $0 < |u(x) - a| < e$ by $|u(x) - a| < e$. The result is a proof of Theorem 20.1.

A2. Continuity Theorems. Throughout this section, f is a function which is continuous on a closed interval I$[a,b]$; that is,

$$\text{If} \quad a < c < b, \quad \text{then} \quad \lim_{x \to c} f(x) = f(c), \quad \text{while} \tag{1}$$

$$\lim_{x \to a+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b-} f(x) = f(b). \tag{2}$$

Before proving Theorem 31.2 we prove a lemma.

LEMMA. *The function f is bounded on I$[a,b]$.*

PROOF. Corresponding to $\epsilon = 1$ let $d_1 > 0$ and $d_2 > 0$ be such that from (2)

$$\text{if } a \le x < a + d_1 \quad \text{then} \quad |f(x) - f(a)| < 1 \quad \text{and}$$

$$\text{if } b - d_2 < x \le b, \quad \text{then} \quad |f(x) - f(b)| < 1.$$

Hence, f is bounded on the closed intervals I$[a,a + d_1/2]$ and I$[b - d_2/2,b]$.

Let A be the set defined by

$$A = \{x \mid a \le x \le b \text{ and } f \text{ is bounded on I}[a,x]\}.$$

The set A is bounded above by b and A is not empty since $a + d_1/2$ is in A because f is bounded on $I[a,a + d_1/2]$. Thus, by the axiom on p. 10 the set A has a least upper bound. Denote this least upper bound by c. Thus

$$a + \frac{d_1}{2} \leq c \leq b \quad \text{so that} \quad a < c \leq b.$$

We now show that $c = b$. For:

(3) Suppose $c < b$.

Then $a < c < b$. Corresponding to $\epsilon = 1$ let $\delta > 0$ be such that [by (1)]

$$\text{if } |x - c| < \delta \quad \text{then} \quad |f(x) - f(c)| < 1$$

with δ less than the smaller of $c - a$ and $b - c$. Hence $I\left[c - \frac{\delta}{2}, c + \frac{\delta}{2}\right] \subset I[a,b]$ and

$$f \text{ is bounded on } I\left[c - \frac{\delta}{2}, c + \frac{\delta}{2}\right].$$

But $c - \delta/2 < c$ so $c - \delta/2$ is in A (since c is the least upper bound of A); that is,

$$f \text{ is bounded on } I\left[a, c - \frac{\delta}{2}\right].$$

Thus, f is bounded on the union of $I\left[a, c - \frac{\delta}{2}\right]$ and $I\left[c - \frac{\delta}{2}, c + \frac{\delta}{2}\right]$ so

$$f \text{ is bounded on } I\left[a, c + \frac{\delta}{2}\right].$$

Consequently, $c + \frac{\delta}{2}$ is in the set A which contradicts the fact that c is the least upper bound of A. Hence, the supposition (3) is wrong and therefore $c = b$.

Thus, b is the least upper bound of the set A. Hence, $b - d_2/2$ is in A so

$$f \text{ is bounded on } I\left[a, b - \frac{d_2}{2}\right] \text{ and on } I\left[b - \frac{d_2}{2}, b\right]$$

and consequently f is bounded on $I[a,b]$ as we wished to prove.

THEOREM 31.2. *The function f has a maximum and a minimum on* $I[a,b]$.

PROOF. Let B be the set defined by

$$B = \{y \mid y = f(x) \text{ for some } x \text{ on } I[a,b]\}.$$

The lemma then states "The set B is bounded." Hence, by the axiom of p. 10 B has a least upper bound. Let

(4) M be the least upper bound of B.

We shall now show that $f(x) = M$ has at least one solution on $I[a,b]$. To do so:

(5) Assume $f(x) \neq M$ for $a \leq x \leq b$.

Since M is an upper bound of B, then $f(x) < M$ for $a \leq x \leq b$ so that under assumption (5)

(6) $M - f(x) > 0$ for $a \leq x \leq b$.

Let g be the function defined by

$$g(x) = \frac{1}{M - f(x)} \quad \text{for} \quad a \leq x \leq b$$

which is permissible since the denominator is $\neq 0$ by (6). Now g is the reciprocal of a continuous function so is itself a continuous and hence by the lemma is bounded on I$[a,b]$. Let $N > 0$ be such that

$$N \geq g(x) = \frac{1}{M - f(x)} \quad \text{for} \quad a \leq x \leq b.$$

Consequently

$$M - f(x) \geq \frac{1}{N}, M - \frac{1}{N} \geq f(x) \quad \text{for} \quad a \leq x \leq b.$$

This, since $N > 0$, says that $M - \dfrac{1}{N}$ is a smaller upper bound of f on I$[a,b]$ than the least upper bound M of f on I$[a,b]$ and we have a contradiction. Thus, the assumption (5) is wrong so $f(x) = M$ has a solution on I$[a,b]$. Let x_1 be such that $a \leq x_1 \leq b$ and $f(x_1) = M$. Since M is an upper bound of f on I$[a,b]$ it then follows that:

$$\text{If } a \leq x \leq b \quad \text{then} \quad f(x) \leq M = f(x_1).$$

Hence $f(x_1)$ is the maximum value of f on I$[a,b]$.

A similar proof shows that f has a minimum on I$[a,b]$.

Problem. As an alternative proof that "f has a minimum on I$[a,b]$," show that the function g defined by $g(x) = -f(x)$, for $a \leq x \leq b$, has a max. on I$[a,b]$, and that $-$(max. of g on I$[a,b]$) $=$ (min. of f on I$[a,b]$).

A3. The Number *e*. In Sec. 51 it was proved that, over integer values, $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n$ exists and then this limit was denoted by e:

(4) $$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Hence, also $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n+1}\right)^{n+1} = e.$

We shall now prove the existence of the limit and the equality

(5) $$\lim_{h\to 0} (1 + h)^{1/h} = e.$$

To accomplish this, first let x be any number such that $x > 1$ and denote by n_x the greatest integer $\leq x$ so that n_x is an integer and

$$1 \leq n_x \leq x < n_x + 1.$$

Therefore
$$\frac{1}{n_x} \geq \frac{1}{x} > \frac{1}{n_x + 1},$$

$$1 + \frac{1}{n_x} \geq 1 + \frac{1}{x} > 1 + \frac{1}{n_x + 1} > 1$$

$$\left(1 + \frac{1}{n_x}\right)^{n_x+1} \geq \left(1 + \frac{1}{x}\right)^{n_x+1} > \left(1 + \frac{1}{x}\right)^x > \left(1 + \frac{1}{n_x + 1}\right)^x \geq \left(1 + \frac{1}{n_x + 1}\right)^{n_x},$$

$$\left(1 + \frac{1}{n_x}\right)\left(1 + \frac{1}{n_x}\right)^{n_x} > \left(1 + \frac{1}{x}\right)^x > \left(1 + \frac{1}{n_x + 1}\right)^{n_x+1}\left(1 + \frac{1}{n_x + 1}\right)^{-1}.$$

Also, $n_x \to \infty$ if and only if $x \to \infty$. Consequently, as $x \to \infty$ it follows that $\left(1 + \frac{1}{n_x}\right) \to 1$, $\left(1 + \frac{1}{n_x}\right)^{n_x} \to e$, $\left(1 + \frac{1}{n_x + 1}\right)^{n_x+1} \to e$ and $\left(1 + \frac{1}{n_x + 1}\right)^{-1} \to 1$ and thus that

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x \text{ exists and } = e, \text{ and then also} \tag{6}$$

$$\lim_{x\to\infty} \left(1 + \frac{1}{x-1}\right)^{x-1} = e. \tag{7}$$

Next, let x be such that $x < -1$, set $\bar{x} = -x$, notice that $\bar{x} > 1$, that

$$\left(1 + \frac{1}{x}\right)^x = \left(1 - \frac{1}{\bar{x}}\right)^{-\bar{x}} = \left(\frac{\bar{x} - 1}{\bar{x}}\right)^{-\bar{x}} = \left(\frac{\bar{x}}{\bar{x} - 1}\right)^{\bar{x}} = \left(1 + \frac{1}{\bar{x} - 1}\right)^{\bar{x}}$$
$$= \left(1 + \frac{1}{\bar{x} - 1}\right)^{\bar{x}-1}\left(1 + \frac{1}{\bar{x} - 1}\right)$$

and that $\bar{x} \to \infty$ if and only if $x \to -\infty$. Hence, as $x \to -\infty$ the expression after the last equality sign in the preceding display approaches $e \cdot 1$ [because of (7)] and therefore

$$\lim_{x\to-\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

This fact together with (6) may be combined as

$$\lim_{|x|\to\infty} \left(1 + \frac{1}{x}\right)^x = e.$$

Finally, for $h \neq 0$ we set $x = 1/h$ and obtain

$$\lim_{h\to 0} (1 + h)^{1/h} = e$$

which shows the existence of the limit and the equality (5), as we wished to prove.

A4. Darboux and Riemann Integrals. Let a and b be numbers such that $a < b$ and let P be a finite set of numbers such that a and b are in P but otherwise each number in P is greater than a and less than b. Such a set P is thought of geometrically as a **partition** of the closed interval $I[a,b]$. If P consists of the $n + 1$ numbers $x_0, x_1, x_2, \cdots, x_n$, then these numbers will be considered as arranged in increasing order so that

$$x_{k-1} < x_k \quad \text{for} \quad k = 1, 2, 3, \cdots, n$$
$$\text{with} \quad x_0 = a \quad \text{and} \quad x_n = b,$$

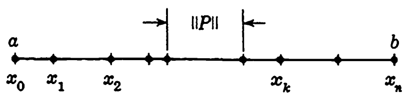


Figure A4.1

and thus $I[a,b]$ may be thought of as being divided by the points x_k into n subintervals

$$I[x_0,x_1],\ I[x_1,x_2],\ I[x_2,x_3],\ \cdots,\ I[x_{n-1},x_n]$$

which may or may not have the same length. The length of the longest subinterval is denoted by $||P||$ and is called the **norm** of P. Thus $0 < x_k - x_{k-1} \leq ||P||$ for $k = 1, 2, 3, \cdots, n$ and the equality holds at least once.

For example, the set P consisting of the $n + 1$ numbers

$$x_k = a + k\frac{b-a}{n} \quad \text{for} \quad k = 0, 1, 2, \cdots, n$$

is a partition of $I[a,b]$, but is a special type of partition since it divides $I[a,b]$ into subintervals all of the same length

$$\|P\| = \frac{b-a}{n}.$$

A partition P consisting of $x_0, x_1, x_2, \cdots, x_n$ is also denoted by using square brackets as $[x_0,x_1,x_2, \cdots, x_n]$ so that

$$P = [x_0,x_1,x_2, \cdots, x_n],$$

and each of the numbers $x_0, x_1, x_2, \cdots, x_n$ will be called an element (or point) of the set. Also, we shall use $\Delta_k x$ to denote the difference $x_k - x_{k-1}$; that is,

$$\Delta_k x = x_k - x_{k-1} \quad \text{for} \quad k = 1, 2, 3, \cdots, n.$$

If P_1 and P_2 are two partitions of $I[a,b]$ such that $P_1 \subset P_2$, then P_2 consists of all points of P_1 together with possibly some more points. The partition P_2 is then said to be a **refinement** of P_1.

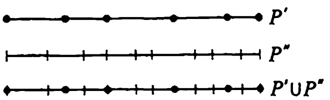


Figure A4.2

If P' and P'' are any two partitions of $I[a,b]$, then the union $P = P' \cup P''$ is also a partition of $I[a,b]$ since P certainly contains a and b together with a finite number of points between a and b. Notice that $P = P' \cup P''$ is a refinement of P' and also is a refinement of P''.

Let f be a function whose domain includes $I[a,b]$ and is such that f is bounded on $I[a,b]$. This means that the set

$$\{y \mid y = f(x) \text{ for some } x \text{ on } I[a,b]\}$$

is bounded. Let $\bar{B}$ be the least upper bound and $\underline{B}$ the greatest lower bound of this set ($\bar{B}$ and $\underline{B}$ are known to exist by the axiom on p. 10). Then for x any number such that $a \le x \le b$ it follows that $\underline{B} \le f(x) \le \bar{B}$, but also given any number $\epsilon > 0$ there are numbers $\underline{x}$ and $\bar{x}$ satisfying

$$a \le \underline{x} \le b,\ a \le \bar{x} \le b \quad \text{for which} \quad f(\underline{x}) < \underline{B} + \epsilon \quad \text{and} \quad f(\bar{x}) > \bar{B} - \epsilon.$$

With $P = [x_0,x_1,x_2, \cdots, x_n]$ a partition of $I[a,b]$, let $\underline{B}_k$ and $\bar{B}_k$ be the g.l.b. and l.u.b., respectively, of f on $I[x_{k-1},x_k]$ for $k = 1, 2, \cdots, n$. Hence $\underline{B} \le \underline{B}_k \le \bar{B}_k \le \bar{B}$,

$$\underline{B}\Delta_k x \le \underline{B}_k\Delta_k x \le \bar{B}_k\Delta_k x \le \bar{B}\Delta_k x, \quad \text{and}$$

$$\underline{B}\sum_{k=1}^{n} \Delta_k x \le \sum_{k=1}^{n} \underline{B}_k\Delta_k x \le \sum_{k=1}^{n} \bar{B}_k\Delta_k x \le \bar{B}\sum_{k=1}^{n} \Delta_k x.$$

The extreme left and right members are equal to $\underline{B} \cdot (b - a)$ and $\bar{B} \cdot (b - a)$, respectively. The middle terms are denoted by $\underline{S}(P)$ and $\bar{S}(P)$:

$$\underline{S}(P) = \sum_{k=1}^{n} \underline{B}_k\Delta_k x, \quad \bar{S}(P) = \sum_{k=1}^{n} \bar{B}_k\Delta_k x$$

and are called the **lower** and **upper Darboux sums** for f on $I[a,b]$ over P. Consequently

(1) $$\underline{B}(b - a) \le \underline{S}(P) \le \bar{S}(P) \le \bar{B}(b - a).$$

Now let $x_1^*, x_2^*, x_3^*, \cdots, x_n^*$ be such that $x_{k-1} \le x_k^* \le x_k$ for $k = 1, 2, \cdots, n$. The sum

(2) $$\sum_{k=1}^{n} f(x_k^*)\Delta_k x$$

is called a **Riemann sum** for f on $I[a,b]$ over P. Notice that with the same function f, the same interval $I[a,b]$, and the same partition P of $I[a,b]$, there are many Riemann sums depending upon choices of $x_1^*, x_2^*, \cdots, x_n^*$. Since, however,

$$x_{k-1} \le x_k^* \le x_k, \quad \text{then} \quad \underline{B}_k \le f(x_k^*) \le \bar{B}_k$$

so that $\underline{B}_k\Delta_k x \le f(x_k^*)\,\Delta_k x \le \bar{B}_k\Delta_k x$ for $k = 1, 2, \cdots, n$ and therefore

(3) $$\underline{S}(P) \le \sum_{k=1}^{n} f(x_k^*)\,\Delta_k x \le \bar{S}(P).$$

Hence, for a given function f and a given partition P of $I[a,b]$ all Riemann sums are bounded below by the lower Darboux sum $\underline{S}(P)$ and above by the upper Darboux sum $\bar{S}(P)$.

We now prove four lemmas about upper and lower Darboux sums. For all these lemmas we restrict the function

(4) *f to be bounded on $I[a,b]$.*

LEMMA 1. *If P is a partition and P_1 is a refinement of P consisting of only one more point than P, then*

(5) $$\underline{S}(P_1) \ge \underline{S}(P), \quad \bar{S}(P_1) \le \bar{S}(P),$$

$$0 \le \underline{S}(P_1) - \underline{S}(P) \le (\bar{B} - \underline{B})\,||P||, \quad \text{and} \quad 0 \le \bar{S}(P) - \bar{S}(P_1) \le (\bar{B} - \underline{B})\,||P||.$$

PROOF. Let $P = [x_0,x_1,x_2, \cdots, x_n]$. Let the additional point of P_1 be denoted by z. Then z falls between say x_{i-1} and x_i so

$$x_{i-1} < z < x_i.$$

Now $\bar{B}_i$ is the l.u.b. of f on $I[x_{i-1},x_i]$ so that if $x_{i-1} \le x \le x_i$, then $f(x) \le \bar{B}_i$. Thus

$$\text{if} \quad x_{i-1} \le x \le z \quad \text{then} \quad f(x) \le \bar{B}_i$$

so $\bar{B}_i$ is an upper bound of f on $I[x_{i-1},z]$. Hence, upon letting $\bar{B}_i'$ be the l.u.b. of f on $I[x_{i-1},z]$ then

$$\bar{B}_i' \le \bar{B}_1.$$

In a similar way $\bar{B}_i'' \le \bar{B}_i$ where $\bar{B}_i''$ is the l.u.b. of f on $I[z,x_i]$. Thus

$$\begin{aligned}\bar{B}_i'(z - x_{i-1}) + \bar{B}_i''(x_i - z) &\le \bar{B}_i(z - x_{i-1}) + \bar{B}_i(x_i - z)\\ &= \bar{B}_i(z - x_{i-1} + x_i - z) = \bar{B}_i(x_i - x_{i-1}).\end{aligned}$$

Since all subintervals other than $I[x_{i-1},x_i]$ contribute the same to $\bar{S}(P_1)$ as to $\bar{S}(P)$ it follows that $\bar{S}(P_1) \le \bar{S}(P)$. Also

$$\begin{aligned}\bar{S}(P) - \bar{S}(P_1) &= \bar{B}_i(x_i - x_{i-1}) - [\bar{B}_i''(x_i - z) + \bar{B}_i'(z - x_{i-1})]\\ &\le \bar{B}(x_i - x_{i-1}) - [\underline{B}(x_i - z) + \underline{B}(z - x_{i-1})]\\ &= \bar{B}(x_i - x_{i-1}) - \underline{B}(x_i - x_{i-1})\\ &= (\bar{B} - \underline{B})\,(x_i - x_{i-1}) \le (\bar{B} - \underline{B})\,||P||.\end{aligned}$$

A similar proof holds for lower sums.

Lemma 2. *If P and P_m are partitions with P_m a refinement of P (so that $P \subset P_m$), then*

$$\underline{S}(P_m) \geq \underline{S}(P) \quad \text{and} \quad \bar{S}(P_m) \leq \bar{S}(P) \tag{6}$$

and if P_m has m more points than P, then

$$0 \leq \underline{S}(P_m) - \underline{S}(P) \leq m(\bar{B} - \underline{B})\,||P||, \quad \text{and} \quad 0 \leq \bar{S}(P) - \bar{S}(P_m) \leq m(\bar{B} - \underline{B})\,||P||. \tag{7}$$

Proof. Consider the points of P_m which are not points of P and denote these points by $z_1, z_2, \cdots, z_m$. Let P_1 be P augmented by z_1, let P_2 be P_1 augmented by z_2, etc., until $z_1, z_2, \cdots, z_m$ are used up. Then $P_1, P_2, \cdots, P_m$ are partitions, P_1 is a refinement of P, P_2 is a refinement of P_1, etc. Thus, by Lemma 1

$$\bar{S}(P) \geq \bar{S}(P_1) \geq \bar{S}(P_2) \geq \cdots \geq \bar{S}(P_m)$$

which is the second inequality of (6); the first also follows in a similar way. The second inequality of (7) follows by addition from the m inequalities (which themselves follow from Lemma 1).

$$\begin{gathered} 0 \leq \bar{S}(P) - \bar{S}(P_1) \leq (\bar{B} - \underline{B})\,||P|| \\ 0 \leq \bar{S}(P_1) - \bar{S}(P_2) \leq (\bar{B} - \underline{B})\,||P|| \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \leq \bar{S}(P_{m-1}) - \bar{S}(P_m) \leq (\bar{B} - \underline{B})\,||P||. \end{gathered}$$

By replacing upper sums by lower sums, the first of (7) follows.

Lemma 3. *If P' and P'' are any partitions, then*

$$\underline{S}(P') \leq \bar{S}(P'') \quad \text{and} \quad \underline{S}(P'') \leq \bar{S}(P'). \tag{8}$$

Proof. Let $P = P' \cup P''$. Then

$$\begin{aligned} &P' \subset P \quad \text{so} \quad \underline{S}(P) \geq \underline{S}(P') \quad \text{and} \quad \bar{S}(P) \leq \bar{S}(P') \quad \text{and} \\ &P'' \subset P \quad \text{so} \quad \underline{S}(P) \geq \underline{S}(P'') \quad \text{and} \quad \bar{S}(P) \leq \bar{S}(P''). \end{aligned} \tag{9}$$

But for the partition P itself, $\underline{S}(P) \leq \bar{S}(P)$ which, together with some of the inequalities of (9), allows us to state

$$\underline{S}(P') \leq \underline{S}(P) \leq \bar{S}(P) \leq \bar{S}(P'') \quad \text{so} \quad \underline{S}(P') \leq \bar{S}(P'')$$

which is the first inequality of (8) and the second follows by interchanging the prime and the double-prime.

Now let P be any partition. From Lemma 3 we may thus state:

The upper sum $\bar{S}(P)$ is an upper bound of all lower Darboux sums. Hence, all lower Darboux sums are bounded above by $\bar{S}(P)$ so the least upper bound of all lower Darboux sums exists and is $\leq \bar{S}(P)$. The l.u.b. of all lower Darboux sums depends only upon the function f and the numbers a and b; it is denoted by $\underline{\int_a^b} f$ and is called the **lower Darboux integral** of f from a to b. Thus

$$\underline{\int_a^b} f \leq \bar{S}(P). \tag{10}$$

The P in (10) was any partition whatever. Thus, the result (10) may be stated:

The lower Darboux integral is a lower bound of all upper Darboux sums.

Hence, the greatest lower bound of all upper Darboux sums is not less than $\underline{\int}_a^b f$. This g.l.b. of upper Darboux sums is denoted by $\overline{\int}_a^b f$, is called the **upper Darboux integral** of f from a to b, and therefore

$$\underline{\int}_a^b f \leq \overline{\int}_a^b f. \tag{11}$$

LEMMA 4. *Let ϵ be an arbitrary number. Corresponding to this number ϵ there is a partition P such that both*

$$\underline{\int}_a^b f - \epsilon < \underline{S}(P) \leq \underline{\int}_b^a f \quad \text{and} \tag{12}$$

$$\overline{\int}_a^b f \leq \bar{S}(P) < \overline{\int}_a^b f + \epsilon. \tag{13}$$

PROOF. We first consider the left inequality of (12). Since $\underline{\int}_a^b f$ is the l.u.b. of lower Darboux sums, then $\underline{\int}_a^b f - \epsilon$ is not an upper bound of lower Darboux sums. Hence, select a partition P' such that

$$\underline{\int}_a^b f - \epsilon < \underline{S}(P').$$

In a similar way select a partition P'' such that

$$\bar{S}(P'') < \overline{\int}_a^b f + \epsilon.$$

Now let $P = P' \cup P''$ so P is a partition which is a refinement of both P' and P''. Thus, from Lemma 2,

$$P' \subset P \quad \text{so} \quad \underline{S}(P) \geq \underline{S}(P') \quad \text{and}$$

$$P'' \subset P \quad \text{so} \quad \bar{S}(P) \leq \bar{S}(P'') \quad \text{and hence}$$

$$\underline{\int}_a^b f - \epsilon < \underline{S}(P') \leq \underline{S}(P) \quad \text{and} \quad \bar{S}(P) \leq \bar{S}(P'') < \overline{\int}_a^b f + \epsilon. \tag{14}$$

But, since P is a partition, $\underline{S}(P) \leq \underline{\int}_a^b f$ and $\bar{S}(P) \geq \overline{\int}_a^b f$ and these inequalities combined with (14) give both (12) and (13).

THEOREM A4.1. (Darboux's Theorem). *Let f be a bounded function on $I[a,b]$ and let ϵ be an arbitrary positive number. Corresponding to this number ϵ there is a number $\delta > 0$ such that if P is any partition with $||P|| < \delta$, then both*

$$\underline{\int}_a^b f - \epsilon < \underline{S}(P) \leq \underline{\int}_a^b f \quad \text{and} \tag{15}$$

$$\overline{\int}_a^b f \leq \bar{S}(P) < \overline{\int}_a^b f + \epsilon. \tag{16}$$

PROOF. Corresponding to $\epsilon/2$ let (by using Lemma 4) P' be a partition such that

$$\underline{\int}_a^b f - \frac{\epsilon}{2} < \underline{S}(P') \quad \text{and} \tag{17}$$

$$\bar{S}(P') < \overline{\int}_a^b f + \frac{\epsilon}{2}. \tag{18}$$

P' then has a finite number of points. Let m be the number of points in P'. Set

$$\delta = \frac{\epsilon}{2m(\bar{B} - \underline{B}) + 1}$$

(where the 1 is added in the denominator just in case $\bar{B} = \underline{B}$). Thus δ is a positive number.

Let P be a partition with $||P|| < \delta$. We now claim:

(19) *The inequalities* (15) *and* (16) *hold for this partition* P.

Toward establishing this claim, let

$$P'' = P \cup P'.$$

Then P'' is a refinement of P and contains at most m more points than P does, so by Lemma 2

$$0 \leq \underline{S}(P'') - \underline{S}(P) \leq m(\bar{B} - \underline{B})||P||$$
$$\leq m(\bar{B} - \underline{B})\frac{\epsilon}{2m(\bar{B} - \underline{B}) + 1} < \frac{\epsilon}{2}$$

so that

$$\underline{S}(P'') < \underline{S}(P) + \epsilon/2.$$

But P'' is also a refinement of P' so from Lemma 2

$$\underline{S}(P') \leq \underline{S}(P'').$$

These last two inequalities together with (17) show that

$$\underline{\int_a^b} f - \frac{\epsilon}{2} < \underline{S}(P') \leq \underline{S}(P'') < \underline{S}(P) + \epsilon/2,$$
$$\underline{\int_a^b} f - \epsilon < \underline{S}(P)$$

which is the left inequality of (15). The right inequality of (15) follows since $\underline{\int_a^b} f$ is an upper bound of all lower sums.

The inequalities (16) are left for the reader to prove.

Thus, our claim (19) is established and hence the theorem is proved.

COROLLARY. *Let ϵ be an arbitrary positive number. Corresponding to this number ϵ there is a number $\delta > 0$ such that if P is any partition with $||P|| < \delta$, then any Riemann sum over P lies between*

$$\underline{\int_a^b} f - \epsilon \quad \textit{and} \quad \overline{\int_a^b} f + \epsilon.$$

PROOF. Let δ be the number shown to exist in Darboux's theorem and let $P = [x_0, x_1, \cdots, x_n]$ be a partition with $||P|| < \delta$ so that

(20) $$\underline{\int_a^b} f - \epsilon < \underline{S}(P) \quad \text{and} \quad \bar{S}(P) < \overline{\int_a^b} f + \epsilon.$$

Choose any numbers $x_1^*, x_2^*, \cdots, x_n^*$ such that $x_{k-1} \leq x_k^* \leq x_k$ for $k = 1, 2, \cdots, n$. Thus by (3)

$$\underline{S}(P) \leq \sum_{k=1}^{n} f(x_k^*)\, \Delta_k x \leq \bar{S}(P)$$

which together with (20) says that this Riemann sum lies between $\underline{\int_a^b} f - \epsilon$ and $\overline{\int_a^b} f + \epsilon$. Since P was any partition with $||P|| < \delta$ and the star-points were any choice, then any Riemann sum over any partition with norm less than δ lies between $\underline{\int_a^b} f - \epsilon$ and $\overline{\int_a^b} f + \epsilon$ as we wished to prove.

The following theorem shows an additive property of upper and lower integrals.

THEOREM A4.2. *Let f be a bounded function on $I[a,b]$ and let c be a number such that $a < c < b$. Then*

$$\underline{\int_a^c} f + \underline{\int_c^b} f = \underline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^c} f + \overline{\int_c^b} f = \overline{\int_a^b} f. \tag{21}$$

PROOF. Let $\epsilon > 0$ be an arbitrary number. Choose partitions P_1 of $I[a,c]$, P_2 of $I[c,b]$, and P_3 of $I[a,b]$ such that

i) $$\underline{\int_a^c} f - \frac{\epsilon}{2} < \underline{S}(P_1) \le \underline{\int_a^c} f,$$

ii) $$\underline{\int_c^b} f - \frac{\epsilon}{2} < \underline{S}(P_2) \le \underline{\int_c^b} f, \quad \text{and}$$

iii) $$\underline{\int_a^b} f - \epsilon < \underline{S}(P_3) \le \underline{\int_a^b} f.$$

Then let $P = P_1 \cup P_2 \cup P_3$, $P' = P \cap I[a,c]$, and $P'' = P \cap I[c,b]$. Notice that P' is a refinement of P_1, so that i) holds with P_1 replaced by P'. In the same way ii) holds with P_2 replaced by P'' and iii) holds with P_3 replaced by P. Also

$$\underline{S}(P) = \underline{S}(P') + \underline{S}(P'').$$

Hence

$$\begin{aligned} \underline{\int_a^b} f - \epsilon < \underline{S}(P) &= \underline{S}(P') + \underline{S}(P'') \\ &\le \underline{\int_a^c} f + \underline{\int_c^b} f \quad \text{so that} \end{aligned}$$

$$\underline{\int_a^b} f \le \underline{\int_a^c} f + \underline{\int_c^b} f. \tag{22}$$

But also

$$\begin{aligned} \underline{\int_a^b} f \ge \underline{S}(P) &= \underline{S}(P') + \underline{S}(P'') \\ &\ge \underline{\int_a^c} f - \frac{\epsilon}{2} + \underline{\int_c^b} f - \frac{\epsilon}{2} \quad \text{so that} \end{aligned}$$

$$\underline{\int_a^b} f \ge \underline{\int_a^c} f + \underline{\int_c^b} f. \tag{23}$$

From (22) and (23) the left hand equality of (21) follows. The right hand equality of (21) may be proved in a similar way.

DEFINITION 1. *If f is a function whose domain includes $I[a,b]$, if f is bounded on $I[a,b]$ and if*

$$\underline{\int_a^b} f = \overline{\int_a^b} f,$$

then f is said to be **Darboux integrable** *on $I[a,b]$ with Darboux integral $\int_a^b f$ where*

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f.$$

DEFINITION 2. *If f is a function whose domain includes $I[a,b]$, if f is bounded on $I[a,b]$, and if a number R exists having the property:*

"Corresponding to each positive number $\epsilon > 0$ there is a number $\delta > 0$ such that whenever P is any partition of $I[a,b]$ with norm $\|P\| < \delta$ it follows that every Riemann sum for f over P differs from R by less than ϵ,"

then f is said to be **Riemann integrable** *with* **Riemann integral** *denoted by $\int_a^b f(x)\,dx$ where*

$$R = \int_a^b f(x)\,dx.$$

THEOREM A4.3. *A function f is Riemann integrable if and only if it is Darboux integrable. Also, if f is integrable in either sense, then its Riemann and Darboux integrals are equal:*

$$\int_a^b f(x)\,dx = \int_a^b f.$$

PROOF. Let f be Darboux integrable on I$[a,b]$ so that f is bounded on I$[a,b]$ and

$$\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f.$$

Let $\epsilon > 0$ be arbitrary. Use the corollary of Theorem A4.1 to choose a number $\delta > 0$ such that if P is a partition with $||P|| < \delta$, then any Riemann sum over P lies between $\int_a^b f - \epsilon$ and $\int_a^b f + \epsilon$. This states, "Over any partition with norm less than δ any Riemann sum differs from $\int_a^b f$ by less than ϵ," and hence (by Definition 2) f is Riemann integrable on I$[a,b]$ with Riemann integral

$$\int_a^b f(x)\,dx = \int_a^b f.$$

Next, let f be Riemann integrable and, for short, denote its Riemann integral by R. Let $\epsilon > 0$ be an arbitrary number and choose a partition P with $||P||$ sufficiently small so that

$$R - \frac{\epsilon}{2} < \sum_{k=1}^{n} f(x_k^*)\,\Delta_k x < R + \frac{\epsilon}{2} \tag{24}$$

for any Riemann sum for f over P. Now choose $\underline{x}_k$ such that

$$x_{k-1} \le \underline{x}_k \le x_k \quad \text{and} \quad f(\underline{x}_k) < \underline{B}_k + \frac{\epsilon}{2(b-a)} \tag{25}$$

and choose $\bar{x}_k$ such that

$$x_{k-1} \le \bar{x}_k \le x_k \quad \text{and} \quad \bar{B}_k - \frac{\epsilon}{2(b-a)} < f(\bar{x}_k). \tag{26}$$

Hence, for this particular subdivision P

$$\begin{aligned} \underline{S}(P) &= \sum_{k=1}^{n} \underline{B}_k\,\Delta_k x > \sum_{k=1}^{n} \left[f(\underline{x}_k) - \frac{\epsilon}{2(b-a)} \right] \Delta_k x \\ &= \sum_{k=1}^{n} f(\underline{x}_k)\,\Delta_k x - \frac{\epsilon}{2(b-a)} \sum_{k=1}^{n} \Delta_k x \\ &= \sum_{k=1}^{n} f(\underline{x}_k)\,\Delta_k x - \frac{\epsilon}{2(b-a)}(b-a) \\ &= \sum_{k=1}^{n} f(\underline{x}_k)\,\Delta_k x - \frac{\epsilon}{2} \end{aligned}$$

so that

$$\sum_{k=1}^{n} f(\underline{x}_k)\,\Delta_k x < \underline{S}(P) + \frac{\epsilon}{2}.$$

Since $\underline{x}_k$ is on I$[x_{k-1},x_k]$ and x_k^* of (24) was any point on I$[x_{k-1},x_k)$ it follows that

$$R - \frac{\epsilon}{2} < \sum_{k=1}^{n} f(\underline{x}_k)\,\Delta_k x.$$

From these last two inequalities $R - \epsilon/2 < \underline{S}(P) + \epsilon/2$ so that

$$R - \epsilon < \underline{S}(P).$$

Moreover $\underline{S}(P) \leq \underline{\int_a^b} f$ since $\underline{\int_a^b} f$ is an upper bound (in fact the least upper bound) of all lower sums and therefore

$$R - \epsilon < \underline{\int_a^b} f.$$

Since R and $\underline{\int_a^b} f$ are constants (certainly independent of ϵ) and ϵ is any positive number, then

$$R \leq \underline{\int_a^b} f.$$

By repeating such arguments, starting with (26) instead of (25), it follows that

$$R \geq \overline{\int_a^b} f.$$

Hence $\overline{\int_a^b} f \leq R \leq \underline{\int_a^b} f$. But $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ from (11), so that

$$\overline{\int_a^b} f \leq R \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f$$

which means that all equality signs hold:

$$R = \underline{\int_a^b} f = \overline{\int_a^b} f.$$

Thus, the lower and upper Darboux integrals are not only equal, so f is Darboux integrable, but since we have been using R for the Riemann integral, then

$$\int_a^b f(x)\,dx = \int_a^b f,$$

and this completes a proof of the theorem.

Since a function is Riemann integrable if and only if it is Darboux integrable, we shall henceforth use the adjective "integrable" or the noun "integral" without either of the appellatives Riemann or Darboux.

The following theorem gives a definite procedure for computing the integral of an integrable function.

THEOREM A4.4. *For f an integrable function on $I[a,b]$,*

$$\text{(27)} \qquad \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\,\frac{b-a}{n}\right)\frac{b-a}{n}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} f\left(a + k\,\frac{b-a}{n}\right)\frac{b-a}{n}.$$

PROOF. Let $\epsilon > 0$ be arbitrary. Corresponding to this number ϵ let $\delta > 0$ be such that if $P = [x_0, x_1, \cdots, x_n]$ is a partition of $I[a,b]$ with $\|P\| < \delta$, then

$$\text{(28)} \qquad \int_a^b f - \epsilon < \sum_{k=1}^{n} f(x_k^*)\,\Delta_k x < \int_a^b f + \epsilon$$

for any choice of x_k^* such that $x_{k-1} \leq x_k^* \leq x_k$ for $k = 1, 2, \cdots, n$.

With $\delta > 0$ so determined, let N be a positive integer such that

$$\text{if} \quad n > N, \quad \text{then} \quad \frac{b-a}{n} < \delta.$$

Choose a number n such that $n > N$ and let x_k be defined by

$$x_k = a + k\frac{b-a}{n} \quad \text{for} \quad k = 0, 1, 2, \cdots, n.$$

Then $a = x_0$, $b = x_n$, and $P = [x_0, x_1, x_2, \cdots, x_n]$ is a partition of I$[a,b]$. Moreover

$$\Delta_k x = x_k - x_{k-1} = \frac{b-a}{n} \quad \text{for} \quad k = 1, 2, 3, \cdots, n$$

so $||P|| = \dfrac{b-a}{n} < \delta$ and thus (28) holds for this partition P and any choice of x_k^* such that $x_{k-1} \leq x_k^* \leq x_k$ for $k = 1, 2, \cdots, n$. Hence, by choosing $x_k^* = x_k$ we have

$$\int_a^b f - \epsilon < \sum_{k=1}^{n} f(x_k)\,\Delta_k x < \int_a^b f + \epsilon; \quad \text{that is,}$$

$$\int_a^b f - \epsilon < \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n} < \int_a^b f + \epsilon.$$

Since this holds for any number $n > N$, then (by definition of a limit) the first equation of (27) follows. The second equation of (27) is obtained by choosing $x_k^* = x_{k-1}$ for $k = 1, 2, 3, \cdots, n$.

THEOREM A4.5. *Let c be such that $a < c < b$. Then f is integrable on I$[a,c]$ and on I$[c,b]$ if and only if f is integrable on I$[a,b]$. Also, if f is integrable on I$[a,b]$, then*

$$\int_a^c f + \int_c^b f = \int_a^b f. \tag{28}$$

PROOF. From Theorem A4.2 (whether f is integrable on I$[a,b]$ or not)

$$\underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f \quad \text{and} \quad \overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f.$$

But also $\underline{\int_a^c} f \leq \overline{\int_a^c} f$ and $\underline{\int_c^b} f \leq \overline{\int_c^b} f$ so that

$$\underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f \leq \left\{\begin{matrix} \underline{\int_a^c} f + \overline{\int_c^b} f \\ \overline{\int_a^c} f + \underline{\int_c^b} f \end{matrix}\right\} \leq \overline{\int_a^c} f + \overline{\int_c^b} f = \overline{\int_a^b} f. \tag{29}$$

First let f be integrable on I$[a,b]$. Then the extreme left and right terms are equal (and both $= \int_a^b f$) so all equalities hold; in particular (using the upper terms enclosed by braces)

$$\int_a^b f = \underline{\int_a^c} f + \underline{\int_c^b} f = \underline{\int_a^c} f + \overline{\int_c^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f = \int_a^b f. \tag{30}$$

From the second of these equations it follows that

$$\underline{\int_c^b} f = \overline{\int_c^b} f \quad \text{whereas} \quad \underline{\int_a^c} f = \overline{\int_a^c} f$$

follows from the third equation. From the first of these equations f is integrable on I$[c,b]$ and from the second f is integrable on I$[a,c]$. Then all lower and upper bars may be removed in (30) and thus (28) follows.

Next, let f be integrable on both I$[a,c]$ and I$[c,b]$. It is left for the reader to check (starting with (29)) that f is then integrable on I$[a,b]$ and that (28) again holds.

Let a function f be defined and bounded on a closed interval with end points a and b. The upper and lower Darboux integrals $\overline{\int}_a^b f$ and $\underline{\int}_a^b f$ have been defined only in case the lower limit of integration a was less than the upper limit of integration b. We now extend the definition to allow the upper limit to be less than or equal to the lower limit by setting

$$\underline{\int}_a^b f = -\underline{\int}_b^a f \quad \text{and} \quad \overline{\int}_a^b f = -\overline{\int}_b^a f \quad \text{if} \quad b < a \quad \text{and} \tag{31}$$

$$\underline{\int}_a^a f = \overline{\int}_a^a f = 0. \tag{32}$$

The following theorem is known as the Mean Value Theorem for upper and lower integrals.

THEOREM A4.6. *Let f be bounded on the closed interval having end points a and b where $a \neq b$. With $\bar{B}$ and $\underline{B}$ the least upper and greatest lower bounds of f on this interval. then*

$$\underline{B} \leq \frac{\underline{\int}_a^b f}{b-a} \leq \bar{B} \quad \textit{and} \quad \underline{B} \leq \frac{\overline{\int}_a^b f}{b-a} \leq \bar{B}. \tag{33}$$

PROOF. CASE 1. $a < b$. For P any partition of $I[a,b]$, then (see (1) p. 528)

$$\underline{B}(b-a) \leq \underline{S}(P) \leq \bar{S}(P) \leq \bar{B}(b-a).$$

Hence, the least upper bound of all lower sums is not greater than $\bar{B}(b-a)$ nor less than $\underline{B}(b-a)$; i.e.,

$$\underline{B}(b-a) \leq \underline{\int}_a^b f \leq \bar{B}(b-a).$$

Upon dividing by the positive number $b - a$, the first part of (33) is obtained. The second part of (33) follows in a similar way.

CASE 2. $b < a$. The interval is now $I[b,a]$ so by Case 1

$$\underline{B} \leq \frac{\underline{\int}_b^a f}{a-b} \leq \bar{B} \quad \text{and} \quad \bar{B} \leq \frac{\overline{\int}_b^a f}{a-b} \leq \bar{B}.$$

Since $\underline{\int}_b^a f = -\underline{\int}_a^b f$, $\overline{\int}_b^a f = -\overline{\int}_a^b f$ and $a - b = -(b - a)$, the middle terms retain their same values if a and b are interchanged, so (33) again holds.

THEOREM A4.7. *For $a < b$, let f be bounded on $I[a,b]$ and let $\underline{F}$* and *$\bar{F}$ be the functions defined by*

$$\underline{F}(x) = \underline{\int}_a^x f \quad \text{and} \quad \bar{F}(x) = \overline{\int}_a^x f, \quad \text{for} \quad a \leq x \leq b. \tag{34}$$

If t is such that $a \leq t \leq b$ and f is continuous at t, then $\underline{F}'(t)$ and $\bar{F}'(t)$ both exist and

$$\underline{F}'(t) = \bar{F}'(t) = f(t). \tag{35}$$

PROOF. First consider $a \leq t < b$. Let $\epsilon > 0$ be arbitrary and determine $\delta > 0$ such that if $t \leq x \leq t + \delta$ then $x < b$ and $|f(x) - f(t)| < \epsilon$. Let h be such that $0 < h < \delta$. Hence, upon applying Theorem A4.2 to the interval $I[a,t + h]$ we have

$$\underline{F}(t+h) = \underline{\int}_a^{t+h} f = \underline{\int}_a^t f + \underline{\int}_t^{t+h} f = \underline{F}(t) + \underline{\int}_t^{t+h} f,$$

so that, upon transposing $\underline{F}(t)$ and then dividing by h

$$\frac{\underline{F}(t+h)-\underline{F}(t)}{h}=\frac{\underline{\int}_t^{t+h} f}{h}. \tag{36}$$

Since $f(t)-\epsilon<f(x)<f(t)+\epsilon$ for $t \le x < t+h$, we apply Theorem A4.6 to the interval $I[t,t+h]$ to obtain

$$f(t)-\epsilon \le \frac{\underline{\int}_t^{t+h} f}{(t+h)-t} \le f(t)+\epsilon.$$

These inequalities together with (36) show that

$$-\epsilon \le \frac{\underline{F}(t+h)-\underline{F}(t)}{h}-f(t) \le \epsilon$$

We have thus shown that corresponding to each number $\epsilon > 0$ there is a number δ such that if $0 < h < \delta$ then

$$\left|\frac{\underline{F}(t+h)-\underline{F}(t)}{h}-f(t)\right| \le \epsilon.$$

Hence, if f is continuous at t where $a \le t < b$, then

$$\lim_{h\to 0^+}\frac{\underline{F}(t+h)-\underline{F}(t)}{h}=f(t).$$

The reader should now check that if f is continuous at t where $a < t \le b$, then

$$\lim_{h\to 0^-}\frac{\underline{F}(t+h)-\underline{F}(t)}{h}=f(t).$$

It therefore follows that if f is continuous at t where $a \le t \le b$, then $\underline{F}'(t)$ exists and is equal to $f(t)$.

The above portion of the proof was written so it could be reread verbatim with every lower bar changed to an upper bar, and thus (35) holds.

The next theorem is a direct corollary of Theorem A4.7, but the result is too important to be given as a corollary.

THEOREM A4.8. *If $a < b$ and f is continuous on the closed interval $I[a,b]$, then f is integrable on $I[a,b]$.*

PROOF. Let f be continuous at each x such that $a \le x \le b$. Then f is bounded on $I[a,b]$ (see p. 524) and in the notation of the above theorem

$$\underline{F}'(x)=\overline{F}'(x)=f(x) \quad \text{for} \quad a \le x \le b.$$

Since these derivatives are equal at each x such that $a \le x \le b$, there is a constant c such that

$$\underline{F}(x)=\overline{F}(x)+c \quad \text{for} \quad a \le x \le b,$$

(see Theorem 39). But $\underline{F}(a)=\overline{F}(a)=\underline{\int}_a^a f=\overline{\int}_a^a f=0$ [see (32)] so that $c=0$. Hence $\underline{F}(x)=\overline{F}(x)$; that is,

$$\underline{\int}_a^x f=\overline{\int}_a^x f \quad \text{for} \quad a \le x \le b.$$

In particular $\underline{\int}_a^b f=\overline{\int}_a^b f$ which, by definition, means that f is (Darboux) integrable on $I[a,b]$.

A5. Rectifiability. PROOF of Theorem 86, p. 272. From the continuity of f' on $I[a,b]$ the function g defined by

$$g(x) = \sqrt{1 + f'^2(x)} \quad \text{for} \quad a \leq x \leq b$$

is continuous on $I[a,b]$. Thus, the integral in (2) exists (see Theorem A4.8).

Let $\epsilon > 0$ be arbitrary. Corresponding to this number ϵ, let $\delta > 0$ be such that if P is any partition of $I[a,b]$ with $||P|| < \delta$, then the lower and upper Darboux sums for the function g relative to P satisfy the inequalities

$$\text{(3)} \qquad \int_a^b \sqrt{1 + f'^2(x)}\, dx - \epsilon < \underline{S}(P) \leq \bar{S}(P) < \int_a^b \sqrt{1 + f'^2(x)}\, dx + \epsilon.$$

Next, let N be such that if $n > N$ then $\Delta_n x = (b - a)/n < \delta$.

Choose an integer n such that $n > N$ and form the partition

$$P = [x_0, x_1, x_2, \cdots, x_n] \quad \text{where} \quad x_k = a + k\,\Delta_n x.$$

Next, for $k = 1, 2, \cdots, n$ choose x_k^* such that (by the Law of the Mean)

$$x_{k-1} < x_k^* < x_k \quad \text{and} \quad \frac{f(x_k) - f(x_{k-1})}{\Delta_n x} = f'(x_k^*).$$

Now the lower and upper Darboux sums for g are such that

$$\underline{S}(P) \leq \sum_{k=1}^{n} \sqrt{1 + f'^2(x_k^*)}\, \Delta_n x \leq \bar{S}(P).$$

Consequently, whenever n is such that $n > N$, then

$$\int_a^b \sqrt{1 + f'^2(x)}\, dx - \epsilon < \sum_{k=1}^{n} \sqrt{1 + \left[\frac{f(x_k) - f(x_{k-1})}{\Delta_n x}\right]^2}\, \Delta_n x < \int_a^b \sqrt{1 + f'^2(x)}\, dx + \epsilon.$$

This states not only that the limit as $n \to \infty$ of the lengths of inscribed polygons exist (so the arc is rectifiable) but also that this limit is the integral in (2), as we wished to prove.

A6. Double Integrals. Throughout this section a, b, c, and d are numbers with $a < b$ and $c < d$ and R is the rectangular region

$$R = \{(x,y) \mid a \leq x \leq b \quad \text{and} \quad c \leq y \leq d\}.$$

Let $P_1 = [x_0, x_1, x_2, \cdots, x_m]$ be a partition (recall that $x_0 = a$ and $x_m = b$) of the interval $I[a,b]$ and let $P_2 = [y_0, y_1, y_2, \cdots, y_n)$ be a partition of the interval $I[c,d]$. By taking the union of the $m + n + 2$ sets

$$\{(x,y) \mid x = x_i, c \leq y \leq d\} \quad \text{and} \quad \{(x,y) \mid a \leq x \leq b, y = y_j\}$$

for $i = 0, 1, 2, \cdots, m$ and $j = 0, 1, 2, \cdots, n$ we obtain what is called a **partition** P of R. This partition P is represented by

$$\text{(1)} \qquad P = [x_0, x_1, x_2, \cdots, x_m;\ y_0, y_1, y_2, \cdots, y_n].$$

Thus, P may be thought of as a net on R dividing R into $m \cdot n$ rectangles

$$\text{(2)} \qquad R_{ij} = \{(x,y) \mid x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$$

for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. With $\Delta_i x = x_i - x_{i-1}$ and $\Delta_j y = y_j - y_{j-1}$ the area of R_{ij} is equal to $\Delta_i x\, \Delta_j y$.

Let f be a function of two variables whose domain includes R and such that f is bounded on R. Also, let $\underline{B}$ and $\bar{B}$ be the greatest lower and least upper bounds of f on R. With P the partition (1) let $\underline{B}_{ij}$ and $\bar{B}_{ij}$ be the greatest lower and least upper bounds of f on the rectangle R_{ij}. We then let

$$\underline{S}(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} \underline{B}_{ij}\, \Delta_i x\, \Delta_j y \quad \text{and} \quad \bar{S}(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} \bar{B}_{ij}\, \Delta_i x\, \Delta_j y,$$

and call $\underline{S}(P)$ and $\bar{S}(P)$ the **lower** and **upper Darboux sums** for f over P. Also, for x_i^* and y_j^* such that $x_{i-1} \leq x_i^* \leq x_i$ and $y_{j-1} \leq y_j^* \leq y_j$ then (x_i^*,y_j^*) is a point of R_{ij} so $\underline{B}_{ij} \leq f(x_i^*,y_j^*) \leq \bar{B}_{ij}$. Thus

$$\underline{S}(P) \leq \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*,y_j^*)\, \Delta_i x\, \Delta_j y \leq \bar{S}(P)$$

and the middle term is called a **Riemann sum** for f over P.

The least upper bound of all lower Darboux sums is called the **lower Darboux double integral** of f on R and is represented by

$$\underline{\iint}_R f.$$

The g.l.b. of all upper Darboux sums is called the **upper Darboux double integral** of f on R and is represented by

$$\overline{\iint}_R f.$$

If these upper and lower Darboux double integrals are equal, then f is said to be **Darboux double integrable** on R with **Darboux double integral** $\iint_R f$ this common value.

The longest diagonal of all the rectangles R_{ij} determined by a partition P is called the norm of P and is represented by $||P||$.

Let T be a number. If "Corresponding to each number $\epsilon > 0$ there is a number $\delta > 0$ such that whenever P is a partition of R with norm $||P|| < \delta$, then all Riemann sums of f over P differ from T by less than ϵ," then f is said to be **Riemann double integrable** over R with **Riemann double integral**

$$T = \iint_{(a,c)}^{(b,d)} f(x,y)\, d(x,y)$$

Section A4 was written in such a way that, with only minor alterations, it could be reread to obtain results for double integrals analogous to most of the results through Theorem A4.4 for single integrals. It is recommended that such a rereading be done to see that for f a bounded function on R then:

1. *f is Riemann double integrable if and only if f is Darboux double integrable.*
2. *If f is double integrable on R in either sense, then its Riemann and Darboux double integrals have the same value and this value is equal to*

$$\text{(3)} \qquad \lim_{(m,n)\to(\infty,\infty)} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(a + i\frac{b-a}{m}, c + j\frac{d-c}{n}\right) \frac{b-a}{m}\frac{d-c}{n}.$$

The results of Sec. A4 through Theorem A4.4 have thus been extended to double integrals. Theorem A4.5 is, however, not easily extendable to double integrals, but Theorem A4.8 (continuity of f on a closed interval implies integrability on this interval) does have an analogue which is proved, however, by methods differing considerably from those used in the proof of Theorem A4.8. For these different methods, the notion of uniform continuity of a function is required which we introduce first for functions of one variable.

DEFINITION A6.1. *Let f be a function of one variable and let A be a subset of the domain of f (so A might be all of the domain of f). The function f is said to be* **uniformly continuous** *on A if for each positive number ϵ there is a number $\delta > 0$ (which depends both on ϵ and A) such that if x and s are points of A and*

$$|s - x| < \delta, \quad \textit{then} \quad |f(s) - f(x)| < \epsilon. \tag{4}$$

An illusive distinction between the notions of "continuity at a point" and "uniform continuity on a set" should be observed. Recall that f is continuous at a point x, in the domain of f, if for each positive number ϵ there is a number $\delta > 0$ (which depends upon both ϵ and x) such that whenever s is in the domain of f and

$$|s - x| < \delta, \quad \text{then} \quad |f(s) - f(x)| < \epsilon. \tag{5}$$

Even though (4) and (5) are identical, it is the words above these respective displays which convey the distinction between continuity at a point and uniform continuity on a set; the crux being the respective parenthetic phrases following $\delta > 0$.

An example of a function which is continuous at each point of its domain, but not uniformly continuous on its domain, may help to clarify the distinction between these notions. Such an example is the function f defined by

$$f(x) = \frac{1}{x} \quad \text{for} \quad 0 < x < 1. \tag{6}$$

The domain is therefore the open interval $I(0,1)$. Even for $\epsilon = \frac{1}{2}$ and δ chosen as any positive number, there are points of $I(0,1)$ differing by less than δ where the values of f differ by more than ϵ. To see this, choose a positive number x less than 1 such that also $x \leq \delta$, and let $s = x/2$. Then $|s - x| = x/2 < \delta$ and

$$|f(s) - f(x)| = \left|\frac{2}{x} - \frac{1}{x}\right| = \frac{1}{x} > 1 > \epsilon.$$

Hence, there is a positive number ϵ, such that no matter what positive number δ is given, then there exist points s and x in the domain of f with $|s - x| < \delta$ but $|f(s) - f(x)| > \epsilon$; that is, f is **not** uniformly continuous on $I(0,1)$.

We now show, on the other hand, that the function f defined by (6) is continuous at each point of $I(0,1)$. To do so select any point x on $I(0,1)$. Thus, x is a number (to emphasize this we even say that x is a "fixed number") such that $0 < x < 1$. Let ϵ be an arbitrary positive number. Next choose

$$\delta = \frac{\epsilon x^2}{1 + \epsilon x},$$

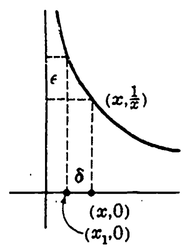


Figure A.6

which certainly depends upon both ϵ and x. This choice of δ might seem cryptic without the aid of Fig. A6 and the hint that we chose $\delta = x - x_1$ where $f(x_1) = f(x) + \epsilon$. Now, let s be such that $|s - x| < \delta$, so that

$$x - \frac{\epsilon x^2}{1 + \epsilon x} < s < x + \frac{\epsilon x^2}{1 + \epsilon x} \quad \text{and hence} \quad \frac{x}{1 + \epsilon x} < s < \frac{x + 2\epsilon x^2}{1 + \epsilon x}$$

Consequently $\dfrac{1 + \epsilon x}{x} - \dfrac{1}{x} > \dfrac{1}{s} - \dfrac{1}{x} > \dfrac{1 + \epsilon x}{x + 2\epsilon x^2} - \dfrac{1}{x}$; that is,

$$\epsilon > \frac{1}{s} - \frac{1}{x} > \frac{-\epsilon}{1 + 2\epsilon x} > -\epsilon \quad \text{so that} \quad \left|\frac{1}{s} - \frac{1}{x}\right| < \epsilon.$$

Thus, f is continuous at this point x and, since x was any point on $I(0,1)$, f is continuous at each point of $I(0,1)$.

DEFINITION A6.2. *Let f be a function of two variables and let A be a subset of the domain of f. The function f is said to be* **uniformly continuous** *on A if corresponding to each positive number ϵ there is a number $\delta > 0$ (depending upon both ϵ and A) such that if (x,y) and (s,t) are two points of A and*

$$\sqrt{(s-x)^2+(t-y)^2} < \delta, \quad \textit{then} \quad |f(s,t) - f(x,y)| < \epsilon.$$

The following theorem is numbered A6.8 to emphasize its analogy with Theorem A4.8.

THEOREM A6.8. *Let f be a function of two variables whose domain includes the rectangle R. If f is bounded and uniformly continuous*† *on R, then f is (double) integrable on R.*

PROOF. For f any bounded function on R, the upper and lower Darboux integrals exist and for any partition P of R

(7) $$\overline{\iint}_R f \le \bar{S}(P), \quad \underline{S}(P) \le \underline{\iint}_R f, \quad \text{and} \quad \underline{\iint}_R f \le \overline{\iint}_R f.$$

Throughout the rest of this proof we take f *to be bounded and uniformly continuous on the rectangle*

$$R = \{(x,y) \mid a \le x \le b \quad \text{and} \quad c \le y \le d\}$$

and we arbitrarily choose a positive number ϵ. In terms of this number ϵ, let $\eta > 0$ be defined by

(8) $$\eta = \frac{\epsilon}{2(b-a)(d-c)}.$$

From the uniform continuity of f on R, let $\delta > 0$ (depending upon R and η) be such that if (x,y) and (s,t) are in R and

(9) $$\sqrt{(s-x)^2+(t-y)^2} < \delta \quad \text{then} \quad |f(s,t) - f(x,y)| < \eta.$$

Having determined $\delta > 0$ in this manner, we select a partition

$$P[x_0, x_1, \cdots, x_m;\ y_0, y_1, \cdots, y_n] \quad \text{with} \quad \|P\| < \delta.$$

Also, let $R_{ij} = \{(x,y) \mid x_{i-1} \le x \le x_i \text{ and } y_{j-1} \le y \le y_j\}$ for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. If (x,y) and (s,t) both belong to the same rectangle R_{ij} then, since the diagonal of this rectangle is less than or equal to $\|P\| < \delta$, we use (9) to see that $f(s,t) - \eta < f(x,y) < f(s,t) + \eta$ of which we use only

$$f(x,y) < f(s,t) + \eta.$$

By thinking of (s,t) as any fixed point of R_{ij} and letting (x,y) run over R_{ij} it follows that the l.u.b. $\bar{B}_{ij}$ of f on R_{ij} is such that

$$\bar{B}_{ij} \le f(s,t) + \eta \quad \text{or} \quad \bar{B}_{ij} - \eta \le f(s,t).$$

With this inequality established for (s,t) any point of R_{ij} it therefore follows that

(10) $$\bar{B}_{ij} - \eta \le \underline{B}_{ij}$$

† One purpose of Sec. A7 which follows is to prove that if f is merely continuous on R, then f is both bounded and uniformly continuous on R. Hence, we will eventually have: *If f is continuous on R, then f is integrable on R.*

and therefore, from $\Delta_i x = x_i - x_{i-1}$ and $\Delta_j y = y_j - y_{j-1}$, we have

$$\bar{B}_{ij}\,\Delta_i x\,\Delta_j y - \eta\,\Delta_i x\,\Delta_j y \le \underline{B}_{ij}\,\Delta_i x\,\Delta_j y.$$

Since this holds for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$ then from the definitions of the upper and lower Darboux sums over P

$$\bar{S}(P) - \eta \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta_i x\,\Delta_j y \le \underline{S}(P).$$

Since the sum multiplying η is the area of R then

$$\bar{S}(P) - \eta(b-a)(d-c) \le \underline{S}(P) \quad \text{so that} \quad \bar{S}(P) - \epsilon/2 \le \underline{S}(P)$$

from the definition (8) of η. The inequalities (7) hold for any partition whatever, and thus from the first of these and the inequality we have just obtained

$$\overline{\iint}_R f - \frac{\epsilon}{2} \le \bar{S}(P) - \frac{\epsilon}{2} \le \underline{S}(P) \le \underline{\iint}_R f.$$

We have thus shown: If ϵ is any positive number whatever (and f is bounded and uniformly continuous on R), then

$$\overline{\iint}_R f - \epsilon < \underline{\iint}_R f.$$

The upper and lower integrals are constants and ϵ is an arbitrary number and therefore

$$\overline{\iint}_R f \le \underline{\iint}_R f.$$

But the lower integral is always less than or equal to the upper integral and thus (for the function f we are considering) these integrals are equal:

$$\overline{\iint}_R f = \underline{\iint}_R f$$

which, by definition, means that our function f is integrable on R. This, therefore, establishes the theorem.

A7. Uniform Continuity. Before studying uniform continuity further we will discuss some auxiliary geometric properties of intervals and circles.

Recall that an interval is said to be **closed** if it includes both its end points and all points between them, and is said to be **open** if it includes all points between its end points but not the end points themselves. We shall be considering a **family** of intervals. A family of intervals is merely a collection of intervals. Instead of "family of intervals" we could say "set of intervals," but prefer "family" instead of "set" to avoid possible confusion with sets of points.

Given a closed interval $\bar{I}$ and a family of intervals, the family will be said to **cover** $\bar{I}$ if each point of $\bar{I}$ is also a point of at least one of the intervals of the family; in other words, if $\bar{I}$ is a subset of the union of all the intervals of the family.

Let $\bar{I}$ be a closed interval with end points a and b so that

$$\bar{I} = \{x \mid a \le x \le b\}.$$

Given a point x of $\bar{I}$ take an open interval with center at x and length < 1. (The number 1 is chosen as a convenient upper bound.) Do this for each point of $\bar{I}$ and let $\mathscr{I}$ denote the family of all such open intervals. The family $\mathscr{I}$ covers $\bar{I}$ since each point of $\bar{I}$ is even the center of one of the open intervals of $\mathscr{I}$. Our first theorem is a special case of what is known as the Heine-Borel theorem for linear sets.

THEOREM A7.1. *There is a finite subfamily of the family $\mathscr{I}$ defined above which also covers $\bar{I}$; that is, there is a finite number of open intervals $I_1, I_2, \cdots, I_n$ selected from the family $\mathscr{I}$ such that*

$$\bar{I} \subset I_1 \cup I_2 \cup \cdots \cup I_n.$$

PROOF. In the proof of this theorem we shall use the set S defined as follows:

$$S = \{x \mid a < x \text{ and the closed interval } I[a,x] \text{ can be covered with a finite number of intervals selected from } \mathscr{I}\}.$$

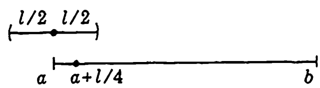


Figure A7.1

The set S is non-empty since the single open interval of $\mathscr{I}$ with center at a covers a closed interval with lower end point a. (To be specific, if the interval of $\mathscr{I}$ which is centered at a has length l, then the closed interval $I[a, a + l/4]$ is covered and $l/4$ is a point of S.) Also, the set S is bounded above (certainly by $b + 1$ since every interval of $\mathscr{I}$ has length <1 and center on $\bar{I}$). Thus, by the axiom on p. 10, the set has a l.u.b. which we call $\bar{x}$. We now claim that

$$\bar{x} > b. \tag{1}$$

In order to establish (1) assume it is not so; that is,

$$\text{assume } \bar{x} \leq b.$$

Under this assumption $\bar{x}$ is a point of $\bar{I}$ and as such there is an open interval of $\mathscr{I}$ with center at $\bar{x}$; we call this open interval J. Let x' and x'' be points of J (see Fig. A7.2) such that $a < x' < \bar{x} < x''$. Then x' is a point of S (since $\bar{x}$ = l.u.b. of S and $x' < \bar{x}$). Hence, the closed interval with end points a and x' can be covered with a finite number of intervals of $\mathscr{I}$ and then exactly these intervals together with J covers the closed interval with end points a and x''. Thus, x'' is a point of S, but this is a contradiction since $\bar{x} < x''$ and $\bar{x}$ is an upper bound (in fact the l.u.b.) of S.

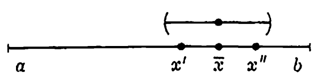


Figure A7.2

The above contradiction shows that (1) holds. Thus, b is a point of S which means that the closed interval $\bar{I}$ with end points a and b can be covered with a finite number of open intervals selected from the family $\mathscr{I}$.

We now turn to considerations of sets in the plane. By a **circle** we shall mean what might better be called a circular disk and shall also say, for (h,k) a given point and $r > 0$, that

$$\{(x,y) \mid (x - h)^2 + (y - k)^2 \leq r^2\}$$

is a **closed circle** and $\{(x,y) \mid (x - h)^2 + (y - k)^2 < r^2\}$ is an **open circle**. Thus, a "closed circle" is a circular disk including its rim, whereas an "open circle" is a circular disk without its rim.

What we have been referring to as a rectangle:

$$R = \{(x,y) \mid a \leq x \leq b \quad \text{and} \quad c \leq y \leq d\},$$

we now call a **closed rectangle** to emphasize it is a rectangular region *including its boundary*. As another convenient notation we shall also denote this closed rectangle by

$$R = [(a,c);\ (b,d)].$$

LEMMA 1. *In the plane take a closed interval parallel to the x-axis and let its end points be (a,y) and (b,y). Let $\mathscr{D}$ be a family of open circles each of diameter <1 and such that each circle of $\mathscr{D}$ has its center on the interval and each point of the interval is the center of a circle of $\mathscr{D}$. Then there are a finite number of circles $C_1, C_2, \cdots, C_n$ selected from $\mathscr{D}$ which cover the interval.*

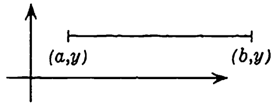


Figure A7.3

PROOF. The line through (a,y) and (b,y) intersects each of the open circles of $\mathscr{D}$ in an open interval centered at a point of the interval and we let $\mathscr{I}$ be the family of such open intervals. This family $\mathscr{I}$ satisfies all conditions of Theorem A7.1 and hence there is a finite number of them $I_1, I_2, \cdots, I_n$ whose union contains the given closed interval. With C_k the circle of $\mathscr{D}$ having I_k as a diameter, then $C_1 \cup C_2 \cup \cdots \cup C_n$ also contains the given closed interval.

LEMMA 2. *Given the open circles $C_1, C_2, \cdots, C_n$ of Lemma 1, there are numbers y' and y'', with $y' < y < y''$, such that the closed rectangle $[(a,y'); (b,y'')]$ is also included in the union $C_1 \cup C_2 \cup \cdots \cup C_n$.*

PROOF. Look at the intersections of the rims of the circles $C_1, C_2, \cdots, C_n$ which are above or below the interval. (If two of these circles have rims intersecting on the interval, then this point belongs to neither of these open circles so there must be a third circle among $C_1, C_2, \cdots, C_n$ containing this point.) Also, draw the lines through (a,y) and (b,y) perpendicular to the segment and note the intersections of these lines with rims of those circles which contain (a,y) or (b,y). The closest of all these (finite number of) intersections to the interval is a positive distance above or below the interval. See Fig. A7.4. We take y' and y'' such that $y' < y < y''$ and such that both $y - y'$ and $y'' - y$ are less than this distance. Notice that these numbers y' and y'' satisfy the conclusion of the lemma.

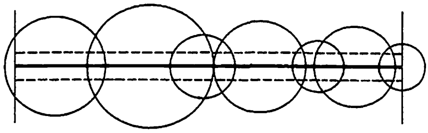


Figure A7.4

The following theorem is a special case of the Heine-Borel theorem for plane sets.

THEOREM A7.2. *Let R be the closed rectangle $[(a,c); (b,d)]$ and let $\mathscr{C}$ be a family of open circles each of diameter <1 such that each point of R is the center of a circle of $\mathscr{C}$, and each circle of $\mathscr{C}$ has its center at a point of R. Then there is a finite subfamily of $\mathscr{C}$ which covers R; that is, there are a finite number of open circles $C_1, C_2, \cdots, C_n$ selected from the family $\mathscr{C}$ such that*

$$R \subset C_1 \cup C_2 \cup \cdots \cup C_n.$$

PROOF. We first define a set T by:

$$T = \{y \mid c < y \text{ and the closed rectangle } [(a,c);\ (b,y)] \text{ may be covered by a finite subfamily of } \mathscr{C}\}.$$

First T is non-empty. For by Lemma 1 there are a finite number of circles of $\mathscr{C}$ centered on the closed interval with end points (a,c) and (b,c) which covers this interval. Then by Lemma 2 there is a number $y'' > c$ such that the rectangle $[(a,c); (b,y'')]$ is also covered by this same subfamily. Hence, y'' is in T so T is non-empty.

Also, T is bounded above by $d + 1$ and we let $\bar{y}$ be the l.u.b. of T. We leave it to the reader to follow arguments similar to those used in the proof of Theorem A7.1 to show that $\bar{y} > d$ and hence prove this theorem.

We now use the geometric Theorem A7.2 to prove the following theorem which has many significant consequences one of which is given in Theorem A7.4. Unless the definition of uniform continuity is well in mind, this definition (see p. 541) should now be reviewed.

THEOREM A7.3. *If a function f of two variables is continuous at each point of a closed rectangle, then f is uniformly continuous on this rectangle.*

PROOF. Let f be continuous on the closed rectangle R and let ϵ be an arbitrary positive number. Select a point (x,y) of R. Since f is continuous at (x,y), let $\rho(x,y)$ be a positive number (depending upon both (x,y) and $\epsilon/2$) such that if (u,v) is a point of R and

(2) $$\sqrt{(u-x)^2 + (v-y)^2} < \rho(x,y), \quad \text{then} \quad |f(u,v) - f(x,y)| < \epsilon/2.$$

Draw the open circle with center (x,y) and (not with radius $\rho(x,y)$, but)

(3) radius the smaller of $\frac{1}{2}\rho(x,y)$ and $\frac{1}{2}$.

Consider this done for each point (x,y) of R and let $\mathscr{C}$ be the family of all circles so obtained. Hence, each point of R is the center of a circle of $\mathscr{C}$, each circle of $\mathscr{C}$ has its center at a point of R, and each circle of $\mathscr{C}$ has radius <1.

By using Theorem A7.2, let $C_1, C_2, \cdots, C_n$ be a finite number of circles of $\mathscr{C}$ such that

$$R \subset C_1 \cup C_2 \cup \cdots \cup C_n.$$

Let $r_1, r_2, \cdots, r_n$ be the respective radii of these circles. Each radius is positive and, being finite in number, there is a smallest among them. Let δ be the smallest of $r_1, r_2, \cdots, r_n$ so that

(4) $$\delta > 0 \quad \text{and also} \quad \delta \le r_1, \delta \le r_2, \cdots, \delta \le r_n.$$

We now assert that:

(5) *If (s,t) and (u,v) are points of R such that*

$$\sqrt{(s-u)^2 + (t-v)^2} < \delta \quad \textit{then} \quad |f(s,t) - f(u,v)| < \epsilon.$$

Notice that (5) says "f is uniformly continuous on R" and thus the proof of this assertion will finish the proof of the theorem.

Let (s,t) and (u,v) be points of R such that

(6) $$\sqrt{(s-u)^2 + (t-v)^2} < \delta.$$

Now (s,t), being a point of R, lies in at least one of the circles $C_1, C_2, \cdots, C_n$. We denote by C_k one of these circles containing (s,t) and also specify its center as (x_k,y_k) and its radius as r_k. Hence, the distance between (s,t) and (x_k,y_k) is $<r_k$:

$$\sqrt{(s-x_k)^2 + (t-y_k)^2} < r_k = \tfrac{1}{2}\rho(x_k,y_k) < \rho(x_k,y_k)$$

so by (2)

(7) $$|f(s,t) - f(x_k,y_k)| < \epsilon/2.$$

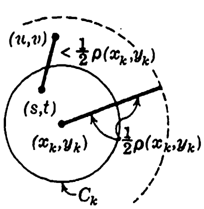


Figure A7.5

From (6), $\sqrt{(s-u)^2 + (t-v)^2} < \delta \le r_k = \frac{1}{2}\rho(x_k,y_k)$. Hence, the distance from (u,v) to (s,t) is $<\frac{1}{2}\rho(x_k,y_k)$ and also the distance from (s,t) to (x_k,y_k) is $<\frac{1}{2}\rho(x_k,y_k)$ so the distance from (u,v) to (x_k,y_k) is less than $\rho(x_k,y_k)$:

$$\sqrt{(u-x_k)^2 + (v-y_k)^2} \le \sqrt{(u-s)^2 + (t-v)^2} + \sqrt{(s-x_k)^2 + (t-y_k)^2}$$
$$< \tfrac{1}{2}\rho(x_k,y_k) + \tfrac{1}{2}\rho(x_k,y_k) = \rho(x_k,y_k).$$

Thus, by (2)

$$|f(u,v) - f(x_k,y_k)| < \epsilon/2. \tag{8}$$

Hence, under the condition (6), both (7) and (8) hold; that is, under the condition (6)

$$\begin{aligned}|f(s,t) - f(u,v)| &\leq |f(s,t) - f(x_k,y_k)| + |f(x_k,y_k) - f(u,v)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

This, however, is the assertion (5) and thus the proof of the theorem is complete.

COROLLARY. *The function f in the proof of Theorem* A7.3 *is bounded on R; that is, if a function is continuous at each point of a closed rectangle, then the function is bounded on the rectangle.*

PROOF. In the notation in the proof of Theorem A7.3, notice the largest and smallest of the finite number of values

$$f(x_1,y_1), f(x_2,y_2), \cdots, f(x_n,y_n). \tag{9}$$

Any point (x,y) of R lies in at least one of the circles $C_1, C_2, \cdots, C_n$ so the value $f(x,y)$ differs from at least one of the values in (9) by less than $\epsilon/2$. Hence, the largest value in (9) plus $\epsilon/2$ and the smallest value in (9) minus $\epsilon/2$ are upper and lower bounds of f on R.

A purpose of Theorem A7.3 and its corollary is so the hypotheses of Theorem A6.8 may be weakened to such an extent that most functions met in practice are known to be integrable on closed rectangles of their domains. Thus, the results of this section together with Theorem A6.8 yield the following theorem (already stated in the footnote of p. 542) which is the complete analogue of Theorem A4.8 for one variable.

THEOREM A7.8. *If a function f of two variables is continuous on a closed rectangle R, then f is integrable on R.*

A set A in the plane is said to have **zero area** if corresponding to each positive number ϵ there is a finite collection of rectangles whose union contains A and the sum of their areas is less than ϵ.

Problem 1. Let f be a function which is bounded on a rectangle R and such that the set of discontinuities of f on R has zero area. Prove that f is double integrable on R.

A **plane curve** is, by definition, the continuous image of a closed interval; that is, there are continuous functions φ and ψ such that

$$C = \{(x,y) \mid x = \varphi(t), y = \psi(t) \quad \text{for} \quad \alpha \leq t \leq \beta\}$$

consists of those points and only those points of the curve. In case the set C has zero area, then the curve is said to have **zero area**†. The curve is said to be **closed** if

$$\varphi(\alpha) = \varphi(\beta) \quad \text{and} \quad \psi(\alpha) = \psi(\beta)$$

and is said to be **simple closed** if, in addition, the points

$$(\varphi(t_1), \psi(t_1)) \quad \text{and} \quad (\varphi(t_2), \psi(t_2))$$

are different whenever $\alpha \leq t_1 < t_2 < \beta$ or $\alpha < t_1 < t_2 \leq \beta$.

† For an example of a curve which does not have zero area see I. J. Schoenberg, "On the Peano Curve of Lebesgue," *Bulletin of the American Mathematical Society*, Vol. 44 (1938), p. 519.

A plane set is said to be **bounded** if there is some rectangle which includes all points of the set.

We assume the Jordan Curve Theorem; namely:

Every simple closed curve divides the plane into two regions, one and only one of which is bounded.

Given a simple closed curve, of the two regions into which it divides the plane, the bounded one is said to be **surrounded** by the curve and the curve is said to be the **boundary** of this region.

Let G be a region surrounded by a simple closed curve and let f be a function defined at each point of G (not necessarily at points of its boundary). Let R be a closed rectangle which includes all points of G and its boundary. Let f^* be the function defined by

$$f^*(x,y) = \begin{cases} f(x,y) & \text{if} \quad (x,y) \text{ is a point of } G \\ 0 & \text{if} \quad (x,y) \text{ is not a point of } G. \end{cases}$$

If f^* is integrable on R, then f is said to be **integrable** on G with, by definition,

$$\iint_G f = \iint_R f^*.$$

Problem 2. Assume that f is bounded and continuous at each point of G and that the boundary of G has zero area. Give a geometric argument to indicate that f is integrable on G.

A8. Iterated Integrals. Let f be a function of two variables defined on the closed rectangle

$$R = \{(x,y) \mid a \leq x \leq b, \ c \leq y \leq d\}.$$

We consider the respective possibilities a) and b) below:

a) It may be that for each (fixed) y such that $c \leq y \leq d$ the function f (considered as a function of x alone) is integrable on the closed interval joining the points (a,y) and (b,y); that is, it may be that

$$\int_a^b f(x,y)\,dx \quad \text{exists for each } y \text{ such that} \quad c \leq y \leq d. \tag{1}$$

If (1) holds, then we define a function F on $I[c,d]$ by setting

$$F(y) = \int_a^b f(x,y)\,dx \quad \text{for} \quad c \leq y \leq d. \tag{2}$$

b) It may then be that this function F is integrable on $I[c,d]$; that is, it may be that

$$\int_c^d F(y)\,dy \quad \text{exists.} \tag{3}$$

If both (1) and (3) hold, then the result of performing first the integration in (1) and then the integration in (3) is denoted by

$$\int_c^d \int_a^b f(x,y)\,dx\,dy \tag{4}$$

and (4) is said to be an **iterated integral** (x first, y second) of f on R.

By assuming the existence of integrals in reverse order, then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx \tag{5}$$

is also an **iterated integral** (y first, x second) of f on R.

If both iterated integrals (4) and (5) exist, a natural question is "Are these iterated integrals equal?" Also, if f is double integrable on R, so

$$\iint_R f \tag{6}$$

exists "Do the iterated integrals (4) and (5) exist and, if they do, are their values the same as the value of the double integral in (6)?"

These questions will not be answered here, but we shall prove the following special case of what is known as the Fubini Theorem.

THEOREM A8.1. *If the function f is continuous on the rectangle R, then*† *both iterated integrals* (4) *and* (5) *exist and have the same value which is also the value of the double integral:*

$$\iint_R f = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx. \tag{7}$$

PROOF. With f continuous on R we shall prove the existence of the first iterated integral and the first equality in (7).

Since f, as a function of two variables, is continuous on R then for each y, where $c \le y \le d$, the function g defined by $g(x) = f(x,y)$ for $a \le x \le b$ is continuous and hence (by Theorem A4.8) is integrable on $I[a,b]$. We define the function F by (2) and shall prove:

(8) *The function F is continuous on* $I[c,d]$.

Let ϵ be an arbitrary positive number. We again use the fact that the continuity of f on the closed rectangle R implies the uniform continuity of f on R and accordingly determine $\delta > 0$ (depending upon $\epsilon/(b-a)$ and R) such that if (u,v) and (s,t) are points of R and

$$\sqrt{(s-u)^2 + (t-v)^2} < \delta, \quad \text{then} \quad |f(s,t) - f(u,v)| < \frac{\epsilon}{b-a}. \tag{9}$$

Let y be any number such that $c \le y \le d$ and let h be a number such that

(10) $|h| < \delta$ and $c \le y + h \le d$.

For n a positive integer let

$$x_k = a + k\frac{b-a}{n} \quad \text{and}$$

$$\Delta_k x = \frac{b-a}{n} \quad \text{for} \quad k = 1, 2, \cdots, n.$$

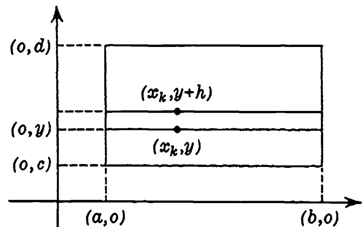


Figure A8

Hence (x_k,y) and $(x_k,y+h)$ are points of R and the distance between them is $<\delta$, because of (10), and therefore by (9)

$$f(x_k,y) - \frac{\epsilon}{b-a} < f(x_k,y+h) < f(x_k,y) + \frac{\epsilon}{b-a}.$$

We now multiply each term by $\Delta_k x$, sum from $k = 1$ to n, and then (by relying upon Theorem A4.4) take the limit as $n \to \infty$ to obtain

$$\int_a^b f(x,y)\,dx - \epsilon \le \int_a^b f(x,y+h)\,dx \le \int_a^b f(x,y) + \epsilon,$$

$$F(y) - \epsilon \le F(y+h) \le F(y) + \epsilon.$$

Thus, under the conditions (10) we have $|F(y+h) - F(y)| \le \epsilon$ and hence F is continuous at y. Since y was any point of $I[c,d]$, the statement (8) has been proved.

Now that we know F is continuous on $I[c,d]$ we know the first iterated integral in (7) exists and we now prove the first equality of (7).

† We already know the double integral (6) exists by Theorem A7.8.

Let $\epsilon > 0$ be arbitrary. Since f is double integrable on R, choose m and n so large (as justified by Formula (3) on p. 540) that

$$\iint_R f - \epsilon < \sum_{j=1}^{n} \sum_{i=1}^{m} \underline{B}_{ij} \, \Delta_m x \, \Delta_n y \le \sum_{j=1}^{n} \sum_{i=1}^{m} \bar{B}_{ij} \, \Delta_m x \, \Delta_n y < \iint_R f + \epsilon, \tag{11}$$

where $\quad \Delta_m x = (b - a)/m, \; x_i = a + i \, \Delta_m x \quad \text{for} \quad i = 0, 1, 2, \cdots, m,$

$\Delta_n y = (d - c)/n, \; y_j = c + j \, \Delta_n y \quad \text{for} \quad j = 0, 1, 2, \cdots, n,$

and $\underline{B}_{ij}$ and $\bar{B}_{ij}$ are the g.l.b. and l.u.b. of f on

$$R_{ij} = \{(x,y) \mid x_{i-1} \le x \le x_i, \, y_{j-1} \le y \le y_j\}.$$

By using the Law of the Mean (for single integrals) let x_{ij}^* be such that

$$x_{i-1} \le x_{ij}^* \le x_i \quad \text{and} \quad \int_{x_{i-1}}^{x_i} f(x,y_j) \, dx = f(x_{ij}^*, y_j) \, \Delta_m x.$$

Hence

$$F(y_j) = \int_a^b f(x,y_j) \, dx = \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} f(x,y_j) \, dx = \sum_{i=1}^{m} f(x_{ij}^*, y_j) \, \Delta_m x. \tag{12}$$

Since (x_{ij}^*, y_j) is a point of R_{ij}, then $\underline{B}_{ij} \le f(x_{ij}^*, y_j) \le \bar{B}_{ij}$ in which multiplication by $\Delta_m x$ and summation from $i = 1$ to m, together with (12), yields

$$\sum_{i=1}^{m} \underline{B}_{ij} \, \Delta_m x \le F(y_j) \le \sum_{i=1}^{m} \bar{B}_{ij} \, \Delta_m x.$$

Now multiply by $\Delta_n y$ and sum from $j = 1$ to n;

$$\sum_{j=1}^{n} \sum_{i=1}^{m} \underline{B}_{ij} \, \Delta_m x \, \Delta_n y \le \sum_{j=1}^{n} F(y_j) \, \Delta_n y \le \sum_{j=1}^{n} \sum_{i=1}^{m} \bar{B}_{ij} \, \Delta_m x \, \Delta_n y.$$

Hence, from (11),

$$\iint_R f - \epsilon < \sum_{j=1}^{n} F(y_j) \, \Delta_n y < \iint_R f + \epsilon.$$

The outside terms do not depend upon n while the limit as $n \to \infty$ of the middle term exists and is (by the continuity of F on $I[c,d]$) the integral of F from c to d:

$$\iint_R f - \epsilon \le \int_c^d F(y) \, dy \le \iint_R f + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this establishes the first equality of (7).

The existence of the second iterated integral and the second equality of (7) may be established in the same manner. Thus, the theorem is proved.

Uniform continuity is also used in proving the following theorem which is frequently used in more advanced mathematics and applications.

THEOREM A8.2. *Let a function f and its partial derived function f_y both be continuous on the closed rectangle R. Then*

$$D_y \int_a^b f(x,y) \, dx = \int_a^b f_y(x,y) \, dx \quad \text{for} \quad c < y < d. \tag{13}$$

PROOF. Because of continuity, both integrals appearing in (1) exist, but we have to prove the existence of the derivative and the equality.

Let $\epsilon > 0$ be arbitrary. Since the function f_y is continuous on the closed rectangle R, it is uniformly continuous on R and we let $\delta > 0$ (depending upon $\epsilon/(b - a)$ and R) be such that if (u,v) and (s,t) are points of R and

$$\sqrt{(u - s)^2 + (v - t)^2} < \delta \quad \text{then} \quad |f_y(u,v) - f_y(s,t)| < \epsilon/(b - a). \tag{14}$$

Now choose a number y such that $c < y < d$ and select a number h such that

(15) $$0 < |h| < \delta \quad \text{and} \quad c < y + h < d.$$

We next determine a function g defined on $I[a,b]$ as follows: For x such that $a \leq x \leq b$, let $g(x)$ be such that, by the Law of the Mean for derivatives of functions of one variable,

$$g(x) \text{ is between } y \text{ and } y + h \text{ and } f(x,y+h) - f(x,y) = f_y(x,g(x))h.$$

Notice that the distance between the points $(x,g(x))$ and (x,y) is less than δ so that, from (14),

$$-\epsilon/(b-a) < f_y(x,g(x)) - f_y(x,y) < \epsilon/(b-a),$$

and therefore

$$-\frac{\epsilon}{b-a} < \frac{f(x,y+h) - f(x,y)}{h} - f_y(x,y) < \frac{\epsilon}{b-a}.$$

By integrating (with respect to x) from a to b we obtain

$$-\epsilon < \frac{1}{h}\left\{\int_a^b f(x,y+h)\,dx - \int_a^b f(x,y)\,dx\right\} - \int_a^b f_y(x,y)\,dx < \epsilon.$$

Since these inequalities were obtained under the conditions (15), this is the statement that the derivative in (13) exists and the equality holds. Thus, the proof of the theorem is complete.

Notice that, under the conditions of the theorem, the following interchange of integral and limit

$$\lim_{\Delta y \to 0} \int_a^b \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}\,dx = \int_a^b \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y}\,dx$$

is permissible.

A9. Rearrangements of Series. On p. 472 it was shown that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots, \quad \text{for} \quad -1 < x \leq 1.$$

Notice in particular that $x = 1$ is a value for which the series converges and the equality holds. Upon setting $x = 1$ we thus have

(1) $$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots + (-1)^{n+1}\frac{1}{n} + \cdots.$$

For illustrative purposes we rearrange this series to show what seems at first to be a paradox. We use each term of the series (1) once and only once, but rearrange them as

(2) $$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \frac{1}{9} - \cdots.$$

From (1) we have taken the first positive term, then the first two negative terms, then the first remaining positive term, then the first two remaining negative terms, etc. Notice that each term of (1) appears once and only once in (2). Call the sequence of partial sums of (2)

(3) $$s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, \cdots, s_n, \cdots$$

but first concentrate on the subsequence $s_3, s_6, s_9, \cdots, s_{3m}, \cdots$. Notice that

$$s_3 = 1 - \frac{1}{2} - \frac{1}{4}, \quad s_6 = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8}, \quad s_9 = s_6 + \frac{1}{5} - \frac{1}{10} - \frac{1}{12},$$

and

$$\begin{aligned} s_{3m} &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{2m-1} - \frac{1}{4m-2} - \frac{1}{4m} \\ &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots + \left(\frac{1}{2m-1} - \frac{1}{4m-2}\right) - \frac{1}{4m} \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{2(2m-1)} - \frac{1}{4m} \\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2m-1} - \frac{1}{2m}\right). \end{aligned}$$

The parenthetic expression is a partial sum of the series in (1) and hence

$$\lim_{m\to\infty} s_{3m} = \tfrac{1}{2} \ln 2.$$

But s_{3m+1} and s_{3m+2} differ from s_{3m} by either one or two terms each of which approaches 0 as $m \to \infty$. Thus, for the whole sequence (3)

$$\lim_{n\to\infty} s_n = \tfrac{1}{2} \ln 2.$$

Hence, although the series (1) and (2) both have exactly the same terms, the series (2) converges to only one-half the sum of the series (1).

Further discussions of rearrangements of series depend upon the following theorems.

THEOREM A9.1. *Let Σu_n be a convergent series all of whose terms are non-negative (or else all of whose terms are non-positive), let L be the sum of the series, and let ΣU_n be a rearrangement of Σu_n. Then ΣU_n also converges to L.*

PROOF. First consider all terms non-negative. Let n be a positive integer and select a positive integer m such that each of the terms $U_1, U_2, \cdots, U_n$ is somewhere among the terms $u_1, u_2, \cdots, u_m$. Since no term is negative, then

$$\sum_{k=1}^{n} U_k \leq \sum_{k=1}^{m} u_k \leq L.$$

Hence, the partial sums of the U-series form a monotonically increasing sequence bounded above by L. Consequently, the U-series converges and has sum, say M, such that

$$M \leq L.$$

This result may be paraphrased as "Given a convergent series of non-negative terms, then any rearrangement is also convergent with sum $\leq$ the sum of the given series." Now by considering the U-series as given, then the u-series is the rearrangement and hence $L \leq M$. Therefore $M = L$, i.e., both series converge and have the same sum.

If the terms are all non-positive, then apply the above result to obtain $\Sigma(-U_n) = \Sigma(-u_n)$ so $\Sigma U_n = \Sigma u_n$.

For any series Σu_n we shall use the notation u_n^+ and u_n^- defined by

$$u_n^+ = \frac{u_n + |u_n|}{2} = \begin{cases} u_n & \text{if } u_n \geq 0 \\ 0 & \text{if } u_n < 0, \end{cases} \qquad u_n^- = \frac{u_n - |u_n|}{2} = \begin{cases} 0 & \text{if } u_n \geq 0 \\ u_n & \text{if } u_n < 0. \end{cases}$$

THEOREM A9.2.

a. *If Σu_n is absolutely convergent, then Σu_n^+ and Σu_n^- are both convergent and*

$$|\textstyle\sum u_n| \leq |\sum u_n|. \tag{4}$$

b. *If* $\Sigma\, u_n^+$ *and* $\Sigma\, u_n^-$ *are both convergent, then* $\Sigma\, u_n$ *is absolutely convergent.*

c. *If* $\Sigma\, u_n$ *is conditionally convergent, then* $\Sigma\, u_n^+$ *diverges to* $+\infty$, $\Sigma\, u_n^-$ *diverges to* $-\infty$, *while*

$$\lim_{n\to\infty} u_n^+ = 0 \quad \text{and} \quad \lim_{n\to\infty} u_n^- = 0. \tag{5}$$

PROOF of a. By hypothesis $\Sigma\, |u_n|$ converges so $\Sigma\, u_n$ converges (by Test 8) and

$$\tfrac{1}{2}\sum u_n + \tfrac{1}{2}\sum |u_n| = \sum \tfrac{1}{2}u_n + \sum \tfrac{1}{2}|u_n| = \sum \frac{u_n + |u_n|}{2} = \sum u_n^+$$

showing the convergence of $\Sigma\, u_n^+$. Similarly $\Sigma\, u_n^-$ converges. Since all of these series are convergent, the inequality (4) may be obtained as follows:

$$\begin{aligned}|\textstyle\sum u_n| &= |\textstyle\sum [u_n^+ + u_n^-]| = |\textstyle\sum u_n^+ + \sum u_n^-| \le |\textstyle\sum u_n^+| + |\textstyle\sum u_n^-| \\ &= \textstyle\sum u_n^+ - \sum u_n^- = \sum [u_n^+ - u_n^-] = \sum |u_n|.\end{aligned}$$

PROOF of b. By hypothesis $\Sigma\, u_n^+$ and $\Sigma\, u_n^-$ are both convergent so $\Sigma\, [u_n^+ - u_n^-]$ is convergent, but $u_n^+ - u_n^- = |u_n|$ so $\Sigma\, |u_n|$ is convergent; that is, $\Sigma\, u_n$ is absolutely convergent.

PROOF of c. By hypothesis $\Sigma\, u_n$ is convergent but $\Sigma\, |u_n|$ is divergent. If $\Sigma\, u_n^+$ were also convergent, then $\Sigma\, [2u_n^+ - u_n] = \Sigma\, |u_n|$ would be convergent (contrary to hypothesis) and hence $\Sigma\, u_n^+$ is not convergent. Since each $u_n^+ \ge 0$ then $\Sigma\, u_n^+$, being divergent, must diverge to $+\infty$. Similarly $\Sigma\, u_n^-$ may be seen to diverge to $-\infty$. Even though $\Sigma\, u_n^+$ and $\Sigma\, u_n^-$ diverge, the limits (5) hold since $\lim_{n\to\infty} u_n = 0$ from the convergence of $\Sigma\, u_n$.

Relative to rearrangements, absolutely convergent series behave much like series of non-negative (or non-positive) series as shown by:

THEOREM A9.3. *Let* $\Sigma\, u_n$ *be an absolutely convergent series and let* $\Sigma\, U_n$ *be a rearrangement of it. Then* $\Sigma\, U_n$ *is also convergent (even absolutely) and both series have the same sum.*

PROOF. $\Sigma\, U_n^+$ is a rearrangement of $\Sigma\, u_n^+$ and since these are series of non-negative terms then $\Sigma\, U_n^+ = \Sigma\, u_n^+$ by Theorem A9.1. In the same way $\Sigma\, U_n^- = \Sigma\, u_n^-$. Hence, since all are convergent series,

$$\left.\begin{matrix}\sum u_n \\ \sum |u_n|\end{matrix}\right\} = \sum [u_n^+ \pm u_n^-]$$

$$= \sum u_n^+ \pm \sum u_n^- = \sum U_n^+ \pm \sum U_n^- = \sum [U_n^+ \pm U_n^-] = \left\{\begin{matrix}\sum U_n \\ \sum |U_n|.\end{matrix}\right.$$

THEOREM A9.4. *Let* $\Sigma\, u_n$ *be a conditionally convergent series and let* λ *be any number. The given series may be rearranged in such a way that the rearranged series will converge to* λ.

PROOF. 1. From the given series pick enough non-negative terms in the order in which they come so their sum just exceeds λ (i.e., so without the last positive term selected the sum would be less than λ). This is always possible since the series of non-negative terms diverges to $+\infty$ (see Theorem A9.2c). In case λ is zero or negative take one positive term.

2. Pick out enough negative terms beginning at the first so when their sum is added to the first group the sum will be just less than λ. This is possible since the series of negative terms diverges to $-\infty$.

3. Pick out enough non-negative terms, beginning with the first such term not already selected, so the sum of all three groups just exceeds λ.

The process of rearranging the given series should now be clear. Since $\lim_{n\to\infty} u_n = 0$, the partial sums of the rearranged series approach λ.

Table 1. Four Place Logarithms

	Mantissas										Proportional Parts								
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	16	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	4	5	7	9	10	12	14	16
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	12	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	5	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	7	8	9	11	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	2	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	4	5	6	7	8	9	11
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	6	7	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	5	6	7	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	3	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	1	2	3	4	5	6	7	8	9

Table 1. Four Place Logarithms

	Mantissas										Proportional Parts								
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	1	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	3	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	5	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	3	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	3	3	4	4	5	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	3	3	4	4	5	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	4	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1	1	2	2	3	3	4	4	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	3	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	1	2	3	4	5	6	7	8	9

Table 2. Trig and Log Trig

[Subtract 10 from logs = $n.xxxx$ if n = 7, 8, or 9]

Radians	Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log		
.0000	0° 00′	.0000	—	.0000	—	—	—	1.0000	0.0000	90° 00′	1.5708
.0029	10	.0029	7.4637	.0029	7.4637	343.77	2.5363	1.0000	.0000	50	1.5679
.0058	20	.0058	.7648	.0058	.7648	171.89	.2352	1.0000	.0000	40	1.5650
.0087	30	.0087	7.9408	.0087	7.9409	114.59	2.0591	1.0000	.0000	30	1.5621
.0116	40	.0116	8.0658	.0116	8.0658	85.940	1.9342	.9999	.0000	20	1.5592
.0145	50	.0145	.1627	.0145	.1627	68.750	.8373	.9999	0.0000	10	1.5563
.0175	1° 00′	.0175	8.2419	.0175	8.2419	57.290	1.7581	.9998	9.9999	89° 00′	1.5533
.0204	10	.0204	.3088	.0204	.3089	49.104	.6911	.9998	.9999	50	1.5504
.0233	20	.0233	.3668	.0233	.3669	42.964	.6331	.9997	.9999	40	1.5475
.0262	30	.0262	.4179	.0262	.4181	38.188	.5819	.9997	.9999	30	1.5446
.0291	40	.0291	.4637	.0291	.4638	34.368	.5362	.9996	.9998	20	1.5417
.0320	50	.0320	.5050	.0320	.5053	31.242	.4947	.9995	.9998	10	1.5388
.0349	2° 00′	.0349	8.5428	.0349	8.5431	28.636	1.4569	.9994	9.9997	88° 00′	1.5359
.0378	10	.0378	.5776	.0378	.5779	26.432	.4221	.9993	.9997	50	1.5330
.0407	20	.0407	.6097	.0407	.6101	24.542	.3899	.9992	.9996	40	1.5301
.0436	30	.0436	.6397	.0437	.6401	22.904	.3599	.9990	.9996	30	1.5272
.0465	40	.0465	.6677	.0466	.6682	21.470	.3318	.9989	.9995	20	1.5243
.0495	50	.0494	.6940	.0495	.6945	20.206	.3055	.9988	9.9995	10	1.5213
.0524	3° 00′	.0523	8.7188	.0524	8.7194	19.081	1.2806	.9986	.9994	87° 00′	1.5184
.0553	10	.0552	.7423	.0553	.7429	18.075	.2571	.9985	.9993	50	1.5155
.0582	20	.0581	.7645	.0582	.7652	17.169	.2348	.9983	.9993	40	1.5126
.0611	30	.0610	.7857	.0612	.7865	16.350	.2135	.9981	.9992	30	1.5097
.0640	40	.0640	.8059	.0641	.8067	15.605	.1933	.9980	.9991	20	1.5068
.0669	50	.0669	.8251	.0670	.8261	14.924	.1739	.9978	.9990	10	1.5039
.0698	4° 00′	.0698	8.8436	.0699	8.8446	14.301	1.1554	.9976	9.9989	86° 00′	1.5010
.0727	10	.0727	.8613	.0729	.8624	13.727	.1376	.9974	.9989	50	1.4981
.0756	20	.0756	.8783	.0758	.8795	13.197	.1205	.9971	.9988	40	1.4952
.0785	30	.0785	.8946	.0787	.8960	12.706	.1040	.9969	.9987	30	1.4923
.0814	40	.0814	.9104	.0816	.9118	12.251	.0882	.9967	.9986	20	1.4893
.0844	50	.0843	.9256	.0846	.9272	11.826	.0728	.9964	.9985	10	1.4864
.0873	5° 00′	.0872	8.9403	.0875	8.9420	11.430	1.0580	.9962	9.9983	85° 00′	1.4835
.0902	10	.0901	.9545	.0904	.9563	11.059	.0437	.9959	.9982	50	1.4806
.0931	20	.0929	.9682	.0934	.9701	10.712	.0299	.9957	.9981	40	1.4777
.0960	30	.0958	.9816	.0963	.9836	10.385	.0164	.9954	.9980	30	1.4748
.0989	40	.0987	8.9945	.0992	8.9966	10.078	1.0034	.9951	.9979	20	1.4719
.1018	50	.1016	9.0070	.1022	9.0093	9.7882	0.9907	.9948	.9977	10	1.4690
.1047	6° 00′	.1045	9.0192	.1051	9.0216	9.5144	0.9784	.9945	9.9976	84° 00′	1.4661
.1076	10	.1074	.0311	.1080	.0336	9.2553	.9664	.9942	.9975	50	1.4632
.1105	20	.1103	.0426	.1110	.0453	9.0098	.9547	.9939	.9973	40	1.4603
.1134	30	.1132	.0539	.1139	.0567	8.7769	.9433	.9936	.9972	30	1.4573
.1164	40	.1161	.0648	.1169	.0678	8.5555	.9322	.9932	.9971	20	1.4544
.1193	50	.1190	.0755	.1198	.0786	8.3450	.9214	.9929	.9969	10	1.4515
.1222	7° 00′	.1219	9.0859	.1228	9.0891	8.1443	0.9109	.9925	9.9968	83° 00′	1.4486
.1251	10	.1248	.0961	.1257	.0995	7.9530	.9005	.9922	.9966	50	1.4457
.1280	20	.1276	.1060	.1287	.1096	7.7704	.8904	.9918	.9964	40	1.4428
.1309	30	.1305	.1157	.1317	.1194	7.5958	.8806	.9914	.9963	30	1.4399
.1338	40	.1334	.1252	.1346	.1291	7.4287	.8709	.9911	.9961	20	1.4370
.1367	50	.1363	.1345	.1376	.1385	7.2687	.8615	.9907	.9959	10	1.4341
.1396	8° 00′	.1392	9.1436	.1405	9.1478	7.1154	0.8522	.9903	9.9958	82° 00′	1.4312
.1425	10	.1421	.1525	.1435	.1569	6.9682	.8431	.9899	.9956	50	1.4283
.1454	20	.1449	.1612	.1465	.1658	6.8269	.8342	.9894	.9954	40	1.4254
.1484	30	.1478	.1697	.1495	.1745	6.6912	.8255	.9890	.9952	30	1.4224
.1513	40	.1507	.1781	.1524	.1831	6.5606	.8169	.9886	.9950	20	1.4195
.1542	50	.1536	.1863	.1554	.1915	6.4348	.8085	.9881	.9948	10	1.4166
.1571	9° 00′	.1564	9.1943	.1584	9.1997	6.3138	0.8003	.9877	9.9946	81° 00′	1.4137
		Value Cosine	Log Cosine	Value Cotangent	Log Cotangent	Value Tangent	Log Tangent	Value Sine	Log Sine	Degrees	Radians

Table 2. Trig and Log Trig

[Subtract 10 from logs = $n.xxxx$ if n = 7, 8, or 9]

Radians	Degrees	Sine		Tangent		Cotangent		Cosine			
		Value	Log	Value	Log	Value	Log	Value	Log		
.1571	9° 00′	.1564	9.1943	.1584	9.1997	6.3138	0.8003	.9877	9.9946	81° 00′	1.4137
.1600	10	.1593	.2022	.1614	.2078	6.1970	.7922	.9872	.9944	50	1.4108
1629	20	.1622	.2100	.1644	.2158	6.0844	.7842	.9868	.9942	40	1.4079
.1658	30	.1650	.2176	.1673	.2236	5.9758	.7764	.9863	.9940	30	1.4050
.1687	40	.1679	.2251	.1703	.2313	5.8708	.7687	.9858	.9938	20	1.4021
.1716	50	.1708	.2324	.1733	.2389	5.7694	.7611	.9853	.9936	10	1.3992
.1745	10° 00′	.1736	9.2397	.1763	9.2463	5.6713	0.7537	.9848	9.9934	80° 00′	1.3963
.1774	10	.1765	.2468	.1793	.2536	5.5764	.7464	.9843	.9931	50	1.3934
.1804	20	.1794	.2538	.1823	.2609	5.4845	.7391	.9838	.9929	40	1.3904
.1833	30	.1822	.2606	.1853	.2680	5.3955	.7320	.9833	.9927	30	1.3875
.1862	40	.1851	.2674	.1883	.2750	5.3093	.7250	.9827	.9924	20	1.3846
.1891	50	.1880	.2740	.1914	.2819	5.2257	.7181	.9822	.9922	10	1.3817
.1920	11° 00′	.1908	9.2806	.1944	9.2887	5.1446	0.7113	.9816	9.9919	79° 00′	1.3788
.1949	10	.1937	.2870	.1974	.2953	5.0658	.7047	.9811	.9917	50	1.3759
.1978	20	.1965	.2934	.2004	.3020	4.9894	.6980	.9805	.9914	40	1.3730
.2007	30	.1994	.2997	.2035	.3085	4.9152	.6915	.9799	.9912	30	1.3701
.2036	40	.2022	.3058	.2065	.3149	4.8430	.6851	.9793	.9909	20	1.3672
.2065	50	.2051	.3119	.2095	.3212	4.7729	.6788	.9787	.9907	10	1.3643
.2094	12° 00′	.2079	9.3179	.2126	9.3275	4.7046	0.6725	.9781	9.9904	78° 00′	1.3614
.2123	10	.2108	.3238	.2156	.3336	4.6382	.6664	.9775	.9901	50	1.3584
.2153	20	.2136	.3296	.2186	.3397	4.5736	.6603	.9769	.9899	40	1.3555
.2182	30	.2164	.3353	.2217	.3458	4.5107	.6542	.9763	.9896	30	1.3526
.2211	40	.2193	.3410	.2247	.3517	4.4494	.6483	.9757	.9893	20	1.3497
.2240	50	.2221	.3466	.2278	.3576	4.3897	.6424	.9750	.9890	10	1.3468
.2269	13° 00′	.2250	9.3521	.2309	9.3634	4.3315	0.6366	.9744	9.9887	77° 00′	1.3439
.2298	10	.2278	.3575	.2339	.3691	4.2747	.6309	.9737	.9884	50	1.3410
.2327	20	.2306	.3629	.2370	.3748	4.2193	.6252	.9730	.9881	40	1.3381
.2356	30	.2334	.3682	.2401	.3804	4.1653	.6196	.9724	.9878	30	1.3352
.2385	40	.2363	.3734	.2432	.3859	4.1126	.6141	.9717	.9875	20	1.3323
.2414	50	.2391	.3786	.2462	.3914	4.0611	.6086	.9710	.9872	10	1.3294
.2443	14° 00′	.2419	9.3837	.2493	9.3968	4.0108	0.6032	.9703	9.9869	76° 00′	1.3265
.2473	10	.2447	.3887	.2524	.4021	3.9617	.5979	.9696	.9866	50	1.3235
.2502	20	.2476	.3937	.2555	.4074	3.9136	.5926	.9689	.9863	40	1.3206
.2531	30	.2504	.3986	.2586	.4127	3.8667	.5873	.9681	.9859	30	1.3177
.2560	40	.2532	.4035	.2617	.4178	3.8208	.5822	.9674	.9856	20	1.3148
.2589	50	.2560	.4083	.2648	.4230	3.7760	.5770	.9667	.9853	10	1.3119
.2618	15° 00′	.2588	9.4130	.2679	9.4281	3.7321	0.5719	.9659	9.9849	75° 00′	1.3090
.2647	10	.2616	.4177	.2711	.4331	3.6891	.5669	.9652	.9846	50	1.3061
.2676	20	.2644	.4223	.2742	.4381	3.6470	.5619	.9644	.9843	40	1.3032
.2705	30	.2672	.4269	.2773	.4430	3.6059	.5570	.9636	.9839	30	1.3003
.2734	40	.2700	.4314	.2805	.4479	3.5656	.5521	.9628	.9836	20	1.2974
.2763	50	.2728	.4359	.2836	.4527	3.5261	.5473	.9621	.9832	10	1.2945
.2793	16° 00′	.2756	9.4403	.2867	9.4575	3.4874	0.5425	.9613	9.9828	74° 00′	1.2915
.2822	10	.2784	.4447	.2899	.4622	3.4495	.5378	.9605	.9825	50	1.2886
.2851	20	.2812	.4491	.2931	.4669	3.4124	.5331	.9596	.9821	40	1.2857
.2880	30	.2840	.4533	.2962	.4716	3.3759	.5284	.9588	.9817	30	1.2828
.2909	40	.2868	.4576	.2994	.4762	3.3402	.5238	.9580	.9814	20	1.2799
.2938	50	.2896	.4618	.3026	.4808	3.3052	.5192	.9572	.9810	10	1.2770
.2967	17° 00′	.2924	9.4659	.3057	9.4853	3.2709	0.5147	.9563	9.9806	73° 00′	1.2741
.2996	10	.2952	.4700	.3089	.4898	3.2371	.5102	.9555	.9802	50	1.2712
.3025	20	.2979	.4741	.3121	.4943	3.2041	.5057	.9546	.9798	40	1.2683
.3054	30	.3007	.4781	.3153	.4987	3.1716	.5013	.9537	.9794	30	1.2654
.3083	40	.3035	.4821	.3185	.5031	3.1397	.4969	.9528	.9790	20	1.2625
.3113	50	.3062	.4861	.3217	.5075	3.1084	.4925	.9520	.9786	10	1.2595
.3142	18° 00′	.3090	9.4900	.3249	9.5118	3.0777	0.4882	.9511	9.9782	72° 00′	1.2566
		Value	Log	Value	Log	Value	Log	Value	Log	Degrees	Radians
		Cosine		Cotangent		Tangent		Sine			

Table 2. Trig and Log Trig

[Subtract 10 from logs = *n.xxxx* if *n* = 7, 8, or 9]

Radians	Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log		
.3142	18° 00′	.3090	9.4900	.3249	9.5118	3.0777	0.4882	.9511	9.9782	72° 00′	1.2566
.3171	10	.3118	.4939	.3281	.5161	3.0475	.4839	.9502	.9778	50	1.2537
.3200	20	.3145	.4977	.3314	.5203	3.0178	.4797	.9492	.9774	40	1.2508
.3229	30	.3173	.5015	.3346	.5245	2.9887	.4755	.9483	.9770	30	1.2479
.3258	40	.3201	.5052	.3378	.5287	2.9600	.4713	.9474	.9765	20	1.2450
.3287	50	.3228	.5090	.3411	.5329	2.9319	.4671	.9465	.9761	10	1.2421
.3316	19° 00′	.3256	9.5126	.3443	9.5370	2.9042	0.4630	.9455	9.9757	71° 00′	1.2392
.3345	10	.3283	.5163	.3476	.5411	2.8770	.4589	.9446	.9752	50	1.2363
.3374	20	.3311	.5199	.3508	.5451	2.8502	.4549	.9436	.9748	40	1.2334
.3403	30	.3338	.5235	.3541	.5491	2.8239	.4509	.9426	.9743	30	1.2305
.3432	40	.3365	.5270	.3574	.5531	2.7980	.4469	.9417	.9739	20	1.2275
.3462	50	.3393	.5306	.3607	.5571	2.7725	.4429	.9407	.9734	10	1.2246
.3491	20° 00′	.3420	9.5341	.3640	9.5611	2.7475	0.4389	.9397	9.9730	70° 00′	1.2217
.3520	10	.3448	.5375	.3673	.5650	2.7228	.4350	.9387	.9725	50	1.2188
.3549	20	.3475	.5409	.3706	.5689	2.6985	.4311	.9377	.9721	40	1.2159
.3578	30	.3502	.5443	.3739	.5727	2.6746	.4273	.9367	.9716	30	1.2130
.3607	40	.3529	.5477	.3772	.5766	2.6511	.4234	.9356	.9711	20	1.2101
.3636	50	.3557	.5510	.3805	.5804	2.6279	.4196	.9346	.9706	10	1.2072
.3665	21° 00′	.3584	9.5543	.3839	9.5842	2.6051	0.4158	.9336	9.9702	69° 00′	1.2043
.3694	10	.3611	.5576	.3872	.5879	2.5826	.4121	.9325	.9697	50	1.2014
.3723	20	.3638	.5609	.3906	.5917	2.5605	.4083	.9315	.9692	40	1.1985
.3752	30	.3665	.5641	.3939	.5954	2.5386	.4046	.9304	.9687	30	1.1956
.3782	40	.3692	.5673	.3973	.5991	2.5172	.4009	.9293	.9682	20	1.1926
.3811	50	.3719	.5704	.4006	.6028	2.4960	.3972	.9283	.9677	10	1.1897
.3840	22° 00′	.3746	9.5736	.4040	9.6064	2.4751	0.3936	.9272	9.9672	68° 00′	1.1868
.3869	10	.3773	.5767	.4074	.6100	2.4545	.3900	.9261	.9667	50	1.1839
.3898	20	.3800	.5798	.4108	.6136	2.4342	.3864	.9250	.9661	40	1.1810
.3927	30	.3827	.5828	.4142	.6172	2.4142	.3828	.9239	.9656	30	1.1781
.3956	40	.3854	.5859	.4176	.6208	2.3945	.3792	.9228	.9651	20	1.1752
.3985	50	.3881	.5889	.4210	.6243	2.3750	.3757	.9216	.9646	10	1.1723
.4014	23° 00′	.3907	9.5919	.4245	9.6279	2.3559	0.3721	.9205	9.9640	67° 00′	1.1694
.4043	10	.3934	.5948	.4279	.6314	2.3369	.3686	.9194	.9635	50	1.1665
.4072	20	.3961	.5978	.4314	.6348	2.3183	.3652	.9182	.9629	40	1.1636
.4102	30	.3987	.6007	.4348	.6383	2.2998	.3617	.9171	.9624	30	1.1606
.4131	40	.4014	.6036	.4383	.6417	2.2817	.3583	.9159	.9618	20	1.1577
.4160	50	.4041	.6065	.4417	.6452	2.2637	.3548	.9147	.9613	10	1.1548
.4189	24° 00′	.4067	9.6093	.4452	9.6486	2.2460	0.3514	.9135	9.9607	66° 00′	1.1519
.4218	10	.4094	.6121	.4487	.6520	2.2286	.3480	.9124	.9602	50	1.1490
.4247	20	.4120	.6149	.4522	.6553	2.2113	.3447	.9112	.9596	40	1.1461
.4276	30	.4147	.6177	.4557	.6587	2.1943	.3413	.9100	.9590	30	1.1432
.4305	40	.4173	.6205	.4592	.6620	2.1775	.3380	.9088	.9584	20	1.1403
.4334	50	.4200	.6232	.4628	.6654	2.1609	.3346	.9075	.9579	10	1.1374
.4363	25° 00′	.4226	9.6259	.4663	9.6687	2.1445	0.3313	.9063	9.9573	65° 00′	1.1345
.4392	10	.4253	.6286	.4699	.6720	2.1283	.3280	.9051	.9567	50	1.1316
.4422	20	.4279	.6313	.4734	.6752	2.1123	.3248	.9038	9561	40	1.1286
.4451	30	.4305	.6340	.4770	.6785	2.0965	.3215	.9026	.9555	30	1.1257
.4480	40	.4331	.6366	.4806	.6817	2.0809	.3183	.9013	.9549	20	1.1228
.4509	50	.4358	.6392	.4841	.6850	2.0655	.3150	.9001	.9543	10	1.1199
.4538	26° 00′	.4384	9.6418	.4877	9.6882	2.0503	0.3118	.8988	9.9537	64° 00′	1.1170
.4567	10	.4410	.6444	.4913	.6914	2.0353	3086	.8975	.9530	50	1.1141
.4596	20	.4436	.6470	.4950	.6946	2.0204	3054	.8962	.9524	40	1.1112
.4625	30	.4462	.6495	.4986	.6977	2.0057	.3023	.8949	.9518	30	1.1083
.4654	40	.4488	.6521	.5022	.7009	1.9912	.2991	.8936	.9512	20	1.1054
.4683	50	.4514	.6546	.5059	.7040	1.9768	.2960	.8923	.9505	10	1.1025
.4712	27° 00′	.4540	9.6570	.5095	9.7072	1.9626	0.2928	.8910	9.9499	63° 00′	1.0996
		Value Cosine	Log Cosine	Value Cotangent	Log Cotangent	Value Tangent	Log Tangent	Value Sine	Log Sine	Degrees	Radians

Table 2. Trig and Log Trig

[Subtract 10 from logs = $n.xxxx$ if n = 7, 8, or 9]

Radians	Degrees	Sine Value	Sine Log	Tangent Value	Tangent Log	Cotangent Value	Cotangent Log	Cosine Value	Cosine Log		
.4712	27° 00′	.4540	9.6570	.5095	9.7072	1.9626	0.2928	.8910	9.9499	63° 00′	1.0996
.4741	10	.4566	.6595	.5132	.7103	1.9486	.2897	.8897	.9492	50	1.0966
.4771	20	.4592	.6620	.5169	.7134	1.9347	.2866	.8884	.9486	40	1.0937
.4800	30	.4617	.6644	.5206	.7165	1.9210	.2835	.8870	.9479	30	1.0908
.4829	40	.4643	.6668	.5243	.7196	1.9074	.2804	.8857	.9473	20	1.0879
.4858	50	.4669	.6692	.5280	.7226	1.8940	.2774	.8843	.9466	10	1.0850
.4887	28° 00′	.4695	9.6716	.5317	9.7257	1.8807	0.2743	.8829	9.9459	62° 00′	1.0821
.4916	10	.4720	.6740	.5354	.7287	1.8676	.2713	.8816	.9453	50	1.0792
.4945	20	.4746	.6763	.5392	.7317	1.8546	.2683	.8802	.9446	40	1.0763
.4974	30	.4772	.6787	.5430	.7348	1.8418	.2652	.8788	.9439	30	1.0734
.5003	40	.4797	.6810	.5467	.7378	1.8291	.2622	.8774	.9432	20	1.0705
.5032	50	.4823	.6833	.5505	.7408	1.8165	.2592	.8760	.9425	10	1.0676
.5061	29° 00′	.4848	9.6856	.5543	9.7438	1.8040	0.2562	.8746	9.9418	61° 00′	1.0647
.5091	10	.4874	.6878	.5581	.7467	1.7917	.2533	.8732	.9411	50	1.0617
.5120	20	.4899	.6901	.5619	.7497	1.7796	.2503	.8718	.9404	40	1.0588
.5149	30	.4924	.6923	.5658	.7526	1.7675	.2474	.8704	.9397	30	1.0559
.5178	40	.4950	.6946	.5696	.7556	1.7556	.2444	.8689	.9390	20	1.0530
.5207	50	.4975	.6968	.5735	.7585	1.7437	.2415	.8675	.9383	10	1.0501
.5236	30° 00′	.5000	9.6990	.5774	9.7614	1.7321	0.2386	.8660	9.9375	60° 00′	1.0472
.5265	10	.5025	.7012	.5812	.7644	1.7205	.2356	.8646	.9368	50	1.0443
.5294	20	.5050	.7033	.5851	.7673	1.7090	.2327	.8631	.9361	40	1.0414
.5323	30	.5075	.7055	.5890	.7701	1.6977	.2299	.8616	.9353	30	1.0385
.5352	40	.5100	.7076	.5930	.7730	1.6864	.2270	.8601	.9346	20	1.0356
.5381	50	.5125	.7097	.5969	.7759	1.6753	.2241	.8587	.9338	10	1.0327
.5411	31° 00′	.5150	9.7118	.6009	9.7788	1.6643	0.2212	.8572	9.9331	59° 00′	1.0297
.5440	10	.5175	.7139	.6048	.7816	1.6534	.2184	.8557	.9323	50	1.0268
.5469	20	.5200	.7160	.6088	.7845	1.6426	.2155	.8542	.9315	40	1.0239
.5498	30	.5225	.7181	.6128	.7873	1.6319	.2127	.8526	.9308	30	1.0210
.5527	40	.5250	.7201	.6168	.7902	1.6212	.2098	.8511	.9300	20	1.0181
.5556	50	.5275	.7222	.6208	.7930	1.6107	.2070	.8496	.9292	10	1.0152
.5585	32° 00′	.5299	9.7242	.6249	9.7958	1.6003	0.2042	.8480	9.9284	58° 00′	1.0123
.5614	10	.5324	.7262	.6289	.7986	1.5900	.2014	.8465	.9276	50	1.0094
.5643	20	.5348	.7282	.6330	.8014	1.5798	.1986	.8450	.9268	40	1.0065
.5672	30	.5373	.7302	.6371	.8042	1.5697	.1958	.8434	.9260	30	1.0036
.5701	40	.5398	.7322	.6412	.8070	1.5597	.1930	.8418	.9252	20	1.0007
.5730	50	.5422	.7342	.6453	.8097	1.5497	.1903	.8403	.9244	10	.9977
.5760	33° 00′	.5446	9.7361	.6494	9.8125	1.5399	0.1875	.8387	9.9236	57° 00′	.9948
.5789	10	.5471	.7380	.6536	.8153	1.5301	.1847	.8371	.9228	50	.9919
.5818	20	.5495	.7400	.6577	.8180	1.5204	.1820	.8355	.9219	40	.9890
.5847	30	.5519	.7419	.6619	.8208	1.5108	.1792	.8339	.9211	30	.9861
.5876	40	.5544	.7438	.6661	.8235	1.5013	.1765	.8323	.9203	20	.9832
.5905	50	.5568	.7457	.6703	.8263	1.4919	.1737	.8307	.9194	10	.9803
.5934	34° 00′	.5592	9.7476	.6745	9.8290	1.4826	0.1710	.8290	9.9186	56° 00′	.9774
.5963	10	.5616	.7494	.6787	.8317	1.4733	.1683	.8274	.9177	50	.9745
.5992	20	.5640	.7513	.6830	.8344	1.4641	.1656	.8258	.9169	40	.9716
.6021	30	.5664	.7531	.6873	.8371	1.4550	.1629	.8241	.9160	30	.9687
.6050	40	.5688	.7550	.6916	.8398	1.4460	.1602	.8225	.9151	20	.9657
.6080	50	.5712	.7568	.6959	.8425	1.4370	.1575	.8208	.9142	10	.9628
.6109	35° 00′	.5736	9.7586	.7002	9.8452	1.4281	0.1548	.8192	9.9134	55° 00′	.9599
.6138	10	.5760	.7604	.7046	.8479	1.4193	.1521	.8175	.9125	50	.9570
.6167	20	.5783	.7622	.7089	.8506	1.4106	.1494	.8158	.9116	40	.9541
.6196	30	.5807	.7640	.7133	.8533	1.4019	.1467	.8141	.9107	30	.9512
.6225	40	.5831	.7657	.7177	.8559	1.3934	.1441	.8124	.9098	20	.9483
.6254	50	.5854	.7675	.7221	.8586	1.3848	.1414	.8107	.9089	10	.9454
.6283	36° 00′	.5878	9.7692	.7265	9.8613	1.3764	0.1387	.8090	9.9080	54° 00′	.9425
		Value Cosine	Log Cosine	Value Cotangent	Log Cotangent	Value Tangent	Log Tangent	Value Sine	Log Sine	Degrees	Radians

Table 2. Trig and Log Trig

[Subtract 10 from logs = *n.xxxx* if *n* = 7, 8, or 9]

Radians	Degrees	Sine		Tangent		Cotangent		Cosine			
		Value	Log	Value	Log	Value	Log	Value	Log		
.6283	36° 00′	.5878	9.7692	.7265	9.8613	1.3764	0.1387	.8090	9.9080	54° 00′	.9425
.6312	10	.5901	.7710	.7310	.8639	1.3680	.1361	.8073	.9070	50	.9396
.6341	20	.5925	.7727	.7355	.8666	1.3597	.1334	.8056	.9061	40	.9367
.6370	30	.5948	.7744	.7400	.8692	1.3514	.1308	.8039	.9052	30	.9338
.6400	40	.5972	.7761	.7445	.8718	1.3432	.1282	.8021	.9042	20	.9308
.6429	50	.5995	.7778	.7490	.8745	1.3351	.1255	.8004	.9033	10	.9279
.6458	37° 00′	.6018	9.7795	.7536	9.8771	1.3270	0.1229	.7986	9.9023	53° 00′	.9250
.6487	10	.6041	.7811	.7581	.8797	1.3190	.1203	.7969	.9014	50	.9221
.6516	20	.6065	.7828	.7627	.8824	1.3111	.1176	.7951	.9004	40	.9192
.6545	30	.6088	.7844	.7673	.8850	1.3032	.1150	.7934	.8995	30	.9163
.6574	40	.6111	.7861	.7720	.8876	1.2954	.1124	.7916	.8985	20	.9134
.6603	50	.6134	.7877	.7766	.8902	1.2876	.1098	.7898	.8975	10	.9105
.6632	38° 00′	.6157	9.7893	.7813	9.8928	1.2799	0.1072	.7880	9.8965	52° 00′	.9076
.6661	10	.6180	.7910	.7860	.8954	1.2723	.1046	.7862	.8955	50	.9047
.6690	20	.6202	.7926	.7907	.8980	1.2647	.1020	.7844	.8945	40	.9018
.6720	30	.6225	.7941	.7954	.9006	1.2572	.0994	.7826	.8935	30	.8988
.6749	40	.6248	.7957	.8002	.9032	1.2497	.0968	.7808	.8925	20	.8959
.6778	50	.6271	.7973	.8050	.9058	1.2423	.0942	.7790	.8915	10	.8930
.6807	39° 00′	.6293	9.7989	.8098	9.9084	1.2349	0.0916	.7771	9.8905	51° 00′	.8901
.6836	10	.6316	.8004	.8146	.9110	1.2276	.0890	.7753	.8895	50	.8872
.6865	20	.6338	.8020	.8195	.9135	1.2203	.0865	.7735	.8884	40	.8843
.6894	30	.6361	.8035	.8243	.9161	1.2131	.0839	.7716	.8874	30	.8814
.6923	40	.6383	.8050	.8292	.9187	1.2059	.0813	.7698	.8864	20	.8785
.6952	50	.6406	.8066	.8342	.9212	1.1988	.0788	.7679	.8853	10	.8756
.6981	40° 00′	.6428	9.8081	.8391	9.9238	1.1918	0.0762	.7660	9.8843	50° 00′	.8727
.7010	10	.6450	.8096	.8441	.9264	1.1847	.0736	.7642	.8832	50	.8698
.7039	20	.6472	.8111	.8491	.9289	1.1778	.0711	.7623	.8821	40	.8668
.7069	30	.6494	.8125	.8541	.9315	1.1708	.0685	.7604	.8810	30	.8639
.7098	40	.6517	.8140	.8591	.9341	1.1640	.0659	.7585	.8800	20	.8610
.7127	50	.6539	.8155	.8642	.9366	1.1571	.0634	.7566	.8789	10	.8581
.7156	41° 00′	.6561	9.8169	.8693	9.9392	1.1504	0.0608	.7547	9.8778	49° 00′	.8552
.7185	10	.6583	.8184	.8744	.9417	1.1436	.0583	.7528	.8767	50	.8523
.7214	20	.6604	.8198	.8796	.9443	1.1369	.0557	.7509	.8756	40	.8494
.7243	30	.6626	.8213	.8847	.9468	1.1303	.0532	.7490	.8745	30	.8465
.7272	40	.6648	.8227	.8899	.9494	1.1237	.0506	.7470	.8733	20	.8436
.7301	50	.6670	.8241	.8952	.9519	1.1171	.0481	.7451	.8722	10	.8407
.7330	42° 00′	.6691	9.8255	.9004	9.9544	1.1106	0.0456	.7431	9.8711	48° 00′	.8378
.7359	10	.6713	.8269	.9057	.9570	1.1041	.0430	.7412	.8699	50	.8348
.7389	20	.6734	.8283	.9110	.9595	1.0977	.0405	.7392	.8688	40	.8319
.7418	30	.6756	.8297	.9163	.9621	1.0913	.0379	.7373	.8676	30	.8290
.7447	40	.6777	.8311	.9217	.9646	1.0850	.0354	.7353	.8665	20	.8261
.7476	50	.6799	.8324	.9271	.9671	1.0786	.0329	.7333	.8653	10	.8232
.7505	43° 00′	.6820	9.8338	.9325	9.9697	1.0724	0.0303	.7314	9.8641	47° 00′	.8203
.7534	10	.6841	.8351	.9380	.9722	1.0661	.0278	.7294	.8629	50	.8174
.7563	20	.6862	.8365	.9435	.9747	1.0599	.0253	.7274	.8618	40	.8145
.7592	30	.6884	.8378	.9490	.9772	1.0538	.0228	.7254	.8606	30	.8116
.7621	40	.6905	.8391	.9545	.9798	1.0477	.0202	.7234	.8594	20	.8087
.7650	50	.6926	.8405	.9601	.9823	1.0416	.0177	.7214	.8582	10	.8058
.7679	44° 00′	.6947	9.8418	.9657	9.9848	1.0355	0.0152	.7193	9.8569	46° 00′	.8029
.7709	10	.6967	.8431	.9713	.9874	1.0295	.0126	.7173	.8557	50	.7999
.7738	20	.6988	.8444	.9770	.9899	1.0235	.0101	.7153	.8545	40	.7970
.7767	30	.7009	.8457	.9827	.9924	1.0176	.0076	.7133	.8532	30	.7941
.7796	40	.7030	.8469	.9884	.9949	1.0117	.0051	.7112	.8520	20	.7912
.7825	50	.7050	.8482	.9942	9.9975	1.0058	0.0025	.7092	.8507	10	.7883
.7854	45° 00′	.7071	9.8495	1.0000	0.0000	1.0000	0.0000	.7071	9.8495	45° 00′	.7854
		Value	Log	Value	Log	Value	Log	Value	Log	Degrees	Radians
		Cosine		Cotangent		Tangent		Sine			

Table 3. Exponential and Hyperbolic Functions

		e^x		e^{-x}		sinh x		cosh x	
x	ln x	Value	log	Value	log	Value	log	Value	log
0.0		1.000	0.000	1.000	0.000	0.000		1.000	0
0.1	−2.303	1.105	0.043	0.905	9.957	0.100	9.001	1.005	0.002
0.2	−1.610	1.221	0.087	0.819	9.913	0.201	9.304	1.020	0.009
0.3	−1.204	1.350	0.130	0.741	9.870	0.305	9.484	1.045	0.019
0.4	−0.916	1.492	0.174	0.670	9.826	0.411	9.614	1.081	0.034
0.5	−0.693	1.649	0.217	0.607	9.783	0.521	9.717	1.128	0.052
0.6	−0.511	1.822	0.261	0.549	9.739	0.637	9.804	1.185	0.074
0.7	−0.357	2.014	0.304	0.497	9.696	0.759	9.880	1.255	0.099
0.8	−0.223	2.226	0.347	0.449	9.653	0.888	9.948	1.337	0.126
0.9	−0.105	2.460	0.391	0.407	9.609	1.027	0.011	1.433	0.156
1.0	0.000	2.718	0.434	0.368	9.566	1.175	0.070	1.543	0.188
1.1	0.095	3.004	0.478	0.333	9.522	1.336	0.126	1.669	0.222
1.2	0.182	3.320	0.521	0.301	9.479	1.509	0.179	1.811	0.258
1.3	0.262	3.669	0.565	0.273	9.435	1.698	0.230	1.971	0.295
1.4	0.336	4.055	0.608	0.247	9.392	1.904	0.280	2.151	0.333
1.5	0.405	4.482	0.651	0.223	9.349	2.129	0.328	2.352	0.372
1.6	0.470	4.953	0.695	0.202	9.305	2.376	0.376	2.577	0.411
1.7	0.531	5.474	0.738	0.183	9.262	2.646	0.423	2.828	0.452
1.8	0.588	6.050	0.782	0.165	9.218	2.942	0.469	3.107	0.492
1.9	0.642	6.686	0.825	0.150	9.175	3.268	0.514	3.418	0.534
2.0	0.693	7.389	0.869	0.135	9.131	3.627	0.560	3.762	0.575
2.1	0.742	8.166	0.912	0.122	9.088	4.022	0.604	4.144	0.617
2.2	0.788	9.025	0.955	0.111	9.045	4.457	0.649	4.568	0.660
2.3	0.833	9.974	0.999	0.100	9.001	4.937	0.690	5.037	0.702
2.4	0.875	11.02	1.023	0.091	8.958	5.466	0.738	5.557	0.745
2.5	0.916	12.18	1.086	0.082	8.914	6.050	0.782	6.132	0.788
2.6	0.956	13.46	1.129	0.074	8.871	6.695	0.826	6.769	0.831
2.7	0.993	14.88	1.173	0.067	8.827	7.406	0.870	7.473	0.874
2.8	1.030	16.44	1.216	0.061	8.784	8.192	0.913	8.253	0.917
2.9	1.065	18.17	1.259	0.055	8.741	9.060	0.957	9.115	0.960
3.0	1.099	20.09	1.303	0.050	8.697	10.018	1.001	10.068	1.003
3.5	1.253	33.12	1.520	0.030	8.480	16.543	1.219	16.573	1.219
4.0	1.386	54.60	1.737	0.018	8.263	27.290	1.436	27.308	1.436
4.5	1.504	90.02	1.954	0.011	8.046	45.003	1.653	45.014	1.653
5.0	1.609	148.4	2.171	0.007	7.829	74.203	1.870	74.210	1.870
6.0	1.792	403.4	2.606	0.002	7.394	201.7	2.305	201.7	2.305
7.0	1.946	1096.6	3.040	0.001	6.960	548.3	2.739	548.3	2.739
8.0	2.079	2981.0	3.474	0.000	6.526	1490.5	3.173	1490.5	3.173
9.0	2.197	8103.1	3.909	0.000	6.091	4051.5	3.608	4051.5	3.608
10.0	2.303	22026.	4.343	0.000	5.657	11013.	4.041	11013.	4.041

Table 4. Constants

π =	3.14159	26535	89793	23846	26433	83280
e =	2.71828	18284	59045	23536	02874	71353
log e =	0.43429	44819	03251	82765	11289	18917
ln 10 =	2.30258	50929	94045	68401	79914	54684
log π =	0.49714	98726	94133	85435	12682	88291
log log e =	9.63778	43113	00536	78912	−10	

Table 5. Indefinite Integrals

Note: In this table x, X, u, v, and w are variables which may be functions. All other letters are constants. Constants of integration have been omitted.

RATIONAL ALGEBRAIC INTEGRALS

1. $\int au(x)\,dx = a\int u(x)\,dx$

2. $\int [u(x) + v(x) - w(x)]\,dx = \int u(x)\,dx + \int v(x)\,dx - \int w(x)\,dx$

3. $\int u(x)D_x v(x)\,dx = u(x)v(x) - \int v(x)D_x u(x)\,dx, \quad \int u\,dv = uv - \int v\,du$

4_1., 4_2.
$$\int u^p\,du = \begin{cases} \dfrac{u^{p+1}}{p+1} & \text{if } p \neq -1 \\ \ln|u| & \text{if } p = -1 \end{cases}$$

5_1., 5_2.
$$\int (au+b)^p\,du = \begin{cases} \dfrac{(au+b)^{p+1}}{a(p+1)} & \text{if } p \neq -1 \\ \dfrac{1}{a}\ln|au+b| & \text{if } p = -1 \end{cases}$$

6. $\displaystyle\int \frac{u\,du}{au+b} = \frac{1}{a^2}\{(au+b) - b\ln|au+b|\}$

7. $\displaystyle\int \frac{u^2\,du}{au+b} = \frac{1}{a^3}\left\{\frac{(au+b)^2}{2} - 2b(au+b) + b^2\ln|au+b|\right\}$

8. $\displaystyle\int \frac{u\,du}{(au+b)^2} = \frac{1}{a^2}\left\{\frac{b}{au+b} + \ln|au+b|\right\}$

9. $\displaystyle\int \frac{u^2\,du}{(au+b)^2} = \frac{1}{a^3}\left\{au+b - \frac{b^2}{au+b} - 2b\ln|au+b|\right\}$

10. $\displaystyle\int \frac{du}{u(au+b)} = \frac{1}{b}\ln\left|\frac{u}{au+b}\right|$

11. $\displaystyle\int \frac{du}{u^2(au+b)} = -\frac{1}{bu} + \frac{a}{b^2}\ln\left|\frac{au+b}{u}\right|$

12. $\displaystyle\int \frac{du}{u(au+b)^2} = \frac{1}{b(au+b)} + \frac{1}{b^2}\ln\left|\frac{u}{au+b}\right|$

13. $\displaystyle\int \frac{du}{u^2(au+b)^2} = -\frac{2au+b}{b^2u(au+b)} + \frac{2a}{b^3}\ln\left|\frac{au+b}{u}\right|$

14. $\displaystyle\int x^m(ax+b)^n\,dx, \int \frac{(ax+b)^n}{x^m}\,dx$, n a positive integer. $\left\{\text{Expand } (ax+b)^n \text{ by the binomial theorem.}\right.$

15. $\displaystyle\int x^m(ax+b)\,dx, \int \frac{x^m}{(ax+b)^n}\,dx, \begin{cases} m \text{ a positive integer} \\ n \text{ not an integer} \end{cases}$

Substitute $ax + b = u$, $x = \dfrac{u-b}{a}$, $dx = \dfrac{1}{a}\,du$.

16. $\displaystyle\int \frac{dx}{x^m(ax+b)^n} = \frac{-1}{b^{m+n-1}} \int \frac{(u-a)^{m+n-2}}{u^n}\,du, \quad \text{where} \quad u = \frac{ax+b}{x}$

17. $\displaystyle\int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a}$ 18. $\displaystyle\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|$

19. $\displaystyle\int \frac{du}{au^2+b} = \frac{1}{a}\int \frac{du}{u^2+(b/a)}$ Use 17 or 18.

20. $\displaystyle\int \frac{du}{(au^2+b)^p} = \frac{u}{2(p-1)b(au^2+b)^{p-1}} + \frac{2p-3}{2(p-1)b}\int \frac{du}{(au^2+b)^{p-1}} \quad \text{if} \quad p \neq 1$

21. $\displaystyle\int \frac{u\,du}{au^2+b} = \frac{1}{2a}\ln|au^2+b|$

22. $\displaystyle\int \frac{u\,du}{(au^2+b)^p} = -\frac{1}{2(p-1)a(au^2+b)^{p-1}} \quad \text{if} \quad p \neq 1$

23. $\displaystyle\int \frac{u^2\,du}{au^2+b} = \frac{u}{a} - \frac{b}{a}\int \frac{du}{au^2+b}$

24. $\displaystyle\int \frac{u^2\,du}{(au^2+b)^p} = -\frac{u}{2(p-1)a(au^2+b)^{p-1}} + \frac{1}{2(p-1)a}\int \frac{du}{(au^2+b)^{p-1}} \quad \text{if} \quad p \neq 1$

25. $\displaystyle\int \frac{du}{u^2(au^2+b)} = -\frac{1}{bu} - \frac{a}{b}\int \frac{du}{au^2+b}$

26. $\displaystyle\int \frac{du}{u^2(au^2+b)^p} = \frac{1}{b}\int \frac{du}{u^2(au^2+b)^{p-1}} - \frac{a}{b}\int \frac{du}{(au^2+b)^p}$

27. $\displaystyle\int \frac{du}{au^2+b} = \frac{k}{3b}\left\{\sqrt{3}\tan^{-1}\frac{2u-k}{k\sqrt{3}} + \ln\left|\frac{k+u}{\sqrt{k^2-ku+u^2}}\right|\right\}$

28. $\displaystyle\int \frac{u\,du}{au^2+b} = \frac{1}{3ak}\left\{\sqrt{3}\tan^{-1}\frac{2u-k}{k\sqrt{3}} - \ln\left|\frac{k+u}{\sqrt{k^2-ku+u^2}}\right|\right\}$

(27, 28) where $k = \sqrt[2]{\dfrac{b}{a}}$

29. $\displaystyle\int \frac{du}{u(au^p+b)} = \frac{1}{bp}\ln\left|\frac{u^p}{au^p+b}\right|$

INTEGRALS INVOLVING $\sqrt{au+b}$

$\displaystyle\int \sqrt{au+b}\,du, \qquad \int \frac{du}{\sqrt{au+b}}$

$\displaystyle\int (au+b)^n\sqrt{au+b}\,du, \qquad \int \frac{du}{(au+b)\sqrt{au+b}}$

These may be integrated by Formula 5_1.

30. $\displaystyle\int u\sqrt{au+b}\,du = \frac{2(3au-2b)\sqrt{(au+b)^3}}{15a^2}$

31. $\displaystyle\int u^2\sqrt{au+b}\,du = \frac{2(15a^2u^2-12abu+8b^2)\sqrt{(au+b)^3}}{105a^3}$

32. $\displaystyle\int u^n\sqrt{au+b}\,du = \frac{2}{a(2n+3)}\left\{u^n\sqrt{(au+b)^3} - nb\int u^{n-1}\sqrt{au+b}\,du\right\}$

if $2n+3 \neq 0$

$$\int \frac{\sqrt{au+b}}{u}\,du = \begin{cases} 2\sqrt{au+b} + \sqrt{b}\ln\left|\dfrac{\sqrt{au+b}-\sqrt{b}}{\sqrt{au+b}+\sqrt{b}}\right| & \text{if } b>0 \quad (33_1) \\[2ex] 2\sqrt{au+b} - 2\sqrt{-b}\tan^{-1}\sqrt{\dfrac{au+b}{-b}} & \text{if } b<0 \quad (33_2) \end{cases}$$

Use Formula 4_1 if $b = 0$.

34. $$\int \frac{\sqrt{au+b}}{u^n}\,du = -\frac{1}{(n-1)b}\left\{\frac{\sqrt{(au+b)^3}}{u^{n-1}} + \frac{(2n-5)a}{2}\int \frac{\sqrt{au+b}}{u^{n-1}}\,du\right\} \quad \text{if } n \neq 1$$

35. $$\int \frac{u\,du}{\sqrt{au+b}} = \frac{2(au-2b)}{3a^2}\sqrt{au+b}$$

36. $$\int \frac{u^2\,du}{\sqrt{au+b}} = \frac{2(3a^2u^2-4abu+8b^2)}{15a^3}\sqrt{au+b}$$

37. $$\int \frac{u^n\,du}{\sqrt{au+b}} = \frac{2}{a(2n+1)}\left\{u^n\sqrt{au+b} - nb\int \frac{u^{n-1}\,du}{\sqrt{au+b}}\right\} \quad \text{if } 2n+1 \neq 0$$

$$\int \frac{du}{u\sqrt{au+b}} = \begin{cases} \dfrac{1}{\sqrt{b}}\ln\left|\dfrac{\sqrt{au+b}-\sqrt{b}}{\sqrt{au+b}+\sqrt{b}}\right| & \text{if } b>0 \quad (38_1) \\[2ex] \dfrac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\dfrac{au+b}{-b}} & \text{if } b<0 \quad (38_2) \end{cases}$$

39. $$\int \frac{du}{u^n\sqrt{au+b}} = -\frac{\sqrt{au+b}}{(n-1)bu^{n-1}} - \frac{(2n-3)a}{(2n-2)b}\int \frac{du}{u^{n-1}\sqrt{au+b}} \quad \text{if } n \neq 1$$

INTEGRALS INVOLVING $\sqrt{u^2 \pm a^2}$, $\sqrt{a^2-u^2}$, AND $\sqrt{2au \pm u^2}$

(These integrals are important special cases of those in the next section.)

40. $$\int \sqrt{u^2 \pm a^2}\,du = \tfrac{1}{2}\{u\sqrt{u^2 \pm a^2} \pm a^2\ln|u+\sqrt{u^2 \pm a^2}|\}$$

41. $$\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\left\{u\sqrt{a^2-u^2} + a^2\sin^{-1}\frac{u}{a}\right\}$$

42. $$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln|u+\sqrt{u^2 \pm a^2}|$$

43. $$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a}$$

44. $$\int u(u^2 \pm a^2)^{n/2}\,du = \frac{(u^2 \pm a^2)^{(n/2)+1}}{n+2} \quad \text{if } n \neq -2$$

45. $$\int u(a^2-u^2)^{n/2}\,du = -\frac{(a^2-u^2)^{(n/2)+1}}{n+2} \quad \text{if } n \neq -2$$

46. $$\int u^2\sqrt{u^2 \pm a^2}\,du = \frac{u}{4}\sqrt{(u^2 \pm a^2)^3} \mp \frac{a^2}{8}\{u\sqrt{u^2 \pm a^2} \pm a^2\ln|u+\sqrt{u^2 \pm a^2}|\}$$

47. $$\int u^2\sqrt{a^2-u^2}\,du = -\frac{u}{4}\sqrt{(a^2-u^2)^3} + \frac{a^2}{8}\left\{u\sqrt{a^2-u^2} + a^2\sin^{-1}\frac{u}{a}\right\}$$

48. $\displaystyle\int \frac{\sqrt{a^2 \pm u^2}}{u}\,du = \sqrt{a^2 \pm u^2} - a \ln\left|\frac{a + \sqrt{a^2 \pm u^2}}{u}\right|$

49. $\displaystyle\int \frac{\sqrt{u^2 - a^2}}{u}\,du = \sqrt{u^2 - a^2} - a \cos^{-1}\frac{a}{u}$

50. $\displaystyle\int \frac{\sqrt{u^2 \pm a^2}}{u^2}\,du = -\frac{\sqrt{u^2 \pm a^2}}{u} + \ln|u + \sqrt{u^2 \pm a^2}|$

51. $\displaystyle\int \frac{\sqrt{a^2 - u^2}}{u^2}\,du = -\frac{\sqrt{a^2 - u^2}}{u} - \sin^{-1}\frac{u}{a}$

52. $\displaystyle\int \frac{u\,du}{\sqrt{u^2 \pm a^2}} = \sqrt{u^2 \pm a^2}$ 53. $\displaystyle\int \frac{u\,du}{\sqrt{a^2 - u^2}} = -\sqrt{a^2 - u^2}$

54. $\displaystyle\int \frac{u^2\,du}{\sqrt{u^2 \pm a^2}} = \frac{u}{2}\sqrt{u^2 \pm a^2} \mp \frac{a^2}{2}\ln|u + \sqrt{u^2 \pm a^2}|$

55. $\displaystyle\int \frac{u^2\,du}{\sqrt{a^2 - u^2}} = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}$

56. $\displaystyle\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 \pm u^2}}{u}\right|$ 57. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\cos^{-1}\frac{a}{u}$

58. $\displaystyle\int \frac{du}{u^2\sqrt{u^2 \pm a^2}} = \mp\frac{\sqrt{u^2 \pm a^2}}{a^2 u}$ 59. $\displaystyle\int \frac{du}{u^2\sqrt{a^2 - u^2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u}$

60. $\displaystyle\int \sqrt{(u^2 \pm a^2)^3}\,du = \frac{1}{4}\left\{u\sqrt{(u^2 \pm a^2)^3} \pm \frac{3a^2u}{2}\sqrt{u^2 \pm a^2} + \frac{3a^4}{2}\ln|u + \sqrt{u^2 \pm a^2}|\right\}$

61. $\displaystyle\int \sqrt{(a^2 - u^2)^3}\,du = \frac{1}{4}\left\{u\sqrt{(a^2 - u^2)^3} + \frac{3a^2u}{2}\sqrt{a^2 - u^2} + \frac{3a^4}{2}\sin^{-1}\frac{u}{a}\right\}$

62. $\displaystyle\int \frac{du}{\sqrt{(u^2 \pm a^2)^3}} = \frac{\pm u}{a^2\sqrt{u^2 \pm a^2}}$ 63. $\displaystyle\int \frac{du}{\sqrt{(a^2 - u^2)^3}} = \frac{u}{a^2\sqrt{a^2 - u^2}}$

64. $\displaystyle\int \sqrt{2au - u^2}\,du = \frac{u - a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u - a}{a}\right)$

65. $\displaystyle\int u^n\sqrt{2au - u^2}\,du = \frac{-u^{n-1}(2au - u^2)^{2/2}}{n + 2} + \frac{(2n + 1)a}{n + 2}\int u^{n-1}\sqrt{2au - u^2}\,du$ if $n \neq -2$

66. $\displaystyle\int \frac{\sqrt{2au - u^2}}{u^n}\,du = \frac{(2au - u^2)^{2/2}}{(3 - 2n)au^n} + \frac{n - 3}{(2n - 3)a}\int \frac{\sqrt{2au - u^2}}{u^{n-1}}\,du$ if $n \neq \frac{3}{2}$

67. $\displaystyle\int \frac{du}{\sqrt{2au - u^2}} = 2\sin^{-1}\sqrt{\frac{u}{2a}} = \cos^{-1}\left(\frac{a - u}{a}\right)$

68. $\displaystyle\int \frac{u^n\,du}{\sqrt{2au - u^2}} = \frac{-u^{n-1}\sqrt{2au - u^2}}{n} + \frac{a(2n - 1)}{n}\int \frac{u^{n-1}}{\sqrt{2au - u^2}}\,du$ if $n \neq 0$

69. $$\int \frac{du}{u^n\sqrt{2au-u^2}} = \frac{\sqrt{2au-u^2}}{a(1-2n)u^n} + \frac{n-1}{(2n-1)a}\int \frac{du}{u^{n-1}\sqrt{2au-u^2}} \quad \text{if} \quad n \neq \tfrac{1}{2}$$

70. $$\int \frac{du}{(2au-u^2)^{3/2}} = \frac{u-a}{a^2\sqrt{2au-u^2}}$$

71. $$\int \frac{u\,du}{(2au-u^2)^{3/2}} = \frac{u}{a\sqrt{2au-u^2}}$$

72. $$\int \frac{du}{\sqrt{2au+u^2}} = \ln|u+a+\sqrt{2au+u^2}|$$

INTEGRALS INVOLVING $ax^2 + bx + c$ AND $\sqrt{\dfrac{mx+n}{rx+s}}$

Let $X = ax^2 + bx + c$ and $q = b^2 - 4ac$:

$7 3_1$, 73_2. $$\int \frac{dx}{X} = \begin{cases} \dfrac{1}{\sqrt{q}} \ln\left|\dfrac{2ax+b-\sqrt{q}}{2ax+b+\sqrt{q}}\right| & \text{if} \quad q > 0 \\ \dfrac{2}{\sqrt{-q}} \tan^{-1}\dfrac{2ax+b}{\sqrt{-q}} & \text{if} \quad q < 0 \end{cases}$$

If $q = 0$, then X is a perfect square; use 5, with $p = -2$

74. $$\int \frac{(mx+n)}{X}\,dx = \frac{m}{2a}\ln|X| + \frac{2an-bm}{2a}\int\frac{dx}{X}$$

75. $$\int \frac{x^2}{X}\,dx = \frac{x}{a} - \frac{b}{2a^2}\ln|X| + \frac{b^2-2ac}{2a^2}\int\frac{dx}{X}$$

76. $$\int \frac{dx}{xX} = \frac{1}{2c}\ln\frac{x^2}{|X|} - \frac{b}{2c}\int\frac{dx}{X}$$

77_1, 77_2. $$\int \frac{dx}{\sqrt{X}} = \begin{cases} \dfrac{1}{\sqrt{a}} \ln\left|\sqrt{X} + \dfrac{2ax+b}{2\sqrt{a}}\right| & \text{if} \quad a > 0 \\ \dfrac{1}{\sqrt{-a}} \sin^{-1}\left(\dfrac{-2ax-b}{\sqrt{q}}\right) & \text{if} \quad a < 0 \end{cases}$$

78. $$\int \frac{mx+n}{\sqrt{X}}\,dx = \frac{m\sqrt{X}}{a} + \frac{2an-bm}{2a}\int\frac{dx}{\sqrt{X}}$$

79. $$\int \frac{x^2\,dx}{\sqrt{X}} = \frac{(2ax-3b)\sqrt{X}}{4a^2} + \frac{3b^2-4ac}{8a^2}\int\frac{dx}{\sqrt{X}}$$

80_1, 80_2, 80_3. $$\int \frac{dx}{x\sqrt{X}} = \begin{cases} -\dfrac{1}{\sqrt{c}} \ln\left|\dfrac{\sqrt{X}+\sqrt{c}}{x} + \dfrac{b}{2\sqrt{c}}\right| & \text{if} \quad c > 0 \\ \dfrac{1}{\sqrt{-c}} \sin^{-1}\dfrac{bx+2c}{x\sqrt{q}} & \text{if} \quad c < 0 \\ -\dfrac{2\sqrt{X}}{bx} & \text{if} \quad c = 0 \end{cases}$$

Let $k = an^2 - bmn + cm^2$.

$$\int \frac{dx}{(mx+n)\sqrt{X}} = \begin{cases} -\dfrac{1}{\sqrt{k}} \ln \left| \dfrac{m\sqrt{X} + \sqrt{k}}{mx+n} + \dfrac{bm-2an}{2\sqrt{k}} \right| & \text{if } k > 0 \quad (81_1) \\ \dfrac{1}{\sqrt{-k}} \sin^{-1} \left[\dfrac{(bm-2an)(mx+n)+2k}{m(mx+n)\sqrt{q}} \right] & \text{if } k < 0 \quad (81_2) \\ -\dfrac{2m\sqrt{X}}{(bm-2an)(mx+n)} & \text{if } k = 0 \quad (81_3) \end{cases}$$

82. $\displaystyle\int \frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{cx} - \frac{b}{2c}\int \frac{dx}{x\sqrt{X}}$

83. $\displaystyle\int \sqrt{X}\, dx = \frac{(2ax+b)\sqrt{X}}{4a} - \frac{q}{8a}\int \frac{dx}{\sqrt{X}}$

84. $\displaystyle\int x\sqrt{X}\, dx = \frac{X\sqrt{X}}{3a} - \frac{b(2ax+b)\sqrt{X}}{8a^2} + \frac{bq}{16a^2}\int \frac{dx}{\sqrt{X}}$

85. $\displaystyle\int X^2\sqrt{X}\, dx = \left(x - \frac{5b}{6a}\right)\frac{X\sqrt{X}}{4a} + \frac{5b^2-4ac}{16a^2}\int \sqrt{X}\, dx$

86. $\displaystyle\int \frac{\sqrt{X}}{mx+n}\, dx = \frac{\sqrt{X}}{m} + \frac{bm-2an}{2m^2}\int \frac{dx}{\sqrt{X}} + \frac{an^2-bmn+cm^2}{m^2}\int \frac{dx}{(mx+n)\sqrt{X}}$

87. $\displaystyle\int \frac{\sqrt{X}}{x^2}\, dx = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int \frac{dx}{x\sqrt{X}} + a\int \frac{dx}{\sqrt{X}}$

88. $\displaystyle\int \frac{dx}{X\sqrt{X}} = -\frac{2(2ax+b)}{q\sqrt{X}}$

89. $\displaystyle\int X\sqrt{X}\, dx = \frac{(2ax+b)\sqrt{X}}{8a}\left(X - \frac{3q}{8a}\right) + \frac{3q^2}{128a^2}\int \frac{dx}{\sqrt{X}}$

90. $\displaystyle\int \sqrt{\frac{mx+n}{rx+s}}\, dx = \int \frac{(mx+n)\, dx}{\sqrt{rmx^2+(sm+rn)x+sn}}$, then use Formula 78.

BINOMIAL REDUCTION FORMULAS

$$\int x^m(ax^n+b)^p\, dx = \begin{cases} \left.\begin{array}{l} \dfrac{x^{m-n+1}(ax^n+b)^{p+1}}{a(m+np+1)} - \dfrac{b(m-n+1)}{a(m+np+1)}\displaystyle\int x^{m-n}(ax^n+b)^p\, dx \quad (91_1) \\ \dfrac{x^{m+1}(ax^n+b)^p}{m+np+1} + \dfrac{bnp}{m+np+1}\displaystyle\int x^m(ax^n+b)^{p-1}\, dx \quad (91_2) \end{array}\right\} \\ \qquad \text{if } m+np+1 \neq 0 \\ \dfrac{x^{m+1}(ax^n+b)^{p+1}}{b(m+1)} - \dfrac{a(m+np+n+1)}{b(m+1)}\displaystyle\int x^{m+n}(ax^n+b)^p\, dx \quad (91_3) \\ \qquad \text{if } m+1 \neq 0 \\ \dfrac{-x^{m+1}(ax^n+b)^{p+1}}{bn(p+1)} + \dfrac{m+np+n+1}{bn(p+1)}\displaystyle\int x^m(ax^n+b)^{p+1}\, dx \quad (91_4) \\ \qquad \text{if } p+1 \neq 0 \end{cases}$$

TRANSCENDENTAL INTEGRALS

92. $\int \sin u\, du = -\cos u$

93. $\int \cos u\, du = \sin u$

94. $\int \sin^2 u\, du = \frac{1}{2}(u - \sin u \cos u)$

95. $\int \cos^2 u\, du = \frac{1}{2}(u + \sin u \cos u)$

96. $\int \sin^n u\, du = -\dfrac{\sin^{n-1} u \cos u}{n} + \dfrac{n-1}{n}\int \sin^{n-2} u\, du$

97. $\int \cos^n u\, du = \dfrac{\cos^{n-1} u \sin u}{n} + \dfrac{n-1}{n}\int \cos^{n-2} u\, du$

In case n is odd, Formulas 98 and 99 may be used.

98. $\int \sin^{2m+1} u\, du = \int (1 - \cos^2 u)^m \sin u\, du$

99. $\int \cos^{2m+1} u\, du = \int (1 - \sin^2 u)^m \cos u\, du$

Expand and use Formula 104 or 105.

100. $\displaystyle\int \frac{du}{\sin^n u} = -\frac{\cos u}{(n-1)\sin^{n-1} u} + \frac{n-2}{n-1}\int \frac{du}{\sin^{n-2} u}$

101. $\displaystyle\int \frac{du}{\cos^n u} = \frac{\sin u}{(n-1)\cos^{n-1} u} + \frac{n-2}{n-1}\int \frac{du}{\cos^{n-2} u}$

If $n \neq 1$. In case n is even, Formulas 102 and 103 may be used.

102. $\displaystyle\int \frac{du}{\sin^{2m} u} = \int \csc^{2m} u\, du$ then use Formula 125.

103. $\displaystyle\int \frac{du}{\cos^{2m} u} = \int \sec^{2m} u\, du$ then use Formula 124.

104. $\displaystyle\int \sin^p u \cos u\, du = \frac{\sin^{p+1} u}{p+1}$

105. $\displaystyle\int \cos^p u \sin u\, du = -\frac{\cos^{p+1} u}{p+1}$

if p is any number $\neq -1$

106. $\displaystyle\int \sin^2 u \cos^2 u\, du = \frac{4u - \sin 4u}{32}$

107. $\displaystyle\int \frac{du}{\sin u \cos u}\, du = \ln |\tan u|$

The integrals $\int \sin^m u \cos^n u\, du$, $\displaystyle\int \frac{du}{\sin^m u \cos^n u}$, $\displaystyle\int \frac{\sin^m u}{\cos^n u}\, du$ and $\displaystyle\int \frac{\cos^n u}{\sin^m u}\, du$ may be reduced to integrals given above by use of the following reduction formulas, in which r and s are any integers positive or negative.

$$\int \sin^r u \cos^s u\, du = \begin{cases} \dfrac{\cos^{s-1} u \sin^{r+1} u}{r+s} + \dfrac{s-1}{r+s}\displaystyle\int \sin^r u \cos^{s-2} u\, du & \text{if } r + s \neq 0 \quad (108_1) \\ \dfrac{-\sin^{r-1} u \cos^{s+1} u}{r+s} + \dfrac{r-1}{r+s}\displaystyle\int \sin^{r-2} u \cos^s u\, du & \text{if } r + s \neq \quad (108_2) \\ \dfrac{\sin^{r+1} u \cos^{s+1} u}{r+1} + \dfrac{s+r+2}{r+1}\displaystyle\int \sin^{r+2} u \cos^s u\, du & \text{if } r \neq -1 \quad (108_3) \\ \dfrac{-\sin^{r+1} u \cos^{s+1} u}{s+1} + \dfrac{s+r+2}{s+1}\displaystyle\int \sin^r u \cos^{s+2} u\, du & \text{if } s \neq -1 \quad (108_4) \end{cases}$$

$$\left.\begin{aligned}
&\textbf{109.}\quad \int \sin mu \sin nu\, du = -\frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)}\\
&\textbf{110.}\quad \int \cos mu \cos nu\, du = \frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)}\\
&\textbf{111.}\quad \int \sin mu \cos nu\, du = -\frac{\cos(m+n)u}{2(m+n)} - \frac{\cos(m-n)u}{2(m-n)}
\end{aligned}\right\}\quad \text{if}\quad m \neq \pm n$$

112. $\int \tan u\, du = -\ln|\cos u|$ **113.** $\int \cot u\, du = \ln|\sin u|$

$$\left.\begin{aligned}
&\textbf{114.}\quad \int \tan^n u\, du = \frac{\tan^{n-1} u}{n-1} - \int \tan^{n-2} u\, du\\
&\textbf{115.}\quad \int \cot^n u\, du = -\frac{\cot^{n-1} u}{n-1} - \int \cot^{n-2} u\, du
\end{aligned}\right\}$$

If $n \neq 1$. In case n is odd, Formulas 116 and 117 may be used.

$$\left.\begin{aligned}
&\textbf{116.}\quad \int \tan^{2m+1} u\, du = \int (\sec^2 u - 1)^m \tan u\, du\\
&\textbf{117.}\quad \int \cot^{2m+1} u\, du = \int (\csc^2 u - 1)^m \cot u\, du
\end{aligned}\right\}$$

Expand and use Formula 126 or 127.

118. $\int \sec u\, du = \ln|\sec u + \tan u|$ **119.** $\int \csc u\, du = \ln|\csc u - \cot u|$

120. $\int \sec^2 u\, du = \tan u$ **121.** $\int \csc^2 u\, du = -\cot u$

$$\left.\begin{aligned}
&\textbf{122.}\quad \int \sec^n u\, du = \int \frac{du}{\cos^n u}\\
&\textbf{123.}\quad \int \csc^n u\, du = \int \frac{du}{\sin^n u}
\end{aligned}\right\}$$

Use Formula 101 or 100. In case n is even, Formula 124 or 125 may be used.

$$\left.\begin{aligned}
&\textbf{124.}\quad \int \sec^{2m} u\, du = \int (\tan^2 u + 1)^{m-1} \sec^2 u\, du\\
&\textbf{125.}\quad \int \csc^{2m} u\, du = \int (\cot^2 u + 1)^{m-1} \csc^2 u\, du
\end{aligned}\right\}$$

Expand and use Formula 128 or 129.

$$\left.\begin{aligned}
&\textbf{126.}\quad \int \sec^p u \tan u\, du = \frac{\sec^p u}{p}\\
&\textbf{127.}\quad \int \csc^p u \cot u\, du = -\frac{\csc^p u}{p}
\end{aligned}\right\}\quad \text{if } p \text{ is any constant} \neq 0$$

$$\left.\begin{aligned}
&\textbf{128.}\quad \int \tan^p u \sec^2 u\, du = \frac{\tan^{p+1} u}{p+1}\\
&\textbf{129.}\quad \int \cot^p u \csc^2 u\, du = -\frac{\cot^{p+1} u}{p+1}
\end{aligned}\right\}\quad \text{if } p \text{ is any constant} \neq -1$$

130₁. / **130₂.**

$$\int \frac{du}{a + b\sin u + c\cos u} = \begin{cases}
\textbf{130}_1.\quad \dfrac{1}{\sqrt{b^2+c^2-a^2}} \ln \left| \dfrac{b - \sqrt{b^2+c^2-a^2} + (a-c)\tan\frac{u}{2}}{b + \sqrt{b^2+c^2-a^2} + (a-c)\tan\frac{u}{2}} \right| & \text{if}\quad a^2 < b^2 + c^2\\[2ex]
\textbf{130}_2.\quad \dfrac{2}{\sqrt{a^2-b^2-c^2}} \tan^{-1} \dfrac{b + (a-c)\tan\frac{u}{2}}{\sqrt{a^2-b^2-c^2}} & \text{if}\quad a^2 > b^2 + c^2
\end{cases}$$

131. $\int \sqrt{1 - \cos u}\, du = -2\sqrt{2} \cos \frac{u}{2}$

132. $\int \sqrt{(1 - \cos u)^3} = 4 \frac{\sqrt{2}}{3} \left(\cos^3 \frac{u}{2} - 3 \cos \frac{u}{2} \right)$

133. $\int u \sin u\, du = \sin u - u \cos u$ 134. $\int u \cos u\, du = \cos u + u \sin u$

135. $\int u^n \sin u\, du = -u^n \cos u + n \int u^{n-1} \cos u\, du$

136. $\int u^n \cos u\, du = u^n \sin u - n \int u^{n-1} \sin u\, du$

137. $\int \sin^{-1} u\, du = u \sin^{-1} u + \sqrt{1 - u^2}$ 138. $\int \cos^{-1} u\, du = u \cos^{-1} u - \sqrt{1 - u^2}$

139. $\int \tan^{-1} u\, du = u \tan^{-1} u - \ln \sqrt{1 + u^2}$

140. $\int \cot^{-1} u\, du = u \cot^{-1} u + \ln \sqrt{1 + u^2}$

141. $\int \sec^{-1} u\, du = u \sec^{-1} u - \ln |u + \sqrt{u^2 - 1}|$

142. $\int \csc^{-1} u\, du = u \csc^{-1} u + \ln |u + \sqrt{u^2 - 1}|$

143. $\int \log_b u\, du = u(\log_b u - \log_b e)$ 144. $\int \ln u\, du = u(\ln u - 1)$

145. $\int u^m \log_b u\, du = u^{m+1} \left\{ \frac{\log_b u}{m + 1} - \frac{\log_b e}{(m + 1)^2} \right\}$ if $m \neq -1$

146. $\int u^m \ln u\, du = u^{m+1} \left\{ \frac{\ln u}{m + 1} - \frac{1}{(m + 1)^2} \right\}$ if $m \neq -1$

$1 47_1.$, $147_2.$

$$\int \frac{(\ln u)^p}{u}\, du = \begin{cases} \dfrac{(\ln u)^{p+1}}{p + 1} & \text{if } p \neq -1 \\ \ln |\ln u| & \text{if } p = -1 \end{cases}$$

148. $\int \sin (\ln u)\, du = \frac{u}{2} [\sin (\ln u) - \cos (\ln u)]$

149. $\int \cos (\ln u)\, du = \frac{u}{2} [\sin (\ln u) + \cos (\ln u)]$

150. $\int b^u\, du = \frac{b^u}{\ln b}$ 151. $\int e^u\, du = e^u$

152. $\int ue^u\, du = e^u(u - 1)$ 153. $\int u^n e^u\, du = u^n e^u - n \int u^{n-1} e^u\, du$

154. $\int \frac{du}{a + be^{nu}} = \frac{1}{an} \{nu - \ln |a + be^{nu}|\}$

155. $\int \frac{du}{ae^{nu} + be^{-nu}} = \frac{1}{n\sqrt{ab}} \tan^{-1} \left(e^{nu} \sqrt{\frac{a}{b}} \right)$

156. $\int e^{au} \sin nu\, du = \frac{e^{au}(a \sin nu - n \cos nu)}{a^2 + n^2}$

157. $\int e^{au} \cos nu\, du = \dfrac{e^{au}(n \sin nu + a \cos nu)}{a^2 + n^2}$

158. $\int \sinh x\, dx = \cosh x$

159. $\int \cosh x\, dx = \sinh x$

160. $\int \sinh^2 x\, dx = \dfrac{\sinh 2x}{4} - \dfrac{x}{2}$

161. $\int \cosh^2 x\, dx = \dfrac{\sinh 2x}{4} + \dfrac{x}{2}$

162. $\int x \sinh x\, dx = x \cosh x - \sinh x$

163. $\int x \cosh x\, dx = x \sinh x - \cosh x$

164. $\int \tanh x\, dx = \ln(\cosh x)$

165. $\int \coth x\, dx = \ln |\sinh x|$

166. $\int \tanh^2 x\, dx = x - \tanh x$

167. $\int \coth^2 x\, dx = x - \coth x$

168. $\int \operatorname{sech} x\, dx = \tan^{-1}(\sinh x)$

169. $\int \operatorname{csch} x\, dx = \ln \left| \tanh \dfrac{x}{2} \right|$

170. $\int \operatorname{sech}^2 x\, dx = \tanh x$

171. $\int \operatorname{csch}^2 x\, dx = -\coth x$

172. $\int \sinh x \cosh x\, dx = \frac{1}{4} \cosh 2x$

173. $\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x$

174. $\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x$

175. $\int \sinh mx \sinh nx\, dx = \dfrac{\sinh (m+n)x}{2(m+n)} - \dfrac{\sinh (m-n)x}{2(m-n)}, \qquad m \neq \pm n$

176. $\int \cosh mx \cosh nx\, dx = \dfrac{\sinh (m+n)x}{2(m+n)} + \dfrac{\sinh (m-n)x}{2(m-n)}, \qquad m \neq \pm n$

177. $\int \sinh mx \cosh nx\, dx = \dfrac{\cosh (m+n)x}{2(m+n)} + \dfrac{\cosh (m-n)x}{2(m-n)}, \qquad m \neq \pm n$

If an integral involves only $\sin u$ and $\cos u$, such as

$$\int \frac{\sin u\, du}{3 \sin u + 4 \cos u + 2},$$

try the substitution $\tan \dfrac{u}{2} = t$. Then

$$1 + t^2 = 1 + \tan^2 \frac{u}{2} = \sec^2 \frac{u}{2}, \quad \text{so } \cos \frac{u}{2} = \frac{1}{\sqrt{1+t^2}}, \quad \sin \frac{u}{2} = \frac{1}{\sqrt{1+t^2}},$$

$$\sin u = 2 \sin \frac{u}{2} \cos \frac{u}{2} = \frac{2t}{1+t^2}, \quad \cos u = 2 \cos^2 \frac{u}{2} - 1 = \frac{1-t^2}{1+t^2},$$

$$dt = \sec^2 \frac{u}{2} \frac{du}{2} = \frac{1+t^2}{2} du, \quad \text{and} \quad du = \frac{2}{1+t^2} dt.$$

For example, the above integral becomes

$$2 \int \frac{t\, dt}{(1+t^2)(-t^2+3t+3)}$$

which may be handled by using partial fractions (Sec. 76) and then Table formula 74.

Review of Analytic Trigonometry

T1. Trigonometric Functions of Angles

The **unit circle** is, by definition, the circle of radius 1 and center at the origin; it has equation

$$x^2 + y^2 = 1.$$

An angle with vertex at the origin and initial side on the positive x-axis is said to be in **standard position.**

Place an angle α in standard position. The terminal side of α then cuts the unit circle at a point. Let c denote the abscissa and s the ordinate of this point. Then, by definition,

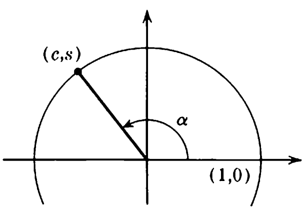


Figure T1.1

$$\sin\alpha = s \qquad \cot\alpha = \frac{c}{s} \quad \text{if } s \neq 0$$

$$\cos\alpha = c \qquad \sec\alpha = \frac{1}{c} \quad \text{if } c \neq 0$$

$$\tan\alpha = \frac{s}{c} \quad \text{if } c \neq 0 \qquad \csc\alpha = \frac{1}{s} \quad \text{if } s \neq 0$$

As direct consequences of these definitions it follows that whenever both sides of the expressions below are defined, then the equalities hold:

(1) $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$ (5) $\csc\alpha = \dfrac{1}{\sin\alpha}$

(2) $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$ (6) $\sin^2\alpha + \cos^2\alpha = 1$ since $s^2 + c^2 = 1$

(3) $\cot\alpha = \dfrac{1}{\tan\alpha}$ (7) $1 + \tan^2\alpha = \sec^2\alpha$ since $1 + \left(\dfrac{s}{c}\right)^2 = \left(\dfrac{1}{c}\right)^2$

(4) $\sec\alpha = \dfrac{1}{\cos\alpha}$ (8) $\cot^2\alpha + 1 = \csc^2\alpha$ since $\left(\dfrac{c}{s}\right)^2 + 1 = \left(\dfrac{1}{s}\right)^2$

These eight relations are called "The eight fundamental identities." By use of these identities other identities may be established without returning to the definitions.

Example 1. Show that $\sin\alpha\cos\beta\sec\alpha\csc\beta = \tan\alpha\ \cancel{\tan}\,\beta$.

Solution.

$$\sin\alpha\cos\beta\sec\alpha\csc\beta = \sin\alpha\cos\beta\frac{1}{\cos\alpha}\frac{1}{\sin\beta} \qquad \text{by (4) and (5)}$$

$$= \frac{\sin\alpha}{\cos\alpha}\frac{\cos\beta}{\sin\beta} = \tan\alpha\cot\beta \qquad \text{by (1) and (2).}$$

In particular, the terminal sides of the angles of 90° and −270° in standard position cut the unit at the point (0,1) so that

$$\sin 90^\circ = \sin(-270^\circ) = 1 \quad \text{and} \quad \cos 90^\circ = \cos(-270^\circ) = 0.$$

Also, directly from Fig. T1.1 $\sin(\pm 180^\circ) = 0$, $\cos(\pm 180^\circ) = -1$, $\sin 0^\circ = \sin(\pm 360^\circ) = 0$ and $\cos 0^\circ = \cos(\pm 360^\circ) = 1$.

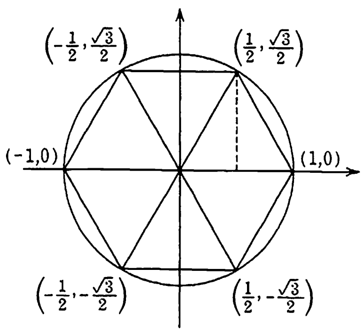


Figure T1.2

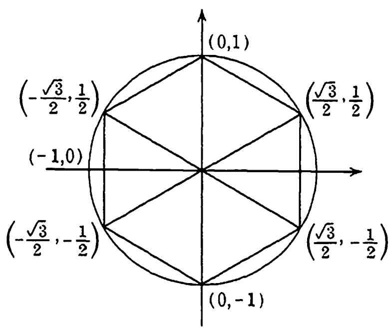


Figure T1.3

From Figs. T1.2 and T1.3 may be read

$$\sin(\pm 60^\circ) = \pm\frac{\sqrt{3}}{2}, \quad \cos(\pm 60^\circ) = \frac{1}{2}, \quad \tan(\pm 60^\circ) = \pm\sqrt{3}$$

$$\sin(\pm 120^\circ) = \pm\frac{\sqrt{3}}{2}, \quad \cos(\pm 120^\circ) = -\frac{1}{2}, \quad \tan(\pm 120^\circ) = \pm\sqrt{3}, \text{ etc.}$$

$$\sin(\pm 30^\circ) = \pm\frac{1}{2}, \quad \cos(\pm 30^\circ) = \frac{\sqrt{3}}{2}, \quad \tan(\pm 30^\circ) = \pm\frac{1}{\sqrt{3}}, \text{ etc.}$$

A square with sides parallel to the axes inscribed in the unit circle has corners at the points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ and thus it follows that

$$\sin(\pm 45^\circ) = \pm\frac{1}{\sqrt{2}}, \quad \cos(\pm 45^\circ) = \frac{1}{\sqrt{2}}, \quad \tan(\pm 45^\circ) = \pm 1, \text{ etc.}$$

PROBLEMS

1. For any angle α, the point $(\cos\alpha, \sin\alpha)$ is on the unit circle. Show that the distance from this point to the point $(1,0)$ is $2 - 2\cos\alpha$.

2. By using the fundamental identities, show that whenever both sides of the expressions are defined

   a. $(1 - \sin\alpha)^2 + \cos^2\alpha = 2(1 - \sin\alpha)$.

   b. $(1 - \cos\alpha)^2 + \sin^2\alpha = 2(1 - \cos\alpha)$.

   c. $(1 + \tan\alpha)^2 - \sec^2\alpha = 2\tan\alpha$.

   d. $\sec^2\alpha + \csc^2\alpha = \sec^2\alpha\csc^2\alpha = (\tan\alpha + \cot\alpha)^2$.

   e. $\dfrac{1}{1+\sin\alpha} + \dfrac{1}{1-\sin\alpha} = 2\sec^2\alpha$.

   f. $\dfrac{1-\sin\alpha}{\cos\alpha} - \dfrac{\cos\alpha}{1+\sin\alpha} = 0$.

   g. $\dfrac{2+\tan^2\alpha}{\sec^2\alpha} = 1 + \cos^2\alpha$.

   h. $\dfrac{\cot^2\alpha}{1+\csc\alpha} = \csc\alpha - 1$.

3. Show that the line containing the terminal side of an angle α in standard position intersects the tangent to the unit circle:

   a. At $(1,0)$ in the point $(1, \tan\alpha)$ if $\alpha \neq 90° + m \cdot 180°$, m an integer.

   b. At $(0,1)$ in the point $(\cot\alpha, 1)$ if $\alpha \neq m \cdot 180°$.

T2. Addition and Subtraction Formulas

For any positive or negative angle α, the terminal sides of α and $-\alpha$ in standard position cut the unit circle in points symmetric to the x-axis so that

$$\sin(-\alpha) = -\sin\alpha \quad \text{and} \quad \cos(-\alpha) = \cos\alpha. \tag{1}$$

Let α and β be any angles whatever. The terminal sides of these angles in standard position cut the unit circle at the points

$$A(\cos\alpha, \sin\alpha) \quad \text{and} \quad B(\cos\beta, \sin\beta),$$

respectively. The square of the distance $\overline{AB}$ is

$$\begin{aligned}\overline{AB}^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2\\ &= (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)\\ &= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta).\end{aligned}$$

If two points on a circle are rotated about the center through a common angle, they remain the same distance apart.

Rotate the points A and B through the same angle $-\beta$. The new positions A' and B' are

$$A'(\cos(\alpha - \beta), \sin(\alpha - \beta)) \quad \text{and} \quad B'(1,0).$$

Then $\overline{A'B'}^2 = [1 - \cos(\alpha - \beta)]^2 + \sin^2(\alpha - \beta) = 2 - 2\cos(\alpha - \beta)$. But $\overline{A'B'}^2 = \overline{AB}^2$ so that

$$(2) \qquad \mathbf{\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta}$$

holds with no restrictions whatever on the angles α and β.

In (2) we may replace β by $-\beta$ and obtain

$$(3) \quad \begin{aligned} \mathbf{\cos(\alpha + \beta)} &= \cos[\alpha - (-\beta)] \\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) && \text{by (2)} \\ &= \mathbf{\cos\alpha\cos\beta - \sin\alpha\sin\beta}. && \text{by (1)} \end{aligned}$$

For different relations we again use (2), but this time with $\alpha = 90°$, and obtain

$$(4) \quad \begin{aligned} \mathbf{\cos(90° - \beta)} &= \cos 90° \cos\beta + \sin 90° \sin\beta \\ &= 0 \cdot \cos\beta + 1 \cdot \sin\beta \\ &= \mathbf{\sin\beta} \end{aligned}$$

for any angle β. Hence we may set $90° - \beta = \alpha$ in (4) and have

$$(5) \qquad \mathbf{\cos\alpha = \sin(90° - \alpha)}.$$

We next write (4) as $\sin\gamma = \cos(90° - \gamma)$ and then set $\gamma = \alpha + \beta$ to obtain

$$(6) \quad \begin{aligned} \mathbf{\sin(\alpha + \beta)} &= \cos[90° - (\alpha + \beta)] \\ &= \cos[(90° - \alpha) - \beta] \\ &= \cos(90° - \alpha)\cos\beta + \sin(90° - \alpha)\sin\beta && \text{from (2)} \\ &= \mathbf{\sin\alpha\cos\beta + \cos\alpha\sin\beta} && \text{by (4) and (5)} \end{aligned}$$

Upon replacing β by $-\beta$ in (6) it follows that

$$(7) \qquad \mathbf{\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}.$$

Example 1. Show that for α any angle whatever and n an integer, then

$$(8) \qquad \cos(\alpha + n \cdot 180°) = (-1)^n \cos\alpha.$$

Solution. From (3) with $\beta = n \cdot 180°$

$$\cos(\alpha + n \cdot 180°) = \cos\alpha\cos(n \cdot 180°) - \sin\alpha\sin(n \cdot 180°).$$

The terminal side of the angle $n \cdot 180°$ in standard position cuts the unit circle at the point

$$(1,0) \text{ if } |n| \text{ is even or zero}$$
$$(-1,0) \text{ if } |n| \text{ is odd}$$

so that $\cos(n \cdot 180°) = (-1)^n$ and $\sin(n \cdot 180°) = 0$. Hence (8) is seen to hold.

In the same way

$$\sin(\alpha + n \cdot 180^\circ) = (-1)^n \sin\alpha \tag{9}$$

Example 2. Establish that

$$\sin A + \cos B = 2\sin\tfrac{1}{2}(A + B)\cos\tfrac{1}{2}(A - B) \tag{10}$$

holds for any angles A and B whatever.

Solution. From (6) and (7) by addition

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta. \tag{11}$$

Make the substitutions $\alpha + \beta = A$, $\alpha - \beta = B$. Then.

$$\alpha = \tfrac{1}{2}(A + B), \quad \beta = \tfrac{1}{2}(A - B),$$

which substituted into (11) yields (10).

A formula expressing $\tan(\alpha + \beta)$ in terms of $\tan\alpha$ and $\tan\beta$ is obtained as follows:

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \tag{12}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

PROBLEMS

1. By using the method of Example 2, show that

   a. $\sin A - \sin B = 2\cos\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$.

   b. $\cos A + \cos B = 2\cos\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B)$.

   c. $\cos A - \cos B = -2\sin\frac{1}{2}(A + B)\sin\frac{1}{2}(A - B)$.

   d. $\dfrac{\sin A - \sin B}{\sin A + \sin B} = \dfrac{\tan\frac{1}{2}(A - B)}{\tan\frac{1}{2}(A + B)}$.

2. Establish the formulas:

   a. $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$. b. $\cot(\alpha + \beta) = \dfrac{\cot\alpha\cot\beta - 1}{\cot\beta + \cot\alpha}$.

   c. $\cot(\alpha - \beta) = \dfrac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$.

3. Establish the following by specializing some of the formulas given previously.

a. $\sin 2\alpha = 2 \sin \alpha \cos \beta$

b. $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$

$= 2 \cos^2 \alpha - 1$

$= 1 - 2 \sin^2 \alpha.$

c. $\tan 2\alpha = \dfrac{2 \tan \alpha}{1 - \tan^2 \alpha}.$

d. $\sin \alpha = 2 \sin \dfrac{\alpha}{2} \cos \dfrac{\alpha}{2}$

$$= 2 \frac{\tan \frac{\alpha}{2}}{\sec^2 \frac{\alpha}{2}}$$

4. Establish each of the following:

a. $(\cos \alpha - \sin \alpha)(\cos \alpha + \sin \alpha) = \cos 2\alpha.$

b. $\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$, $\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha).$

c. $\tan \alpha + \cot \alpha = 2 \csc 2\alpha.$

d. $\dfrac{\sin 3\alpha}{\sin \alpha} - \dfrac{\cos 3\alpha}{\cos \alpha} = 2.$

e. $\dfrac{\sin \alpha + \sin 2\alpha}{1 + \cos \alpha + \cos 2\alpha} = \tan \alpha.$

f. $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha.$

g. $\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha.$

5. Set $\tan \dfrac{\alpha}{2} = t$ and show that

$$\tan \alpha = \frac{2t}{1 - t^2}, \sin \alpha = \frac{2t}{1 + t^2}, \text{ and } \cos \alpha = \frac{1 - t^2}{1 + t^2}.$$

(Hint: Use Problem 3c and d.)

6. Show, for any angles α, β, and γ, that

a. $\sin \alpha \sin (\beta - \gamma) + \sin \beta \sin (\gamma - \alpha) + \sin \gamma \sin (\alpha - \beta) = 0.$

b. $\cos \alpha \sin (\beta - \gamma) + \cos \beta \sin (\gamma - \alpha) + \cos \gamma \sin (\alpha - \beta) = 0.$

c. $\cos \alpha \sin (\beta - \gamma) - \sin \beta \cos (\gamma - \alpha) + \sin \gamma \cos (\alpha - \beta) = 0.$

7. a. Show that if a and b are numbers, not both zero, then there is an angle α such that both

$$\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}} \quad \text{and} \quad \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$$

b. Show that for α an angle as in Part a,

$$a \cos \theta + b \sin \theta = \frac{1}{\sqrt{a^2 + b^2}} \cos (\theta - \alpha)$$

c. Show that there is an angle α such that

$$a \cos \theta + b \sin \theta = \frac{1}{\sqrt{a^2 + b^2}} \sin (\alpha + \theta).$$

T3. Trigonometric Functions of Numbers

On a circle of radius r, lay off an arc of length r. The central angle subtended by this arc is, by definition, an angle of one radian. Since the circumference is of length $2\pi r$, then there are 2π radians in one revolution. Thus

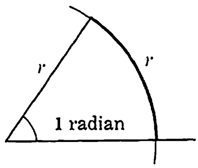


Figure T3.1

$$2\pi \text{ radians} = 360°$$

(1) $$\text{one radian} = \left(\frac{360}{2\pi}\right)^\circ = \left(\frac{180}{\pi}\right)^\circ.$$

Since 3.14159 is an approximation of π, then one radian is approximately

$$\left(\frac{180}{3.14159}\right)^\circ = 57.29583° = 57°\,17.7498' = 57°\,17'\,44.988''.$$

It will be taken for granted that the areas of two sectors of a circle are to each other as their subtended angles. Hence in a circle, two sectors of areas A_1 units^2 and A_2 units^2 whose angles have x_1 radians and x_2 radians, respectively, are such that

$$\frac{A_1}{A_2} = \frac{x_1}{x_2}.$$

A circle of radius r units may be considered as a sector of itself, the sector having area πr^2 units^2 and angle of 2π radians. Hence a sector of this circle with area A units^2 and central angle x radians, $0 \leq x \leq 2\pi$, is such that

(2) $$\frac{A}{\pi r^2} = \frac{x}{2\pi} \quad \text{so that} \quad A = \frac{x}{2}\,r^2.$$

If the central angle had been given as $y°$, $0 \leq y \leq 360$, then

(3) $$\frac{A}{\pi r^2} = \frac{y}{360} \quad \text{so that} \quad A = \frac{y}{360}\,\pi r^2.$$

One of the main reasons for using radians, instead of the sexagesimal system, for measuring angles is because (2) is simpler than (3).

The use of "sin" is extended to $\sin x$, with x a number, by setting

(4) $$\sin x = \sin\,(\text{an angle of } x \text{ radians}).$$

Also, by definition, $\cos x = \cos$ (an angle of x radians) and in the same way $\tan x$, $\cot x$, $\sec x$ and $\csc x$ are defined. The previously derived formulas involving angles may, therefore, be used to obtain analogous formulas involving numbers. For example, with x and y numbers, then

(5) $$\sin\,(x + y) = \sin x \cos y + \cos x \sin y.$$

In all of calculus it is understood that "sin x" carries the implication that x is a number. Thus we write

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x$$

and do not replace $\pi/2$ by 90°. Also

$$\sin 1 \text{ is approximately } \sin\,(57^\circ\, 17'\, 45'') = 0.841\ 3655$$

whereas $\sin 1^\circ$ is approximately 0.017 4524.

With x a number, it is permissible to write either

$$\sin x \quad \text{or} \quad \sin x^\circ,$$

whichever is meant, but these should not be expected to be equal.

It is possible to avoid specific mention of angles and to develop "Trigonometry without Angles" by using a wrapping technique. Assume that you never heard about the sine of an angle and consider the following procedure as your introduction to trigonometry.

Let x be a number. Lay a piece of inelastic flexible string along the x-axis, the piece cut so one end is at (0,0) and the other end at $(x,0)$. Next shift the piece of string along the x-axis so its ends are at (1,0) and $(x + 1,0)$. Keep the end at (1,0) fixed and wrap the string along the rim of the unit circle, going counterclockwise if $x > 0$, but clockwise if $x < 0$. The end which was originally at $(x,0)$, and then at $(x + 1,0)$, will come to rest at a point (c,s) of the unit circle. With x a given number, this procedure uniquely determines numbers c and s. By definition

$$\sin x = s, \quad \cos x = c, \quad \tan x = \frac{s}{c}, \text{ etc.}$$

Without going into all the details, it should be seen that the method used

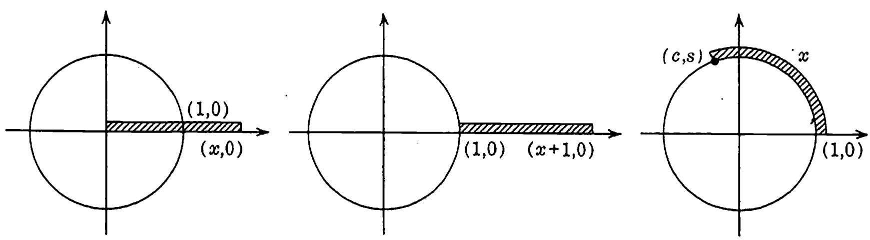


Figure T3.2

with angles could now be duplicated to produce analogous results entirely in terms of numbers.

The following graphs are included for reference.

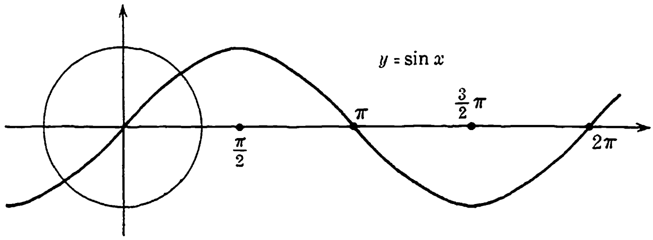


Figure T3.3

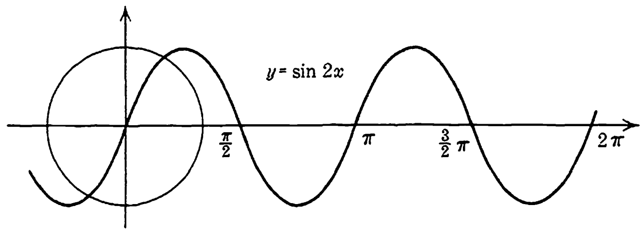


Figure T3.4

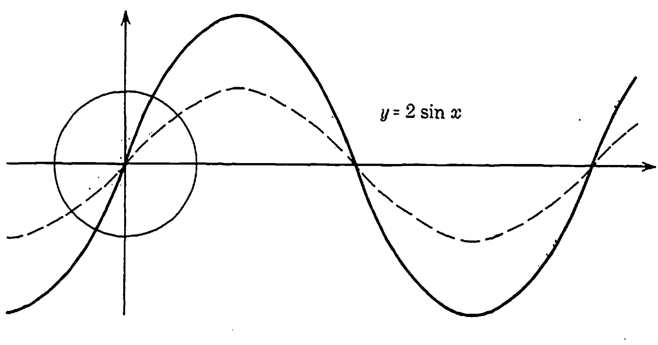


Figure T3.5

Answers

Sec. 2, page 5

1. a. $x < \frac{3}{8}$. c. $x > 11$. e. $x \geq -1$.

2. a. $x = 3$ or $x = -4$. c. $-4 < x < 3$. e. $-\frac{3}{2} < x < 4$.

3. a. $x < -\frac{1}{2}$ or $x > 3$. c. $-4 < x < -\frac{1}{2}$. e. $x > -1$.

4. a. $9 < x < 11$. c. $1 < x < 2$.

6. a. 3. c. 8. e. Jan. 29, 1946.

7. a. $x = 3$ or $x = 11$. The points 3 and 11 are each twice as far from the point -1 as from the point 5. c. $x = -23$ and $x = -\frac{7}{9}$. Each of the points -23 and $-\frac{7}{9}$ is such that its distance to the point -3 is four-fifths of its distance to the point 2.

Sec. 3, page 8

1. a. 0, 1, 2, 3, 4. c. -1.5, 0, 1.5, 3, 4.5, 6, 7.5, 9, 10.5. e. -11.91, -11.82, -11.73, -11.64, etc., to -3.09 where for $k = 1, 2, 3, \cdots, 99$ the kth point is $-12 + k(0.09)$.

2. a. 3, 11, 11, -4. c. $t = 3 + \sqrt{15}$. e. Once when $0 \leq t \leq 2$, again when $4 \leq t \leq 6$. g. -4.1, -3.9, $-4 - h$.

3. a. 6, 55, 72, 57, 10. b. -0.6 ft/sec, 2.6 ft/sec, -31.16 ft/sec, $65 - 32t_1 - 16h$ ft/sec.

4. a. $6t_1^2 + 6t_1h + 2h^2 - 4$. c. $\dfrac{\sqrt{3(t_1 + h) + 4} - \sqrt{3t_1 + 4}}{h}$. e. $\dfrac{-5(2t_1 + h)}{(t_1 + h)^2 t_1^2}$.

Sec. 4, page 11

1. a. Bounded above, l.u.b. = 3; bounded below, g.l.b. = -3.
   c. Bounded above, l.u.b. = 3; not bounded below.
   e. Bounded above, l.u.b. = 4; bounded below, g.l.b. = 0.
   g. Not bounded above; bounded below, g.l.b. 27. i. Not bounded above, g.l.b. = 1.

2. a. $A \cup B = \{x \mid -3 < x < 10\}$, $A \cap B = \{x \mid 2 \leq x \leq 4\}$.
   c. $A \cup B = \{x \mid -2 < x \leq 4\}$, $A \cap B$ is empty.
   e. $A \cup B = \{x \mid x < 5\}$, $A \cap B = \{x \mid -5 < x < 3\}$.

3. a. $\{x \mid -1 < x \leq 3\}$. c. $\{x \mid -2 \leq x < 1.96\}$. e. $\{x \mid x \leq -2\}$. g. empty. i. and k. $\{x \mid -2 < x \leq 1\}$.

Sec. 6, page 17

3. a. (1,2). c. (0,6.5). 4. a. (5,6). c. $(-3,8.5)$.

5. a. $y - 3 = \frac{2}{7}(x - 2)$ or $2x - 7y + 17 = 0$. c. $y = 1$. e. $2x + y - 11 = 0$.

6. a. $2x + y - 2 = 0$. c. $y = 4$.

8. a. $x^3 - x^2y + 4x^2 + 2x - 2y + 8 \equiv (x^2 + 2)(x - y + 4)$ and since $x^2 + 2 \neq 0$ the given expression is zero if and only if $x - y + 4 = 0$.

9. a. $m = \frac{1}{2}, (0,\frac{5}{2})$. c. $m = \frac{1}{3}, (0,-\frac{5}{3})$.

Sec. 9, page 24

1. a. Half-lines each with end point $(-2,0)$; one through $(0,2)$, other through $(-4,2)$.
   c. Half-lines each with end point $(0,1)$; one through $(1,0)$, other through $(-1,0)$.
   e. Square with vertices $(1,0), (0,1), (-1,0), (0,-1)$.
   g. Two parallel lines with slope 1; one through $(0,1)$, other through $(0,-1)$.

2. a. $\{x \mid x \text{ any number}\}, \{y \mid y \geq 0\}$. c. $\{x \mid x \text{ any number}\}, \{y \mid y \leq 1\}$.
   h. $\{x \mid x \text{ any number}\}, \{y \mid y \text{ any number}\}$. Others not functions.

3. a. Horizontal unit intervals closed on left and open on right with left ends at $\cdots, (-3,-1), (-2,0), (-1,1), (0,2), (1,3), \cdots$.
   c. Intervals as in a, left ends at $\cdots, (-1,2), (0,1), (1,0), \cdots$.
   e. Unit squares closed on lower and left edges, open on other edges, with lower left corners at $\cdots, (-1,2), (0,1), (1,0), \cdots$.
   g. A strip bounded by parallel lines of slope 1, one through $(0,1)$, other through $(0,2)$, lower line in set, upper line not in set.

4. a. $-\frac{7}{12}, \frac{7}{12}, \frac{3}{2}, -\frac{3}{2}$. c. $(103)/(93)$. e. 3.55.

Sec. 10, page 28

1. a. $2\sqrt{5}$. c. 13. 2. a. $\frac{15}{2}$. c. $\frac{41}{4}$.

3. a. $\{(x,y) \mid (x + 2)^2 + (y - 3)^2 = 2^2\}$. c. $\{(x,y) \mid (x - 12)^2 + (y + 5)^2 = 13^2\}$.
   e. $\{(x,y) \mid (x + r)^2 + (y - r)^2 = r^2\}$.

4. a. Circle, center $(2,-3)$, radius 4. c. Circle center $(\frac{1}{2},-2)$, radius 1.
   e. Single point $(1,-2)$.

5. a. All points inside the circle having center $(-3,4)$ and radius 5.
   c. All points common to the interiors of two circles of radius 5, one with center at $(0,0)$, the other with center $(5,0)$.
   e. All points in either of two circles, one with center at $(0,0)$ with radius 5, the other with center $(5/\sqrt{2},5/\sqrt{2})$ and radius 5.
   g. All points inside the circle having center $(-3,4)$ and radius 5.

Sec. 11, A, B, C, page 33

1. Not sym. to either axis or origin.
   $\{x \mid x \neq 2\}, \{y \mid y \neq 0\}$. Asym. $\{(x,y) \mid y = 0\}, \{(x,y) \mid x = 2\}$.

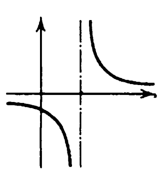


Figure Prob. 1

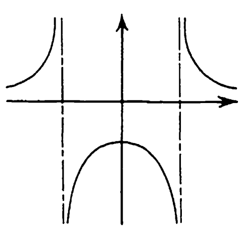


Figure Prob. 5

3. Sym. to y-axis. $\{x \mid x \neq 0\}$, $\{y \mid y > 0\}$. Asym. x-axis and y-axis.

5. Sym. to y-axis. $\{x \mid x \neq \pm\sqrt{2}\}$, $\{y \mid y > 0 \text{ or } y < -\frac{3}{2}\}$.
Asym. x-axis and $\{(x,y) \mid x = \pm\sqrt{2}\}$.

7. Sym. to y-axis. No restriction on x,
$\{y \mid 0 \leq y \leq 1\}$. No asym.

9. Sym. to x-axis, y-axis, origin. $\{x \mid -1 \leq x \leq 1\}$,
$\{y \mid -1 \leq y \leq 1\}$. No asym.

11. Sym. to x-axis, y-axis, origin. $\{x \mid -\sqrt{5} \leq x \leq \sqrt{5}\}$, $\{y \mid -\sqrt{2} \leq y \leq \sqrt{2}\}$.

13. Sym. to x-axis, y-axis, origin. $\{x \mid -\frac{1}{3} \leq x \leq \frac{1}{3}\}$, $\{y \mid -\frac{1}{4} \leq y \leq \frac{1}{4}\}$.

Sec. 11, D, E, page 36

1. a. Not sym. to x-axis, y-axis, or origin. $\{x \mid x \neq 0\}$, no restriction on y. Oblique asym. $\{(x,y) \mid y = x\}$, vertical asym. y-axis.
   c. Sym. to y-axis. $\{x \mid x \neq 0\}$, $\{y \mid y < -1\}$. Asym. $\{(x,y) \mid y = -1\}$, $\{(x,y) \mid x = 0\}$.
   e. No sym. $\{x \mid x \neq +1\}$, $\{y \mid y \geq 3 \text{ or } y \leq -1\}$. Asym. $\{(x,y) \mid y = x\}$, $\{(x,y) \mid x = 1\}$.

$$\left(\text{Hint: } y = x + \frac{1}{x-1} \quad \text{and} \quad x = \frac{y + 1 \pm \sqrt{(y+1)(y-3)}}{2}\right).$$

2. a. $\{(x,y) \mid 2y = \pm\sqrt{6}x\}$. c. $\{(x,y) \mid \sqrt{6}y = \pm 2x\}$. e. $\{(x,y) \mid 4y = \pm 3x\}$.

Sec. 12, page 39

1. a. $\{(X,Y) \mid XY = 0\}$, so graph is X-axis and Y-axis. Translated back, graph is $\{(x,y) \mid x = -\frac{3}{2}\}$and $\{(x,y) \mid y = 2\}$.
   c. $\{(X,Y) \mid Y = 2^X\}$. e. $\{(X,Y \mid Y = aX^2\}$.

2. a. $X = x + 1$, $Y = y - 2$, $\left\{(X,Y) \mid \frac{X^2}{4} + \frac{Y^2}{3} = 1\right\}$.
   c. $X = x - \frac{3}{2}$, $Y = y + 3$, $\{(X,Y) \mid 4X^2 + 6Y^2 = 0\}$. The graph is the single point $(0,0)$ in the XY-system or $(\frac{3}{2},-3)$ in the xy-system.

3. a. $\{(X,Y) \mid XY = -6\}$, $(2,-\frac{3}{2})$. c. $\{(X,Y) \mid XY = 1\}$, $(1,-4)$.

Sec. 14, page 42

1. a. $V(0,0)$, $F(\frac{1}{4},0)$; $(\frac{1}{4},\frac{1}{2})$, $(\frac{1}{4},-\frac{1}{2})$; $\{(x,y) \mid x = -\frac{1}{4}\}$.
   c. $V(0,0)$, $F(0,1)$; $(-2,1)$, $(2,1)$; $\{(x,y) \mid y = -1\}$.
   e. $V(-1,-\frac{1}{2})$, $F(-1,\frac{1}{2})$; $(-3,\frac{1}{2})$, $(1,\frac{1}{2})$; $\{(x,y) \mid y = -\frac{3}{2}\}$.
   g. $V(-2,0)$, $F(0,0)$; $(0,-4)$, $(0,4)$; $\{(x,y) \mid x = -4\}$.
   i. $V(2,-3)$, $F(2,-1)$: $(-2,-1)$, $(6,-1)$; $\{(x,y) \mid y = -5\}$.

2. a. In this case $p = 2$ and this parabola is the graph of $\{(x,y) \mid y^2 = 4 \cdot 2x\} = \{(x,y) \mid y^2 - 8x = 0\}$. Thus, $A = 0, C = 1, D = -8, E = F = 0$.
   c. $A = 1$, $C = 0$, $D = -2$, $E = -12$, $F = 25$. e. $A = 1$, $C = 0$, $D = -4$, $E = 8$, $F = -20$. g. $A = 0$, $C = 9$, $D = -96$, $E = 18$, $F = 73$.

5. $(4,256)$.

6. a. $\delta = \sqrt{1.25} - 1$. c. $\delta = \sqrt{9.1} - 3$. d. $\delta = \sqrt{9 + \epsilon} - 3$.

Sec. 15, page 46

1. a. $e = \frac{5}{3}$, $(0,0)$, $(\pm 10,0)$, $\{(x,y) \mid x = \pm\frac{18}{5}\}$, $(\pm 6,0)$, $\{(x,y) \mid 3y = \pm 4x\}$.
c. $e = \frac{3}{5}$, $(0,0)$, $(0,\pm 6)$, $\{(x,y) \mid y = \pm\frac{50}{3}\}$, $(0,\pm 10)$, $(\pm 8,0)$.
e. $e = \frac{3}{5}$, $(2,-3)$; $(-1,-3)$ and $(5,-3)$, $\{(x,y) \mid x = \frac{31}{3}$ or $x = -\frac{19}{3}\}$, $(7,-3)$ and $(-3,-3)$, $(2,1)$ and $(2,-7)$.
g. $e = \frac{5}{3}$, $(2,-4)$; $(2,1)$ and $(2,-9)$, $\{(x,y) \mid y = -\frac{11}{5}$ or $y = -\frac{29}{5}\}$, $(2,1)$ and $(2,-7)$, $\{(x,y) \mid 3x - 4y = 22$ or $3x + 4y = -10\}$.

2. a. $h = k = 0$, $a = 1$, $e = \frac{3}{4}$. c. $h = 2$, $k = -1$, $a = 5$, $e = \frac{4}{5}$.
e. $h = 2$, $k = -3$, $a = 3$, $e = \frac{5}{3}$.

3. $\sqrt{(x - ae)^2 + y^2} + \sqrt{(x + ae)^2 + y^2} = 2a$, $\sqrt{(x - ae)^2 + y^2} = 2a - \sqrt{(x + ae)^2 + y^2}$,
$x^2 - 2aex + a^2e^2 + y^2 = 4a^2 - 4a\sqrt{(x + ae)^2 + y^2} + x^2 + 2aex + a^2e^2 + y^2$,
$4a\sqrt{(x + ae)^2 + y^2} = 4a^2 + 4aex$, $\sqrt{(x + ae)^2 + y^2} = a + ex$,
$x^2 + 2aex + a^2e^2 + y^2 = a^2 + 2aex + e^2x^2$, $x^2(1 - e^2) + y^2 = a^2(1 - e^2)$,
$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

Sec. 17, page 53

1. a. 10. c. $-\frac{22}{13}$. e. $\frac{44}{5}$. g. 4.

4. a. Let ϵ be an arbitrary positive number. The point in f with ordinate $(\frac{11}{3}) + \epsilon$ has abscissa $x = \dfrac{15 - 3(\frac{11}{3} + \epsilon)}{2} = \dfrac{15 - 11 - 3\epsilon}{2} = 2 - \dfrac{3}{2}\epsilon$. Also, the point $(2 + \frac{3}{2}\epsilon, \frac{11}{3} - \epsilon)$ is in f. The portion of the graph between these points is a line segment. Thus,
$$\left\{(x,y) \,\middle|\, 0 < |x - 2| < \frac{3}{2}\epsilon,\ y = \frac{15 - 2x}{3}\right\}$$
$$\subset \left\{(x,y) \,\middle|\, 0 < |x - 2| < \frac{3}{2}\epsilon,\ \left| y - \frac{11}{3}\right| < \epsilon\right\}.$$
Hence $\delta = 3\epsilon/2$.

c. Let L be any number, take ϵ such that $0 < \epsilon < \frac{1}{2}$, and let δ be any positive number. Take x_1 such that $1 < x_1 < 2$ and $2 - \delta < x_1 < 2$ and take x_2 such that $2 < x_2 < 3$ and $2 < x_2 < 2 + \delta$. Then $f(x_1) = 1$ and $f(x_2) = 2$ so certainly at least one of $f(x_1)$ or $f(x_2)$ differs from L by more than ϵ. Thus, f does not have a limit at $c = 2$.

Given $c = 2.1$, let $\epsilon > 0$ be arbitrary and choose $\delta = 0.1$. Hence, if $|x - 2.1| < \delta$, then $f(x) = 2$ so $|f(x) - 2| = 0 < \epsilon$. Hence,
$$\{(x,y) \mid 0 < |x - 2.1| < \delta,\quad y = f(x)\} \subset \{(x,y) \mid 0 < |x - 2.1| < \delta,\quad |y - 2| < \epsilon\},$$
so $\lim\limits_{x \to 2.1} f(x)$ exists and $= 2$.

Given $c = 1.9$, let $\epsilon > 0$ be arbitrary, again take $\delta = 0.1$, etc.

Sec. 19, page 57

2. a. 2. c. 19. e. 4.

3. a. $f(u(x)) = \dfrac{3(3x + 5)/(x - 3) + 4}{2(3x + 5)/(x - 3) - 6} = \text{etc.} = \dfrac{13x + 3}{28}$, $x \neq 3$, $x \neq -\dfrac{5}{3}$,
$\therefore \lim\limits_{x \to 3} f(u(x)) = \dfrac{3}{2}$. c. 2.

Sec. 21, page 64

1. a. $y - \sqrt{2} = (\sqrt{2}/4)(x - 1), y - 1 = (\frac{1}{2})x$. c. $y - 12 = 12(x - 2)$, $y - 12 = -12(x + 2)$. e. $y - 8 = 10(x - 2), 3y + 1 = 0$. g. $y - 8 = 10(x - 2), y - 8 = -10(x + \frac{4}{5})$.

2. a. 2, all x. c. $\dfrac{-2x}{(x^2 - 1)^2}, \{x \mid x \neq \pm 1\}$. e. $\dfrac{-1}{2(x + 3)^{3/2}}, \{x \mid x > -3\}$.
g. $\dfrac{x + 6}{2(x + 3)^{3/2}}, \{x \mid x > -3\}$. i. $x + 3$, all x. k. $\dfrac{x}{\sqrt{x^2 - 4}} + \dfrac{1}{2\sqrt{x - 1}}, \{x \mid x > 2\}$.

Sec. 23, page 67

1. a. 12 ft/sec; -12 ft/sec; 0 ft/sec. d. 16 ft.

2. a. 6. c. $\dfrac{-1}{(t + 1)^2}$. e. $\dfrac{1}{2(20 - t)^{3/2}}$.

3. a. $f'(x) = 2x$. c. $f'(x) = 4x^3$. e. $f'(x) = \dfrac{1}{3}\dfrac{1}{x^{2/3}}$. g. $f'(x) = 1 + 3x^2$.
i. $f'(x) = 1 + \dfrac{1}{2\sqrt{x + 3}}$. k. $f'(x) = \dfrac{x}{2\sqrt{x + 3}} + \sqrt{x + 3} = \dfrac{3x + 6}{2\sqrt{x + 3}}$.

Sec. 24, page 73

1. a. $20x^3 - 12x$. c. $2(x^2 \cos x + 2x \sin x)$. e. $6x^2(x^3 + 2)$. g. $x \sin x$.

2. a. $\pi \cos \pi t$. c. $u^3 - u$. e. $v \cos v + \sin v$.

Sec. 25, page 75

1. a. $\dfrac{1}{2\sqrt{x}}(-2x \sin x + \cos x)$. c. $-\dfrac{2}{x^3}(x \sin 2x + \cos 2x)$.
e. $\frac{1}{3}x^{-2/3} + \frac{1}{2}x^{-1/2}$. g. $x^3 + x^{-3/4}$.

2. a. $-8 \cos^3 2x \sin 2x$. c. $20 \sin^4 (4x + 2) \cos (4x + 2)$.
e. $5(2x^3 + 7x + 1)^4(6x^2 + 7)$. g. $4x^7 \sin^3 x(x \cos x + 2 \sin x)$.

3. a. $\sin^2 2x(6x \cos 2x + \sin 2x)$. c. $3(x^2 + \cos x)^2(2x - \sin x)$.
e. $x \sin 2x + \sin^2 x$. g. $6x(3x - 6)^6(9x - 4)$.

4. a. $-2x \sin x \sin 2x + x \cos x \cos 2x + \sin x \cos 2x$.
c. $-2\sqrt{x} \sin 5x \cos x \sin x + 5\sqrt{x} \cos 5x \cos^2 x + \dfrac{1}{2\sqrt{x}} \sin 5x \cos^2 x$.

Sec. 26, page 79

2. a. $6x$. c. $\frac{4}{3}(x \sin x^2)^{1/3}(2x^2 \cos x^2 + \sin x^2)$.
e. $\frac{5}{2}(x^2 + x \sin x)^{3/2}(2x + x \cos x + \sin x)$. g. $6x(x^2 + 1)^2 + 1$.
h. $4[6x(x^2 + 1)^2 + 1][(x^2 + 1)^3 + x]^3$.

4. a. $2(x - 2) = 2x - 4$, same derivative since $(x - 2)^2$ and $x^2 - 4x$ differ by a constant.
c. $\dfrac{1}{2} \sin x, 2\left(\sin \dfrac{x}{2} \cos \dfrac{x}{2}\right)\dfrac{1}{2} = \dfrac{1}{2}\left(2 \sin \dfrac{x}{2} \cos \dfrac{x}{2}\right) = \dfrac{1}{2} \sin x$.

5. a. $\dfrac{x \cos x - \sin x}{x^3}$. c. $-\dfrac{x + 2}{2x^2\sqrt{x + 1}}$. e. $\dfrac{\sin x - (1 + x \cos x)}{\sin^2 x}$.

7. a. $-\dfrac{1}{x^2}$. c. $\dfrac{4x}{(x^2 + 1)^2}$. e. $-2\dfrac{\cos x}{\sin^3 x} = -2 \csc^2 x \cot x$.

Sec. 27, page 82

1. a. $\dfrac{2}{x^3}$. c. $x + 1$. e. $\dfrac{3x^2 + 12x + 8}{4x^3(x + 1)^{3/2}}$. g. $\dfrac{2}{(x + 1)^3}$. i. 0.

2. a.
$$\lim_{h\to 0} \frac{1}{h^2} [(x + 2h)^3 - 2(x + h)^3 + x^3]$$
$$= \lim_{h\to 0} \frac{1}{h^2} [x^3 + 3x^2(2h) + 3x(2h)^2 + (2h)^3 - 2(x^3 + 3x^2h + 3xh^2 + h^3) + x^3]$$
$$= \lim_{h\to 0} \frac{1}{h^2} [6xh^2 + 6h^3] = \lim_{h\to 0} (6x + 6h) = 6x.$$

c.
$$\lim_{h\to 0} \frac{1}{h^2} [\sin (x + 2h) - 2 \sin (x + h) + \sin x]$$
$$= \lim_{h\to 0} \frac{1}{h^2} \{[\sin (x + 2h) - \sin (x + h)] - [\sin (x + h) - \sin x]\}$$
$$= \lim_{h\to 0} \frac{1}{h^2}\left\{2 \cos \left(x + \frac{3}{2}h\right) \sin \frac{h}{2} - 2 \cos \left(x + \frac{h}{2}\right) \sin \frac{h}{2}\right\}$$
$$= \lim_{h\to 0} \frac{2}{h^2}\left(\sin \frac{h}{2}\right)\left[\cos \left(x + \frac{3}{2}h\right) - \cos \left(x + \frac{h}{2}\right)\right]$$
$$= \lim_{h\to 0} \frac{2}{h^2}\left(\sin \frac{h}{2}\right)\left[-2 \sin (x + h) \sin \frac{h}{2}\right]$$
$$= \lim_{h\to 0} \left[\frac{\sin (h/2)}{h/2}\right]^2 [-\sin (x + h)] = -\sin x.$$

4. a. CASE 1. For $x > 0$ and $x + h > 0$, then $(x + h)\,|x + h| = (x + h)^2$ and $x\,|x| = x^2$ so
$$\lim_{h\to 0} \frac{(x + h)\,|x + h| - x\,|x|}{h} = \lim_{h\to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h\to 0} (2x + h) = 2x.$$

CASE 2. For $x < 0$ and $x + h < 0$, then $(x + h)\,|x + h| = -(x + h)^2$ and $x|x| = -x^2$ so
$$\lim_{h\to 0} \frac{(x + h)\,|x + h| - x|x|}{h^2} = \lim_{h\to 0} \frac{-(x + h)^2 + x^2}{h} = \lim_{h\to 0} (-2x - h) = -2x.$$

CASE 3. For $x = 0$, then
$$\lim_{h\to 0} \frac{(0 + h)\,|0 + h| - 0\,|0|}{h} = \lim_{h\to 0} \frac{h\,|h|}{h} = \lim_{h\to 0} |h| = 0.$$

Sec. 29, page 86

1. a. $y = 5$, $y + 1 = -10(x - 1)$, $y + 5 = 22(x + 1)$. c. $y + 13 = 6(x - 3)$.

2. a. $y + 3 = 6(x - 1)$. c. $y + 39 = 42(x + 5)$. e. $6x - y + 23 = 0$.

3. a. $y - \frac{8}{9} = - \frac{49}{108}(x - 3)$. c. $x + y = 1$, $x - y = 4$. e. $y + \dfrac{\sqrt{3}}{4} = -\dfrac{7}{4}\left(x - \dfrac{\pi}{3}\right)$.

4. Two points where the graph crosses the x-axis, and a point on the graph between them where the tangent has slope 0 are:

a. $(0,0)$, $(\sqrt{3},0)$, and $(1,-2)$; also $(-\sqrt{3},0)$, $(0,0)$, and $(-1,2)$.

c. $(\pi/4,0)$, $(5\pi/4,0)$, and $(3\pi/4,\sqrt{2})$. e. $(0,0)$, $(\pi,0)$, and $(\pi/3,3\sqrt{3}/2)$.

6. Let $\epsilon > 0$ be arbitrary. By left continuity of f at c choose $\delta_1 > 0$ such that
$\{(x,y) \mid c - \delta_1 < x \le c, y = f(x)\} \subset \{(x,y) \mid c - \delta_1 < x \le c, f(c) - \epsilon < f(x) < f(c) + \epsilon\}$.
By right continuity of f at c choose $\delta_2 > 0$ such that
$\{(x,y) \mid c \le x < c + \delta_2, y = f(x)\} \subset \{(x,y) \mid c \le x < c + \delta_2, f(c) - \epsilon < f(x) < f(c) + \epsilon\}$.
Let δ be the smaller of δ_1 and δ_2. Then $\delta > 0$ and
$$\{(x,y) \mid c - \delta < x < c + \delta, y = f(x)\}$$
$$\subset \{(x,y) \mid c - \delta < x < c + \delta, f(c) - \epsilon < f(x) < f(c) + \epsilon\}.$$
Hence, $\lim_{x \to c} f(x)$ exists and $= f(c)$; i.e., f is continuous at c.

Sec. 30, page 90

1. a. $-0.79, 2.12$. b. 2.33. c. 0.51. d. 1.39.

2. a. $f(x) = x^2 - 12x^2 + 45x - 35$, $x_1 = 1$, $x_2 = 1.042$, $f(x_2) = -0.008$, in absolute value $< 5 \times 10^{-2}$.
b. $f(x) = x^2 - 12x + 22$, $x_1 = -4$, $x_2 = -4.16$, $f(x_2) = -0.072$, $x_2 = -4.158$, $f(x_2) = 0.009$. c. $f(x) = \cos x - x$, $x_1 = \pi/4 = 0.7854$, $x_2 = 0.7393$, $f(x_2) = 0.0004$.
d. $f(x) = x^2 - 1 - \sin x$, x_1 the radian measure of 80°; i.e., $x_1 = 1.3963$.
$f(x_1) = -0.0380$, $x_2 = 1.4065$, $f(x_2) = -0.0080$, $x_2 = 1.4096$,
$f(x_2) = 1.9870 - 1.9870 = 0$ to five decimal places.
For the other "solution" use x_1 the radian measure of $-40°$ so
$x_1 = -0.6981$, $x_2 = -0.6379$, $f(x_2) = 0.0024 < 5 \times 10^{-2}$.

3. With x_1 an approximation of $\sqrt[3]{A}$, then $x_{n+1} = \dfrac{1}{3}\left(2x_n + \dfrac{A}{x_n^2}\right)$, $n = 1, 2, 3, \cdots$.

4. With x_1 an approximation of $1/A$, then $x_{n+1} = 2x_n - Ax_n^2$, $n = 1, 2, 3, \cdots$. (Hint: Use $f(x) = x^{-1} - A$.)

Sec. 31, page 94

1. a. $0, \pi/2, \pi$; max. $f(\pi/2) = 1$, min. $f(0) = f(\pi) = 0$.
c. $-1, 0, 1$; max. $f(-1) = f(1) = 1$, min. $f(0) = 0$.
e. $0, 3, 5, 6$; max. $f(6) = \sqrt{6}$, min. $f(3) = -\sqrt{3}\sqrt[3]{2}$.
g. $0, \pi/4, 5\pi/4, 2\pi$; max. $f(\pi/4) = \sqrt{2}$, min. $f(5\pi/4) = -\sqrt{2}$.
i. $0, 2\pi/3, \pi$; max. $f(2\pi/3) = \pi/3 + \sqrt{3}/2$, min. $f(0) = 0$.

2. a. $5 - \sqrt{7}$, $2 + 2\sqrt{7}$, $8 + 2\sqrt{7}$, $V = 8(10 + 7\sqrt{7})$.

3. a. 2 in. × 4 in. × 8 in.

4. a. $r = R\sqrt{\tfrac{2}{3}}$, $h = 2R/\sqrt{3}$, $V = 4\pi R^3/(3\sqrt{3})$. c. With x the radius of the base
$$A(x) = 2\pi(x^2 + 2x\sqrt{R^2 - x^2}),\ A'(x) = 4\pi\frac{x\sqrt{R^2 - x^2} + R^2 - 2x^2}{\sqrt{R^2 - x^2}}.$$
$A'(x) = 0$ if $x\sqrt{R^2 - x^2} = 2x^2 - R^2$ which seems to have positive solutions $x = R\sqrt{\tfrac{1}{2} \pm \sqrt{5}/10}$, but note that the minus sign gives an extraneous root because, with x positive, $x\sqrt{R^2 - x^2} = 2x^2 - R^2$ requires $2x^2 - R^2 \ge 0$. Maximum of A is $\pi R^2(1 + \sqrt{5})$.
The above difficulties may be avoided by expressing A in terms of an angle in the right triangle having vertex at the center of the sphere, hypotenuse R, and sides the radius and semi-altitude of the cylinder.

5. a. $4\sqrt{2} \times 4\sqrt{2}$. c. $b = 5, h = 5$. e. $b = 6, h = 8$.

6. a. $x = 2$, min. $5\sqrt{2}$. c. $x = \frac{10}{3}$, min. $(116)/3$. e. $x = 5$, min. 5.

7. a. Side $\perp$ to line of revolution is 12, other 6.

8. a. Section through $(1/\sqrt{2}, 0)$, area $\pi/4$ units^2.

9. a. $(1/\sqrt{m}, \sqrt{m})$, $(2/\sqrt{m}, 0)$, $A(m) = 1$ unit^2.

Sec. 32, page 100

1. a. Rel. min. $f(4) = -26$, horiz. infl. $(1,1)$. c. Rel. min. $f(2) = -26$, horiz. infl. $(5,1)$.

3. 15 ft $\times \frac{45}{8}$ ft.

5. $15\sqrt{2}$ ft $\times 5\sqrt{2}$ ft.

Sec. 33, page 104

1. a. Down $x < 1$, up $x > 1$; infl. $(1, -4)$.
c. Down $0 \leq x < \pi/4$ and $3\pi/4 < x \leq \pi$, up $\pi/4 < x < 3\pi/4$; infl. $(\pi/4, 1)$, $(3\pi/4, 1)$.
e. Down $0 < x < 2$ and $2 < x$, up $x < 0$; vertical infl. $(0,0)$.
(Note: The point $(2,0)$ is a cusp for this graph.)

2. a. Rel. min. $f(2) = 0$, rel. max. $f(-\frac{2}{3}) = \frac{256}{27}$.

c.

$f' = 0$	$-\pi$	$-3\pi/4$	$-\pi/4$	0	$\pi/4$	$3\pi/4$	π
f''	$+$	$-$	$+$	$-$	$+$	$-$	$+$
f	min. 2	max. $2\sqrt{2}$	min. $-2\sqrt{2}$	max. -2	min. $-2\sqrt{2}$	max. $2\sqrt{2}$	min. 2

3. a. Base side $\sqrt[3]{2V}$, depth $\sqrt[3]{V/4}$. c. Base side $\sqrt[3]{2bV/a}$, depth $\sqrt[3]{a^2V/(4b^2)}$.

4. a. Radius $\sqrt[3]{V/2\pi}$, alt. $\sqrt[3]{4V/\pi}$.

7. $(-1, -1)$, $(1, 1)$.

8. a. Rectangle $\dfrac{2s}{4 + \pi} \times \dfrac{s}{4 + \pi}$.

Sec. 34, page 107

1. 27. **3.** 948. **5.** a. 29 units at $76.67 apiece.

Sec. 35, page 109

1. a. 3π ft/sec, 15π ft^2/sec. **2.** a. $-4/\sqrt{15}$ ft/sec. c. -3 ft/sec.

3. $(576)/\sqrt{189} = 41.9$ mi/hr, $(592)/\sqrt{61} = 75.8$ mi/hr.

5. a. $6\sqrt[3]{t/\pi}$. c. $0.5\pi^{-1/2}$ ft/min. **6.** a. $h(t) = 4\sqrt[3]{t/\pi}$. c. π min.

7. $50\sqrt{3} = 86.6$ mi, $-90\sqrt{3} = -155.9$ mi/hr; $5\sqrt{223} = 74.6$ mi, $(1920)/\sqrt{223}$
$= 128.5$ mi/hr.

Sec. 36, page 112

1. $0.75\sqrt{A/\pi}$ in.^{2}/min, A in.^{3} is the area at given instant. **3.** $1/(6\pi)$ ft/min.

5. $1/(160)$ lb/in.^{2}/min. **7.** a. ± 600 ft/sec. **9.** If $D_t v = ks$, then $D_t r = k$.

11. About 5.4 ft/min. **13.** About 13.4 ft/sec.

Sec. 37, page 115

1. a. $v(t) = (3t+1)(t-3)\begin{cases} > 0 & \text{if} \quad -5 \le t < -\frac{1}{3} \text{ or } \quad 3 < t \le 5 \\ < 0 & \text{if} -\frac{1}{3} < t < 3, \end{cases}$

$\alpha(t) = 2(3t-4)\begin{cases} > 0 & \text{if} \quad \frac{4}{3} < t \le 5 \\ < 0 & \text{if} \quad -5 \le t < \frac{4}{3}. \end{cases}$

Starts at $s(-5) = -200$ with velocity $v(-5) = 112$, slows down until $s(-\frac{1}{3}) = \frac{284}{27}$ is reached, reverses direction, speeds up to rel. max. speed $|v(\frac{4}{3})| = |-\frac{25}{3}|$ at $s(\frac{4}{3}) = \frac{34}{27}$, then slows down until $s(3) = -8$ is reached, again reverses direction, speeds up to velocity $v(5) = 32$ at $s(5) = 20$.

c. $v(t) = 6\cos 2t \begin{cases} > 0 & \text{if} \quad 0 \le t < \pi/4 \quad \text{or} \quad 3\pi/4 < t \le \pi \\ < 0 & \text{if} \quad \pi/4 < t < 3\pi/4, \end{cases}$

$\alpha(t) = -12\sin 2t \begin{cases} > 0 & \text{if} \quad \pi/2 < t < \pi \\ < 0 & \text{if} \quad 0 < t < \pi/2. \end{cases}$

Starts at $s(0) = 0$ with velocity $v(0) = 6$, slows down until $s(\pi/4) = 3$ is reached, reverses direction, speeds up to max. speed $|v(\pi/2)| = |-6|$ at $s(\pi/2) = 0$, slows down until $s(3\pi/4) = -3$ is reached, again reverses direction and reaches $s(\pi) = 0$ with velocity $v(\pi) = 6$.

2. a. $v(t) = 6(t-3)(t+2)\begin{cases} > 0 & \text{if} \quad -5 \le t < -2 \quad \text{or} \quad 3 < t \le 5 \\ < 0 & \text{if} \quad -2 < t < 3. \end{cases}$

$\alpha(t) = 6(2t-1), > 0$ if $\frac{1}{2} < t \le 5, < 0$ if $-5 \le t < \frac{1}{2}$.

c. $v(t) = \sin 2t \begin{cases} > 0 & \text{if} \quad 0 < t < \pi/2 \quad \text{or} \quad \pi < t < 3\pi/2 \\ < 0 & \text{if} \quad \pi/2 < t < \pi \quad \text{or} \quad 3\pi/2 < t < 2\pi \end{cases}$

$\alpha(t) = 2\cos 2t \begin{cases} > 0 & \text{if} \quad 0 \le t < \pi/4, 3\pi/4 < t < 5\pi/4 \quad \text{or} \quad 7\pi/4 < t \le \pi \\ < 0 & \text{if} \quad \pi/4 < t < 3\pi/4 \quad \text{or} \quad 5\pi/4 < t < 7\pi/4. \end{cases}$

3. a. $v(-2) = v(1) = -8$ rel. min. and min., $v(-1) = 8$ rel. max., $v(3) = 72$ max.
$\alpha(0) = -12$ rel. min. and min., $\alpha(3) = 96$ max., $\alpha(-2) = 36$ rel. max.

c. $v(0) = 1$ rel. max., $v(\pi/8) = 3 - 2\sqrt{2}$ min., $v(5\pi/8) = 3 + 2\sqrt{2}$ max., $v(\pi) = 1$ rel. min.
$\alpha(0) = -4$ rel. min., $\alpha(3\pi/8) = 4\sqrt{2}$ max., $\alpha(7\pi/8) = -4\sqrt{2}$ min.,
$\alpha(\pi) = -4$ rel. max.

Sec. 38, page 117

1. a. $3, \pi, -\frac{1}{2}$. c. $5, 2, 1$. e. $0.5, \frac{1}{60}, \frac{1}{240}$.

2. a. $x = \sqrt{2}\cos(t - \pi/4), \sqrt{2}, 2\pi, \pi/4$; or $x = \sqrt{2}\sin(t + \pi/4), \sqrt{2}, 2\pi, -\pi/4$.

c. $x = 5\cos\pi(t - \alpha/\pi)$ where $\alpha = \tan^{-1}(\frac{4}{3}), 5, 2, \alpha/\pi$; or
$x = 5\sin\pi(t + \beta/\pi)$ where $\beta = \tan^{-1}(0.75), 5, 2, -\beta/\pi$.

e. $x = 0.5 + 0.5\cos 2t$ so $X = 0.5\cos 2t$ where $X = x - 0.5$; $0.5, \pi, 0$.

g. $x = \cos 2t$; $1, \pi, 0$.

3. a. $v(t) = -ab\sin b(t - t_0)$ or $v(t) = ab\cos b(t - t_0)$; $|ab|, 2\pi/|b|, t_0$.

c. $-\frac{1}{2}a^2b\sin 2b(t - t_0)$ or $\frac{1}{2}a^2b\sin 2b(t - t_0)$; $|\frac{1}{2}a^2b|, \pi/|b|, t_0$.

Sec. 40, page 122

1. a. $f(x) = \frac{1}{2}x^2 + 1$. c. $f(x) = \frac{2}{3}x^{3/2} + \frac{4}{3}$. e. $f(x) = -\frac{1}{2}\cos 2x + \frac{3}{4}$.
g. $f(x) = \frac{1}{2}x^2 - \cos x + 3$. i. $f(x) = \frac{1}{6}(25 - \cos^3 2x)$.

2. a. $s(t) = \frac{3}{2}t^2 - 2$. c. $s(t) = 4t - t^3 - 3$. e. $s(t) = -\frac{1}{3}\cos 3t + \frac{13}{6}$.
g. $s(t) = \frac{3}{2}\sin\frac{2}{3}t - 3\cos\frac{5}{3}t + 4$.

3. a. $f(x) = \frac{3}{2}x^2 - x$. c. $f(x) = 1 + x - \cos x$. e. $f(x) = \frac{1}{2}\sin 2x - 2\sin x + 2$.

4. a. $f(2) = -\frac{10}{3}$ min., $f(-3) = 17.5$ max. c. $f(1) = 2$ min., no max.
e. $f(x) = 1 + x/\sqrt{x+1}$ has no rel. max. or min.

5. a. $s(t) = -16.1t^2 + 10t + 25$. c. $s(t) = 2 - \cos 3t + \sin 2t$.

Sec. 41, page 126

1. a. -1. c. 1. e. 0. **2.** a. $4,5.5$. c. $-\pi/180,0$.

3. a. $[(2x + 1)\cos x + 2\sin x]\,dx$. c. $(8x\sin^3 x^2 \cos x^2)\,dx$. e. $2\sin 2x\,dx$.
g. $(-2\sin^3 x + \cos^2 x)\cos x\,dx$. i. $2\cos 2x\,dx$. k. $-\sin x\cos(\cos x)\,dx$.

4. a. $(18x + 8)(dx)^2$. c. $2(1 + 1/x^3)(dx)^2$. e. $[2\cos 2x + \csc x(2\csc^2 x - 1)](dx)^2$.

Sec. 42, page 128

1. a. $\frac{1}{6}[\cos 1 - \cos(3x^2 + 1)] + 2$. c. $\frac{1}{9}[(3x^2 + 1)^{3/2} + 19]$. e. $\frac{1}{3}(\sqrt{3x^2 + 1} - 2)$.

2. a. $x\sin x\,dx$. c. $\dfrac{-4^2}{x^2\sqrt{x^2 + 4^2}}\,dx$. e. $y = \dfrac{1}{4^2}\left(\dfrac{91}{3} - \dfrac{\sqrt{x^3 + 4^2}}{x}\right)$.
g. $y = \sin x - x\cos x + 2$.

3. a. $y = \frac{1}{6}(x^4 + 3)^{3/2} + c$. c. $y = \frac{1}{8}x^8 + \frac{3}{4}x^4 + c$. e. $y = \sin\left(x + \dfrac{1}{x}\right) + c$.

Sec. 44, page 133

1. a. $\frac{8}{7} = 1.143^-$. c. $(0.71)/0.7 = 1.014^+$. **2.** a. -4.25. c. 31.20. e. 3.033.

4. $dg = -(8\pi^2 l/T^3)\,dT$, $(dg)/g = -2(dT)/T$.

5. a. 0.3 units. **6.** $1.25 + 0.01/32$.

Sec. 45, page 136

1. a. $\dfrac{-\csc^2 x}{2\sqrt{\cot x}}$. c. $x\sec^2 x + \tan x$. e. $\sec x(2\sec^2 x - 1)$. g. $\sin x(1 + \sec^2 x)$.
i. $-4x\csc(2x^2 + 1)\cot(2x^3 + 1)$ k. $-2\csc^2 x\cot x$.

3. $60°$. **7.** $(a^{3/3} + b^{2/3})^{3/2}$ ft. **9.** $(\frac{8}{3})a^2\pi$, a = radius of sphere.

Sec. 46, page 141

1. a. $\dfrac{5}{\sqrt{1 - 25x^3}}$. c. $\dfrac{1}{2\sqrt{x}(1 + x)}$. e. $\dfrac{\cot x}{\sqrt{\sin^2 x - 1}}$, obtained formally, but actually $f(x) = \sec^{-1}(\sin x)$ is defined only at isolated values so this function does not have a derivative.

3. a. $\frac{1}{2}\tan^{-1}\dfrac{x}{2} + 5 - \dfrac{\pi}{8}$. c. $\dfrac{1}{2}\left(x\sqrt{3 - x^2} + 3\sin^{-1}\dfrac{x}{\sqrt{3}}\right) + \dfrac{\pi}{2} + \dfrac{3\sqrt{3}}{8}$.

Sec. 49, page 148

2. a. $y = 2(\frac{3}{2})^x$. c. $y = \frac{4}{3}(\frac{3}{2})^x$. e. $y = 1.76(\frac{4}{3})^x$. **3.** c. $y = 6^x$.

4. $\lim\limits_{x\to\infty}[10\log(2^x + 3) - 10\log 2^x] = 10\lim\limits_{x\to\infty}\log\left(1 + \dfrac{3}{2^x}\right) = 0$.

5. a. $(0,2.1)$, $(7,35)$, $y = 2.1(1.495)^x$. c. $(10,5.4)$, $(66,0.1)$, $y = 11.01(0.9316)^x$.

Sec. 51, page 153

1. All complete straight lines through the origin (1,1) with respective slopes 1, 2, $\frac{1}{2}$, -1, $-\frac{1}{2}$, -2.
2. a. $y = 2x^{0.585}$. c. $y = 1.5x^2$. 3. c. $y = x^5$. 5. c. $y = 3.45x^5$.
7. a. (1,23), (10,3.8), $y = 23x^{-0.7819}$. c. (0.7,1.6), (20,15), $y = 2.03x^{0.868}$.
8. a. Semi-log, (0,2.5), (5.7,32), $y = 2.5(1.56)^x$. b. Rect. $y = (-\frac{2}{7})x + 2$.
   c. Semi-log, (0.5,3), (5.7,0.2), $y = 3.89(0.594)^x$.
   d. log-log, (0.1,0.62), (1.5,3), $y = 0.237x^{0.582}$.

Sec. 52, page 157

1. a. $\dfrac{3}{x}$. c. $1 + \ln|3x|$. e. $\dfrac{2}{x^2+1} - \dfrac{1}{x^2}\ln(x^2+1)$. g. $\dfrac{1}{x(1+\ln^2|x|)}$. i. $\dfrac{2}{x}\ln|x|$.
4. a. $f(x) = \dfrac{1}{6}\ln\left|\dfrac{x-3}{x+3}\right| + 1 + \dfrac{1}{6}\ln 5$.
   c. $f(x) = \ln(x + \sqrt{x^2-9}) - 2\ln 3$. e. $y = x(\ln|x| - 1) + 1$.
5. a. $D_xy = 1/x$, $D_x^2y = -1/x^2$, hence graph is concave downward.
6. a. $D_xy = \dfrac{1}{x-1} + \dfrac{3}{2}\dfrac{x^2}{x^2+3} - \dfrac{1}{2}\dfrac{x}{x^2+1}$.

Sec. 53, page 159

1. a. $-e^{-x}$, e^{-x}. c. $\sec^2 x e^{\tan x}$, $\sec^2 x e^{\tan x}(\sec^2 x + 2\tan x)$.
   e. $(-1/x^2)e^{1/x}$, $(1/x^4)e^{1/x}(2x+1)$. g. $(2x + x^2)e^x$, $(2 + 4x + x^2)e^x$.
2. a. $(2\cos 2x - \sin 2x)\,e^{-x}\,dx$. c. $(e^x + e^{-x})\,dx$. e. $3(\ln 10)10^{3x}\,dx$.
4. a. $\left(1, \dfrac{1}{e}\right)$ max. c. $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}e^{-1/2}\right)$ max., $\left(-\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}e^{-1/2}\right)$ min.
5. a. (0,1). c. $\{(x,y) \mid y = 0\}$. 6. a. $y = ex$. c. $y = 2ex$.
7. a. $y' = e^{-x}(\cos x - \sin x)$, $y'' = -2e^{-x}\cos x$. Concave down for $0 \le x < \pi/2$ and $3\pi/2 + 2m\pi < x < 5\pi/2 + m2\pi$, concave up for $\pi/2 + 2m\pi < x < 3\pi/2 + 2m\pi$. Rel. max. and min. $\left(\dfrac{\pi}{4} + 2m\pi, \dfrac{1}{\sqrt{2}}e^{-(\pi/4+2m\pi)}\right)$, $\left(\dfrac{5\pi}{4} + 2m\pi, -\dfrac{1}{\sqrt{2}}e^{-(5\pi/4+2m\pi)}\right)$.
   Asymptote is x-axis.
8. a. $f(x) = \frac{1}{2}e^{2x} + c$. c. $f(x) = \dfrac{10^x}{\ln 10} + c = 10^x \log e + c$.

Sec. 54, page 162

1. a. $x^x(1 + \ln x)$, $x^x(1 + \ln x)^2 + x^{x-1}$. c. $3(10x)^{2x}(1 + \ln 10x)$.
   $3(10x)^{2x}[x^{-1} + 3(1 + \ln 10x)^2]$.
   e. $(\ln x)^x\left[\dfrac{1}{\ln x} + \ln(\ln x)\right]$, $(\ln x)^x\left[\dfrac{-1}{x(\ln x)^2} + \dfrac{1}{x\ln x} + \left(\dfrac{1}{\ln x} + \ln\ln x\right)^2\right]$.
   g. $|\sin x|^x(x\cot x + \ln|\sin x|)$, $|\sin x|^x[-x\csc^2 x + 2\cot x + (x\cot x + \ln|\sin x|)^2]$.
   i. $2x^{2x}(1 + \ln|x|)$, $2x^{2x}[x^{-1} + 2(1 + \ln|x|)^2]$.
2. a. $\dfrac{\sin x}{x^2+1}\left(\cot x - \dfrac{2x}{x^2+1}\right)$. c. $\dfrac{\sin x\sqrt{1+\cos^2 x}}{\tan^2 x}\left(\cot x - \dfrac{\sin x\cos x}{1+\cos^2 x} - \dfrac{3}{\sin x\cos x}\right)$.

Sec. 57, page 173

1. a. $\int_{-1}^{1} x^2\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(-1 + \frac{2k}{n}\right)^2 \frac{2}{n} = 2 \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(1 - \frac{4}{n}k + \frac{4}{n^2}k^2\right)$

$$= 2 \lim_{n\to\infty} \frac{1}{n}\left[n - \frac{4}{2n}(n^2 + n) + \frac{4}{6n^2}(2n^3 + 3n^2 + n)\right]$$

$$= 2 \lim_{n\to\infty}\left[1 - 2\left(1 + \frac{1}{n}\right) + \frac{2}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)\right] = 2\left[1 - 2 + \frac{4}{3}\right] = \frac{2}{3},$$

$$\int_{1}^{2} x^2\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(1 + \frac{k}{n}\right)^2 \frac{1}{n} = \text{etc.} = \frac{7}{3},$$

$$\int_{-1}^{2} x^2\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(-1 + \frac{3}{n}k\right)^2 \frac{3}{n} = \text{etc.} = 3, \quad \frac{2}{3} + \frac{7}{3} = 3.$$

c. $\int_{2}^{4} (t-1)^2\,dt = \lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{2}{n}k - 1\right)^2 \frac{2}{n} = \text{etc.} = \frac{26}{3},$

$$\int_{2}^{4} u^2\,du - 2\int_{2}^{4} v\,dv + \int_{2}^{4} 1\cdot ds = \lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{2}{n}k\right)^2 \frac{2}{n}$$

$$- 2 \lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{2}{n}k\right)\frac{2}{n} + \lim_{n\to\infty} \sum_{k=1}^{n} (1)\frac{2}{n}$$

$$= 2 \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[4 + \frac{8}{n}k + \frac{4}{n^2}k^2 - 2\left(2 + \frac{2}{n}k\right) + 1\right]$$

$$= 2 \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[1 + \frac{4}{n}k + \frac{4}{n^2}k^2\right]$$

$$= 2 \lim_{n\to\infty} \frac{1}{n}\left[n + \frac{4}{2n}(n^2 + n) + \frac{4}{6n^2}(2n^3 + 3n^2 + n)\right]$$

$$= \text{etc.} = 2(1 + 2 + \tfrac{4}{3}) = 2(\tfrac{13}{3}) = \tfrac{26}{3}.$$

e. $(63)(\frac{15}{2}) \neq \frac{765}{4}$.

2. a. $\frac{125}{3}$. c. $\frac{1}{3}$.

Sec. 59, page 178

1. a. 55. c. $\ln|t+2|\big]_{-3}^{-4} = \ln 2$. e. $\frac{u^2}{2} - \cos u\big]_0^{\pi} = \frac{\pi^2}{2} + 2$.

g. $-\frac{2}{3}\cos^{3/2} x\big]_0^{\pi/2} = \frac{2}{3}$. i. $\sec x\big]_0^{\pi/4} = \sqrt{2} - 1$. k. $-\ln|\cos x|\big]_{\pi/4}^{\pi/3} = \ln\sqrt{2}$.

2. a. $\frac{1}{2}(x^2 - 1)$. c. $21x$. e. $\sin x$. g. $\int_1^3 (x^2 - 8)\,dx = 4$. i. $\frac{133}{6}$.

3. a. $\frac{16}{3}$ units². c. $\pi^2/2$ units².

4. $\ln 2 = \int_1^2 \frac{dx}{x} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1 + k\frac{2-1}{n}} \frac{1}{n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n+k}\cdot\frac{1}{n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}$

$$= \lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right].$$

5. a. $\int_1^2 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}]_1^3 = \frac{2}{3}(2\sqrt{2} - 1)$ or $\int_0^1 \sqrt{1 + x}\,dx = \frac{2}{3}(1 + x)^{3/2}]_0^1 = \frac{2}{3}(2\sqrt{2} - 1)$.

c. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n + 2k} = \frac{1}{2} \lim_{n\to\infty} \sum_{n=1}^{n} \frac{1}{1 + 2k/n} \cdot \frac{2}{n} = \frac{1}{2}\int_0^2 \frac{dx}{1 + x}$ or $\frac{1}{2}\int_1^3 \frac{dx}{x} = \frac{1}{2}\ln 3$.

e. $\int_0^{3\pi/4} \sin\left(\frac{\pi}{4} + x\right) dx$ or $\int_{\pi/4}^{\pi} \sin x\,dx = 1 + \frac{1}{\sqrt{2}}$.

Sec. 60, page 182

1. a. $\frac{94}{15}$. c. $\frac{20}{3}$. e. $\frac{403}{84}$.

g. $\frac{7}{9}\left(\text{Hint: } \frac{d}{dx}\frac{1}{9}(1 + 3x^2)^{3/2} = x\sqrt{1 + 3x^2}\right)$. i. $\frac{128}{35}$.

2. a. $c = 3$, area of $\{(x,y) \mid 0 \le x \le 3, 0 \le y \le x^2\}$ is 9 units^2.

c. $c = 10/\sqrt[3]{2}$. Area of $\{(x,y) \mid 0 \le x \le 10/\sqrt[3]{2}, 0 \le y \le x^2\}$ is equal to area of $\{(x,y) \mid 10/\sqrt[3]{2} \le x \le 10, 0 \le y \le x^2\}$.

e. $c = 4.5$. $\{(x,y) \mid 0 \le x \le 1, 0 \le y \le x + 4.5\}$ has area 5 units^2.

3. a. $c = \frac{30}{13}$. c. $c = \frac{1}{3}$.

4. a. $= \int_1^2 (x - \ln\sin x + \ln\sin x)\,dx = \int_1^2 x\,dx = \frac{3}{2}$.

c. $= \left[\int_1^2 x\sin x\,dx - \int_1^3 x\sin x\,dx + \int_2^3 x\sin x\,dx\right] + \int_1^3 x\,dx + \int_2^3 x\,dx$
$= [0] + [x^2/2]_1^3 + [x^2/2]_2^3 = \frac{13}{2}$.

Sec. 61, page 186

1. a. $\frac{1}{6}$. c. $\frac{1}{3}$. e. $\frac{125}{6}$. g. $\sqrt{2} - 1$. i. $6 - 8\ln 2$.

2. Parts a, b, and c should not be worked out by calculus since a and c were used in deriving the derivative (and hence integral) formulas for trigonometric and inverse trigonometric functions. Also, b can be obtained from c by subtracting the area of a triangle. d. πab. e. $(\frac{8}{3})p^3$, $|p|$ = vertex to focus distance. f. $(1 - c + c\ln c)(a^{-1} + b^{-1})$.

6. a. $\{(x,y) \mid 0 \le x \le \sin^{-1} t, 0 \le y \le \sin x\} = \{(x,y) \mid 0 \le y \le t, \sin^{-1} y \le x \le \sin^{-1} t\}$ so

$\int_0^{\sin^{-1} t} \sin x\,dx = \int_0^t (\sin^{-1} t - \sin^{-1} y)\,dy,$

$-\cos x]_0^{\sin^{-1} t} = \sin^{-1} t[y]_0^t - \int_0^t \sin^{-1} y\,dy,$

$-\cos\sin^{-1} t + \cos 0 = t\sin^{-1} t - 0 \cdot \sin^{-1} t - \int_0^t \sin^{-1} y\,dy,$

$\int_0^t \sin^{-1} y\,dy = t\sin^{-1} t + \cos\sin^{-1} t - \cos 0 = t\sin^{-1} t + \sqrt{1 - t^2} - 1.$

Sec. 62, page 190

2. a. $62.5\pi 8 \times 10^3$. c. $\pi r^3 h(H/3 + h/4)62.5$. e. $6.25\pi 3.8 \times 10^4$.
g. $6.25\pi 2.5 \times 10^4$. i. $(62.5)32$. j. $62.5(32 + 60\pi)$. k. 8.20×10^4.

3. $62.5\pi(7.2) \times 10^3 = 1.41 \times 10^5$.

Sec. 64, page 194

1. a. $62.5(3968)/3 = 8.27 \times 10^4$. c. $62.5(2048)/3 = 4.27 \times 10^4$.

2. a. 375. c. $(62.5)24 = 1.5 \times 10^3$. **3.** a. $62.5(27\pi)$. c. $62.5\pi abc$.

4. $\int_0^4 (10 + \frac{1}{2}x)\{(4 - x)62.5 + 5 \cdot 50\}\,dx + \int_4^9 50(10 + \frac{1}{2}x)(9 - x)\,dx$, x measured up.

5. a. $-\pi/2$. c. $-\frac{3}{4} + 2 \ln 2$. (Hint: Set $u(x) = \ln x$.) e. $\frac{20}{3}$.
g. $-\frac{1}{3}[(25 - x^2)^{3/2}]_0^3 = \frac{61}{3}$. i. $e - 2$.

6. a. $L = 4\sqrt{2} = 5.656$. c. $d = 8/\sqrt[3]{2} = 4^{4/3} = 6.35$.

7. a. $[x \sin x + \cos x]_0^{\pi/2} + \int_0^{\pi/2} 0 \cdot (-\cos x)\, dx = \dfrac{\pi}{2} - 1$.

Sec. 65, page 199

1. a. $(1,\frac{8}{5})$. c. $(\frac{2}{5}\sqrt[3]{b},\frac{4}{7}b)$. e. $(\frac{17}{5},1)$.
g. $\bar{x} = \dfrac{\ln 4 - (\frac{3}{4})}{\ln 4 - 1}, \bar{y} = \dfrac{\ln^2 2 - \ln 4 + 1}{\ln 4 - 1}$ (Hint: Find the area as the denominator of $\bar{y}$. See Prob. 5(c), Sec. 64 for the numerator of $\bar{x}$.)

3. a. On radius of symmetry $(\frac{4}{3})r/\pi$ from center. c. $(0,\pi/8)$.

4. 3 ft 9 in. from top of gate.

6. a. $(\frac{11}{6}, \frac{7}{6})$. c. $\left(\dfrac{25 + 8\pi}{10 + 4\pi}, \dfrac{47 + 12\pi}{30 + 12\pi}\right)$ or about (2.22,1.25).
e. $\bar{x} = (27 - \pi)r/(18 - \pi) = 1.61r$, $\bar{y} = 0$.

Sec. 66, page 203

1. $\dfrac{a^3b}{3}$. **3.** $\dfrac{ab}{12}(a^3 + ac + c^2)$. **5.** $\dfrac{a^3b}{16}\pi$. **7.** $\frac{16}{15}p^4$.

Sec. 67, page 206

1. a. $\frac{1024}{5}\pi$, $\bar{x} = \frac{10}{3}$. c. $\frac{128}{3}\pi$, $\bar{y} = \frac{16}{5}$. e. $\frac{22}{15}4^4\pi$, $\bar{x} = \frac{35}{11}$.

2. a. $V = (\frac{4}{3})\pi r^3$, centroid at the center.

c. $V = \dfrac{\pi}{3} h(R^2 + Rr + r^2)$, centroid $\dfrac{h}{4}\,\dfrac{R^2 + 2Rr + 3r^2}{R^2 + Rr + r^2}$ from R-base.

e. $V = \pi h\left(r^2 - a^2 - ah - \dfrac{h^2}{3}\right)$, $\bar{x} = \left(a + \dfrac{h}{2}\right)\dfrac{r^2 - a^2 - ah - h^2/2}{r^2 - a^2 - ah - h^2/3}$.

g. $V = (\pi/2)^2$, $\bar{x} = \pi/4 + 1/\pi$.

3. a. $\pi r^2 \dfrac{H^3}{3}\rho$. c. $\dfrac{\pi r^2 H^3}{30}\rho, \dfrac{\pi r^2 H^3}{5}\rho$. e. $\dfrac{1}{2}bh\dfrac{H^3}{30}\rho, \dfrac{1}{2}bh\dfrac{H^3}{5}\rho$.

4. Centroid 2 in. above center of base. 144ρ, 432ρ.

Sec. 68, page 209

1. a. 2. b. Does not exist. c. 6. d. Does not exist. e. 3. f. 2.

g. Does not exist. h. $\frac{3}{2}(2^{2/3} - 1)$. i. $\dfrac{\pi}{2}$. $\left[\text{Hint: } D_x \sin^{-1} x = \dfrac{1}{\sqrt{1 - x^2}}.\right]$

4. $\int_1^t x^{-2}\, dx = -\dfrac{1}{t} + 1$ if $t > 0$. $-\dfrac{1}{t} + 1 = 4$ has solution $t = -\frac{1}{3}$. $\int_t^{-1/3} x^{-2}\, dx$ is improper and does not exist.

Sec. 69, page 214

1. a. $\frac{1}{4}x^4 - \frac{1}{2}x^2 + 5x + c$. c. $\frac{2}{5}x^{5/2} - 2x^{3/2} + c$. e. $\frac{1}{3}(x^2 + 1)^{2/2} + c$.
g. $x - \ln|x + 1| + c$.

$$\left(\text{Hint: Set } u = x + 1 \text{ or else write } \frac{x}{x+1} = \frac{x+1-1}{x+1} = 1 - \frac{1}{x+1}.\right)$$

i. $-\ln|\cos x| + c$. k. $\frac{1}{b}\frac{2}{3}(a + bx)^{2/2} + c$. m. $\frac{2}{3}x^{3/2} - 2\sqrt{x} + c$. o. $\frac{2}{3}\sin^2\frac{x}{2} + c$.
q. $\frac{1}{\pi}\sin(\pi x + 2) + c$. s. $\frac{1}{2}\sqrt{a^2 + x^4} + c$.

2. a. $2a\sqrt{t} + bt + c$. c. $-\frac{2}{3}\cos^3 t + c$. e. $\frac{2\sqrt{2}}{5} z^{5/2} + c$. g. $\frac{1}{a}\ln(e^{at} + e^{-at}) + c$.

Sec. 70, page 216

1. a. $(-\frac{1}{6})\cos 3x^2 + c$. c. $\ln|\sec e^x + \tan e^x| + c$.
e. $2\sec\sqrt{x} + c$. g. $-\cot x + \csc x + c$.

2. a. See 1g. c. $-\csc x - \cot x - x + c$. e. $\sec x - \tan x + x + c$.

3. a. $-\frac{1}{2}(\frac{1}{3}\cos 3x + \cos x) + c$. c. $\frac{1}{2}(\frac{1}{8}\sin 8x + \frac{1}{2}\sin 2x) + c$.

4. a. $\frac{1}{2}(\frac{1}{3}\cos^2 2x - \cos 2x) + c$. c. $\frac{1}{7}\sin^7 x - \frac{2}{5}\sin^5 x + \frac{1}{3}\sin^2 x + c$.
e. $2\sqrt{\sin x} - (\frac{2}{5})(\sin x)^{5/2} + c$.

5. $$\int \tan^2 x\,dx = \frac{\tan^{2-1} x}{2-1} - \int \tan^{2-2} x\,dx = \tan x - \int dx = \tan x - x + c.$$

$$\int \tan^4 x\,dx = \frac{\tan^3 x}{3} - \int \tan^2 x\,dx = \frac{\tan^2 x}{3} - \tan x + x + c.$$

Sec. 71, page 219

1. a. $\sin^{-1}\frac{x}{2} + c$. c. $\frac{1}{3}\sec^{-1}\frac{x+3}{3} + c$.
e. $\frac{1}{4}x - \frac{1}{8}\ln(4x^2 + 4x + 2) + \frac{1}{2}\tan^{-1}(2x + 1) + c$.

2. a. $2\ln|x^2 + 4x - 3| - \frac{5}{2\sqrt{7}}\ln\left|\frac{x + 2 - \sqrt{7}}{x + 2 + \sqrt{7}}\right| + c$. c. $\sin^{-1}\frac{x+2}{\sqrt{7}} + c$.

e. $x - 2\ln|x^2 + 4x - 12| + \frac{5}{2}\ln\left|\frac{x-2}{x+6}\right| + c$

g. $-\frac{1}{6}\ln(3x^2 - 2x + 1) - \frac{1}{3\sqrt{2}}\tan^{-1}\frac{3x-1}{\sqrt{2}} + c$.

Sec. 73, page 223

1. a. $\frac{1}{3}e^{2x} + c$. c. $\frac{10^x}{\ln 10} + c$. e. $\frac{(2e)^x}{1 + \ln 2} + c$. g. $2\sinh x + c$. i. $\frac{1}{5}e^{5x} + c$.

2. a. $\frac{x}{25\sqrt{25 + x^2}} + c$. c. $-\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{a} + c$. e. $\frac{3x^2 - 2}{15}(1 + x^2)^{2/2} + c$.

3. a. $4\pi - 3\sqrt{3}$. c. $16\pi + 6\sqrt{3}$.

Sec. 74, page 225

2. a. $x \sin x + \cos x + c$. c. $\frac{1}{2} \sin x^2 + c$. e. $x \cosh x - \sinh x + c$.
g. $e^{2x}(\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8}) + c$. i. $\frac{1}{2}x^2 \ln |x| - \frac{1}{4}x^2 - x \cos x + \sin x + c$.
k. $x(\ln^2 |x| - 2 \ln |x| + 2) + c$.

3. a. $\frac{2}{13}e^{3x}(\sin 2x + \frac{3}{2} \cos 2x) + c$. c. $\frac{1}{8}(\sin 3x \cos x - 3 \sin x \cos 3x) + c$.

4. $A = \dfrac{p}{a^2 + p^2} e^{ax}\left(\sin px + \dfrac{a}{p} \cos px\right) + c$.

Sec. 75, page 228

1. a. 33_1 with $a = 3, b = 2$. c. 73_2 with $a = 3, b = 2, c = 1, q = -8$. e. 148.

2. a. $-\frac{1}{4} \sin^3 x \cos x + \frac{3}{8}(x - \sin x \cos x) + c$.
c. $(x + 2)^{3/2}(\frac{2}{9}x^3 - \frac{8}{21}x^2 + \frac{64}{105}x - \frac{256}{315}) + c$.
e. $\dfrac{3x - 1}{6} \sqrt{-3x^2 + 2x + 1} + \dfrac{2}{3\sqrt{3}} \sin^{-1} \left(\dfrac{3x - 1}{2}\right) + c$.

3. a. $2 \ln (\sqrt{x} + 1) + c$. c. $\ln |x + 3\sqrt{x} - 2| - \dfrac{3}{\sqrt{17}} \ln \left|\dfrac{2\sqrt{x} + 3 - \sqrt{17}}{2\sqrt{x} + 3 + \sqrt{17}}\right| + c$.
e. $12\left\{\sum\limits_{k=1}^{26} \dfrac{1}{k} x^{k/12} + \ln (x^{1/12} - 1)\right\} + c$.

Sec. 76, page 232

1. a. $\frac{1}{6} \ln |2x + 1| + \frac{1}{3} \ln |x - 1| + c$. b. $\dfrac{1}{2} \ln |2x + 3| - \ln |x + 1| - \dfrac{3}{x + 1} + c$.
c. $\dfrac{9}{2\sqrt{15}} \tan^{-1} \dfrac{4x + 3}{\sqrt{15}} + \dfrac{5}{4} \ln (2x^2 + 3x + 3) - 2 \ln |x| + c$.
d. $\ln |x| - \frac{1}{2} \ln (2x^2 + 3x + 3) - \dfrac{3}{5} \dfrac{2x - 1}{2x^2 + 3x + 3} - \dfrac{27}{5\sqrt{15}} \tan^{-1} \dfrac{4x + 3}{\sqrt{15}} + c$.

2. a. $\ln \left|\dfrac{x + 1}{x + 2}\right| + c$. c. $\ln |x| - \ln \sqrt{x^2 + 1} + c$.
e. $\ln \left|\dfrac{x}{x - 1}\right| - \dfrac{3}{2} \dfrac{1}{x - 1} + \dfrac{1}{2} \dfrac{1}{x + 1} + c$.

3. $\frac{1}{5}(\ln 3 - 3\ln 2) = -0.196$.

5. a. $A = \frac{1}{6}$; $\sum\limits_{k=1}^{n} k^2 = \frac{1}{6}n(n + 1)(2n - 1)$.
c. $A = \frac{1}{10}$, $B = -\frac{1}{30}$; $\sum\limits_{k=1}^{n} k^4 = \frac{1}{30}n(n + 1)(2n + 1)(3n^2 + 3n - 1)$.

Sec. 77, page 236

1. a. $4 + 25 \sin^{-1} \frac{4}{5}$, $25\pi - 4 - 25 \sin^{-1} \frac{4}{5}$. c. $\frac{1}{2}(\tan^{-1} \frac{5}{2} + \tan^{-1} \frac{3}{2})$.
e. $a^2(e - e^{-1})$. g. $a^2/6$. i. $20 \ln 3$.

2. a. $\dfrac{\pi}{4}(181e^{20} - 1)$. c. $2\pi^2$. e. $\dfrac{\pi}{4} a^3(e^2 - e^{-2} + 4)$. g. $4\pi\ln 2 - \frac{3}{2}\pi$.

3. a. $\bar{x} = \dfrac{b-1}{\ln b}, \bar{y} = \dfrac{b-1}{2b \ln b}$. c. $\bar{x} = \dfrac{\pi}{2}, \bar{y} = \dfrac{7\pi}{8(\pi - 1)}$.

e. $\bar{x} = \dfrac{2}{\ln 3}, \bar{y} = \dfrac{25(\pi/2 - \sin^{-1} 0.6)}{30 \ln 3}$. g. $\bar{x} = \bar{y} = \dfrac{256a}{315\pi}$.

i. $\bar{x} = \dfrac{28}{9\pi} a, \bar{y} = \dfrac{28}{9\pi} b$.

4. a. (i) $62.5\pi 2070$. (ii) $62.5(2250)\pi$.

5. a. (i) $r^4\pi/4$. (ii) $5r^4\pi/4$. c. $2r^4\left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right)$.

7. a. $(7.5)62.5$. b. $62.5[60 + (\frac{5}{2})^3(\pi/2 - \frac{2}{3})]$.

Sec. 78, page 242

1. a. Let P be the mid-point of $\overrightarrow{BC}$. Then $\overrightarrow{AM} = \frac{2}{3}\overrightarrow{AP} = \frac{2}{3}(\overrightarrow{AB} + \overrightarrow{BP}) = \frac{2}{3}(\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC})$. Also, $\overrightarrow{BM} = \frac{2}{3}(\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA})$, $\overrightarrow{CM} = \frac{2}{3}(\overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB})$, and by addition
$\overrightarrow{AM} + \overrightarrow{BM} + \overrightarrow{CM} = \frac{2}{3}[(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) + \frac{1}{2}(\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB})] = \frac{2}{3}[\vec{0} + \frac{1}{2}\vec{0}] = \vec{0}$.

b. $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$, $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$, $\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM}$, and by addition
$3\overrightarrow{OM} = (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) + (\overrightarrow{AM} + \overrightarrow{BM} + \overrightarrow{CM}) = (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) + \vec{0}$
by Part a.

c. $\overrightarrow{MM'} = \overrightarrow{MA} + \overrightarrow{AA'} + \overrightarrow{A'M'}$ with two similar equations and by addition the result follows by Part a.

d. $a = \frac{6}{7}, b = \frac{5}{7}$. (Hint: $\overrightarrow{AC} + \overrightarrow{CQ} = \overrightarrow{AQ}$, now substitute the given expressions for $\overrightarrow{AQ}$ and $\overrightarrow{CQ}$, then express everything in terms of $\overrightarrow{AB}$ and $\overrightarrow{BC}$, and then equate the coefficients of each of these vectors on both sides of the equation.)

2. a. Let Q_1 and Q_2 be the mid-points of $\overrightarrow{AC}$ and $\overrightarrow{BD}$, resp. Then
$\overrightarrow{AQ_1} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$, $\overrightarrow{AQ_2} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CD} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} - \frac{1}{2}\overrightarrow{AB}$
$= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$ so $\overrightarrow{AQ_1} = \overrightarrow{AQ_2}$ and hence $Q_1 = Q_2$.

b. $a = b = \frac{1}{3}$. (Hint: $\overrightarrow{AP} + \overrightarrow{PQ} = \overrightarrow{AQ}$.)

c. $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$, etc., $4\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} + (\overrightarrow{AM} + \overrightarrow{BM} + \overrightarrow{CM} + \overrightarrow{DM})$. Parenthetic terms have resultant $\vec{0}$ since $\overrightarrow{CM} = -\overrightarrow{AM}$ and $\overrightarrow{DM} = -\overrightarrow{BM}$.

3. Let Q be the intersection of the diagonals so $\overrightarrow{AQ} = \overrightarrow{QC}$ and $\overrightarrow{BQ} = \overrightarrow{QD}$. Then
$\overrightarrow{AB} = \overrightarrow{AQ} + \overrightarrow{QB} = \overrightarrow{QC} - \overrightarrow{BQ} = \overrightarrow{QC} - \overrightarrow{QD} = -(\overrightarrow{CQ} + \overrightarrow{QD}) = -\overrightarrow{CD} = \overrightarrow{DC}$, etc. for other sides.

4. a. Write $\vec{v}_1 + 2\vec{v}_2 - 3\vec{v}_3 = \vec{0}$ as $-\vec{v}_3 + \vec{v}_1 = 2(-\vec{v}_2 + \vec{v}_3)$ and see that $\overrightarrow{CA} = 2(\overrightarrow{BC})$ showing that $\overrightarrow{CA}$ and $\overrightarrow{BC}$ have the same direction and hence lie on a line, since C is common to both segments.

Sec. 80, page 247

1. a. $(\vec{u} \cos \alpha + \vec{v} \sin \alpha)^2 = \vec{u}^2 \cos^2 \alpha + 2\vec{u} \cdot \vec{v} \cos \alpha \sin \alpha + \vec{v}^2 \sin^2 \alpha = \cos^2 \alpha + 0 + \sin^2 \alpha = 1$ since $|\vec{u}| = |\vec{v}| = 1$ and $\vec{u} \cdot \vec{v} = 0$. Also, $(\vec{u} \sin \alpha - \vec{v} \cos \alpha)^2 = 1$.
$(\vec{u} \cos \alpha + \vec{v} \sin \alpha) \cdot (\vec{u} \sin \alpha - \vec{v} \cos \alpha) = \vec{u}^2 \cos \alpha \sin \alpha + \vec{u} \cdot \vec{v}(-\cos^2 \alpha + \sin^2 \alpha) - \vec{v}^2 \sin \alpha \cos \alpha = \cos \alpha \sin \alpha + 0(-\cos^2 \alpha + \sin^2 \alpha) - \sin \alpha \cos \alpha = 0$.

c. $a^2 + b^2 = 1, c^2 + d^2 = 1, ac + bd = 0$.

3. (Hint: With $\vec{u}$ and $\vec{v}$ two adjacent sides, the sum of the squares of the moduli of the diagonals is $(\vec{u} + \vec{v})^2 + (\vec{u} - \vec{v})^2$.)

4. a. With $\vec{u}$, $\vec{v}$, and $\vec{u} + \vec{v}$ the sides of the triangle with $|\vec{u}| = |\vec{v}|$, then the median from the initial end of $\vec{u}$ is $\vec{u} + \frac{1}{2}\vec{v}$, whereas the median from the mid-point of $\vec{u}$ is $\frac{1}{2}\vec{u} + \vec{v}$. Since $(\vec{u} + \frac{1}{2}\vec{v})^2 = \vec{u}^2 + \vec{u}\cdot\vec{v} + \frac{1}{4}\vec{v}^2 = \frac{5}{4}|\vec{u}|^2 + \vec{u}\cdot\vec{v}$ and
$(\frac{1}{2}\vec{u} + \vec{v})^2 = \frac{1}{4}\vec{u}^2 + \vec{u}\cdot\vec{v} + \vec{v}^2 = \frac{5}{4}|\vec{u}|^2 + \vec{u}\cdot\vec{v}$ the diagonals have the same length.

5. c. $(\vec{v} + t\vec{u})\cdot\vec{v} = 0$ has solution $t = -\dfrac{\vec{v}^2}{\vec{u}\cdot\vec{v}} = -\dfrac{|\vec{v}|}{|\vec{u}|}\sec\theta$.

7. (Hint: With M_1, M_2, M_2 the mid-points of $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{AC}$, let O be the intersection of the perpendiculars at M_1 and M_2, designate $\overrightarrow{M_1O} = v_1$, $\overrightarrow{M_2O} = v_2$, and $\overrightarrow{M_2O} = \vec{v}_2$. From triangles M_1M_2O, M_2M_3O, and M_2M_1O
$$\overrightarrow{M_1M_2} + \vec{v}_2 = \vec{v}_1, \overrightarrow{M_2M_2} + \vec{v}_2 = \vec{v}_2, \overrightarrow{M_2M_1} + v_1 = v_3.$$
Use $\overrightarrow{M_1M_2} = \frac{1}{2}\overrightarrow{AC}$, etc. Now dot both sides of the first of these equations by $\vec{v}_2$, the second by $\vec{v}_1$, the third by $\vec{v}_2$. The fact that $\overrightarrow{AB}\cdot\vec{v}_1 = 0$ and $\overrightarrow{BC}\cdot\vec{v}_2 = 0$ will reveal that $\overrightarrow{AC}\cdot\vec{v}_3 = 0$, thus showing that $\overrightarrow{AC}$ and $\vec{v}_2$ are perpendicular.)

Sec. 81, page 251

1. a. $-4\vec{i} + 7\vec{j}$. c. $\overrightarrow{OP} = \overrightarrow{OP_1} + \frac{1}{2}\overrightarrow{P_1P_2} = (\vec{i} - 2\vec{j}) + \frac{1}{2}(4\vec{i} + 9\vec{j}) = 3\vec{i} + 2.5\vec{j}$.
e. $-\sqrt{3}\vec{i} + \vec{j}$. g. $\dfrac{a\vec{i} + b\vec{j}}{\sqrt{a^2 + b^2}} = \dfrac{1}{\sqrt{17}}\vec{i} + \dfrac{4}{\sqrt{17}}\vec{j}$ since $\dfrac{b}{a} = D_x x^2]_{x=2} = 4$ and $a > 0$.

2. a. $\sqrt{2}$, $-45°$. c. 5, $-126° 52'$. e. 13, $67° 23'$.

3. a. 6. c. 0, the three points lie on a line. e. 6.

5. a. 45°. c. $\cos\theta = -1/\sqrt{65}$, $\theta = 97° 08'$.

6. a. $\cos\theta = \frac{15}{17}$, $\theta = 28° 03'$. c. $\cos\theta = \frac{1}{3}$, $\theta = 70° 32'$.
e. At the origin 90°; at (1,1) $\cos\theta = \frac{4}{5}$, $\theta = 36° 52'$.

8. a. $y' = 3x^2 - 12x + 12$, $m = 3 - 12 + 12 = 3$, $\vec{T} = \vec{i} + 3\vec{j}$.
$y'']_{x=1} = 6x - 12]_{x=1} < 0$ so curve concave down. Hence, rotate $\vec{T}$ through $-90°$ to obtain internal normal $\vec{N} = 3\vec{i} - \vec{j}$ and unit internal normal $\vec{n} = \dfrac{1}{\sqrt{10}}(3\vec{i} - \vec{j})$.
(Note: If curve is concave up, rotate forward tangent vector through 90° to obtain internal normal vector.) c. $\dfrac{2}{\sqrt{4 + e^2}}\vec{i} + \dfrac{e}{\sqrt{4 + e^2}}\vec{j}$.
e. $\dfrac{1}{\sqrt{7}}(\sqrt{3}\vec{i} + 2\vec{j})$.

Sec. 82, page 256

1. a. $x = a(\theta - \sin\theta)$, $y = a(-1 + \cos\theta)$.
b. $\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$, $\overrightarrow{CP} = b\vec{I}$ where $\vec{I}$ is $-\vec{i}$ rotated through ϕ.
$x = (a + b)\cos\theta - b\cos\dfrac{a + b}{b}\theta$, $y = (a + b)\sin\theta - b\sin\dfrac{a + b}{b}\theta$.
c. $x = (a - b)\cos\theta + b\cos\dfrac{a - b}{b}\theta$, $y = (a - b)\sin\theta - b\sin\dfrac{a - b}{b}\theta$.
d. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

e. $x = \frac{3}{4} a \cos \theta + \frac{1}{4} a \cos 3\theta = \frac{a}{4}[3 \cos \theta + \cos 3\theta] = \frac{a}{4}[2 \cos \theta + (\cos 3\theta + \cos \theta)]$

$= \frac{a}{4}[2 \cos \theta + 2 \cos 2\theta \cos \theta] = \frac{a}{2} \cos \theta\,(1 + \cos 2\theta) = a \cos^3 \theta$. Similarly

$y = a \sin^3 \theta$. $x^{2/3} + y^{2/3} = (a \cos^3 \theta)^{2/3} + (a \sin^3 \theta)^{2/3} = a^{2/3}(\cos^2 \theta + \sin^2 \theta) = a^{2/3}$.

2. All lie on the graph of $y = (\frac{3}{2})\,x + 1$. a. Half-line in first quadrant and end point, (0,1). c. Interval with end points (2,4), (−2,−2). e. Interval with end points (4,7) (0,1).

4. a. $\sqrt{5}/3$. c. $1 + \sqrt{5}/3$. **5.** $\pi/6$.

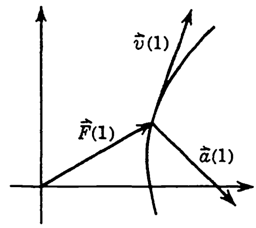
Sec. 83, page 261

1. a. $x\vec{i} + y\vec{j} = (5 - 2t)\vec{i} + (6 - 2t)\vec{j}$; $x = 5 - 2t, y = 6 - 2t$; $y = x + 1$.
c. $x\vec{i} + y\vec{j} = (\frac{1}{2})\vec{i} + (3 + t)\vec{j}$; $x = \frac{1}{2}, y = 3 + t$; $x = \frac{1}{2}$.
e. $x\vec{i} + y\vec{j} = (3 + t)\vec{i} + (-2 + mt)\vec{j}$; $x = 3 + t, y = -2 + mt$; $y + 2 = m(x - 3)$.

2. a. 4, same. c. 0, point is on line. e. $\frac{14}{3}$, opposite.

3. a. $\frac{5}{2}$. c. 7. **4.** a. 11. c. 29.

5. a. $t = -\dfrac{1.51}{2.01}, \left(\dfrac{11.57}{2.01}, -\dfrac{9.05}{2.01}\right)$. c. (−1,3).



Prob. 1a

Sec. 84, page 266

1. a. $\vec{F}(1) = 3\vec{i} + \frac{5}{3}\vec{j}$, $\vec{v}(1) = \vec{i} + 3\vec{j}$, $\vec{a}(1) = 2(\vec{i} - \vec{j})$,
$\vec{a}_T(1) = -\frac{2}{5}(\vec{i} + 3\vec{j})$, $\vec{a}_N(1) = \frac{4}{5}(3\vec{i} - \vec{j})$.
c. $\vec{F}(1) = \vec{i} + 2\vec{j}$, $\vec{v}(1) = -\vec{i} + 2\vec{j}$, $\vec{a}(1) = \vec{i} + 2\vec{j}$,
$\vec{a}_T(1) = \frac{3}{5}(-\vec{i} + 2\vec{j})$, $\vec{a}_N(1) = \frac{4}{5}(2\vec{i} + \vec{j})$.

2. a. $\vec{v}(t) = \vec{i}\,3 \cos t - \vec{j}\,2 \sin t$,
$\vec{a}(t) = -(\vec{i}\,3 \sin t + \vec{j}\,2 \cos t) = -\vec{F}(t)$,

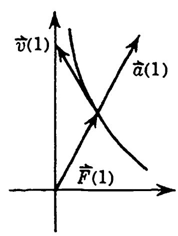
$$\vec{a}_T(t) = \frac{-5 \sin t \cos t}{9 \cos^2 t + 4 \sin^2 t}(\vec{i}\,3 \cos t - \vec{j}\,2 \sin t),$$

$$\vec{a}_N(t) = \frac{-6}{4 + 5 \cos^2 t}(\vec{i}\,2 \sin t + \vec{j}\,3 \cos t).$$

c. $\vec{v}(t) = 2\vec{i} + \vec{j}(2t + 1)$, $\vec{a}(t) = 2\vec{j}$,

$$(\vec{a}_T t) = \frac{2(2t + 1)}{4t^2 + 4t + 5}[2\vec{i} + \vec{j}(2t + 1)],$$

$$\vec{a}_N(t) = \frac{-4}{4t^2 + 4t + 5}[(2t + 1)\vec{i} - 2\vec{j}]$$



Prob. 1c

e. $\vec{v}(t) = \vec{i} \cosh t + \vec{j} \sinh t$, $\vec{a}(t) = \vec{F}(t)$, $\vec{a}_T(t) = (\tanh 2t)\,\vec{v}(t)$,
$\vec{a}_N(t) = (\operatorname{sech} 2t)\,(-\vec{i} \sinh t + \vec{j} \cosh t)$.

g. $\vec{v}(t) = \vec{i}t \cos t + \vec{j}t \sin t$, $\vec{a}(t) = \vec{i}(\cos t - t \sin t) + \vec{j}(\sin t + t \cos t)$,

$$\vec{a}_T(t) = \frac{1}{t}\,\vec{v}(t),\ \vec{a}_N(t) = -\vec{i}t \sin t + \vec{j}t \cos t.$$

5. a. $\vec{F}(t) = \vec{i}\,30t + \vec{j}\,(8t^2 + 100)$. c. $\vec{F}(t) = \vec{i}\,(e^t + 1) + \vec{j}e^{-2t}$.

7. $\vec{F}(t) = \vec{i}\,40t \cos \alpha + \vec{j}\,(40t \sin \alpha - \frac{1}{2}gt^2 + 6.5)$.

Sec. 85, page 271

1. a. $(x - 32)^2 + \left(y - \dfrac{25}{2}\right)^2 = \dfrac{17^3}{4}$. c. $(x + 4)^2 + (y - \frac{8}{3})^2 = \frac{250}{9}$.

e. $\left(x - \dfrac{\pi}{4} - \dfrac{3}{2}\right)^2 + (y + \sqrt{2})^2 = \dfrac{27}{4}$. g. $(x - 3)^2 + (y + 2)^3 = 8$.

i. $\overrightarrow{OC} = \vec{F}(2) + \vec{R}(2) = (3\vec{i} + 4\vec{j}) + 10(-2\vec{i} + \vec{j}) = -17\vec{i} + 14\vec{j}$ so equation of circle is $(x + 17)^2 + (y - 14)^2 = |R(2)|^2 = 10^2[(-2)^2 + 1] = 500$.

k. $(x + 2)^2 + (y + \frac{3}{4})^2 = \frac{125}{16}$.

3. a. (0,4). c. (0,4).

4. $y = 2^{-9}(9x^3 - \frac{1}{2}x^4 + \frac{1}{16}x^5)$.

5. $\kappa'(x) = 0$ for $x < 0$ or $x > 1$ whereas $\kappa'(x) = \dfrac{2}{(1 + x^4)^{5/2}}(1 - 5x^4)$ for $0 < x < 1$. Thus, neither $\lim\limits_{x\to 0} \kappa'(x)$ nor $\lim\limits_{x\to 1} \kappa'(x)$ exists.

Sec. 86, page 275

1. a. $e - e^{-1}$. c. $-0.9 + \ln 19 = 2.045$. e. $2\sqrt{2} - \sqrt{3}$.

3. a. $3\sqrt{10} - \sqrt{2} + \ln \dfrac{3 + \sqrt{10}}{1 + \sqrt{2}}$. c. $\dfrac{122}{27}$. e. $2\sqrt{2}$. **4.** a. $8a$.

Sec. 87, page 278

1. a. $D_x y = \frac{1}{2}t^2$, $D_x^2 y = \frac{1}{2}t$. c. $\dfrac{\cos\theta}{2\cos 2\theta}, \dfrac{2\cos\theta \sin 2\theta - \cos 2\theta \sin\theta}{4\cos^3 2\theta}$.

e. $\dfrac{e^{2t}}{1 - t}, \dfrac{e^{3t}(3 - 2t)}{(1 - t)^3}$. g. $\dfrac{e^{2t}}{t}, \dfrac{e^{2t}(2t - 1)}{2t^3}$.

2. a. (1,2), (−1,−2). c. (0,1).

3. c. With $x' = D_t x$, etc., since $\vec{\tau} = (x'^2 + y'^2)^{-1/2}(\vec{i}x' + \vec{j}y')$,

$$\frac{d\vec{\tau}}{dt} = (\vec{i}x' + \vec{j}y')\frac{d}{dt}(x'^2 + y'^2)^{-1/2} + (x'^2 + y'^2)^{-1/2}\frac{d}{dt}(\vec{i}x' + \vec{j}y')$$

$$= \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}(-\vec{i}y' + \vec{j}x'),$$

$$\frac{d\vec{\tau}}{ds} = \frac{d\vec{\tau}}{dt}\frac{dt}{ds} = \frac{d\vec{\tau}}{dt}\frac{1}{\sqrt{x'^2 + y'^2}} = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^2}(-\vec{i}y' + \vec{j}x').$$

$$\frac{d\vec{\tau}}{ds}\cdot\vec{R} = \left[\frac{x'y'' - x''y'}{(x'^2 + y'^2)^2}(-\vec{i}y' + \vec{j}x')\right]\cdot\left[\frac{x'^2 + y'^2}{x'y'' - x''y'}(-\vec{i}y' + \vec{j}x')\right]$$

$$= \frac{1}{x'^2 + y'^2}(-\vec{i}y' + \vec{j}x')^2 = \frac{1}{x'^2 + y'^2}[(-y')^2 + x'^2] = 1.$$

Sec. 88, page 281

1. a. and c. $3X^2 - Y^2 = 1$. e. $|X| + |Y| = 1$. g. No graph. i. $X = \pm 1$.

3. a. Center (0,0), vertices $(\mp 3, \pm 3)$, foci $(\mp\sqrt{2}, \pm\sqrt{2})$, ends of minor axis $(\pm\sqrt{7}, \pm\sqrt{7})$ directrices $y = x \pm 9\sqrt{2}$.

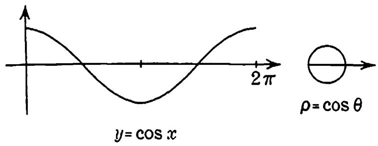
c. Center (0,0) vertices $(\pm 4/\sqrt{13}, \pm 6/\sqrt{13})$, foci $(\pm 2, \pm 3)$, asymptotes $12x + 5y = 0$, $y = 0$.

e. Vertex (0,0), focus $(\frac{3}{20}, \frac{1}{5})$, ends of right focal chord $(-\frac{1}{4}, \frac{1}{2})$, $(\frac{11}{20}, -\frac{1}{10})$.

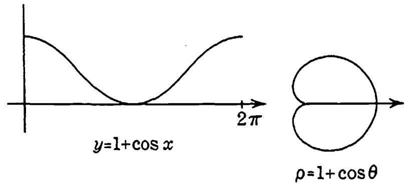
Sec. 89, page 285

1. a. Fig. 89.2 rotated 45°. c. Three leaved clover traced twice for $0 \leq \theta \leq 360°$.
   e. Spiral expanding as θ increases, approaching the pole as $\theta \to -\infty$.

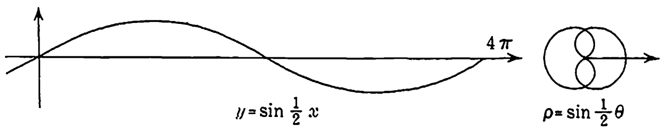
2.



Prob. 2a

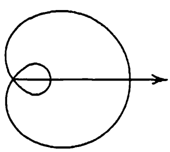


Prob. 2c

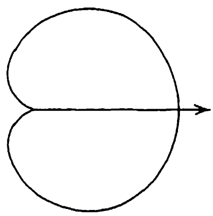


Prob. 2e

3.

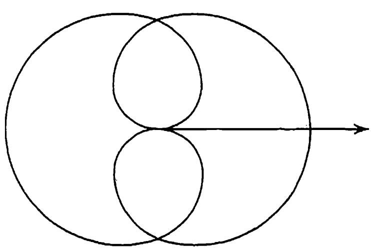


Prob. 3a



Prob. 3c

4.



Prob. 4

5. a. $(x^2 + y^2)^2 = 4(x^3 - 3xy^2)$. c. $(x - 2)^2 + (y - \frac{3}{2})^2 = (\frac{5}{2})^2$.

6. a. $\rho^2 - 4\rho\cos\theta + 6\rho\sin\theta + 9 = 0$. c. $\rho = \pm b + a\sec\theta$.

Sec. 90, A, B, page 287

1. a. Only $\rho = 4\cos\theta$. c. $\rho = \cos\frac{3}{2}\theta$, $\rho = -\sin\frac{3}{2}\theta$, $\rho = -\cos\frac{3}{2}\theta$, $\rho = \sin\frac{3}{2}\theta$.
e. $\rho = |\sin\theta|$, $\rho = -|\sin\theta|$.

2. a. $(1,0°)$, $(1,180°)$. c. $\rho = 1$ and angles $10°$, $50°$, $130°$, $170°$, $250°$, $290°$.
e. The pole and $(\sqrt{3}/2, 30°)$.

3. a. The graph has equations $\rho = \sin\frac{1}{2}\theta$, $\rho = -\cos\frac{1}{2}\theta$, $\rho = -\sin\frac{1}{2}\theta$, $\rho = \cos\frac{1}{2}\theta$, and when solved simultaneously in pairs give the pole and the simplest designations of the other points $(1/\sqrt{2}, 90°)$, $(1/\sqrt{2}, -90°)$. c. The pole.

Sec. 90, C, D, E, page 291

1. a. *Method* I. $\rho[5(\frac{3}{5}\cos\theta + \frac{4}{5}\sin\theta)] = 10$. Let ϕ be an angle such that $\cos\phi = \frac{3}{5}$ and $\sin\phi = \frac{4}{5}$. Then the equation is $\rho(\cos\phi\cos\theta + \sin\phi\sin\theta) = 2$ so that $\rho\cos(\phi - \theta) = 2$ which is in the form (5) with $\theta_1 = \phi$ and $\rho_1 = 2$. Thus, the graph is the straight line perpendicular to $\vec{v}_1 = 2(\vec{i}\cos\phi + \vec{j}\sin\phi)$ and passing through its terminal end. *Method* II. Change to rectangular coordinates: $3x + 4y = 10$.
c. $\sin\theta = \cos(\theta - 90°)$ so equation is $\rho = (\frac{2}{5})\cos(\theta - 90°)$ which is in the form (4). Graph is a circle with center $(\frac{1}{5}, 90°)$, radius $\frac{1}{5}$.
e. $\cos\theta + \sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right) = \sqrt{2}(\cos\theta\cos 45° + \sin\theta\sin 45°)$
$= \sqrt{2}\cos(\theta - 45°)$. Thus, the equation is $\rho\left[1 + \frac{4\sqrt{2}}{3}\cos(\theta - 45°)\right] = \frac{10}{3}$ which has the form of the second equation of (8). The eccentricity is $4\sqrt{2}/3 > 1$ so the graph is a hyperbola with transverse axis making an angle of $45°$ with the polar axis.
g. With ϕ as in Part (a), equation is $10 = \rho^2 - 5\rho\cos(\theta - \phi) = \rho^2 - 2(\frac{5}{2})\rho\cos(\theta - \phi)$, $10 + (\frac{5}{2})^2 = \rho^2 + (\frac{5}{2})^2 - 2(\frac{5}{2})\rho\cos(\theta - \phi)$ which is in the form (3) so the graph is a circle with center $(\frac{5}{2}, \phi)$ and radius $\sqrt{65}/2$.

2. a. $e = \frac{1}{2}$, so an ellipse. $q = 6$, $a = 4$, $b = 2\sqrt{3}$. Center $(2, 180°)$, vertices $(2, 0°)$, $(6, 180°)$, foci at pole and $(4, 180°)$, directrices perpendicular to polar axis at $(6, 0°)$, $(10, 180°)$.
c. $\rho[1 + \cos(\theta - 90°)] = \frac{5}{4}$. Since $e = 1$ graph is a parabola. Axis is vertical with vertex up. $eq = 1 \cdot q = \frac{5}{4} = 2p$ where, in the usual parabola notation, p is the distance between the focus and vertex. Hence, $p = \frac{5}{8}$ and the vertex is $(\frac{5}{8}, 90°)$. Right focal chord has length $4p$ and ends $(\frac{5}{4}, 0°)$, $(\frac{5}{4}, 180°)$.
e. $\rho(1 + \frac{5}{3}\cos\theta) = \frac{32}{3}$. Hyperbola since $e = \frac{5}{3} > 1$. $q = \frac{32}{5}$, $a = 6$, $ae = 10$, $\frac{a}{e} = \frac{18}{5}$, $b = 8$. Center $(10, 0°)$, foci at pole and $(20, 0°)$, vertices $(4, 0°)$, $(16, 0°)$, directrices $\perp$ polar axis at $(\frac{32}{5}, 0°)$, $(\frac{68}{5}, 0°)$.

3. a. $\rho(2 - \cos\theta) = 9$ or $\rho(2 + \cos\theta) = -9$. c. $\rho(1 - \cos\theta) = -6$ or $\rho(1 + \cos\theta) = 6$.
e. $\rho(8 - 5\cos\theta) = -39$ or $\rho(8 + 5\cos\theta) = 39$.

Sec. 91, A, page 293

1. a. $60°$. c. $40°\,53'$.

2. a. (Hint: Find alternate equation for first graph.)

3. a. $70°\,32'$; $90°$ at pole and at $\theta = \pi/4$. c. $90°$; $43°\,36'$ at $\theta = \frac{1}{2}$ radians.

4. a. [Hint: Use $f(\theta) = \sin\frac{1}{2}\theta$, $g(\theta) = \cos\frac{1}{2}\theta$]. $53°\,08'$ at $\theta = \pi/2$, $0°$ at pole.
c. At pole $0°$ and $60°$.

Sec. 91, B, page 296

2. a. $\frac{\pi^3}{3}$, $\pi\sqrt{1+\pi^2} + \frac{1}{2}\ln\frac{\pi+\sqrt{1+\pi^2}}{-\pi+\sqrt{1+\pi^2}} = \pi\sqrt{1+\pi^2} + \ln(\pi+\sqrt{1+\pi^2})$.
c. $3\pi/2$, 8.

3. a. 25π. c. $9\pi/2$.

4. a. $\sqrt{2} + \ln(1+\sqrt{2})$.

5. a. $\frac{1}{2}\int_{2\pi/3}^{4\pi/3}\rho^2\,d\theta = \pi - \frac{3}{2}\sqrt{3}$. c. $2(\frac{1}{2})\int_{-\pi/4}^{\pi/4}\rho^2\,d\theta = 1$. (There are two loops.)

Sec. 93, page 301

1. a. $(1, -2, 5)$. c. $(-2, -1, -3)$. e. $(5, -6, -3)$.

2. a. From origin to other two points $\vec{u} = \vec{i} + 2\vec{j} - 3\vec{k}$, $\vec{v} = 3\vec{i} + 3\vec{j} + 3\vec{k}$, from second to third point $\vec{w} = 2\vec{i} + \vec{j} + 6\vec{k}$. $\vec{u}\cdot\vec{v} = 1(3) + 2(3) + (-3)3 = 0$ so $\vec{u}$ and $\vec{v}$ perpendicular.
$\theta_1 = \cos^{-1}\sqrt{\frac{27}{41}} = 35° 45'$, $\theta_2 = \cos^{-1}\sqrt{\frac{14}{41}} = 54° 15'$.
c. $46° 55'$, $43° 05'$.

3. a. $\sqrt{y^2+z^2}$. c. $|x|$.

4. a. With (0,1,2) initial end $\vec{u} = -\vec{i} - 3\vec{j} + 5\vec{k}$, $\vec{v} = 5\vec{i} - \vec{j} - 3\vec{k}$ are such that $\cos\theta = \frac{-5+3-15}{\sqrt{1+9+25}\sqrt{25+1+9}}$ showing these are the equal sides and $\cos\theta = -\frac{17}{35}$. Thus $\theta = 180° - (60° 57') = 119° 03'$ is the desired angle.
c. At $(-1,17,1)$, $\theta = \cos^{-1}(\frac{70}{98}) = 44° 25'$.

5. a. Vector from third to first is parallel to vector from second to fourth; and these vectors have the same length. Also other sides are parallel.

6. a. (1,8,4), (5,4,0), (−3,0,6). (Hint: Let $\vec{u}$ be the vector from the origin to one of the given points and let $\vec{v}_1$ and $\vec{v}_2$ be the vectors from this point to the other given points. Then fourth vertex is the terminal end of $\vec{u} + \vec{v}_1 + \vec{v}_2$.

Sec. 95, page 306

1. a. $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. c. $\frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}$. **2.** a. $\frac{6}{7}, \frac{-3}{7}, \frac{2}{7}$. c. $\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}$.

3. a. 60° or 120°. c. 30° or 150°.

4. a. $\cos^{-1}\frac{25}{9\sqrt{11}} = 33° 07.5'$. c. $\cos^{-1}\frac{4+\sqrt{2}}{6} = 25° 33'$.

5. a. $(4,3,-5)$. c. $(x_0 + at_0, y_0 + bt_0, z_0 + ct_0)$, $t_0 = \frac{a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)}{a^2+b^2+c^2}$.

6. a. $(5,-1,3)$; $x = 5 - 3t$, $y = -1 - t$, $z = 3 + 4t$.

7. a. $(6,5,-1)$, $(-2,1,3)$.

Sec. 96, page 310

1. a. $x = -1 - 2t$, $y = 2 + 3t$, $z = 3 - t$. c. $x = 2t$, $y = 4t$, $z = 0$.

2. a. $3(x-2) + 8(y+1) - 7(z-3) = 0$ or $3x + 8y - 7z = -23$.
c. $-3(x-0) + 1(y-0) + 5(z+4) = 0$ or $3x - y - 5z = 20$. (Hint: Find two points common to both planes, from these points obtain direction numbers of normal to desired plane.)

3. a. $3x - 2y + 5z = 6$. c. $4y - z = 3$.

4. a. and c. $3x - 4y + z = 5$.

5. a and c. $3x + y - 4z = 2$.

6. a. $3x - y + 2z = 17$. c. $19x + 17y + 7z = 0$.

Sec. 97, pages 315, 318, 322

1. a. 7. b. 23. c. -7. d. 0. e. 50. f. 1. g. 1.
h. $b^2 - 4ac$. i. 148. j. 148. k. 30.

6. a. $13\vec{i} + 14\vec{j} + 16\vec{k}$. b. $3\vec{i} + 4\vec{j}$. c. $-8\vec{i} + 6\vec{j} + 4\vec{k}$.

8. a. 56. b. 322. c. -9.

9. a. -220. b. 16.

10. b. $2x - 4y - 9z + 17 = 0$.

13. a. -1. b. 34.

Sec. 98, page 326

1. a. $-2\vec{i} - \vec{k}$. c. $\vec{0}$.

2. a. $\vec{u} \times \vec{v} = -2\vec{i} + 6\vec{j} + 4\vec{k}$, $(\vec{u} + \vec{v}) \times \vec{w} = -2\vec{i} + 14\vec{j} - 22\vec{k}$.
$\vec{u} \cdot \vec{w} = 9 + 10 - 6 = 13$, $\vec{v} \cdot \vec{w} = 3 + 6 - 4 = 5$,
$(\vec{u} \cdot \vec{w})\vec{v} = 13(\vec{i} + 3\vec{j} - 4\vec{k}) = 13\vec{i} + 39\vec{j} - 52\vec{k}$,
$(\vec{v} \cdot \vec{w})\vec{u} = 5(3\vec{i} + 5\vec{j} - 6\vec{k}) = 15\vec{i} + 25\vec{j} - 30\vec{k}$,
$(\vec{u} \cdot \vec{w})\vec{v} - (\vec{v} \cdot \vec{w})\vec{u} = -2\vec{i} + 14\vec{j} - 22\vec{k} = (\vec{u} \times \vec{v}) \times \vec{w}$.
b. $(\vec{u} \times \vec{v}) \times \vec{w} = 3\vec{i} - 5\vec{j} - 6\vec{k} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{v} \cdot \vec{w})\vec{u}$.

3. a. $x - 16y - 13z = -2$. c. $z = -20$.

4. a. $x = 2 + t, y = -3 - 16t, z = 4 - 13t$. c. $x = 10, y = 3, z = -20 + t$.

5. a. $T = (0.5)\sqrt{269}$. c. $T = (8.5)\sqrt{5}$.

6. a. $t_1 = 6, t_2 = 1, t_3 = 5.5$. c. $t_1 = 0, t_2 = 8.5, t_3 = 17$.

Sec. 99, page 329

2. a. and b. $\frac{220}{3}$. c. 3. d. $\dfrac{abc}{6}$.

Sec. 100, page 332

1. a. $\vec{\tau}(t) = \dfrac{1}{\sqrt{36\pi^2 + 1}}(-6\pi\vec{i}\sin\pi t + 6\pi\vec{j}\cos\pi t + \vec{k})$,

$\kappa(t) = \dfrac{12\pi^2}{36\pi^2 + 1}$, $\vec{\eta}(t) = -\vec{i}\cos\pi t - \vec{j}\sin\pi t$,

$\vec{\beta}(t) = \dfrac{1}{\sqrt{36\pi^2 + 1}}(\vec{i}\sin\pi t + \vec{j}\cos\pi t + 6\pi\vec{k})$, $\pi\sqrt{36\pi^2 + 1}$ = dist. traveled.

c. $\vec{\tau}(t) = \dfrac{1}{\sqrt{2}}(\vec{i}\tanh 3t + \vec{j} + \vec{k}\,\mathrm{sech}\, 3t)$, $\kappa(t) = \dfrac{\mathrm{sech}^2\, 3t,}{4}$,
$\vec{\eta}(t) = \vec{i}\,\mathrm{sech}\, 3t - \vec{k}\tanh 3t$, $\vec{\beta}(t) = \dfrac{1}{\sqrt{2}}(-\vec{i}\tanh 3t + \vec{j} - \vec{k}\,\mathrm{sech}\, 3t)$, $2\sqrt{2}\sinh 6\pi$.

2. $\vec{a} = \vec{i}f'' + \vec{j}g'' + \vec{k}h''$, $\vec{\tau} = \dfrac{d\vec{F}}{dt}\dfrac{dt}{ds} = (\vec{i}f' + \vec{j}g' + \vec{k}h')(f'^2 + g'^2 + h'^2)^{-1/2}$.

$$\vec{N} = \left\{(\vec{i}f' + \vec{j}g' + \vec{k}h')\frac{(f'f'' + g'g'' + h'h'')}{-(f'^2 + g'^2 + h'^2)^{2/2}} + \frac{\vec{i}f'' + \vec{j}g'' + \vec{k}h''}{\sqrt{f'^2 + g'^2 + h'^2}}\right\}\frac{1}{\sqrt{f'^2 + g'^2 + h'^2}}$$
$$= \frac{\vec{i}(-f'g'g'' - f'h'h'' + g'^2f'' + h'^2f'') + \vec{j}(\qquad) + \vec{k}(\qquad)}{(f'^2 + g'^2 + h'^2)^2}.$$
$$\vec{N}\cdot\vec{a} = \frac{(-f'g'g''f'' - f'h'h''f'' + g'^2f''^2 + h'^2f''^2) + (\qquad) + (\qquad)}{(f'^2 + g'^2 + h'^2)^2}$$
$$= \frac{(-2f'g'g''f'' + g'^2f''^2 + f'^2g''^2) + (\qquad) + (\qquad)}{(f'^2 + g'^2 + h'^2)^2}$$
$$= \frac{(g'f'' - f'g'')^2 + (\qquad)^2 + (\qquad)^2}{(f'^2 + g'^2 + h'^2)^2} \geq 0.$$

Since also $\vec{N}\cdot\vec{a} = |\vec{N}|\,|\vec{a}|\cos\theta$ for θ the angle from $\vec{a}$ to $\vec{N}$, it follows that $\cos\theta \geq 0$ so $|\theta| \leqslant 90°$.

Sec. 101, page 338

1. a. $\{(x,y,z) \mid 1 \leq x \leq 2, 0 \leq y \leq 2, 0 \leq z \leq x + y\}$, $\{(x,y,z) \mid 0 \leq y \leq 2, 1 \leq x \leq 2, 0 \leq z \leq x + y\}$. c. $\{(x,y,z) \mid -4 \leq x \leq 4, 0 \leq y \leq \sqrt{16 - x^2}, 0 \leq z \leq 2y\}$, $\{(x,y,z) \mid 0 \leq y \leq 4, -\sqrt{16 - y^2} \leq x \leq \sqrt{16 - y^2}, 0 \leq z \leq 2y\}$.
   e. $\{(x,y,z) \mid 0 \leq x \leq 4, x^2/4 \leq y \leq 2\sqrt{x}, 0 \leq z \leq x^2 + y\}$, $\{(x,y,z) \mid 0 \leq y \leq 4, y^2/4 \leq x \leq 2\sqrt{y}, 0 \leq z \leq x^2 + y\}$.

2. a. A solid right circular cylinder but with slanting top.
   c. All points inside a right circular cylinder except the points on its central axis; radius 1, altitude 2.
   e. The solid common to a beam, with square base and side $\sqrt{2}$, and the space enclosed by the nose of a circular paraboloid.

3. $\{(x,y,z) \mid -a \leq x \leq a, -\sqrt{a^2 - x^2} \leq z \leq \sqrt{a^2 - x^2}, -\sqrt{a^2 - z^2} \leq y \leq \sqrt{a^2 - z^2}\}$ or $\{(x,y,z) \mid -a \leq y \leq a, -\sqrt{a^2 - y^2} \leq z \leq \sqrt{a^2 - y^2}, -\sqrt{a^2 - z^2} \leq x \leq \sqrt{a^2 - z^2}\}$ or $\{(x,y,z) \mid -a \leq z \leq a, -\sqrt{a^2 - z^2} \leq x \leq \sqrt{a^2 - z^2}, -\sqrt{a^2 - z^2} \leq y \leq \sqrt{a^2 - z^2}\}$

4. a. $\{(x,y,z) \mid 0 \leq x \leq 2, 0 \leq y^2 + z^2 \leq x^4\}$ or $\{(x,y,z) \mid -4 \leq z \leq 4, -\sqrt{16 - z^2} \leq y \leq \sqrt{16 - z^2}, \sqrt[4]{y^2 + z^2} \leq x \leq 2\}$. From first form $V = \int_0^2 \pi x^4\,dx = \frac{32}{5}\pi$.
   c. $\{(x,y,z) \mid 0 \leq z \leq 4, \sqrt{z} \leq \sqrt{x^2 + y^2} \leq 2\} = \{(x,y,z) \mid 0 \leq z \leq 4, z \leq x^2 + y^2 \leq 4\}$ or $\{(x,y,z) \mid -2 \leq x \leq 2, -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}, 0 \leq z \leq x^2 + y^2\}$, $V = \int_0^4 \pi(4 - z)\,dz = 8\pi$.

Sec. 102, page 343

1. a. $\lim\limits_{x\to 0} f(x,mx) = \lim\limits_{x\to 0}\dfrac{x}{x + mx} = \dfrac{1}{1 + m}$, different for different values of m.

$$\lim_{x\to 1} f(x,-1 + m(x - 1)) = \lim_{x\to 1}\frac{x}{x - 1 + m(x - 1)} = \lim_{x\to 1}\frac{x}{(x - 1)(1 + m)} = \infty$$

for $m \neq -1$, hence, there is at least one line (actually any line with slope $\neq -1$) along which the limit at $(1,-1)$ does not exist, so the limit at $(1,-1)$ does not exist.

Also at $(-1,1)$

$$\lim_{x\to -1} f(x,1+m(x+1)) = \infty \quad \text{for} \quad m \neq -1.$$

c. $\displaystyle \lim_{x\to 4} f(x,2+m(x-4)) = \lim_{x\to 4} \frac{(x-4)}{x-[2+m(x-4)]^2}$

$$= \lim_{x\to 4} \frac{x-4}{(x-4)[1-4m+m^2(x-4)]}$$

$\displaystyle = \frac{1}{1-4m}$ if $m \neq \frac{1}{4}$ but ∞ if $m = \frac{1}{4}$, so no limit at $(4,2)$.

e. $\displaystyle \lim_{x\to 1} f(x,-2+m(x-1)) = \frac{m}{1+m^2}$, so f has no limit at $(1,-2)$.

2. a. $\displaystyle \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x^2mx}{x^4+m^2x^2} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = 0$ so at $(0,0)$ the limit along any line, except y-axis, is zero. But also the limit along the y-axis is zero since

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0^2y}{0^4+y^2} = 0.$$

b. For $k \neq 0$, consider approach to $(0,0)$ along the parabola $\{(x,y) \mid y = kx^2\}$: $\displaystyle \lim_{x\to 0} f(x,kx^2) = \lim_{x\to 0} \frac{x^2(kx^2)}{x^4+(kx^2)^2} = \frac{k}{1+k^2}$ and, hence, different limits are obtained along different parabolas. It follows that f does not have a limit at $(0,0)$.

Sec. 103, page 345

1. a. Solid sphere of radius 2 and center at the origin.
   c. plane. e. Ice cream cone and ice cream. g. Hyperbola.
2. a. $\{(\rho,\theta,z) \mid \rho(2\cos\theta - 3\sin\theta) + 4z = 6\}$, $\{(r,\theta,\phi) \mid r\sin\phi(2\cos\theta - 3\sin\theta) + 4r\cos\phi = 6\}$. c. $\{(\rho,\theta,z) \mid 4\rho^2 - 9z^2 = 36\}$, $\{(r,\theta,\phi) \mid 4r^2\sin^2\phi - 9r^2\cos^2\phi = 36\} = \{(r,\theta,\phi) \mid r^2(4 - 13\cos^2\phi) = 36\}$.
   e. $\{(\rho,\theta,z) \mid \rho^2 = 4 - 4z^2\}$, $\{(r,\theta,\phi) \mid r^2(1 + 3\cos^2\phi) = 4\}$.
3. a. $\left(\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}, 6\right)$. c. $\left(\dfrac{3\sqrt{3}}{2}, -\dfrac{3}{2}, 4\right)$.
4. a. $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)$. c. $\left(\dfrac{-3\sqrt{3}}{4}, -\dfrac{3}{4}, \dfrac{3\sqrt{3}}{2}\right)$. 5. a. $60°, 45°, 120°$.

Sec. 104, page 350

1. $\frac{47}{105}$. 3. $\frac{5}{4}\pi$.

5. $\displaystyle \int_1^2 \int_y^{10y} \sqrt{xy-y^2}\,dx\,dy = \int_1^2 \frac{2}{3} \frac{(xy-y^2)^{2/2}}{y}\Bigg]_{x=y}^{x=10y} dy = \text{etc.} = 42.$

7. $R = \{(x,y) \mid 0 \le y \le 1, y \le x \le 2-y\}$ so $\int_0^1 \int_y^{2-y} (x+1)y\,dx\,dy = \frac{2}{3}$ or $R = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\} \cup \{(x,y) \mid 1. \le x \le 2, 0 \le y \le 2-x\}$ and $\int_0^1 \int_0^x (x+1)\,y\,dy\,dx + \int_1^2 \int_0^{2-x} (x+1)\,y\,dy\,dx = \frac{2}{3}$.

Sec. 105, page 353

2. a. $\int_1^2 \int_0^2 (x+y)\,dy\,dx = 5$ or $\int_0^2 \int_1^2 (x+y)\,dx\,dy = 5$.

   c. $\displaystyle \int_{-4}^4 \int_0^{\sqrt{16-x^2}} 2y\,dy\,dx = \frac{256}{3}$ or $\displaystyle \int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 2y\,dx\,dy = \frac{256}{3}$.

e. $\int_0^4 \int_{x^2/4}^{2\sqrt{x}} (x^2 + y)\, dy\, dx = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} (x^2 + y)\, dx\, dy = \dfrac{1104}{35}$.

3. a. $\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (x + 5)\, dy\, dx = 80\pi$. c. $\int_0^2 \int_{2-\sqrt{2x-x^2}}^{2+\sqrt{2x-x^2}} (7 - 5)\, dy\, dx = 2\pi$.

e. $4\int_0^1 \int_0^{1-x} (4 - x^2 - y^2)\, dy\, dx = \dfrac{22}{3}$.

Sec. 106, page 355

1. a. $|R| = \frac{125}{6}$, $M_y = -\frac{125}{12}$, $M_x = \frac{625}{6}$, $(-\frac{1}{2},5)$.

c. $|R| = \dfrac{5}{2} + 6 \ln \dfrac{2}{3}$, $M_y = M_x = \dfrac{1}{6}$, $\bar{x} = \bar{y} = \dfrac{1}{15 + 36 \ln (\frac{2}{3})}$.

e. $|R| = 1 + 10 \sin^{-1} \dfrac{3}{\sqrt{10}}$, $M_y = \dfrac{102}{5}$, $M_x = 0$.

$$\bar{x} = \frac{102}{5 + 50 \sin^{-1} (3/\sqrt{10})} = 1.51, \bar{y} = 0.$$

2. a. $\int_{-15/8}^{3} \int_{4x^2/9}^{\frac{1}{2}(x+5)} dy\, dx = \frac{2197}{256}$ c. $\int_0^{25/16} \int_{2y-5}^{-\frac{3}{2}\sqrt{y}} dx\, dy = \frac{875}{256}$.

3. a. $I_x = \int_0^{\pi/2} \int_0^{\sin y} xy^2\, dx\, dy = \dfrac{\pi^3}{96} + \dfrac{\pi}{16}$, $I_y = \int_0^{\pi/2} \int_0^{\sin y} x^3\, dx\, dy = \dfrac{3\pi}{64}$.

c. $I_x = \dfrac{7e^8 + 1}{64}$, $I_y = \dfrac{17e^4 + 3}{16}$.

e. Since $|x - y| = \begin{Bmatrix} x - y \text{ if } x \geq y \\ y - x \text{ if } x < y \end{Bmatrix}$, divide the region by the graph of $y = x$ and have

$R = \{(x,y) \mid 0 \leq x \leq 2, 0 \leq y \leq x\} \cup \{(x,y) \mid 0 \leq x \leq 2, x \leq y \leq \sqrt{2x}\}$,

$I_x = \int_0^2 \left\{ \int_0^x (x - y)y^2\, dy + \int_x^{\sqrt{2x}} (y - x)y^2\, dy \right\} dx = \frac{24}{35}$, $I_y = \frac{148}{45}$.

4. a. $I_y(t) = \iint_R (x - t)^2\, \delta(x,y)\, dR = \iint_R (x^2 - 2xt + t^2)\, \delta(x,y)\, dR$

$= \iint_R x^2\, \delta(x,y)\, dR - 2t \iint_R x\, \delta(x,y)\, dR + t^2 \iint_R \delta(x,y)\, dR$

$= I_y(0) - 2tM_y + t^2\mu$.

Sec. 107, page 359

1. a. 4π. c. $\int_0^\pi \int_0^{2\cos 3\theta} \rho\, d\rho\, d\theta = \pi$. e. $\int_0^{2\pi} \int_0^{2+\cos\theta} \rho\, d\rho\, d\theta = \frac{9}{2}\pi$.

2. a. $M_x = \int_0^{\pi/2} \int_0^{2a\cos\theta} \rho \sin\theta\, \rho\, d\rho\, d\theta = \dfrac{2}{3} a^3$, $M_y = \int_0^{\pi/2} \int_0^{2a\cos\theta} \rho \cos\theta\, \rho\, d\rho\, d\theta = \dfrac{\pi}{2} a^2$,

$|G| = \dfrac{\pi}{2} a^2$, $\bar{y} = \dfrac{4}{3} \dfrac{a}{\pi}$, $\bar{x} = a$.

c. $|G| = \dfrac{a^2}{8}\pi$, $M_x = 0$, $M_y = \dfrac{16\sqrt{2}}{105} a^3$, $\bar{y} = 0$, $\bar{x} = \dfrac{128\sqrt{2}}{105} \dfrac{a}{\pi}$.

3. a. $\int_0^{\pi/2} \int_0^{2\cos\theta} \rho^2\, d\rho\, d\theta = \frac{16}{9}$.

c. $2 \int_0^{2\pi} \int_0^b \sqrt{a^2 - \rho^2}\, \rho\, d\rho\, d\theta = \frac{4}{3}\pi\, [a^3 - (a^2 - b^2)^{3/2}]$.

Sec. 108, page 361

1. a. $\frac{1}{4}(1 - \cos 4)$. c. $\int_0^2 \int_1^{x+1} e^{-x^2}\, dy\, dx = \frac{1}{2}\left(1 - \frac{1}{e^4}\right)$.

e. $\int_0^{\pi/4} \int_{\tan y}^1 x\, dx\, dy = \frac{1}{2} \int_0^{\pi/4} [x^2]_{\tan y}^1\, dy = \text{etc.} = \frac{1}{2}\left(\frac{\pi}{2} - 1.\right)$.

g. $\int_0^2 \int_{y^2}^{2y} f(y)\, dx\, dy = \int_0^2 (2y - y^2) f(y)\, dy = \int_0^2 (2 - y) \sin y\, dy = \text{etc.} = 2 - \sin 2$.

2. a. $\frac{\pi}{4}(e^{a^2} - 1)$. c. $\int_0^{\pi/2} \int_0^{2\cos\theta} \rho \sin\theta\, \rho\, d\rho\, d\theta = \frac{2}{3}$.

3. a. $\int_0^{\pi/2} \int_0^{\cos y} y\, dx\, dy = \frac{\pi}{2} - 1$. b. $\int_0^{\pi/2} \int_1^2 \rho \cos\theta\, \rho\, d\rho\, d\theta = \frac{7}{3}$.

c. $\int_0^2 \int_0^x e^x\, dy\, dx$ or $\int_0^2 \int_y^2 e^x\, dx\, dy = e^2 + 1$. d. $\dfrac{e^{\pi/2} - 3e^{-\pi/2}}{2}$.

Sec. 109, page 364

1. a. $\frac{124}{3}$. c. 0. e. $\int_0^{\pi/6} \int_0^{2\pi} \int_0^{a \sec\phi} \rho \cos\phi\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{\pi}{12} a^4$.

2. a. $\mu = \int_0^{\pi/2} \int_0^{2\pi} \int_0^a k\, \rho \cos\phi\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{k\pi}{4} a^4$.

$M_{xy} = \frac{2k\pi}{15} a^5,\ \bar{x} = \bar{y} = 0,\ \bar{z} = \frac{8}{15} a$.

c. $\bar{x} = \bar{y} = 0,\ \bar{z} = \frac{3}{4} h$, assuming the cylinder is setting with its rare end on the xy-plane with center at the origin.

Sec. 110, page 366

1. a. $-\vec{k}Km\,\delta \int_0^{2\pi} \int_0^a \int_0^h \dfrac{c + h - z}{[\rho^2 + (c + h - z)^2]^{2/2}}\, dz\, \rho\, d\rho\, d\theta$

$$= +\vec{k}Km\,\delta 2\pi \int_0^a \left[\frac{1}{\sqrt{\rho^2 + (c+h)^2}} - \frac{1}{\sqrt{\rho^2 + c^2}}\right] \rho\, d\rho$$

$$= \vec{k}Km\delta\, 2\pi\, [\sqrt{a^2 + (c+h)^2} - \sqrt{a^2 + c^2} - h].$$

c. $\vec{k}Km\,\delta \int_0^{2\pi} \int_0^h \int_0^a \dfrac{z + c}{[\rho^2 + (z + c)^2]^{3/2}}\, \rho\, d\rho\, dz\, d\theta$

$$= -\vec{k}Km\,\delta 2\pi \int_0^h \left[\frac{z + c}{\sqrt{a^2 + (z + c)^2}} - 1\right] dz = -\vec{k}Km\,\delta 2\pi\, [\sqrt{a^2 + (z + c)^2} - z]_0^h$$

$$= -\vec{k}Km\,\delta 2\pi\, [\sqrt{a^2 + (h + c)^2} - \sqrt{a^2 + c^2} - h].$$

2. a. The cone is $\{(\rho,\theta,z) \mid 0 \le \rho \le 1,\ \rho \le z \le 1,\ 0 \le \theta \le 2\pi\}$. The answer is $\vec{k}Km\,\delta$ multiplied by the integral

$$\int_0^1 \int_\rho^1 \int_0^{2\pi} \frac{z + 1}{[\rho^2 + (z + 1)^2]^{3/2}}\, d\theta\, dz\, \rho\, d\rho = 2\pi \int_0^1 \left[\frac{-1}{\sqrt{\rho^2 + (z+1)^2}}\right]_{z=\rho}^{z=1} \rho\, d\rho$$

$= 2\pi \int_0^1 \left[\dfrac{\rho}{\sqrt{2\rho^2 + 2\rho + 1}} - \dfrac{\rho}{\sqrt{\rho^2 + 4}}\right] d\rho$ which, with the aid of Table formula **78**

evaluates as $\pi\left[-\sqrt{5} + 3 - \dfrac{1}{\sqrt{2}} \ln \dfrac{\sqrt{10} + 3}{\sqrt{2} + 1}\right]$.

Sec. 111, page 368

1. a. $f_x(x,y) = 2xy^2 + y^2, f_y(x,y) = 3x^2y^2 + 2xy.$

c. $\dfrac{\partial f(x,y)}{\partial x} = \dfrac{y}{x^2+y^2}, \dfrac{\partial f(x,y)}{\partial y} = \dfrac{-x}{x^2+y^2}.$

e. $\dfrac{\partial f(u,v)}{\partial u} = 2u \ln \left|\dfrac{v}{u}\right| - u, \dfrac{\partial f(u,v)}{\partial v} = \dfrac{u^2}{v}.$

2. a. $\dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}}.$ c. $-\dfrac{y}{x^2}\cos\dfrac{y}{x}, \dfrac{1}{x}\cos\dfrac{y}{x}.$ e. $6t \sin(2s^2 - 3t^2).$

3. a. $\dfrac{-2y}{(x-y)^2}, \dfrac{2x}{(x-y)^2}.$ c. $\dfrac{2x}{a^2}, \dfrac{2y}{b^2}.$ e. $e^{y/x}\left(1 - \dfrac{y}{x}\right), e^{y/x}.$

Sec. 112, page 372

1. a. $4x - 4y + z = 8;\ x = 2 - 4t, y = -1 + 4t, z = -4 - t$ or $x - 2 = -y - 1 = 4z + 16.$

c. $x + y + 4z + \pi = 0;\ x = -2 + t, y = 2 + t, z = -\pi/4 + 4t$ or $4(x+2) = 4(y-2) = z + \pi/4.$

e. $3x + 5y + 2z = 8;\ x = 3 + 3t, y = -1 + 5t, z = 2 + 2t.$

4. a. $D_1 = 76, (-3,2,4);\ D_2 = -24, (2,-2,-20).$ c. $D = -2, (2,-1,2).$

5. a. $(0,\frac{3}{2},-\frac{3}{2}).$ c. $(2,3,15), (-2,-3,-15).$

Sec. 114, page 377

1. a. $8\int_0^r\int_0^{\sqrt{r^2-x^2}} \sqrt{\dfrac{(-x)^2+(-y)^2}{r^2-x^2-y^2}+1}\, dy\, dx = 8\int_0^r\int_0^{\sqrt{r^2-x^2}} \dfrac{r}{\sqrt{r^2-x^2-y^2}}\, dy\, dx$

$= 8r\int_0^r \sin^{-1}\dfrac{y}{\sqrt{r^2-x^2}}\Bigg]_0^{\sqrt{r^2-x^2}} dx = 8r\int_0^r (\sin^{-1}1 - \sin^{-1}0)\, dx = 8r\dfrac{\pi}{2}r = 4\pi r^2.$

3. $\int_0^1\int_0^x \dfrac{3}{\sqrt{9-x^2-y^2}}\, dx = 3\int_0^1 \sin^{-1}\dfrac{x}{\sqrt{9-x^2}}\, dx$ or $\int_0^{\pi/4}\int_0^{\sec\theta} \dfrac{3}{\sqrt{9-\rho^2}}\rho\, d\rho\, d\theta$

$= 3\int_0^{\pi/4}(3 - \sqrt{9-\sec^2\theta}\, d\theta.$ 4. $\dfrac{\pi}{6}(2\sqrt{2}-1).$ 5. $16\sqrt{2}.$ 6. $\dfrac{\sqrt{A^2+B^2+C^2}}{|C|}\pi a^2$

8. $kK\delta\int_0^1\int_0^x \dfrac{x^2+y}{[x^2+(y-1)^2+(x^2+y)^2]^{3/2}}\sqrt{4x^2+2}\, dy\, dx.$ 9. $\sqrt{6}\pi.$

Sec. 115, page 379

1. a. $f(x,y) = x^2y - x^2y^2 + y^2 + \sin y.$ c. $f(x,y) = \tan^{-1}\dfrac{y}{x} + \sqrt{x^2+1} - \tan^{-1}\dfrac{1}{x}.$

e. $f(x,y) = \dfrac{\sqrt{x^2-y^2}}{xy^2} - \dfrac{\sqrt{16-y^2}}{4y^2}.$ g. $f(x,y) = e^{2x/y}.$

2. a. $f(x,y) = \dfrac{x}{y} + y + c.$ c. $f(x,y) = \tan^{-1}\dfrac{x}{y} + e^x + c.$ e. No function.

Sec. 117, page 384

1. a. $-0.87, -0.9, 0.967.$

c. $\cos 65° 36' - \cos 60° = -0.0869, -0.0836, 1.04.$

2. a. $(6x + 4y)\,dx + (4x + 2y)\,dy$. c. $\dfrac{\sin y\,dx + x\cos y\,dy}{1 + x^2\sin^2 y}$.
e. $x^{\sin y}\left[\sin y\dfrac{dx}{x} + \ln x\cos y\,dy\right]$.

5. a. Second $d\left(\dfrac{x}{y}\right)$. c. First $d(xy)$. e. Second $d(\sqrt{x^2 + y^2} + \frac{1}{2}x^2)$.

Sec. 119, page 388

1. a. $D_xy = -\dfrac{y + 2xy^3}{x + 3x^2y^2}$. c. Formally we obtain $D_xy = -x/y$, but note that there is no pair of numbers (x,y) satisfying $x^2 + y^2 + 1 = 0$.
e. $D_xy = -\dfrac{y}{x\ln x} = -\dfrac{y}{x}\log_x e$ for $0 < x < 1$ or $1 < x$.

2. a. $y = -1, x = -1$. c. $2x - y = 2, x + y = 2$.

3. a. $y^2\,dx + (8 - xy)\,dy = 0$. c. $(ye^x - \sin x)\,dx + e^x\,dy = 0$.
e. $y(1 - \cos x)\,dx - (x - \sin x)\,dy = 0$.

4. a. $2y(x - 1)\,dx + [y^2 - (x - 1)^2]\,dy = 0$.
c. Primitive $x^2/4 + y^2/c = 1$; diff. eq. $xy\,dx + (4 - x^2)\,dy = 0$.
e. $(y - 2)\,dx - (x + 1)\,dy = 0$.

Sec. 120, page 391

1. a. $\dfrac{\partial z}{\partial x} = \dfrac{2xy - z}{x + 3yz^2}, \dfrac{\partial z}{\partial y} = \dfrac{x^2 - z^3}{x + 3yz^2}$. c. $\dfrac{\partial z}{\partial x} = yze^{xy}, \dfrac{\partial z}{\partial y} = xze^{xy}$.

2. a. $\dfrac{\partial x}{\partial y} = \dfrac{x^2 - z^3}{z - 2xy}, \dfrac{\partial x}{\partial z} = \dfrac{x + 3yz^2}{2xy - z}$. c. $\dfrac{\partial x}{\partial y} = -\dfrac{x}{y}, \dfrac{\partial x}{\partial z} = \dfrac{1}{yze^{xy}}$.

3. a. $x - 2y - 3z + 14 = 0$; $x = -1 + t, y = 2 - 2t, z = 3 - 3t$; $(0,0,0)$.
c. $x - 2y + 9z = 0$; $x = 3 + t, y = -3 - 2t, z = -1 + 9t$; $(\frac{28}{9},-\frac{29}{9},0,)$ $(\frac{3}{2},0,-\frac{29}{2})$, $(0,3,-28)$. e. $2x - y + 2z = 4$; $x = 1 + 2t, y = -t, z = 1 + 2t$; $(0,\frac{1}{2},0)$, $(1,0,1)$.

4. $8x + 6y - 5z = 25$.

5. Line has direction numbers 2,2,1, passes through (0,0,0), so has parametric equations $x = 2t$, $y = 2t$, $z = t$, pierces the surface when $2(2t)^2 + 4(2t)^2 + t^2 = 100$ so when $t = \pm 2$. Thus, one of the points is (4,4,2). At this point normal has direction numbers 16, 32, 4 or better 4, 8, 1. Thus, desired angle θ has

$$\cos\theta = \frac{2(4) + 2(8) + 1(1)}{\sqrt{2^2 + 2^2 + 1^2}\,\sqrt{4^2 + 8^2 + 1^2}} = \frac{25}{27}.$$

By symmetry the angle at the other point is the same.

7. a. $a = 3, b = -1, c = 4$.

8. a. $\vec{i}(3x^2y + yz) + \vec{j}(x^3 + xz + 3z^2) + \vec{k}(xy + 6yz)$.

Sec. 121, page 394

2. a. (1) $z = e^{2t} - e^t\ln t$, $D_tz = 2e^{2t} - e^t\ln t - e^t/t$.
(2) $D_tz = (2x - y)D_tx + (-x)D_ty = (2e^t - \ln t)\,e^t + (-e^t)(1/t)$
$= 2e^{2t} - e^t\ln t - e^t/t$.
c. (1) $z = \sin 2e^t\cos 2t$, $D_tz = -2\sin 2e^t\sin 2t + 2e^t\cos 2e^t\cos 2t$.
(2) $D_tz = (2\cos 2x\cos y)D_tx + (-\sin 2x\sin y)D_ty$
$= 2(\cos 2e^t\cos 2t)\,e^t - (\sin 2e^t\sin 2t)\,2$.

d. (1) $z = \sqrt{1 + \tan^2 t} + \ln|\sec^2 t - \tan^2 t| = |\sec t| + \ln 1 = |\sec t|$,

$$D_t z = \begin{Bmatrix} \sec t \tan t \text{ if } \sec t > 0 \\ -\sec t \tan t \text{ if } \sec t < 0 \end{Bmatrix}. \quad (\text{Note: } D_t z \neq |\sec t \tan t|.)$$

(2) $$D_t z = \frac{2x}{x^2 - y^2} D_t x + \left[\frac{y}{\sqrt{1 + y^2}} - \frac{2y}{x^2 - y^2}\right] D_t y$$

$$= \frac{2 \sec t}{\sec^2 t - \tan^2 t} \sec t \tan t + \left[\frac{\tan t}{\sqrt{1 + \tan^2 t}} - \frac{2 \tan t}{\sec^2 t - \tan^2 t}\right] \sec^2 t$$

$$= \frac{2 \sec^2 t \tan t}{1} + \frac{\tan t \sec^2 t}{\sqrt{\sec^2 t}} - \frac{2 \tan t \sec^2 t}{1}$$

$$= \frac{\tan t \sec^2 t}{|\sec t|} = \tan t\, |\sec t|.$$

3. a. (1) $z = e^{2t} \cos^2 s - e^{2t} \cos s \sin s = e^{2t}(\cos^2 s - \cos s \sin s)$,

$$\frac{\partial z}{\partial t} = 2e^{2t}(\cos^2 s - \cos s \sin s),$$

$$\frac{\partial z}{\partial s} = e^{2t}(-2 \cos s \sin s - \cos^2 s + \sin^2 s) = -e^{2t}(\sin 2s + \cos 2s).$$

(2) $$\frac{\partial z}{\partial t} = (2x - y)\frac{\partial x}{\partial t} + (-x)\frac{\partial y}{\partial t} = e^t(2 \cos s - \sin s)e^t \cos s - (e^t \sin s)e^t \cos s,$$

$$\frac{\partial z}{\partial s} = (2x - y)\frac{\partial x}{\partial s} + (-x)\frac{\partial y}{\partial s} = e^t(2 \cos s - \sin s)(-e^t \sin s) - (e^t \cos s)e^t \cos s.$$

c. (1) $z = \sin(2se^t) \cos st$,

$$\frac{\partial z}{\partial t} = 2se^t \cos(2se^t) \cos st - s \sin(2se^t) \sin st,$$

$$\frac{\partial z}{\partial s} = 2e^t \cos(2se^t) \cos st - t \sin(2se^t) \sin st.$$

(2) $$\frac{\partial z}{\partial t} = 2 \cos 2x \cos y \frac{\partial x}{\partial t} - \sin 2x \sin y \frac{\partial y}{\partial t}$$

$$= [2 \cos 2\,se^t \cos st]\, se^t - [\sin 2se^t \sin st]\, s,$$

$$\frac{\partial z}{\partial s} = [2 \cos 2se^t \cos st]\, e^t - [\sin 2se^t \sin st]\, t.$$

Sec. 122, page 398

1. a. $z_{xx} = 6x$, $z_{xy} = -4y$, $z_{yx} = -4y$, $z_{yy} = -4x$.

c. $z_{xx} = -4 \cos(2x + 3y)$, $z_{xy} = \dfrac{\partial}{\partial y}[-2 \sin(2x + 3y)] = -6 \cos(2x + 3y)$,

$z_{yx} = \dfrac{\partial}{\partial x}[-3 \sin(2x + 3y)] = -6 \cos(2x + 3y)$, $z_{yy} = -9 \cos(2x + 3y)$.

e. $z_{xx} = \dfrac{y}{x^3}\left(2 + \dfrac{y}{x}\right)e^{y/x}$, $z_{xy} = -\dfrac{1}{x^2}\left(1 + \dfrac{y}{x}\right)e^{y/x} = z_{yx}$, $z_{yy} = \dfrac{1}{x^2} e^{y/x}$.

5. a. $f_{xxx}, f_{xxy}, f_{xyx}, f_{yxx}, f_{yyy}, f_{yyx}, f_{yxy}, f_{xyy}$.

b. (1) $z_{xxx} = 6$, $z_{xxy} = -8y^3 = z_{xyx} = z_{yxx}$, $z_{yyy} = -24x^2 y$,

$z_{yyx} = -24xy^2 = z_{yxy} = z_{xyy}$.

(3) $z_{xxx} = e^x \cos y$, $z_{xxy} = -e^x \sin y$, $z_{xyy} = -e^x \cos y$, $z_{yyy} = e^x \sin y$.

Sec. 123, page 401

1. a. $-\dfrac{1}{5\sqrt{2}}$. c. $\dfrac{e^2}{\sqrt{2}}$. e. $\dfrac{-2ab}{\sqrt{a^2+b^2}}$.

2. a. $\mathscr{D}_\alpha f(0,2\sqrt{3}) = \sqrt{3}\cos\alpha + \sin\alpha$. b. Value 0 if $\alpha = 120°$ or $-60°$, max. value 2 when $\alpha = 30°$, min. value -2 when $\alpha = -150°$.

4. a. At (2,4); 0 if $\alpha = -45°$ or $135°$, max. $8\sqrt{2}e$ if $\alpha = 45°$, min. $-8\sqrt{2}e$ if $\alpha = -135°$. At $(2,-4)$; 0 if $\alpha = 45°$ or $-135°$, max. $8\sqrt{2}e$ if $\alpha = -45°$, min. $-8\sqrt{2}e$ if $\alpha = 135°$.
c. At (3,4); $\theta = \tan^{-1}\frac{4}{3} = 53°08'$. Value 0 if $\alpha = 53°08' - 90° = -36°52'$ or $\alpha = 53°08' + 90° = 143°08'$, max. $\frac{1}{5}$ if $\alpha = 53°08'$, min. $-\frac{1}{5}$ if $\alpha = -126°52'$.
At $(-3,4)$; $\theta = \tan^{-1}\dfrac{4}{-3} = -53°08'$, etc.

Sec. 125, page 404

1. a. $\dfrac{100}{\sqrt{14}}$. (Hint: Write the direction vector $\vec{v} = \dfrac{2}{\sqrt{14}}\vec{i} - \dfrac{3}{\sqrt{14}}\vec{j} + \dfrac{1}{\sqrt{14}}\vec{k}$ so the coefficients are direction cosines and not merely direction numbers.)
c. 3. e. $-2/\sqrt{14}$.

2. a. $\frac{1}{3}$. c. $\sqrt{6}/4$.

4. a. $x = 3 + 9t, y = 4 - 8t, z = 5 - t$.
c. $x = -2 + t, y = 2 + t, z = 1$.

5. $(-1,0,-\frac{1}{2})$. (Hint: Use s as the parameter on the new line.)

Sec. 126, page 409

1. a. $\int_0^1 (x + 2x)\,dx - \int_0^1 \sqrt{x}\,d(2x) = \frac{1}{6}$. c. $\frac{1}{6}$. e. $-\frac{13}{30}$.

2. a. $\frac{2}{3}$ for all parts. **3.** a. 24.1. c. $\left(\dfrac{4}{\pi}\right)^2 + \dfrac{17e^2}{4} + \dfrac{1}{4}$.

4. a. $0 + \int_0^2 2\,dy + \int_2^0 4\,dy + 0 = -4$. c. $\int_0^{2\pi}[4\sin^2 t(-2\sin t) + 2\cos t(2\cos t)]\,dt = 4\pi$.

Sec. 127, page 415

2. a. $\dfrac{\partial(xy\cos x + \sin y)}{\partial y} = x\cos x + \cos y = \dfrac{\partial(x\cos y + x\sin x + \cos x)}{\partial x}$ so answer follows from Theo. 126.2.
c. $\dfrac{\partial}{\partial y}\dfrac{2x}{\sqrt{1+y^2}} = -\dfrac{2xy}{(1+y^2)^{3/2}} = \dfrac{\partial}{\partial x}\dfrac{-x^2y}{(1+y^2)^{3/3}}$ so ans. follows from Theo. 126.2.
e. $\dfrac{\partial}{\partial y}F_x = F_{xy} = F_{yx} = \dfrac{\partial}{\partial x}F_y$ so ans. follows from Theo. 126.2.

3. With C a circle with center (0,0) and radius r

$$\int_C P\,dx + Q\,dy = \int_C x\,dx + xy\,dy = \int_0^{2\pi} r\cos\theta\,dr\cos\theta + \int_0^{2\pi}(r\cos\theta)(r\sin\theta)\,dr\sin\theta$$
$$= \frac{(r\cos\theta)^2}{2}\Bigg]_0^{2\pi} + r^3\int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta = 0 + r^3\frac{\cos^3\theta}{3}\Bigg]_0^{2\pi} = 0.$$

Sec. 128, page 418

1. $e^2 = 1 + 2 + \dfrac{2^2}{2!} + \dfrac{2^3}{3!} + \dfrac{2^4}{4!} + \dfrac{2^5}{5!}e^{\xi_5}, 0 < \xi_5 < 2$.

3. $\ln 1.2 = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4\xi_4^4}$, $1 < \xi_4 < 1.2$.

5. $\frac{\pi}{4} = 1 - \frac{2}{3!}\frac{(1 - 3\xi_3^2)}{(1 + \xi_3^2)^3}$, $0 < \xi_3 < 1$.

7. $\frac{2}{3} = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4}\frac{1}{(1 + \xi_4)^5}$, $0 < \xi_4 < \frac{1}{2}$.

9. $(2.01)^4 = 2^4 + 4(2^3)(0.01) + \frac{4\cdot 3}{2!}(2^2)(0.01)^2 + \frac{4\cdot 3\cdot 2}{3!}(2)(0.01)^3 + \frac{4!}{4!}(0.01)^4.$

$= 16 + 0.32 + 0.0024 + 0.000\,008 + 0.000\,000\,01 = 16.322\,408\,01.$

Sec. 129, page 420

1. $1.0955 \pm 0.000\,062\,5$. 3. $n = 9$. 5. 2.08.

Sec. 131, page 423

1. a. $\cos 40° = 1 - \frac{1}{2!}\left(\frac{2\pi}{9}\right)^2 + \frac{1}{4!}\left(\frac{2\pi}{9}\right)^4 - \frac{1}{6!}\left(\frac{2\pi}{9}\right)^6 \pm 5 \times 10^{-5}$.

2. a. 4.125. c. -0.105. e. 0.0023 by using $f(x) = x^{-2}$, $b = 21$, $a = 20$, $n = 2$.

Sec. 132, page 425

1. 0.1157 ± 0.0002. 3. 0.46365 ± 0.00003 5. a. 0.4940. c. 1.4626 e. 1.318.

6. $\theta = (15/R)^{1/3}$, $x = R\theta^2/2$ which is about 840 ft.

Sec. 134, page 431

1. a. 0.693 159.

2. a. $\frac{1}{4}\int_0^3 \sqrt{\frac{400 - 15x^2}{25 - x^2}}\,dx$. c. $\frac{1}{4}\frac{3}{6}\left[4 + 4\sqrt{\frac{366.25}{22.75}} + \frac{\sqrt{265}}{4}\right] = 3.015,$

$\frac{1}{12}\left[1 + 4\sqrt{\frac{40}{24}} + 2\sqrt{\frac{85}{21}} + 4\sqrt{\frac{160}{16}} + \sqrt{\frac{265}{9}}\right] = 2.347.$

3. $\frac{\pi}{2}\frac{1}{12}\left[\sqrt{2} + 4\sqrt{1 + \cos^2\frac{\pi}{8}} + 2\sqrt{1 + \cos^2\frac{\pi}{4}} + 4\sqrt{1 + \cos^2\frac{3\pi}{8}} + 1\right]$

$= \frac{\pi}{24}[1.4142 + 5.4460 + 2.4494 + 4.4284] = 1.929.$

5. $|f^4(x)| \le e + 7e^2 + 6e^3 + e^4 < 231$. $\frac{(b - a)M}{180n^4} < \frac{1.(231)}{180n^4} < 5 \times 10^{-4}$ if $n \ge 8$.

6. a. $\frac{4 - 0}{3\cdot 4}(346.5) = 115.5$.

Sec. 135, page 436

2. a. $\frac{1}{8}$. c. 0. e. $\frac{1}{2}$. g. $\frac{1}{4}!$, i. 1. k. 6. m. 1. o. 4.

4. a. $OE = \frac{3h[1 - \cos(h/3)]}{h - 3\sin(h/3)}$. b. 9.

Sec. 136, page 439

3. a. 0. c. 4. e. 0 $\left(\text{Note: } \frac{\ln|x|}{|x|} \to -\infty \text{ as } x \to 0\right)$. g. 1.

i. 0 $\left(\text{Hint: } e^{\ln(e^x+1)-x} = \frac{e^{\ln(e^x+1)}}{e^x} = \frac{e^x + 1}{e^x} = 1 + \frac{1}{e^x}\right)$. k. $e^{-a^2/2}$.

4. a. $y = a$. c. $y = 1$. e. $y = e^{-1/2}$.

6. a. -1. c. 2. e. 4!.

Sec. 137, page 443

1. c. $\cos x \cos y - [\sin x \cos y\, dx + \cos x \sin y\, dy] + \dfrac{1}{2!}[-\cos x \cos y(dx)^2$

$+ 2 \sin x \sin y\, dx\, dy - \cos x \cos y(dy)^2]$.

2. c. $e^{dx} \sin y\, dy \sim dy + dx\, dy + \frac{1}{2}(dx)^2\, dy - \frac{1}{6}(dy)^3$.

$e^{dx} \ln(1 + dy) \sim dy + dx\, dy - \frac{1}{2}(dy)^2 + \frac{1}{2}(dx)^2\, dy - \frac{1}{2}\, dx(dy)^2 + \frac{1}{3}(dy)^3$.

Sec. 138, page 446

1. Base 2 ft $\times$ 4 ft, depth 6 ft. **2.** a. $2 \times 3 \times 4 = 24$.

3. a. At (0,0,2) and (0,0,−2) dist. $= 2$. c. At $(\pm 2\sqrt[6]{2}, 2/\sqrt[3]{2}, 0)$, dist. $= 2^{2/3}\sqrt{3}$. (Hint: Same situation as Ex. 2). e. At $(1, \frac{3}{2}, \frac{1}{2})$ dist. $= \sqrt{14}/2$.

4. a. Occurs when $s = -1$, $t = 2$; points (3,6,8), (1,2,2), dist. $= 2\sqrt{14}$.

5. a. If $f_{xx}(x_0, y_0) = 0$, then (2) becomes $0 \cdot f_{yy}(x_0, y_0) - [f_{xx}(x_0, y_0)]^2 \le 0$ contrary to (2).

Sec. 139, page 454

1. a. $\frac{1}{4}$. c. $\frac{5}{18}$. e. $\frac{1}{256}$.

2. a. $\frac{5}{2}$. c. $\frac{5}{2}$. e. $\frac{1}{5}$. g. 1. i. $\frac{1}{2}$. k. $\frac{1}{2}$.

3. a. $u_{n+1} = \dfrac{e^{n+1}}{1 + 3^{n+1}}, \dfrac{e}{3}$. c. $\dfrac{e}{2}$. e. $\dfrac{1000}{999}$. g. ∞. i. 1.

Sec. 140, page 461

1. Conv. *a, b, c, d, e, g, i*. Div. *f, h, j*.

2. Conv. *b, c, d, f, h*. Div. *a, e, g, i*.

Sec. 142, page 466

2. Conv. *a, b, c, e, f, h, j*.

Sec. 144, page 470

3. a. Div. c. Abs. conv. e. Abs. conv. g. Cond. conv. i. Abs. conv. k. Abs. conv.

4. a. $\{x \mid -2 < x < 2\}$. c. $\{x \mid -1 \le x \le 1\}$. e. $\{x \mid -1 \le x < 1\}$.
g. $\{x \mid -2 < x < 2\}$. i. $\{x \mid x > 2 \text{ or } x < -2\}$. k. $\{x \mid x \ge 1 \text{ or } x < -1\}$.
m. $\{x \mid x \ne 0\}$. o. $\{x \mid -1 < x < 5\}$. g. $\{x \mid -5 \le x < 1\}$. s. $\{x \mid 0 \le x \le 1\}$.
u. $\{x \mid -0.1 < x < 0.1\}$. w. $\{x \mid -\infty < x < \infty\}$.

5. a. $\{x \mid -1 < x < 1\}$, $\{x \mid -1 < x < 1\}$, $\{x \mid -1 \le x < 1\}$.
c. $\{x \mid 3 \le x \le 5\}$, $\{x \mid -3 \le x < 5\}$, $\{x \mid 3 \le x \le 5\}$.

Sec. 146, page 480

1. a.
$$\frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots\right)$$
$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \cdots + \frac{(-1)^n (2n+1) x^{3n}}{(2n+1)!} + \cdots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots.$$

Sec. 148, page 487

1. a. $\left(\dfrac{dy}{dx}\right)^3 - x\dfrac{dy}{dx} + y = 0.$ c. $(x^2 + x)\dfrac{dy}{dx} + y = 0.$ e. $\dfrac{dy}{dx} + 2y = e^{-x}.$

g. $(1 + x^2)\,dy = (1 + xy)\,dx.$ i. $x(y')^2 - 2yy' - x = 0.$

2. a. $|1 + y| = e^c\,|2 - x|$ or $1 + y = C(2 - x).$

c. $\ln|x| - y - 3\ln|y - 3| = c$ or $x = Ce^y(y - 3)^3.$

e. $(\frac{2}{3})^x - (\frac{3}{2})^{-y} = C.$ g. $-x^{-1} + \tan^{-1} y = c.$ i. $\dfrac{x^2}{2} - x + \ln|x + 1| - \frac{1}{2}\cot y = c.$

3. a. $(y + 2)^2 = (x - 3)^2 + 24.$ c. $e^x - \ln(e^x + 1) + \ln y^2 = 8.$

e. $x = \dfrac{\pi}{3} - \tan^{-1}\sqrt{y}.$ (Hint: Set $\sqrt{y} = t$.)

Sec. 149, page 490

1. a. $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{y}{\sqrt{3}x} + \frac{1}{2}\ln(3x^2 + y^2) = c.$ c. $(x + y)^2 = y^2(c + \ln y^2).$

e. $\dfrac{y}{x} + \ln|x| - \ln\left|1 + \dfrac{y}{x}\right| = c.$

3. a. $\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{y + 2}{\sqrt{3}(x - 1)} + \frac{1}{2}\ln[3(x - 1)^2 + (y + 2)^2] = c.$ c. $2x - 3y + \ln|x - 2y| = c.$

Sec. 150, page 493

1. a. $y = -x^{-3} + cx^{-2}.$ c. $y = cx^{-1} - x^{-1}\cos x.$ e. $xy = \frac{1}{2}\ln^2 x + c.$

2. a. $\rho = 2(-1 + \sin\theta) + ce^{-\sin\theta}.$ c. $s = t^2(t^2 + c).$ e. $x = c\csc y - \cot y.$

3. a. $y = x^{-2} + x^{-1}.$ c. $y = e^{-\sin x} - 1 + \sin x.$

Sec. 151, page 494

1. a. $y^2 = \dfrac{x^2}{c - x^2}.$ c. $y^2 = x^2 + x + \frac{5}{2} + ce^{2x}.$

2. a. $x^4y^3 + 3x = c.$ (See Sec. 116). c. $7x + 14y + 9\ln|21x - 7y + 8| = c.$

e. $(x - 3)(y - 2) = c.$ g. $2y^{-2}(x + 1)^2 = c - e^{2x}(2x^2 + 2x + 1).$

i. $y + \sqrt{x^2 + y^2} = cx^2$

k. $x = \sin t + c\cos t.$ m. $\ln[(u - 1)^2 + (v + 3)^2] = 2\tan^{-1}\dfrac{v + 3}{u - 1} + c.$

Sec. 152, page 497

1. a. $y = c_1e^{2x} + c_2e^{-3x}.$ c. $y = (c_1 + c_2x)e^{5x/2}.$ e. $y = e^{2x}(c_1\cos 3x + c_2\sin 3x).$

2. a. $s = e^t(c_1\cos 2t + c_2\sin 2t).$ c. $x = c_1 + c_2e^t.$ e. $u = e^{-v/2}(c_1e^{\sqrt{5}v/2} + c_2e^{-\sqrt{5}v/2}).$

3. a. $y = e^{2x}(\cos x - 3\sin x).$ c. $y = e^{5x} + 3e^{-x}.$

4. The problem is translated into: Solve the derivative system

$$\frac{d^2s}{dt^2} + 4s = 0;\ s = 2 \text{ and } \frac{ds}{dt} = 6 \text{ when } t = 0.$$

The general solution (see Ex. 2) is $s = c_1 \cos 2t + c_2 \sin 2t$. Hence, $\dfrac{ds}{dt} = 2(-c_1 \sin 2t + c_2 \cos 2t)$ and the initial conditions give $c_1 = 2$, $c_2 = 3$. Thus,

$$s = 2\cos 2t + 3 \sin 2t = \sqrt{2^2 + 3^2}\left(\frac{2}{\sqrt{13}}\cos 2t + \frac{3}{\sqrt{13}}\sin 2t\right)$$

$$= \sqrt{13}\cos(2t - \theta)\left[\text{where } \cos\theta = \frac{2}{\sqrt{13}},\ \sin\theta = \frac{3}{\sqrt{13}},\ \tan\theta = \frac{3}{2} = 1.5\right]$$

$= \sqrt{13}\cos 2(t - \theta/2)$. Since $\tan 56^\circ 20' = 1.501$, then in terms of radians $\theta = \dfrac{56.33}{180}\pi = 0.98$ and the phase $\dfrac{\theta}{2} = 0.49$.

5. All are simple harmonic. a. $s = (\frac{4}{3}) \sin 3t$, amp. $\frac{4}{3}$, period $2\pi/3$, phase 0.
   c. $s = 2\cos\dfrac{3}{2}\left(t - \dfrac{\pi}{9}\right)$, amp. 2, period $\dfrac{4}{3}\pi$, phase $\dfrac{\pi}{9}$.

Sec. 153, page 500

1. a. $y = \frac{1}{4}x + \frac{1}{2} + e^x + (c_1 + c_2x)e^{2x}$. c. $y = -\frac{1}{2}(\sin x + \cos x) + c_1 + c_2e^x$.
   e. $y = \frac{1}{5}(\cos x - 2\sin x) + e^x(c_1 \cos x + c_2 \sin x)$.
   g. $y = \frac{1}{8}(1 - x \sin 2x) + c_1 \sin 2x + c_2 \cos 2x$. (Hint: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.)

2. a. $y = 0.1(8e^x - 8\cos 2x + \sin 2x)$. c. $s = \sin t - \cos t + e^{-2t}(3\cos t + 4 \sin t)$.

Sec. 154, page 503

1. In all parts $y_H = c_1 + c_2x + c_2x^2 + c_4 \sin x + c_5 \cos x$.
   a. $y = \dfrac{x^3}{3} + y_H$. b. $y = \dfrac{x^4}{24} + y_H$. c. $y = \dfrac{x^5}{60} - \dfrac{x^2}{3} + y_H$. d. $y = \dfrac{x}{2}\sin x + y_H$.
   e. $y = x \sin x + \frac{3}{2}x \cos x + y_H$. f. $y = \frac{1}{2}e^x + y_H$.
3. a. $y = c_1e^x + (c_2 + x)\cos x + (c_2 - x)\sin x$.
   c. $y = (c_1 + c_2x + c_2x^2)e^{-2x} + 3 - 3x + x^2$.
   e. $y = -x^2 - 3x^2 + c_1 + c_2x + (c_2 - 4x + x^2)e^x$.
   g. $y = -8e^{-0.5x} + (c_1e^{-0.5x} - 64)\cos\dfrac{\sqrt{3}}{2}x + (c_2e^{-0.5x} - 24\sqrt{3})\sin\dfrac{\sqrt{3}}{2}x + c_2e^x$.

Sec. 155, page 505

1. a. $y = (c_1 + \ln|\cos x|)\cos x + (c_2 + x)\sin x$.
   c. $y = -2 + c_1 \cos x + (c_2 + \ln|\sec x + \tan x|)\sin x$.

2. a. $y = \cos x + c_1 \cos 2x + (c_2 - \frac{1}{2}\ln|\sec x + \tan x|)\sin 2x$.
   c. $y = -\frac{1}{4} - \sin^2 x + (c_1 + \ln|\cos x|)\cos 2x + (c_2 + x)\sin 2x$.

3. a. $y = e^{-2x}(-\ln|x| + c_1 + c_2x)$. c. $y = (e^x + e^{2x})[-x + \ln(e^x + 1)] + c_1e^x + c_2e^{2x}$.

4. a. $y = \sin x + \cos x + c_1e^{2x} + c_2e^{2x}$.

Sec. 156, page 508

1. a. $y = -\dfrac{x^2}{2} - c_1x - c_1^2 \ln|x - c_1| + c_2$. [Hint: The differential equation in u and x is a Bernoulli equation (see Sec. 151.)].

c. $y = \begin{cases} -x - \dfrac{1}{c_1}\ln\left|\dfrac{c_1x-1}{c_1x+1}\right| + c_2 \\ -x + \dfrac{2}{c_1}\tan^{-1}(c_1x) + c_2 \end{cases}$ according to whether we set $\dfrac{dy/dx - 1}{dy/dx + 1}$ equal to $(c_1x)^2$ or $-(c_1x)^2$.

e. $y^2 = c_1x + c_2$, also $y = c$.

Sec. 157, page 510

1. a. $y = x^2 + 3 + cx$. c. $\frac{1}{3}(x^2 + y^2) - \dfrac{y}{x} = c$.

e. $x + \tan^{-1}(x/y) = c$. g. $xy = -x\cos x + \sin x + c$.

2. a. $x = y\ln|x| + cy$. c. $\frac{1}{2}e^{2x} - e^x + \ln(e^x + 1) = \ln|y| + c$.

e. $\ln(x^2 + y^2) + 6\tan^{-1}\dfrac{x}{y} = c$. g. $-\dfrac{1}{x} = \ln\left|\dfrac{y-1}{y}\right| + c$. i. $y = e^{-ax}(x + c)$.

k. $y = c_1e^x + e^{-x/2}\left(c_2\cos\dfrac{\sqrt{3}}{2}x + c_2\sin\dfrac{\sqrt{3}}{2}x\right)$. m. $y = c_1e^x + c_2e^{2x/2} + c_2e^{-5x}$.

o. $5\dfrac{y}{x} - 11\ln\left|5\dfrac{y}{x} + 1\right| = 25\ln|x| + c$. q. $y = \displaystyle\sum_{k=0}^{n}\frac{a_{n-k}}{(k+1)(k+2)}x^{k+2} + c_1x + c_2$.

s. $-\cos\dfrac{y}{x} = \ln|x| + c$. u. $x^2 - 3x^2y - y^2 = c$.

Sec. 158, page 515

1. a. $y = a_0\left(1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \cdots + \dfrac{x^{2n}}{n!} + \cdots\right) = a_0\displaystyle\sum_{n=0}^{\infty}\frac{(x^2)^n}{n!} = a_0e^{x^2}$.

c. $y = a_0\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k}}{(2k)!} + a_1\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k-1}}{(2k-1)!} = a_0\cos x + a_1\sin x$.

e. $y = a_0\displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}(2k+3)x^{2k} + a_1\sum_{k=0}^{\infty}(-1)^k(k+1)x^{2k+1}$, $-1 < x < 1$.

g. $y = a_0(1 - x^2) + a_1\left[x - \dfrac{1}{3!}x^2 - \displaystyle\sum_{k=2}^{\infty}\frac{1}{(2k-1)(2k+1)2^kk!}x^{2k+1}\right]$, all x.

2. a. $y = 1 + x^2 - 2\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k-1)(2k+1)2^kk!}x^{2k+1}$, all x.

Sec. 159, page 517

1. a. $y_1 = 1 + \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n2^n(n+1)}{1\cdot3\cdot5\cdots(2n-1)}x^n$, $y_2 = \sqrt{x}\left[1 + \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(2n+3)}{3n!}x^n\right]$.

c. $y_1 = 1 + \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n2^nx^n}{1\cdot3\cdots(2n-1)}$, $y_2 = \sqrt{x}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n = \sqrt{x}e^{-x}$.

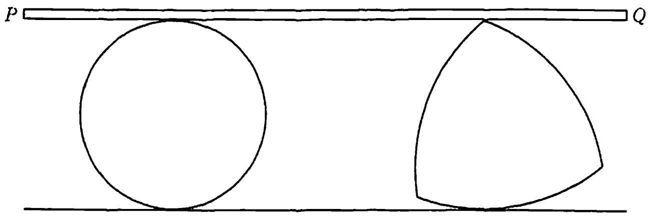
2. a. $y = a_0\left[1 + \displaystyle\sum_{k=0}^{\infty}\frac{(x-\frac{1}{2})^{2k+2}}{3^k(k+1)!\,2\cdot5\cdots(3k+2)}\right]$

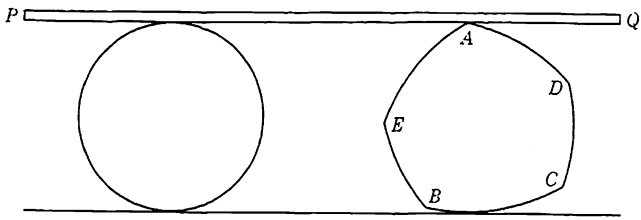
$$+ a_1\left[(x - \tfrac{1}{2}) + \sum_{k=0}^{\infty}\frac{(x-\frac{1}{2})^{2k+4}}{3^k(k+1)!\,4\cdot7\cdots(3k+4)}\right].$$

Sec. 160, page 519

1. a. $y = 1 + \frac{1}{2}x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{40}x^5 + \cdots.$
   c. $y = 1 + \frac{1}{2}x + \frac{1}{2}x^2 + 0 \cdot x^3 - \frac{1}{24}x^4 + \frac{1}{60}x^5 + \cdots, -1 < x < 1.$
2. a. $y = 0 + \frac{1}{2}(x - 1) - \frac{1}{4}(x - 1)^2 + \frac{1}{3}(x - 1)^3 + - \frac{7}{24}(x - 1)^4 + \cdots, 0 < x < 2.$

Examples of rollers (or curves of constant width). See page 118.





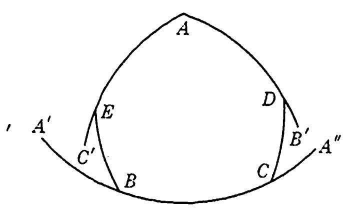


Figure $ADCBE$ was constructed as follows:

With A as center draw arc $A'A''$.

With B on arc $A'A''$ draw arc AB'.

With C on arc $A'A''$ draw arc AC'.

With D on arc AB' draw arc BE.

With E as center draw arc CD.

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