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Asymptotic Dependence Modeling for Spatio-temporal Max-stable Processes

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Abstract. Max-stability is the foundation of multivariate extreme values analysis. This paper investigates the asymptotic dependence modeling of max-stable processes both with spatial and temporal variables. Specifically the paper provides new characterizations of extremal distributions via a dependence measure of the stochastic joint behavior at given locality s and date t. The analytical forms of spatio-temporal asymptotic dependence structures are provided for the main bivariate and trivariate models of max-stable processes.

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1. Introduction

In a spatial framework, modeling the extremes of multivariate phenomenas is an important adequate risk management in environment sciences. Indeed, many environmental extremal problems such as hurricanes, floods, droughts, heat waves, sea height, annual maxima and daily rainfall have an inherent spatial character or are time varying events. Likewise a lot of climate change's problematics and high impact events climatic phenomenas include a spatial component and can be modeled by extreme values approach. This kind of prospect, such as climate change, have provided modeling technics and spatial tools of extreme event statistics and their characterization are often of fundamental interest.

Multivariate extreme values (MEV) theory is often presented in the framework of coordinatewise maxima, so the importance of distinction diminishes. Towards a multivariate analogue of Fisher-Tippett we are looking for some sort of multivariate limit distribution for conveniently normalized vectors of multivariate maxima. For an arbitrary index of set T denoting generally a space of time, a random vector $Y_t = \{Y_j(t); 1 \leq j \leq m, t \in T\}$

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in $\mathbb{R}^m$ is said to be max-stable if, for all $n \in \mathbb{N}$, every $Y_j(t) = (Y_j^{(1)}(t); \ldots; Y_j^{(n)}(t))$ is a n-dimensionnal max-stable vector, that is, there exists suitable and time-varying non-random sequences $\{a_n(t) > 0\}$ and $\{b_n(t) \in \mathbb{R}^d\}$ such as

$$\frac{1}{a_n(t)}\left[M_n(t) - b_n(t)\right] \overset{f.d.d}{\longrightarrow} X(t); t \in T, \tag{1}$$

where $\overset{f.d.d}{\longrightarrow}$ denotes the convergence for the finite-dimensional distributions while $M_n(t) = \max\limits_{1 \le i \le n}(X_i(t)); t \in T$ being the component-wise maxima of the vector $X(t)$.

The theory of extreme values is well elaborated in a statistic and mono-site scheme (see Lo et al [10]). Let suppose now that our random variables both depend on the time indexed by $\{t \in T, T \neq 0\}$; and are studied on multiple sites and then is also indexed by an area index $s \in S$. This leads to study the collections of random vectors $\left(Y_j^s(t)\right); j \geq 0; t \in T; s \in S$ such that for each fixed couple (t, s), the sequence is independent and identically distributed according to a joint cumulative function G_t^s. Under the assumption that this function is max-stable, every univariate margins $G_{t,i}^s$ lies its own domain of attraction and is expressed by on the space of interest $S^+_{\xi_i,t,s} = \left\{z \in \mathbb{R}; \sigma_{i,t,s} + \xi_{i,t,s}\left(y_{t,i}^s - \mu_{i,t,s}\right) > 0; 1 \leq i \leq n\right\}$ by

$$G_i(y_i(s)) = \begin{cases} \exp\left\{-\left[1 + \xi_i(s)\left(\frac{y_i(s)-\mu_i(s)}{\sigma_i(s)}\right)\right]^{\frac{-1}{\xi_i(s_i)}}\right\} & if\ \xi_i(s) \neq 0 \\ \exp\left\{-\exp\left\{-\left(\frac{y_i(s)-\mu_i(s)}{\sigma_i(s)}\right)\right\}\right\} & if\ \xi_i(s) = 0 \end{cases}; \tag{2}$$

and for all site s, the parameters $\{\mu_{i,t,s} \in \mathbb{R}\}$, $\{\sigma_{i,t,s} > 0\}$ and $\{\xi_{i,t,s} \in \mathbb{R}\}$ are referred to as the location, the scale and the shape parameters respectively. Particularly, the different values of $\xi_i(s) \in \mathbb{R}$ allows (2) to be a spatial EV model, that is, to belong either to Fréchet family, the Weibull one or Gumbel one.

The peak-over-threshold approach is common used, fitting data with a generalized Pareto distributions (GPD). This approach, like the coordinatewise one is concerned with asymptotic stochastic behavior of sample of identical copies of variables. In particular, the multivariate GP Distribution of a sample of i.i.d of sequences of variables, is closely linked the underlying MEV one (see Tajvidi (2006)). More precisely, if H defined the MEV model, then the associated multivariate GP distribution G is defined for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and for a given $x_0 = \left(x_0^{(1)}; ...; x_0^{(n)}\right)$ in the support of G,

$$H(x_1, ..., x_n) = \left\{\frac{-1}{\log G(x_0)}\right\} \log\left[\frac{G(x_0 + x)}{G(\min(x, x_0))}\right]. \tag{3}$$

The major contribution of this paper is to propose new model of stochastic dependence for max-stable processes in spatial and temporal framework. Specifically, section 2 gives the preliminaries of the study. Section 3 deals a new characterization of asymptotic models of time-varying models of dependence of spatial processes. In section 4 the analytical forms these spatio-temporal models of dependence for the main usual extremal distributions both for spatial and time varying contexts.

2. Preliminaries

This section summaries definitions and properties on the generalized Pareto processes and the copulas of multivariate joint processes dependence which turn out to be necessary for our approach. For this purpose the definition of multivariate copula is necessary.

Definition 1. *A n-dimensional copula is a non-negative function C_n defined on $\mathbb{R}^n$ satisfying the following properties.*

i) $C_n(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_n) = 0$; for all $(u_1, ..., u_{i-1}, u_{i+1}, ..., u_n) \in I^{n-1}$.

ii) $C_n(u_1, ..., u_{i-1}, 1, u_{i+1}, ..., u_n) = C_{n-1}(u_1, ..., u_{i-1}, u_{i+1}, ..., u_n)$, that is, an (n-1) copula for all i.

iii) The volume V_B of any rectangle $B = [a, b] \subseteq [0, 1]^n$ is positive, that is,

$$V_B = \sum_{\varepsilon=(\varepsilon_1,...,\varepsilon_k)\in\{0,1\}^k} (-1)^{s(\varepsilon)_1+...+i_n} C_n\left(b_1 + \varepsilon_1\left(a_1 - b_1\right), ...b_k + \varepsilon_k\left(a_k - b_k\right)\right) \geq 0. \quad (4)$$

where $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$.

The use of copulas in stochastic analysis whas justified by the canonical parametrization of Sklar, see Joe [9] or Nelsen [12], such that the n-dimensional copula C associated to a random vector $(X_1, ..., X_n)$ with cumulative distribution F and with continuous marginal $F_1, ..., F_n$ is given, for $(u_1, ..., u_n) \in [0, 1]^n$ by

$$C(u_1, ..., u_n) = F[F_1^{-1}(u_1), ..., F_n^{-1}(u_n)]; \quad (5)$$

F^{-1} being the generalized inverse such as $F^{-1}(x) = \inf\{t \in [0, 1], F(t) \leq x\}$.

Even in spatial analysis, stochastic phenomenas can be modeled via copulas. Particularly in a spatial context, Schmitz [14] showed that a collection of copulas and marginal distributions also define a stochastic process. So, the above property ii) is given such as, for all collection $\{C_{t_1,...,t_n}; t_1 < ... < t_n, n \in \mathbb{N}\}$ of copulas satisfying the consistent condition

$$\lim_{u_k \to 1^-} C_{t\,1,...,t\,n}\left(u_1, ..., u_n\right) = C_{t\,1,...,t\,n}\left(u_1, ..., u_{k-1}, u_{k+1}, ..., u_n\right);$$

there exists a probability space $(\Omega, , P)$ and a stochastic processes $\{(Y_x), x \in T\}$ such that

$$P\left(Y_{t_1} < x_1, ..., Y_{t_n} < x_n\right) = C_{t\,1,...,t\,n}\left(F_{t_1}\left(x_1\right), ..., F_{t_n}\left(x_n\right)\right); \quad (6)$$

and $\{(Y_t), t \in T\}$ is measurable for all $t \in T$.

While studying conditional dependence of GPD models, Ferreira et al. (see [7]) have proposed the GP processes as follows. Let $C^+(S)$ be the space of non-negative real continuous functions equipped with the supremium norm where S is compact subset of $\mathbb{R}^d$.

Theorem 1. *A stochastic process W is a generalized Pareto process if the following statement are satisfied.*

(a) The expectation $E\left(W(s)/\sup u \in S\right)$ is positive for all $s \in S$,

(b) $P\left(\sup_{s\in S} W(s)/w_0 > x\right) = x^{-1}\right)$ for $x > 1$ (standard Pareto distribution),

(c) For all $r > w_0$ and $B \in B\left(\bar{C}^{+}(S)\right)$

$$P\left(\frac{w_0 W}{\sup_{s\in S} W(s)} \in B \middle| \sup_{s\in S} W(s) > r\right) = P\left(\frac{w_0 W}{\sup_{s\in S} W(s)} \in B\right); \tag{7}$$

where

$$C^{+}_{w_0}(S) = \left\{ f \in C^{+}(S) : \sup_{s\in S} f(s) = w_0 \right\}.$$

.

In the relation (7) the probability $\rho(B) = P\left(\frac{w_0 W}{\sup_{s\in S} W(s)} \in B\right)$ is refered as the spectral measure. In a discrete set for S, $S = \{s_1, ..., s_n\}$ if $W = (W_1, ..., W_n)$ it provides instead:

$$\rho(B) = P\left(\frac{w_0 (W_1, ..., W_n)}{\max\limits_{1 \le i \le n} (W_i)} \in B\right).$$

3. Asymtotic Dependence for Spatio-temporal Processes

Even in spatial stochastic context, three possible distributions can describe the asymptotic behavior of conveniently normalized extremal distributions at a given geographical locality s. These distributions are instead described by a class of dependence models. Specially in a spatial framework, let $D_N = \{s_1, ..., s_N\} \subset \mathbb{R}^2$ be the set of locations (geographical ereas, mines localities, ...), sampled over a $\left[0, \frac{1}{m}\right] \times \left[0, \frac{1}{m}\right]$ rectangle ($m \in \mathbb{N}$), where the phenomenas are observed. Let Y a variable of interest, observed at given site s and date t.

Let consider the following notation of component-wise vector of spatio-temporal process.

$$Y^{\check{s}}_t(s) = Y(t, s) = \left\{ \left(Y_{t_1}(s_1); ...; Y_{t_n}(s_n)\right), s \in S, t \in T \right\}$$

is the response vector at a given time t from a spatio-temporal and max-stable model.

So, under this notation a realisation $y(t, s) = y_t(s)$ of $Y_t(s)$ is obtained as

$$y_{i,t}(s) = \mu_{i,t}(s_i) + \frac{\sigma_{i,t}(s_i)}{\xi_{i,t}(s_i)} \left[s_t(s)^{\xi^{(i)}_t(s)} - 1 \right] \text{ for } i = 1, ..., m. \tag{8}$$

Equivalently, it comes that, for a given site s $D_N = \{s_1, ..., s_N\} \subset \mathbb{R}^2$

$$P\left(\frac{Y_1(s) - b^{(1)}_n(s)}{a^{(1)}_n(s)} \le y_1(s); ...; \frac{Y_N(s) - b^{(N)}_n(s_n)}{a^{(N)}_n(s_n)} \le y_N(s)\right)^n = H\left(y_1(s); ...; y_N(s)\right). \tag{9}$$

For simplicity reasons, let denote, like in the paper [6] that $Y(t,s) = Y_t^{\check{s}}$ (which is different from Y_t^s, the s-th power of Y_t). Then, under this notational assumption the spatialized version of the joint distribution function F of Y is given by $F_t^{\check{s}}$ for given vector of realization $y_t^{\check{s}} = \left(y_t^{(1)}(s), ..., y_t^{(m)}(s)\right)$ such as

$$F_t^{\check{s}}(y_1(t,s), ..., y_m(t,s)) = F\left(y_1^{\check{s}_1}(t); ...; y_m^{s_m}(t)\right) = F(y_1(t,s); ...; y_m(t,s)).$$

In the same vein, the spatio-temporal copula associated to the distribution G via Sklar parametrization (1) will be denoted as $C_t^{\check{s}} = \left(C_{1,t}^{\check{s}}; ...; C_{m,t}^{\check{s}}\right)$.

So, the relation (5) provides, for all $x_t = \left(x_t^{(1)}; ...; x_t^{(m)}\right)$ in $\mathbb{R}^m \times T$ the relation

$$C_t^{\check{s}}(u_1; ...; u_m) = F_t^{\check{s}}\left[{}_t^{\check{s}}\left(F_t^{\check{s}_1}(u_1)\right)^{-1}; ...; \left(F_t^{\check{s}_m}(u_m)\right)^{-1}\right]. \tag{10}$$

Note that, for all $m \in \mathbb{N}$ and for all geographical locality s, the spatio-temporal unit simplex of $\mathbb{R}^{(m-1}$ is given, under the notational by

$$\Delta_{t,m}^{\check{s}} = \left\{\lambda_t^{\check{s}} = \left(\lambda_1^{\check{s}_1}; ...; \lambda_t^{\check{s}_m}\right) \in \mathbb{R}_+^m; \left\|\lambda_t^{\check{s}}\right\| =_{i=1}^m \lambda_t^{\check{s}_i} = 1\right\}. \tag{11}$$

The following theorem provides an other characterization of the spatio-temporal extreme values distribution associated the process $\{Y_s; s \in S\}$. It is a spatio-temporal parameters version of a key result of extreme values theory, see Resnick [13] or Beirlant[1].

Theorem 2. *Let $\left\{Y_t^{\check{s}}, s \in S, t \in T\right\}$ be a spatio-temporal (ST) process with parametric joint distribution $H_t^{\check{s}} = (H_t^{\check{s}_1}; ...; H_t^{\check{s}_m})$. The following statements are satisfied*

(a) A sufficient condition for the process $H_t^{\check{s}}$ to be a ST-MEV distribution is that there exists two spatio-temporal non-random sequences $\left\{\alpha_n^{\check{s}}(t) > 0\right\}$ and $\left\{\beta_n^{\check{s}}(t) \in \mathbb{R}\right\}$ such that

$$\lim_{n \uparrow \infty} P\left(\frac{M_t^{\check{s}} - \beta_n^{\check{s}}(t)}{\alpha_n^{\check{s}}(t)} \leq y_t^{\check{s}}\right) = \left(H_1\left(y_t^{\check{s}_1}\right), ..., H_m\left(y_t^{\check{s}_m}\right)\right).$$

where $M_t^{\check{s}(i)}$ is univariate margins of the spatio-temporal componentwise vector of maxima.

(b) Under the condition (a) there exists a ST vector of coefficient $\lambda^s(t)$ and ST-dependence function $B_t^{\check{s}}$ mapping $\Delta_{t,m-1}^{\check{s}} \times S$ to $\left[\frac{1}{m-1}, 1\right]$ such that, for all $y_t^{\check{s}} = \left(y_t^{\check{s}(1)}, ..., y_t^{\check{s}(m)}\right) \in [0,1]^m$,

$$H_t^{\check{s}}\left(y_1^{\check{s}}(t); ...; y_m^{\check{s}}(t)\right) = \exp\left[-\sum_{i=1}^m y_i^{\check{s}}(t) B_t^{\check{s}}\left[\lambda_1^{\check{s}}(t), ...,, \lambda_m^{\check{s}}(t)\right]\right]; \tag{12}$$

where $\left\{\lambda_i^{\check{s}}; 1 \leq i \leq m\right\}$ are spatial coefficients.

Proof. (a) Let $\{\alpha_n > 0\}$ and $\{\beta_n \in \mathbb{R}\}$ be the non-random normalizing sequences of H. Then, their corresponding space and time extensions $\left\{\alpha_n^{\check{s}}(t) > 0\right\}$ and $\left\{\beta_n^{\check{s}}(t) \in \mathbb{R}\right\}$ are defined on the set, $\mathbb{N}^* \times S \times T$, such that

$$\lim_{n \to\infty} P\left(\frac{M_t^{\check{s}} - \beta_n^{\check{s}}(t)}{\alpha_n^{\check{s}}(t)} \leq y_t^{\check{s}}\right) = \lim_{n \to\infty} P\left[\underset{i=1}{\overset{n}{}}\left(\frac{M_t^{\check{s}_i} - \beta_i^{\check{s}_i}(t)}{\alpha_i^{\check{s}_i}(t)} \leq y_t^{\check{s}_i}\right)\right]$$

Then,

$$\lim_{n \to\infty} P\left(\frac{M_t^{\check{s}} - \beta_n^{\check{s}}(t)}{\alpha_n^{\check{s}}(t)} \leq y_t^{\check{s}}\right) = \lim_{n \to\infty} P\left[\underset{i=1}{\overset{m}{}}\left(Y_t^{\check{s}_i} \leq \alpha_i^{\check{s}}(t)\, y_i^{\check{s}}(t) + \beta_i^{\check{s}}(t)\right)\right].$$

That is equivalent, due to independence, to

$$\lim_{n \to\infty} P\left(\frac{M_t^{\check{s}} - \beta_n^{\check{s}}(t)}{\alpha_n^{\check{s}}(t)} \leq y_t^{\check{s}}\right) = \lim_{n \to\infty}\left(\prod_{i=1}^{m} P\left[\left(X_i^{\check{s}} \leq \alpha_i^{\check{s}}(t)\, y_i^{\check{s}}(t) + \beta_i^{\check{s}}(t)\right)\right]\right).$$

So, there exists a max-stable distribution G whose max-domain of attraction contains the MEV H. Then,

$$\lim_{n \to\infty} P\left(\frac{M_t^{\check{s}} - \beta_n^{\check{s}}(t)}{\alpha_n^{\check{s}}(t)} \leq y_t^{\check{s}}\right) = \lim_{n \to\infty}\left[G\left(\alpha_i^{\check{s}} y_i{}^{\check{s}}(t) + \beta_i^{\check{s}}(t)\right), ...\alpha_i^{\check{s}}(t)\, y^{\check{s}}(t) + \beta_i^{\check{s}}\right]^n.$$

Finally, since the distribution G is max-stable

$$\lim_{n \to\infty} P\left(\frac{M_t^{\check{s}} - \beta_n^{\check{s}}(t)}{\alpha_n^{\check{s}}(t)} \leq y_t^{\check{s}}\right) = \left(H_1\left(y_t^{\check{s}(1)}\right), ..., H_n\left(y_t^{\check{s}(n)}\right)\right).$$

(b) Assume that the distribution H is a MEV model, that is its univariable marginal H_i satisfies relation (9). Therefore, it is sufficient to show for a given site s and date t, that, $H_t^{\check{s}}$ satisfies the spatio-temporal version of max-stability property.

It comes from Coles ([7]) that, at a given site s and date t the MEV model H has the following representation Y_i such as $y_i = \dfrac{-1}{\log\left[1 - \lambda_i t_i(s_i)\right]}$ with $s_i > u_i$.

Note moreover that it not be restrictive to assume in the following that the spatio-temporal multivariate process $\left\{Y_k^{\check{s}}(t), s \in S, t \in T\right\}$ has spatio-temporal unit Fréchet margin, which it is more convenient to work with.

$$Y_t^{\check{s}} \sim \Phi_{\theta,t}^{\check{s}} \Leftrightarrow \ln\left(Y_t^{\check{s}}\right)^{\theta} \sim \Lambda_t^{\check{s}} \Leftrightarrow \tfrac{-1}{Y_t^{\check{s}}} \sim \Psi_{\theta,t}^{\check{s}} \Leftrightarrow Y_t^{\check{s}} = \mu\left(y_t^{\check{s}}\right) + \tfrac{\sigma\left(y_t^{\check{s}}\right)}{\xi\left(y_t^{\check{s}}\right)}\left[\left(y_t^{\check{s}}\right)^{\xi(y_t)} - 1\right].$$

Therefore,

$$H\left(y_1, ..., y_m\right) = \exp\left[-_{S_m} \max\left(q_1\lambda_1 t_1(s_1), ..., q_m\lambda_m t_m(s_m)\right)\mu d(q)\right] + o(\max(\lambda_i).$$

If, in particularly, for all $i = 1, ..., n$ we set $\lambda_i t_i(s_i) = \lambda_t(s)$, then it follows that there exists a spatio-temporal dependence function $B_t^{\check{s}} = B(\lambda, q, y)$ such as:,

$$B_t^{\check{s}}(\lambda) = \lambda \sum_{i=1}^{i=m} t_i(x_i)_{S_m} \max\left(\frac{q_1 t_1(x_1)}{\sum_{i=1}^{i=m} t_i(x_i)}, ..., q_m\left(1 - \frac{\sum_{i=1}^{i=m-1} t_i(x_i)}{\sum_{i=1}^{i=m} t_i(x_i)}\right)\right)\mu d(q).$$

Particularly under the above component-wise notation

$$H_t^{\check{s}}(y_1^{\check{s}}(t),...,y_n^{\check{s}}(t)) = \exp\left[-\left(\sum_{i=1}^{m} y_i^{\check{s}}(t)\right) B_t^{\check{s}}\left(\frac{-q_11y_1^{\check{s}}(t)}{\sum_{i=1}^m y_i^{\check{s}}(t)};...;\frac{-q_{m-1}y_{m-1}^{\check{s}}(t)}{\sum_{i=1}^m y_i^{\check{s}}(t)}\right)\right]. \quad (13)$$

Moreover, taking into account Dossou et al., it follows that

$$\begin{aligned} & B_t^{\check{s}}\left(\frac{-q_11y_1^{\check{s}}(t)}{\sum_{i=1}^m y_i^{\check{s}}(t)};...;\frac{-q_{m-1}y_{m-1}^{\check{s}}(t)}{\sum_{i=1}^m y_i^{\check{s}}(t)}\right) \\ = \; & D\left(\frac{-q_11y_1^{\check{s}}(t)}{\sum_{i=1}^m y_t^{\check{s}(i)}},\ldots,\frac{-q_{m-1}1y_t^{\check{s}(m-1)}}{\sum_{i=1}^m y_t^{\check{s}(i)}}\right)+\left(1\text{-}\frac{-q_11y_1^{\check{s}}(t)}{\sum_{i=1}^m y_t^{\check{s}(i)}}\right)D_{\bar{N}_1}\left(\frac{\frac{-q_11y_1^{\check{s}}(t)}{\sum_{i=1}^m y_t^{\check{s}(i)}}}{1-\frac{-q_11y_1^{\check{s}}(t)}{\sum_{i=1}^m y_t^{\check{s}(i)}}};\ldots;\frac{\frac{-q_{m-1}1y_t^{\check{s}(m-1)}}{\sum_{i=1}^m y_t^{\check{s}(i)}}}{1-\frac{-q_11y_1^{\check{s}}(t)}{\sum_{i=1}^m y_t^{\check{s}(i)}}}\right). \end{aligned}$$

Finally, by noting $\lambda_i^s(t) = \dfrac{-q_i}{\sum_{i=1}^m y_i^{\check{s}}(t)}$ it follows that

$$B_t^{\check{s}}\left(\lambda_1^s,...,\lambda_{m-1}^s\right) = D\left(\lambda_1^s y_1^{\check{s}}(t),...,\lambda_1^s y_1^{\check{s}}(t)\right) + (1-t)\,D\left(\lambda_1^s y_1^{\check{s}}(t),...,\lambda_1^s y_1^{\check{s}}(t)\right)$$

where $B_t^{\check{s}}$ is the spatialized Pickands dependence function, mapping the simplex $\Delta_{s,m-1}$ to $\left[\frac{1}{m-1};1\right]$ (see Beirlant [1] . Thus, we obtain the result as asserted

Definition 2. *The space and time dependent function $B_t^{\check{s}}\left(\lambda_1^s,...,\lambda_{m-1}^s\right)$ is called the Spatio-temporal Asymptotic Dependence (STAD) function associated to the process $\{Y_s\}$.*

Particularly, in a the following and with a parameter θ we can set $B_{\theta,t}^{\check{s}}(\lambda^s) = B_\theta^{\check{s}}(\lambda_t)$ where $\lambda_t \in \Delta_{t,m}^{\check{s}}$. For example, for the bivariate and one parametric negative logistic model (see Joe [9]) defined for $y_t^{\check{s}} = \left(y_t^{\check{s}(1)}, y_t^{\check{s}(2)}\right)$ and $\theta = (\theta_1,\theta_2) \geq 0$ by

$$G_\theta^{\check{s}}\left(y_t^{\check{s}}\right) = \exp\left\{-\left(\frac{1}{y_t^{\check{s}(1)}}+\frac{1}{y_t^{\check{s}(2)}}-\left[\left(y_t^{-\check{s}(1)\theta_1}y_t^{-\check{s}(2)\theta_1}\right)^{-\theta_1}\right]^{\frac{-1}{\theta}}\right)\right\};$$

then, it follows that the corresponding ST dependence function is given by

$$B_\theta^{\check{s}}(\lambda_t^s) = \frac{1}{1+\lambda_t^s}\left[1-\left(1+\lambda_t^{s-\theta}\right)^{\frac{-1}{\theta}}\right] \text{ with } \lambda_t^s \in [0,1]$$

The following theorem, proposes a spatial characterization the multivariate GP distribution associated to the spatial MEV of the same process Y.

Theorem 3. *Let $\left\{G_t^{\check{s}}, s\in S, t\in T\right\}$ be a MEV distribution of a sample of copies of a spatio-temporal max-stable process X_t^s and $H_t^{\check{s}}$ the multivariate GP associated to the same sample. Then, for a given site s_0 and date t_0 ,*

$$H_t^{\check{s}}(y_t^s) = \frac{-1}{\log G_t^{\check{s}}\ (y_{t_0}^s)}\log\left(\frac{G_t^{\check{s}}\ (y_{t_0}^s+y_t^s)}{G_t^{\check{s}}\ \left(\min(y_t^s,y_{t_0}^s)\right)}\right) = 1-\log\left(\frac{G_t^{\check{s}}\ (y_t^s)}{G_t^{\check{s}}\ \left(\min(y_t^s,y_{t_0}^s)\right)}\right). \quad (14)$$

for all $y_{t_0}^s \in support(G_t^{\check{s}}\ )$.

Proof. It should be noted that the normalizing sequences $\{\alpha_n > 0\}$ and $\{\beta_n \in \mathbb{R}\}$ in Theorem 5 are given by

$$\sigma_n = F^{-1}(1 - \frac{1}{n}) \text{ and } \beta_n = \frac{f(\sigma_n)}{1 - F(\sigma_n)};$$

where f is the common density function of the sample. Moreover, it should be considered that in this section operations on vectors are componentwisely, that is, for a given location s of S;

$$\begin{cases} a_k(s_0) & = \left(a_k^{(1)}(s_{0,}t); ...; a_k^{(m)}(s_{0,}t)\right) \\ a_k(s) + b_k(s) & = \left(a_k^{(1)}(s) + a_k^{(1)}(s); ...; a_k^{(m)}(s) + a_k^{(m)}(s)\right) \\ \dfrac{y_t^s}{y_{t_0}^s} & = \left(\dfrac{y_t^{s(1)}}{y_{t_0}^{s(1)}}; ...; \dfrac{y_t^{s(m)}}{y_{t_0}^{s(m)}}\right) \end{cases} \quad (15)$$

Let H be a given multivariate Pareto distribution. So for a given point $y_{t_0}^s = \left(y_{t_0}^{s_1}, ..., y_{t_0}^{s_m}\right)$ with $y_{t_0}^{s_i} > 0$ and $\alpha > 0$, it follows that

$$H_t^{\check{s}}(y_t^s) = 1 - \left[\frac{y_t^s}{y_{t_0}^{s(1)}}\right]^{-\alpha}.$$

In particular, for $y_t^{s(1)} > y_{t_0}^{s(1)}$, it commes that

$$H^{-1}(y_t^s) = (1 - y_t^s)^{-1/\alpha} y_{t_0}^{s(1)}.$$

Therefore, we have: $a_n(s) = G^{-1}\left(\left[1 - \dfrac{1}{n}\right]\right) = n^{1/\alpha} y_{t_0}^s$.

Otherwise, asymptotically, it comes that

$$\lim_{n \uparrow +\infty} P(M_n(s) \leq a_n(s_0) y_t^s) = \lim_{n \uparrow +\infty} \left[1 - \left(\frac{n^{1/\alpha} y_t^{\check{s}}}{y_{t_0}^{\check{s}}}\right)^{-\alpha}\right]^n$$

that is

$$\lim_{n \uparrow +\infty} P(M_n(s) \leq a_n(s_0) y_t^s) = \lim_{n \uparrow +\infty} \left[1 - n^{1/\alpha} y_t^{-s\alpha}\right]$$

which gives marginally

$$\lim_{n \longrightarrow +\infty} P(M_n(s_i) \leq a_n(s_0) y_t^{s_i}) = \exp\left(\frac{-1}{y_t^{\check{s}(i)}}\right) = H_t^{\check{s}_i}\left(y_t^{\check{s}_i}\right)$$

Furthermore,

$$H_t^{\check{s}}(y_t^s) = \{$$

$$1 - \log\left(\frac{G(y_t^s)}{G\left(\min(y_t^s, y_{t_0}^s)\right)}\right) \; si \; y_t^s \geq 0 \quad .(16)$$

$$0 \qquad elsewhere$$

Finally, using simultanously the relations (4) and (22) we obtain (20) as asserted

4. Analytical Characterization of STAD

Note that it should be noted that even in spatio-temporal context, the dual relation (see [8]) relying the vectors of maxima and minima holds. So, ,

$$\left\{\min_{1\leq k\leq m}\left\{Y_k^{\check{s}}(t)\right\}\right\} = -\left\{\max_{1\leq k\leq m}\left\{-Y_k^{\check{s}}(t)\right\}\right\}\ for\ all\ site\ s\in S \text{ and date } t\in T.$$

Most of these families arise from symmetric, asymmetric or mixed extensions of a known differentiable parametric model: the logistic family (see Degen [4]).

4.1. The Pseudo-Power function of STAD

Theorem 4. *Let $H_{\theta,t}^{\check{s}}$ be the parametric and max-stable distribution modeling the stochastic behavior of a space and time varyng process. Then there exists a mutivariate parametric pseudo-power function $P_\theta^{\check{s}}$ such as*

$$G_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}_1},...,\tilde{y}_t^{\check{s}_m}\right) = \exp\left\{-P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}_1},...,\tilde{y}_t^{\check{s}_m}\right)\right\},$$

where $P_\theta^{\check{s}}$ is defined on $\mathbb{R}\times S\times T$.

Proof. In the proof of theorem 5, the relation (13) shows that Particularly under the above component-wise notation

$$H_t^{\check{s}}(\tilde{y}_t^{\check{s}_1},...,\tilde{y}_t^{\check{s}_m}) = \exp\left[-\left(\sum_{i=1}^m y_t^{\check{s}_i}\right)B_t^{\check{s}}\left(\frac{-q_1 y_t^{\check{s}_1}}{\sum_{i=1}^m y_t^{\check{s}_i}};...;\frac{-q_{m-1}y_t^{\check{s}_{m-1}}}{\sum_{i=1}^m y_t^{\check{s}_i}}\right)\right] \tag{17}$$

By setting

$$P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}_1},...,\tilde{y}_t^{\check{s}_m}\right) = \left(\sum_{i=1}^m y_t^{\check{s}_i}\right)B_t^{\check{s}}\left(\frac{-q_1 y_t^{\check{s}_1}}{\sum_{i=1}^m y_t^{\check{s}_i}};...;\frac{-q_{m-1}y_t^{\check{s}_{m-1}}}{\sum_{i=1}^m y_t^{\check{s}_i}}\right).$$

On obtain a pseudo power function $P_\theta^{\check{s}}$ in $y_t^{\check{s}_i}$ for $i=1,...,n$

Remark 1. *To characterize a spatio-temporal max-stable model consists simply to provides the underlying pseudo-power function*

4.2. Analytical Form of Bivariate STAD

This section, we provide the analytical forms of the ST models of the dependence of the main usual families of extreme distributions. According to remark 7, it is sufficient to gives the corresponding pseudo-power function $P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right)$ where $\tilde{y}_t^{\check{s}} = \left(\tilde{y}_t^{\check{s}_1},\tilde{y}_t^{\check{s}_2}\right)$.

STAD of Logistic model and symmetric entensions

1	Logistic model (Gumbel family) with $\theta \geq 1$ (voir Joe [9]) $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right) = \left(\left(\tilde{y}_t^{\check{s}_1}\right)^\theta + \left(\tilde{y}_t^{\check{s}_2}\right)^\theta\right)^{\frac{1}{\theta}}$; $\cdot\ B_\theta^{\check{s}}(\lambda_t) = \dfrac{\lambda_t}{1+\lambda_t}\left[\left(1+\lambda_t^{-\theta}\right)^{\frac{1}{\theta}} - 1\right]$
2	Negative one-parametric logistic model (Galambos family) with $\theta \geq 0$ $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right) = \left(\frac{1}{\tilde{y}_t^{\check{s}_1}} + \frac{1}{\tilde{y}_t^{\check{s}_2}} - \left[\frac{1}{\tilde{y}_t^{\check{s}_1\theta_1}} + \frac{1}{\tilde{y}_t^{\check{s}_1\theta_1}}\right]^{\frac{-1}{\theta}}\right)$; $\cdot\ B_\theta^{\check{s}}(\lambda_t) = \dfrac{\lambda_t}{1+\lambda_t}\left[1 - \left(1+\lambda_t^{-\theta}\right)^{\frac{-1}{\theta}}\right]$
3	Negative two-parametric logistic model or model of Joe (see [9]); $\theta = (\theta_1, \theta_2)$ $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right) = \left(y_t^{\check{s}_1} + y_t^{\check{s}_2} - \left[y_t^{-\check{s}_1\theta_1} + y_t^{-\check{s}_2\theta_1} - \left(\tilde{y}_t^{\check{s}_1\theta_1\theta_2} + y_t^{\check{s}_2\theta_1\theta_2}\right)^{-\frac{1}{\theta_2}}\right]^{\frac{1}{\theta_1}}\right)$ $\cdot\ B_\theta^{\check{s}}(\lambda_t) = \dfrac{\lambda_t}{1+\lambda_t}\left(\left[\lambda_t^{-\theta_1} + 1 - \left(\lambda_t^{\theta_1\theta_2}+1\right)^{\frac{-1}{\theta_2}}\right]^{\frac{1}{\theta_1}} - 1\right).$
4	Gaussian bivariate model (or model of Hüsler-Réïss) with $\theta \geq 0$ (see [8]) $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right) = \left[\tilde{y}_t^{\check{s}_1}\Phi\left(\frac{1}{\theta} + \frac{\boldsymbol{\theta}}{2}\log\left(\frac{\tilde{y}_t^{\check{s}_1}}{\tilde{y}_t^{\check{s}_2}}\right)\right) + \tilde{y}_t^{\check{s}_1}\Phi\left(\frac{1}{\theta} + \frac{\boldsymbol{\theta}}{2}\log\left(\frac{\tilde{y}_t^{\check{s}_2}}{\tilde{y}_t^{\check{s}_1}}\right)\right)\right]$ Φ being the cumulative distribution function of N(0,1). $\cdot\ B_\theta^{\check{s}}(\lambda_t) = \dfrac{\lambda_t}{1+\lambda_t}\left[\frac{1}{\lambda_t}\Phi\left(\frac{2-\theta^2\log(\lambda_t)}{2\theta}\right) - \Phi\left(-\frac{2+\theta^2\log(\lambda_t)}{2\theta}\right)\right].$
5	Symmetric extension of logistic model or model of Tajvidi (see [12]) $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right) = \exp\left\{-\left[\left(\left(\tilde{y}_t^{\check{s}_1}\right)^{\theta_1} + \left(\tilde{y}_t^{\check{s}(2)}\right)^{\theta_1}\right) + \theta_2\left(\tilde{y}_t^{\check{s}_1}\tilde{y}_t^{\check{s}_2}\right)^{\frac{\theta_2}{2}}\right]^{\frac{1}{\theta_1}}\right\}$;with $\theta = (\theta_1, \theta_2)$ $\cdot\ B_\theta^{\check{s}}(\lambda_t) = \dfrac{\lambda_t}{1+\lambda_t}\left[\lambda_t{}^{-\theta_1} + 1 + \theta_2\lambda_t{}^{\frac{-\theta_1}{2}}\right]^{\frac{1}{\theta}}$ where $0 < \theta_2 \leq 2(\theta_1 - 1); \theta_2 \geq 2$
6	Symmetric two-parametric extension of logistic model $\theta = (\theta_1, \theta_2)$ (Joe [9]) $\circ\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right) = \left[\left(\left(\tilde{y}_t^{\check{s}_1}\right)^{\theta_1} + \left(\tilde{y}_t^{\check{s}_2}\right)^{\theta_1}\right) - \theta_2\left(\left(\tilde{y}_t^{\check{s}_1}\right)^{\theta_1\theta_2} + \left(\tilde{y}_t^{\check{s}_2}\right)^{\theta_1\theta_2}\right)^{\frac{1}{\theta_2}}\right]^{\frac{1}{\theta_1}}$ $\cdot\ B_\theta^{\check{s}}(\lambda_t) = \dfrac{\lambda_t}{1+\lambda_t}\left(\left[\lambda_t{}^{-\theta_1} + 1 - \left(\lambda_t{}^{\theta_1\theta_2}+1\right)^{\frac{-1}{\theta_2}}\right]^{\frac{1}{\theta_1}}\right)$ where $\theta_2 > 0; \theta_1 \geq 1$
7	Symmetric Extension of bilogistic model, proposed by Smith (see Michel [11]) $\circ\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right) = \left(\tilde{y}_t^{\check{s}_1}q^{1-\theta_1} + \tilde{y}_{t2}^{\check{s}_2}(1-q)^{1-\theta_2}\right)^{\frac{1}{\theta_1}}$ with $\theta = (\theta_1, \theta_2)$; $0 < \theta_1$; $\theta_2 < 1$ where $q = q(\theta_1, \theta_2)$ are the roots of equation: $1 - \theta_1)\tilde{y}_t^{\check{s}(1)}(1-q)^{\theta_2} - (1-\theta_2)\tilde{y}_t^{\check{s}(2)}q^{\theta_1} = 0$ $\cdot\ B_\theta^{\check{s}}(\lambda_t) = \dfrac{\lambda_t}{1+\lambda_t}\left[q^{1-\theta_1} + \lambda_t(1-q)^{1-\theta_2} + 1\right].$

While studying max-stable models Joe (see [9]) and Tajvidi (see [15]) have proposed many asymmetric extension of logistic model.

STAD of logistic model and asymmetric generalizations

1	Asymmetric three parametric extension of logistic model with $\theta=(\theta_1,\theta_2,\theta_3)$ $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right)=(1-\theta_2)\,\tilde{y}_t^{\check{s}_1}-(1-\theta_1)\,\tilde{y}_t^{\check{s}_2}-\left[\left(\theta_1\tilde{y}_t^{\check{s}_1}\right)^{\theta_3}+\left(\theta_2\tilde{y}_t^{\check{s}_2}\right)^{\theta_3}\right]^{\frac{1}{\theta_3}}$; $\cdot\ B_\theta^{\check{s}}(\lambda_t)=\dfrac{\lambda_t}{1+\lambda_t}\left[1-\theta_1+\theta_2\lambda_t+\left(\theta_1^{\theta_3}+(\theta_2\lambda_t)^{\theta_3}\right)\right]^{\frac{1}{\theta_3}}$ with $\theta_1\geq 0,\theta_2\leq 1,\theta_3\geq 1$
2	Asymmetric three parametric and negative extension of logistic model $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right)=\left(\tilde{y}_t^{\check{s}_1}+\tilde{y}_t^{\check{s}_2}\right)+\left[\left(\theta_1\tilde{y}_t^{\check{s}_1}\right)^{-\theta_3}+\left(\theta_2\tilde{y}_t^{\check{s}_2}\right)^{-\theta_3}\right]^{\frac{-1}{\theta_3}}$ with $\theta=(\theta_1,\theta_2,\theta_3)$ $\cdot\ B_\theta^{\check{s}}(\lambda_t)=\dfrac{\lambda_t}{1+\lambda_t}\left[1-\left(\theta_1^{-\theta_3}+(\theta_2\lambda_t)^{-\theta_3}\right)\right]^{\frac{-1}{\theta_3}}$ where $0<\theta_1,\theta_2\leq 1,\theta_3>0$
3	Symmetric one parametric, mixed extension (proposed by Tajvidi (see [11]) $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right)=\left[\left(\tilde{y}_t^{\check{s}_1}+\tilde{y}_t^{\check{s}_2}\right)-\theta_2\left(\tilde{y}^{\check{s}_1\theta_1}+\tilde{y}_t^{\check{s}_2\theta_1}\right)^{\frac{1}{\theta_1}}\right]$ with $\theta\geq 0$ $\cdot\ B_\theta^{\check{s}}(\lambda_t)=\dfrac{1}{1+\lambda_t}\left[1-\theta_2\left[1+\lambda_t^{-\theta_1}\right]^{-\frac{1}{\theta_1}}\right]$.
4	Asymmetric two-parametric model (proposed by Coles and Tawn ([10]) $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right)=\left[\left(\tilde{y}_t^{\check{s}_1}+\tilde{y}_t^{\check{s}_2}\right)+\left[1-B(q,\theta_1+1,\theta_2)\right]\tilde{y}_t^{\check{s}_1}+\tilde{y}_t^{\check{s}_2}B(q,\theta_1,1+\theta_2)\right]^{\frac{-1}{\theta_3}}$ where $B(q,\theta_1,\theta_2)$ is Beta distribution at $q(\theta_1,\theta_2)=\frac{\theta_1\tilde{y}_t^{\check{s}_1}}{\theta_1\tilde{y}_t^{\check{s}_1}+\theta_2\tilde{y}_t^{\check{s}_2}}$. $\cdot\ B_\theta^{\check{s}}(\lambda_t)=\dfrac{\lambda_t}{1+\lambda_t}\left[\dfrac{(1-B(q,\theta_1+1,\theta_2))}{\lambda_t}+B(q,\theta_1,1+\theta_2)-1\right]$ with $\theta_1,\theta_2,\theta_3>0$.
5	Bilogistic and negative model (proposed by Müler (see Joe [9]); $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right)=\left(\tilde{y}_t^{\check{s}_1}+\tilde{y}_t^{\check{s}_2}\right)-\tilde{y}_t^{\check{s}_1}q^{1+\theta_1}+\tilde{y}_t^{\check{s}_2}(1-q)^{1+\theta_2}$ where $q=q(\tilde{y}_t^{\check{s}_1},\tilde{y}_t^{\check{s}_2},\theta)$ is root of equation with $\theta=(\theta_1,\theta_2)>0$ $(1+\theta_1)\tilde{y}_t^{\check{s}_1}q^{\theta_1}-(1+\theta_2)\tilde{y}_t^{\check{s}_2}\left(1-q\right)^{\theta_2}=0$ $\cdot\ B_\theta^{\check{s}}(\lambda_t)=\dfrac{\lambda_t}{1+\lambda_t}\left[1-q^{1+\theta_1}-\lambda_t(1-q)^{1+\theta_2}\right]$
6	Symmetric mixed polynomial model of Klüppelberg (see Beirlant [1]) $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right)=-\left(\tilde{y}_t^{\check{s}_1}+\tilde{y}_t^{\check{s}_2}\right)+\theta\ \tilde{y}_t^{\check{s}_1}+\dfrac{\theta\tilde{y}_t^{\check{s}_1}}{\tilde{y}_t^{\check{s}_1}+\tilde{y}_t^{\check{s}_2}}$ $\cdot\ B_\theta^{\check{s}}(\lambda_t)=\dfrac{1}{1+\lambda_t}\left[(1-\theta)+\dfrac{\theta}{1+\lambda_t}\right]$ where $\theta\in[0,1]$
7	Asymmetric, mixed polynomial model of Klüppelberg (see Beirlant [1]) $\cdot\ P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right)=\left(\tilde{y}_t^{\check{s}_1}+\tilde{y}_t^{\check{s}_2}\right)-(\theta_1+\theta_2)\ \tilde{y}_t^{\check{s}_1}-\dfrac{\theta_1\tilde{y}_t^{\check{s}_1}}{\tilde{y}_t^{\check{s}_1}+\tilde{y}_t^{\check{s}_2}}-\dfrac{\theta_2\tilde{y}_t^{\check{s}_2}}{\left(\tilde{y}_t^{\check{s}_1}+\tilde{y}_t^{\check{s}_2}\right)^2}$ with $\theta=(\theta_1,\theta_2,\theta_3)$ where: $\theta_1\geq 0,\theta_1+3\theta_2\geq 0;\theta_1+\theta_2\leq 1;\ \theta_1+2\theta_2\leq 1$ $\cdot\ B_\theta^{\check{s}}(\lambda_t)=\dfrac{1}{1+\lambda_t}\left[(\theta_1+\theta_2)+\dfrac{\theta_1}{1+\lambda_t}+\dfrac{\theta_2}{(1+\lambda_t)^2}\right]$.

4.3. Analytical Form of Tridimensional STAD

We provide analytical form of the STAD function of three dimensional logistic model (see [5] and [4]). In this sub-section, let consider $\tilde{y}_t^{\check{s}} = \left(\tilde{y}_t^{\check{s}_1}, \tilde{y}_t^{\check{s}_2}, \tilde{y}_t^{\check{s}_3}\right)$ and $\lambda_t = \left(\lambda_t^{(1)}, \lambda_t^{(2)}\right)$

1	Trivariate logistic model of ST- max-stable distribution (see [9]) $\cdot\, P_\theta(y_t^{\check{s}}) = \left[\tilde{y}_t^{\check{s}_1\theta} + \tilde{y}_t^{\check{s}_2\theta} + \tilde{y}_t^{\check{s}_3\theta}\right]^{\frac{1}{\theta}}$ where $\theta \geq 1$. $\cdot\, B_\theta^{\check{s}}(\lambda_t) = \left[\lambda_t^{(1)\theta} + \lambda_t^{(2)\theta} + \left(1-\lambda_t^{(1)}-\lambda_t^{(2)}\right)^\theta\right]^{\frac{1}{\theta}} - \left[\lambda_t^{(2)\theta} + \left(1-\lambda_t^{(1)}-\lambda_t^{(2)}\right)^\theta\right]^{\frac{1}{\theta}}$
2	Negative, two parametric extension of trivariate logistic model (see [8]) $\cdot\, P_\theta(y_t^{\check{s}}) = -\left(\left[\tilde{y}_t^{\check{s}_1\theta_1\theta_2} + 2^{-\theta_2}\tilde{y}_t^{\check{s}_2\theta_1\theta_2}\right]^{\frac{1}{\theta_2}} + \left[2^{-\theta_2}\tilde{y}_t^{\check{s}_2\theta_1\theta_2} + \tilde{y}_t^{\check{s}_3\theta_1\theta_2}\right]^{\frac{1}{\theta_2}}\right)^{\frac{1}{\theta_1}}$ $\cdot\, B_\theta^{\check{s}}(\lambda_t) = \left(\left[\lambda_t^{(1)\theta_1\theta_2} + 2^{-\theta_2}\lambda_t^{(2)\theta_1\theta_2}\right]^{\frac{1}{\theta_2}} + \left[2_2^{-\theta_2}\lambda_t^{(2)\theta_1\theta_2} + \left(1-\lambda_t^{(1)}-\lambda_t^{(2)}\right)_3^{\theta_1\theta_2}\right]^{\frac{1}{\theta_2}}\right)^{\frac{1}{\theta_1}}$ $-\left[2^{-\theta_2}\lambda_t^{(2)} + \left(1-\lambda_t^{(1)}-\lambda_t^{(2)}\right)^{-\theta_1\theta_2\frac{1}{\theta_2}}\right]$
3	Trivariate Gaussian model of ST- max-stable distribution (see [8]) $\cdot\, P_\theta(y_t^{\check{s}}) = \sum_{i=1}^3 \tilde{y}_t^{\check{s}_i} + \frac{1}{\tilde{y}_t^{\check{s}_1}}\left[\Phi\left(\frac{1}{\theta_1} + \frac{\theta_1}{2}\log\left(\frac{\tilde{y}_t^{\check{s}_2}}{\tilde{y}_t^{\check{s}_1}}\right)\right) + \Phi\left(\frac{1}{\theta_3} + \frac{\theta_3}{2}\log\left(\frac{\tilde{y}_t^{\check{s}_3}}{\tilde{y}_t^{\check{s}_1}}\right)\right)\right]$ $-\frac{1}{\tilde{y}_t^{\check{s}_2}}\left[\Phi\left(\frac{1}{\theta_1} + \frac{\theta_1}{2}\log\left(\frac{\tilde{y}_t^{\check{s}_1}}{\tilde{y}_t^{\check{s}_2}}\right)\right) + \Phi\left(\frac{1}{\theta_2} + \frac{\theta_2}{2}\log\left(\frac{\tilde{y}_t^{\check{s}_3}}{\tilde{y}_t^{\check{s}_2}}\right)\right)\right]$ $-\frac{1}{\tilde{y}_t^{\check{s}_3}}\left[\Phi\left(\frac{1}{\theta_3} + \frac{\theta_3}{2}\log\left(\frac{\tilde{y}_t^{\check{s}_1}}{\tilde{y}_t^{\check{s}_3}}\right)\right) + \Phi\left(\frac{1}{\theta_2} + \frac{\theta_2}{2}\log\left(\frac{\tilde{y}_t^{\check{s}_2}}{\tilde{y}_t^{\check{s}_3}}\right)\right)\right] + I\left(\bar{\Phi}_2, \theta, \rho\right);$ I being the integral $I =_0^{y_3^{-1}} \bar{\Phi}_2\left(\frac{1}{\theta_2} + \frac{\theta_2}{2}\log\left(\tilde{y}_t^{\check{s}(1)}\right); \frac{1}{\theta_3} + \frac{\theta_3}{2}\log\left(\tilde{y}_t^{\check{s}(4)}\right); \rho\right) dt$ and $\rho = \begin{pmatrix} 0 & 2\left(\frac{1}{\theta_1^2} + \frac{1}{\theta_2^2} + \frac{1}{\theta_3^3}\right) \\ 2\left(\frac{1}{\theta_1^2} + \frac{1}{\theta_2^2} + \frac{1}{\theta_3^3}\right) & 0 \end{pmatrix}$;covariance-matrix. $\cdot\, B_\theta^{\check{s}}(\lambda_t) = 2 - \lambda_t^{(1)}\left[\Phi\left(\frac{1}{\theta_1} + \frac{\theta_1}{2}\log\left(\frac{\lambda_t^{(1)}}{\lambda_t^{(2)}}\right)\right) + \Phi\left(\frac{1}{\theta_3} + \frac{\theta_3}{2}\log\left(\frac{\lambda_t^{(1)}}{1-\lambda_t^{(1)}-\lambda_t^{(2)}}\right)\right)\right]$ $-\lambda_t^{(2)}\left[\Phi\left(\frac{1}{\theta_1} + \frac{\theta_1}{2}\log\left(\frac{\lambda_t^{(1)}}{\lambda_t^{(2)}}\right)\right) + \Phi\left(\frac{1}{\theta_2} + \frac{\theta_2}{2}\log\left(\frac{\lambda_t^{(1)}}{1-\lambda_t^{(1)}-\lambda_t^{(2)}}\right)\right)\right]$ $-\left(1-\lambda_t^{(1)}-\lambda_t^{(2)}\right)\left[\Phi\left(\frac{1}{\theta_3} + \frac{\theta_3}{2}\log\left(\frac{1-\lambda_t^{(1)}-\lambda_t^{(2)}}{\lambda_t^{(1)}}\right)\right) + \Phi\left(\frac{1}{\theta_2} + \frac{\theta_2}{2}\log\left(\frac{1-\lambda_t^{(1)}-\lambda_t^{(2)}}{\lambda_t^{(1)}}\right)\right)\right]$ $-\lambda_t^{(2)}\Phi\left(\frac{1}{\theta_2} + \frac{\theta_2}{2}\log\left(\frac{\lambda_t^{(2)}}{1-\lambda_t^{(2)}}\right)\right) - \left(1-\lambda_t^{(2)}\right)\Phi\left(\frac{1}{\theta_2} + \frac{\theta_2}{2}\log\left(\frac{1-\lambda_t^{(2)}}{\lambda_t^{(2)}}\right)\right) + R(\lambda_t^{(1)}, \lambda_t^{(2)}, \theta).$
4	An asymmetric extension of logistic ST- max-stable distribution (see [9]) $\cdot\, P_\theta^{\check{s}}\left(\tilde{y}_t^{\check{s}}\right) = \sum_{i=1}^{3}\tilde{y}_t^{\check{s}_i} - \left(\tilde{y}_t^{-\check{s}_1\theta_1} + \tilde{y}_t^{-\check{s}_3\theta_2})\right)^{\frac{-1}{\boldsymbol{\theta}_1}} + \left[\tilde{y}_t^{\check{s}_1\boldsymbol{\theta}_1} + \tilde{y}_t^{\check{s}_2\boldsymbol{\theta}_1} - \left(\tilde{y}_t^{\check{s}_2\theta_1\theta_2} + 2^{\theta_1}\tilde{y}_t^{\check{s}(2)\theta_1\theta_2}\right)^{\frac{-1}{\theta_2}}\right]^{\frac{-1}{\theta_1}}$ $+\left[\tilde{y}_t^{\check{s}_2\boldsymbol{\theta}_1} + \tilde{y}_t^{\check{s}_3\boldsymbol{\theta}_1} - \left(\tilde{y}_t^{\check{s}_3\theta_1\theta_2} + 2^{\theta_1}\tilde{y}_t^{\check{s}_2\theta_1\theta_2}\right)^{\frac{-1}{\boldsymbol{\theta}_2}}\right]^{\frac{-1}{\theta_1}}$ $-\left[-\tilde{y}_t^{-\check{s}_1\boldsymbol{\theta}_1} + \tilde{y}_t^{-\check{s}_2\boldsymbol{\theta}_1} + \tilde{y}_t^{-\check{s}_3\boldsymbol{\theta}_1} - \left(\tilde{y}_t^{\check{s}_1\theta_1\theta_2} + 2^{\theta_1}\tilde{y}_t^{\check{s}_2\theta_1\theta_2}\right)^{\frac{-1}{\theta_2}} - \left(\tilde{y}_t^{\check{s}_3\theta_1\theta_2} + 2^{\theta_1}\tilde{y}_t^{\check{s}_2\theta_1\theta_2}\right)^{\frac{-1}{\boldsymbol{\theta}_2}}\right]^{\frac{-1}{\theta_1}}.$ $\cdot\, B_\theta^{\check{s}}(\lambda_t) = \lambda_t^{(1)} - \left(\lambda_t^{(1)\boldsymbol{\theta}_1} + \left(1-\lambda_t^{(1)}-\lambda_t^{(2)}\right)^{\theta_1}\right)^{\frac{-1}{\boldsymbol{\theta}_1}} + \left[\lambda_t^{(1)\boldsymbol{\theta}_1} - \left(\lambda_t^{(1)\boldsymbol{\theta}_1\boldsymbol{\theta}_2} + 2\lambda_2^{(1)\boldsymbol{\theta}_1\boldsymbol{\theta}_2}\right)^{\frac{-1}{\boldsymbol{\theta}_1}}\right]^{\frac{-1}{\theta_2}}$ $+\left[\left(1-\lambda_t^{(1)}-\lambda_t^{(2)}\right) + \lambda_t^{-(1)\boldsymbol{\theta}_1} + \lambda_t^{-(2)\boldsymbol{\theta}_2} - \left(\lambda_t^{-(1)\boldsymbol{\theta}_1\boldsymbol{\theta}_2} + 2\lambda_2^{-(2)\boldsymbol{\theta}_1\boldsymbol{\theta}_2}\right)^{\frac{-1}{\boldsymbol{\theta}_1}}\right]^{\frac{-1}{\theta_2}}.$

5. Conclusion

The results of the study provides important characterizations of parametric max-stable processes. Especially they show that stochasctic dependence is also the property of the spatial and temporal coordonates of the phenomenas observed and modeled by the multivariate max-stable processes. In particular spalized and conditional dependence measure are built for extremal classical structures such that pickands function are clarified both for bivariate and trivariate models of ST stochastic processes.

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