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Symplectic Manifolds and Their Lagrangian Submanifolds

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1. INTRODUCTION

A classical theorem of Darboux may be stated as follows: if Ω is a closed 2-form of maximal rank on a $2n$-dimensional manifold M and p is any point of M, then there exists a coordinate system $x_1, ..., x_n, y_1, ..., y_n$ defined on a neighborhood U of p in M such that $\Omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$ on U. The purpose of this paper is to generalize Darboux's theorem in several directions and to give some applications of the generalizations.

The first direction of generalization is that the manifolds considered are Banach, rather than finite-dimensional, manifolds. Although our results in this context may have some use in theoretical mechanics, the main force of this generalization is that it requires us to use a method of proof other than induction on the dimension of M, which has been the standard proof technique for Darboux's theorem. The new method of proof, first used in this context by Moser [12], enables us to obtain theorems concerning symplectic structures in the neighborhood of a closed submanifold, rather than just a point, of M. Another by-product is an equivariant version of the Darboux theorem (Corollary 4.3).

Our main result, Theorem 4.1, is most useful when applied to the so-called *lagrangian* submanifolds of M. These, roughly speaking, are maximal submanifolds on which Ω pulls back to zero. Essentially, Theorem 6.1 says that these submanifolds have no geometric invariants. This fact, in turn, has the surprising consequence (discussed at the end of Section 6) that a neighborhood of the identity in the symplectic automorphism group of a symplectic manifold may be smoothly parametrized by a neighborhood of zero in the Lie algebra of infinitesimal symplectic automorphisms. Section 7 is devoted to foliations of symplectic manifolds by lagrangian submanifolds. This study turns out to be a generalization of the theory, familiar in mechanics, of systems of functions in involution. The leaves of lagrangian foliations are characterized in Theorem 7.8 as manifolds admitting a torsion-free flat affine connection.

Most of the results contained in this paper have been announced in [19] and [20]. Not all of them are new in the finite-dimensional context, but we feel that the generality of our treatment warrants their repetition.

2. The Action of a Lie Group on a Manifold

Let $\alpha : G \times S \to S$ be a differentiable action of a finite dimensional Lie group G on a finite dimensional manifold S. For $s \in S$, denote by $\alpha_s : G \to S$ the orbit map $g \mapsto \alpha(g, s)$. s is called *stable* if the image of α_s (i.e., the orbit of s) contains a neighborhood of s. s is called *infinitesimally stable* if the differential at the identity e of G, $T_e\alpha_s : T_eG \to T_sS$, is surjective, or, in more picturesque language, the image of α_s contains an infinitesimal neighborhood of s. It follows directly from the implicit function theorem that infinitesimal stability implies stability. If S is connected and every $s \in S$ is infinitesimally stable, then α_s is surjective and the action α is transitive. The linear condition of infinitesimal stability thus provides, in the finite-dimensional situation, a sufficient condition for the transitivity or local transitivity of a group action.

There is a substantial amount of recent research devoted to extending the result that infinitesimal stability implies stability to situations in which G and S are function spaces, and α, as well as the group action, involves functional composition. If G and S were Banach manifolds and α_s a differentiable mapping for each s, one could apply the implicit function theorem for Banach manifolds, but this is almost never the case.

We use, instead, a "path-lifting" method which has also been exploited by Mather [11] with the aid of a technical apparatus more sophisticated than any utilized here. We remark that another very powerful method for deducing stability from infinitesimal stability is the Kolmogorov–Arnold–Moser variation of Newton's method described extensively in [18].

We shall describe, in formal terms, an abstraction of the path-lifting method. It appears that Mather's techniques can be used to get some very general theorems out of this discussion, but we leave this matter to another paper. The method goes as follows.

Given a point $r \in S$, construct a smooth path $\sigma : [0, 1] \to S$ such that

$\sigma(0) = s$ and $\sigma(1) = r$. We seek a path $\gamma : [0, 1] \to G$ such that $\gamma(0) = e$ and $\alpha(\gamma(t), \sigma(t)) \equiv s$, in which case we would have $s = \alpha(\gamma(1), r)$ and $r = \alpha_s(\gamma(1)^{-1})$. Given $\gamma(0) = e$, it suffices to require that

$$\frac{d}{dt}\alpha(\gamma(t), \sigma(t)) \equiv 0.$$

If we suppose $\gamma(t)$ to be the integral curve with $\gamma(0) = e$ of a time-dependent left-invariant vector field $\{Y_t\}_{t\in[0,1]}$ on G, a formal computation shows that the condition $d/dt\, \alpha(\gamma(t), \sigma(t)) \equiv 0$ on γ is equivalent to the condition $T_e\alpha_{\sigma(t)}(Y_t(e)) + d\sigma(t)/dt \equiv 0$. The infinitesimal stability of s implies that we can solve this equation for $t = 0$, for any σ. All the hard work in [11] is devoted to showing that the infinitesimal stability of s implies that one can find a solution $\{Y_t\}_{t\in[0,1]}$ varying smoothly with t, provided that σ is close enough to a constant curve. In the situation considered in the present paper, this conclusion is attained quite simply. There is also a secondary question of showing that the vector field $\{Y_t\}$ can actually be integrated for t going from 0 to 1, which imposes some conditions on the group considered.

Theorem 4.1 of this paper is an application of the path-lifting method to a situation in which S is a space of symplectic structures and G is a group of diffeomorphisms.[1]

3. Local Manifold Pairs

For problems in local differential geometry, it is useful to extend the category of differentiable manifolds to include germ-like objects. We consider pairs (M, N), where M is a C^∞ manifold modeled on a Banach space and $N \subset M$ is a closed submanifold.[2] The pairs (M_1, N_1) and (M_2, N_2) are considered equivalent if $N_1 = N_2 = N$, and there exists a pair (M, N) such that M is simultaneously an open submanifold of M_1 and M_2. Two mappings of pairs $f_i : (M_i, N) \to (P_i, Q)$ $(i = 1, 2)$ are considered equivalent if there exists a mapping $f : (M, N) \to (P, Q)$ such that M is an open submanifold of M_1 and M_2, P is an open sub-

[1] R. Palais, with whom I have had useful discussions on these matters, has used the path-lifting method to give a new proof of the Morse lemma on functions with non-degenerate critical points [14].

[2] We refer the reader to [10] for material on the foundations of differential geometry on Banach manifolds.

manifold of P_1 and P_2, and $f = f_i \mid M$. The equivalence classes of pairs and mappings form the category $\mathscr{L}$ of *local manifold pairs*. The ordinary manifolds are identified with a full subcategory $\mathscr{M}$ of $\mathscr{L}$ by the functor $M \mapsto (M, M)$, and the functor $(M, N) \mapsto N$ is a retraction of $\mathscr{L}$ onto $\mathscr{M}$.

When it is necessary to avoid ambiguity, we denote the equivalence class of (M, N) by $[M, N]$ and that of f by $[f]$. Usually, though, we work with representatives and leave unsaid that the statements and constructions are independent of the representative chosen.

Vector Fields

If $[M, N]$ is a local manifold pair, the pair $[TM, TN]$ plays the role of its tangent bundle. A vector field on $[M, N]$ is, by definition, a section of that bundle. A time-dependent vector field $\{Y_t\}_{t\in[0,1]}$ is called *integrable* if there is a one-parameter family $\{f_t\}_{t\in[0,1]}$ of diffeomorphisms from $[M, N]$ to itself such that f_0 is the identity and $df_t/dt = Y_t \circ f_t$. The following lemma follows immediately from the openness (see [10]) of the domain of definition of the solutions of a system of ordinary differential equations.

LEMMA 3.1. *If $\mathscr{Y} = \{Y_t\}_{t\in[0,1]}$ is a time-dependent vector field on $[M, N]$, then $\mathscr{Y}$ is integrable if and only if the field $\mathscr{Y} \mid N = \{Y_t \mid N\}_{t\in[0,1]}$ on N is integrable. In particular, $\mathscr{Y}$ is integrable if $\mathscr{Y} \mid N$ is identically zero.*

Differential Forms

A p-form on the local manifold pair $[M, N]$ is, by definition, an equivalence class of p-forms on M under the equivalence relation of equality on a neighborhood of N. It will cause us no trouble that this definition is not properly dual to the definition of a vector field on $[M, N]$. We denote by $\mathscr{F}^p(M, N)$ the space of all p-forms on $[M, N]$. Exterior differentiation on M induces an operator $d : \mathscr{F}^p(M, N) \to \mathscr{F}^{p+1}(M, N)$.

Given $\omega \in \mathscr{F}^p(M, N)$, there are two ways of "restricting" it to N. First, we may restrict it to a mapping $\omega \mid N$ from N to the restricted bundle $T^*M \mid N$. We may compose this restriction with the bundle mapping from $T^*M \mid N$ to T^*N to get the *pullback* $\omega_N \in \mathscr{F}^p(N)$.

Our goal for this section, a homotopy operator $I : \mathscr{F}^p(M, N) \to \mathscr{F}^{p-1}(M, N)$, will be constructed in terms of a deformation of $[M, N]$ into N. By the tubular neighborhood theorem, we may assume that M is a vector bundle whose zero-section is N. Let $\pi_t : (M, N) \to (M, N)$ be multiplication by t for $t \in [0, 1]$, and let $X_t : (M, N) \to (TM, TN)$ be the vector field $(d\pi_s/ds)_{s=t}$ along π_t. Then it is a standard result that, for $\omega \in \mathscr{F}^p(M, N)$,

$$\frac{d}{ds}(\pi_s{}^*\omega)_{s=t} = \pi_t{}^*(X_t \lrcorner\, d\omega) + d[\pi_t{}^*(X_t \lrcorner\, \omega)]. \tag{3.1}$$

(See [9, p. 114] for a proof of this formula and interpretation—there is really only one possible—of the symbols used therein.) Integrating (3.1) with respect to t over the interval [0, 1] and using the fact that π_1 is the identity, we obtain

$$\omega - \pi_0{}^*\omega = \int_0^1 \{\pi_t{}^*(X_t \lrcorner\, d\omega) + d(\pi_t{}^*(X_t \lrcorner\, \omega))\}\, dt. \tag{3.2}$$

If we define $I(\omega)$ as $\int_0^1 \pi_t{}^*(X_t \lrcorner\, \omega)\, dt$, then (3.2) becomes

$$\omega - \pi_0{}^*\omega = I(d\omega) + d(I\omega).$$

In particular, if $\pi_0{}^*\omega = 0$ (which is the case if and only if $\omega_N = 0$) and $d\omega = 0$, then $\omega = d(I\omega)$. This is our version of the Poincaré lemma.

We close this section by observing two properties of I. First, because X_t vanishes along N and $\pi_t(N) \subset N$, $I\omega \mid N$ is always zero. Second, if G is a group of diffeomorphisms of (M, N) compatible with the vector bundle structure of M, then I commutes with the induced action of G on the $\mathscr{F}^p(M, N)$'s. In particular, if G leaves the form ω invariant, it also leaves $I\omega$ invariant.

4. Symplectic Manifolds

A symplectic structure on a local manifold pair $[M, N]$ is an element $\Omega \in \mathscr{F}^2(M, N)$ such that $d\Omega = 0$ and the associated bundle mapping $\tilde{\Omega} : TM \to T^*M$ defined by $\tilde{\Omega}(x) = x \lrcorner\, \Omega$ is an isomorphism over a neighborhood of N. Since the isomorphisms from one Banach space to another form an open subset of the space of all continuous linear mappings, it suffices to require that the restriction $\Omega \mid N : TM \mid N \to T^*M \mid N$ be a bundle isomorphism.

Theorem 4.1. *Let Ω_0 and Ω_1 be symplectic structures on $[M, N]$ such that $\Omega_0 \mid N = \Omega_1 \mid N$. Then there exists a diffeomorphism $f : [M, N] \to [M, N]$ such that $f \mid N = 1_N$ (the identity mapping of N) and $f^*\Omega_1 = \Omega_0$.*

Proof. Let $\omega = \Omega_1 - \Omega_0$, $\Omega_t = \Omega_0 + t\omega$ $(t \in [0, 1])$. Since $d\omega = 0$ and $\omega \mid N = 0$, it follows that $d\Omega_t = 0$ and $\tilde{\Omega}_t \mid N = \tilde{\Omega}_0 \mid N$ is an isomorphism, so that Ω_t is a symplectic structure. Let $\phi = I(\omega)$ (see Section 3). Since ω is closed and $\omega \mid N = 0$ implies $\omega_N = 0$, it follows that $d\phi = \omega$. Let $Y_t = -\tilde{\Omega}_t^{-1}(\phi)$. Since $\phi \mid N$ is zero, so is $Y_t \mid N$, and the time-dependent vector field $\{Y_t\}_{t\in[0,1]}$ is integrable, by lemma 3.1, to a one-parameter family $\{f_t\}_{t\in[0,1]}$ of diffeomorphisms of $[M, N]$. Computing, as in [12], we find

$$\begin{aligned}\frac{d}{ds}(f_s{}^*\Omega_s)_{s=t} &= f_t{}^*\left(\frac{d\Omega_s}{ds}\right)_{s=t} + f_t{}^*(d(Y_t \lrcorner \Omega_t)) \\ &= f_t{}^*(\omega + d(-\phi)) = 0.\end{aligned}$$

Let $f = f_1$. Then $f^*\Omega_1 = f_1{}^*\Omega_1 = f_0{}^*\Omega_0 = \Omega_0$. Q.E.D.

An immediate application of Theorem 4.1 is the generalization of Darboux's theorem to symplectic structures on Banach manifolds.

COROLLARY 4.2. *Let M be a Banach space, $0 \in M$ the origin, and Ω a symplectic structure on $[M, 0]$. If Ω_0 is the constant (with respect to the natural parallelization of M) symplectic structure which agrees with Ω at 0, there is a diffeomorphism $f : [M, 0] \to [M, 0]$ such that $f^*\Omega = \Omega_0$.*

The local classification of symplectic structures is thus reduced to the classification of linear symplectic structures on Banach spaces. The linear theory is further developed in Section 5 for this and further applications.

The global classification of symplectic structures still appears to be a very difficult problem, particularly in the compact case. Which compact manifolds admit a symplectic structure? Such a manifold must admit an almost complex structure and have a real cohomology class in dimension 2 which remains nonzero when raised to the power $\frac{1}{2} \cdot \dim M$. All the compact symplectic manifolds known to the author are Kähler manifolds, which have a natural symplectic structure. Beyond these statements, nothing seems to be known. In the noncompact but finite-dimensional case, Gromov [8] has shown that an open manifold admits a symplectic structure (in fact, an *exact* one) if and only if it admits an almost complex structure. Since all Hilbert manifolds are diffeomorphic to open subsets of Hilbert space [6], they all admit symplectic structures.

As for the equivalence question, Moser's reduction [12], of this problem to a homotopy question in the compact case is the best result to date, but the homotopy question, like the existence problem, has not been solved. In the noncompact case, Gromov has shown that two exact symplectic structures are homotopic if they are homotopic as nonsingular (but not necessarily closed) 2-forms, but this does not lead, at least not directly, to the equivalence of the forms. In Hilbert space, the absence of an integration theory to provide a volume invariant suggests the following amusing question. Let H be a Hilbert space with a constant symplectic structure. Is H symplectically diffeomorphic to an open ball in H with the induced symplectic structure? Is a ball of radius 1 symplectically diffeomorphic to a ball of radius 2?

The Equivariant Darboux Theorem

COROLLARY 4.3. *In the situation of Corollary* 4.2, *suppose a group G acts linearly on M and leaves Ω invariant. Then the diffeomorphism f such that $f^*\Omega = \Omega_0$ commutes with the action of G.*

Proof. By the remark at the end of Section 3, the form φ in the proof of Theorem 4.1 is G-invariant; hence, so are the vector fields Y_t, and each f_t commutes with the action of G. Q.E.D.

Corollary 4.3 yields a positive answer to a question posed by I. Segal, as follows:

COROLLARY 4.4. *In the situation of Corollary* 4.3, *if the linear action of G on the tangent space of M at* 0 *leaves invariant a positive-definite bilinear form* (i.e., *a Hilbert space structure*), *then G leaves invariant a Kähler structure on* $[M, 0]$ *whose associated 2-form is Ω.*

Proof. By Corollary 4.3, we may assume that Ω is a constant symplectic structure. By [15], G leaves invariant a linear complex structure and a Hilbert space structure on M whose associated 2-form is $\Omega \mid 0$. Extending these linear structures to constant structures on $[M, 0]$ gives the required (flat) Kähler structure.

5. LINEAR SYMPLECTIC STRUCTURES

A linear symplectic structure on a Banach space V is a skew-symmetric bilinear form $\Omega : V \times V \to R$ such that the associated mapping $\tilde{\Omega} : V \to V^*$ defined by $\tilde{\Omega}(x)(y) = \Omega(x, y)$ is a toplinear isomorphism. There is a natural 1–1 correspondence between linear symplectic structures on V, considered as a vector space, and constant symplectic structures on V, considered as a manifold, but the latter structures have more (nonlinear) mappings between them.

The skew-symmetry of Ω implies that the restriction to V of the dual transformation $\tilde{\Omega}^* : V^{**} \to V^*$ is equal to $-\tilde{\Omega}$. Since $\tilde{\Omega}^*$ and $-\tilde{\Omega}$ are both isomorphisms, we conclude that $V^{**} = V$; i.e., V must be reflexive if it admits a linear symplectic structure.

All the examples known to the author of linear symplectic structures on Banach spaces arise in the following way. Let W be a reflexive Banach space and $V = W \oplus W^*$. Then V^* is naturally isomorphic to $W^* \oplus W$, and the natural symplectic structure Ω_W on V is defined by $\Omega_W(x \oplus x^*, y \oplus y^*) = y^*(x) - x^*(y)$, or $\tilde{\Omega}_W(x \oplus x^*) = (-x^*) \oplus x$. (If W is not reflexive, then $\tilde{\Omega}_W$ is injective but not an isomorphism.)

One may try to construct isomorphisms from a given symplectic space to one of the form $(W \oplus W^*, \Omega_W)$ by the following method. If (V, Ω) is any symplectic space, a subspace W of V is called *isotropic* if $\Omega \mid W \times W$ is identically zero. The isotropic subspace W is called *lagrangian* if there is another isotropic subspace W' such that $V = W \oplus W'$. The term is due to Maslov and Arnold [1] in the finite-dimensional case, where a lagrangian subspace is just an isotropic subspace of dimension $\frac{1}{2}(\dim V)$. The injectivity of $\tilde{\Omega}$ implies that a lagrangian subspace is a maximal element of the set of isotropic subspaces of V, ordered by inclusion. This maximality and the continuity of Ω imply that a lagrangian subspace is always closed.

Given a splitting $V = W \oplus W'$ into lagrangian subspaces, there is a mapping $\psi : W' \to W^*$ defined by $\psi(x)(y) = \Omega(x, y)$. ($\psi$ is just the restriction of $\tilde{\Omega}$ to W', followed by the projection onto W^*.) One may easily check that ψ is an isomorphism. Furthermore, for any $x, y \in W$ and $z, w \in W'$, $\Omega(x + z, y + w) = \Omega(x, w) - \Omega(y, z) = \psi(w)(x) - \psi(z)(y) = \Omega_W(x \oplus \psi(z), y \oplus \psi(w))$. $1_W \oplus \psi$ is, therefore, an isomorphism from (V, Ω) to $(W \oplus W^*, \Omega_W)$ which arises naturally from the lagrangian splitting $V = W \oplus W'$.

The question of whether every symplectic space is isomorphic to one of the form $(W \oplus W^*, \Omega_W)$ is, therefore, reduced to the question of whether every symplectic space admits a lagrangian splitting. By Zorn's lemma, every symplectic space admits a maximal isotropic subspace, and the following proposition closes the question in the hilbertable case.

PROPOSITION 5.1. *If V is a Hilbert space and Ω is a symplectic structure on V, then any maximal isotropic subspace of V is lagrangian.*

Proof. If $\langle\ ,\ \rangle$ denotes the inner product on V, there is an operator $T : V \to V$ such that $\langle Tx, y\rangle = \Omega(x, y)$ for all x and y. Since Ω is antisymmetric, T is a skew-adjoint isomorphism. $-T^2$ is then positive definite and, by the spectral theory of operators on Hilbert space, it has a positive definite square root P which defines a new Hilbert structure $\langle\ ,\ \rangle_P$ on V by $\langle x, y\rangle_P = \langle Px, y\rangle$. Writing J for $P^{-1}T$ one finds, as in [17, p. 213], that J is orthogonal, $T = PJ = JP$, and $J^2 = -I$. Also, for all x and y in V, $\Omega(Jx, Jy) = \langle TJx, Jy\rangle = \langle JTx, Jy\rangle = \langle Tx, y\rangle = \Omega(x, y)$, so J is a symplectic isomorphism. Thus, if W is any maximal isotropic subspace, $W' = JW$ is maximal isotropic as well. Finally, $W \oplus W' = V$. In fact, if $x \in V$ is P-orthogonal to W, then $y \in W' \Rightarrow \Omega(x, y) = \langle Tx, y\rangle = \langle JPx, y\rangle = -\langle Px, Jy\rangle = 0$. By the maximality of W' as an isotropic subspace, $x \in W'$. On the other hand, if $x \in W'$ and $y \in W$, $\langle Px, y\rangle = -\langle TJx, y\rangle = -\Omega(Jx, y) = 0$, since Jx and y are in the lagrangian space W. So W' is the P-orthogonal complement of W, and $W \oplus W' = V$. Q.E.D.

COROLLARY 5.2. *Let V_0 and V_1 be isomorphic Hilbert spaces (possibly finite dimensional). If Ω_0 and Ω_1 are symplectic structures on $[V_0, V]$ and $[V_1, 0]$, respectively, then there is a diffeomorphism $f : [V_0, 0] \to [V_1, 0]$ such that $f^*\Omega_1 = \Omega_0$.*

Proof. Combine the discussion above with Corollary 4.2, noting that "halves" of isomorphic Hilbert spaces are isomorphic.

Lagrangian Complements

If W is a lagrangian subspace of (V, Ω), there exists, by definition, an isotropic subspace W' of (V, Ω) such that $V = W \oplus W'$. For applications to symplectic manifolds, it is useful to know that the space of all such W' is contractible.

Let $P : V \to V$ be the projection onto W along W'. If $\overline{W}$ is any isotropic subspace such that $V = W \oplus \overline{W}$, define $\overline{W}_t = (1_V - tP)\overline{W}$ for $t \in [0, 1]$. One may check that $\overline{W}_0 = \overline{W}$, $\overline{W}_1 = W'$, $\overline{W}_t$ is isotropic and $V = W \oplus \overline{W}_t$ for all $t \in [0, 1]$, and $\overline{W}_t$ depends continuously upon $\overline{W}$ and t (in the usual topology on a Grassman manifold of a Banach space [4].)

In fact (see [1] for the finite-dimensional case), one may parametrize the space of all isotropic $\overline{W}$ such that $V = W \oplus \overline{W}$ by the linear space of operators $T : W' \to W$ whose composition with the natural mapping $\psi : W \to W'^*$ is self-adjoint.

6. Lagrangian Submanifolds

We return now to the study of symplectic manifolds. If (M, Ω) is a symplectic manifold, a submanifold $N \subset M$ is a *lagrangian submanifold* if, for each $x \in N$, the tangent space T_xN is a lagrangian subspace of T_xM. It follows from the discussion at the end of Section 5 that one can find an isotropic subbundle $E \subsetneq TM \mid N$ such that $TM \mid N = TN \oplus E$.

Cotangent Bundles

We discuss here a well-known class of symplectic manifolds which provide models for all local manifold pairs $[M, N]$ with symplectic structures for which N is a lagrangian submanifold.

If N is any Banach manifold, the cotangent bundle T^*N carries a natural 1-form $\omega_N \in \mathscr{F}^1(T^*N)$ characterized by the following property: if $\sigma : N \to T^*N$ is any section (i.e., $\sigma \in \mathscr{F}^1(N)$), then $\sigma^*\omega_N = \sigma$. The 2-form $\Omega_N = -d\omega_N$ is a symplectic structure if and only if N is reflexive. To see this, we may assume that N is a Banach space V, in which case T^*V is naturally diffeomorphic to $V \oplus V^*$, its tangent spaces are naturally isomorphic to $V \oplus V^*$, and its cotangent spaces are naturally isomorphic to $V^* \oplus V^{**}$. The value of ω_v at $z \oplus z^*$ is just $z^* \oplus 0$, and $-d\omega_v$ evaluated on the pair of constant vector fields $(x \oplus x^*, y \oplus y^*)$ gives $y^*(x) - x^*(y)$. Thus, Ω_v is the constant 2-form corresponding to the natural bilinear form on $V \oplus V^*$ which is also called Ω_V. As we saw in Section 5, this is a symplectic structure if and only if V is reflexive. (We remark that some authors use the form $-\Omega_V$ instead of Ω_V, with compensating sign changes elsewhere in the theory.)

Now we may describe some lagrangian submanifolds of T^*N. From the local description, it is clear that the fibres of the cotangent bundle are lagrangian submanifolds. As for submanifolds transversal to the fibres, any such submanifold is locally the graph of a 1-form $\sigma : N \to T^*N$. The graph of σ is isotropic (in which case it follows that it is lagrangian) if and only if $0 = \sigma^*(\Omega_N) = \sigma^*(-d\omega_N) = -d(\sigma^*\omega_N) = -d\sigma$, i.e., if and only if σ is a closed 1-form. In particular, the zero section Z_N is a lagrangian submanifold, and the lagrangian submanifolds "near" the zero section are in natural $1-1$ correspondence with "small" closed 1-forms on N.

Equivalence Theorem for Lagrangian Submanifolds

Theorem 6.1. *Let $[M_i, N]$ $(i = 1, 2)$ be local manifold pairs with symplectic structures Ω_i such that N is a lagrangian submanifold of each*

M_i. Then there is a diffeomorphism $[f] : [M_1, N] \to [M_2, N]$ such that $f \mid N = 1_N$ and $f^*\Omega_2 = \Omega_1$.

Proof. There exist isotropic subbundles $E_i \subseteq TM_i \mid N$ such that $TM_i \mid N = TN \oplus E_i$. Applying the linear theory of Section 5 to each fibre of these bundles, we get bundle isomorphisms $\psi_i : E_i \to T^*N$ such that $1_{TN} \oplus \psi_i : TM_i \mid N \to TN \oplus T^*N$ pulls back the standard symplectic structure (in each fibre) to Ω_i. Thus, we have a bundle isomorphism $\gamma = (1_{TN} \oplus \psi^{-1}) \cdot (1_{TN} \oplus \psi_1) : TM_1 \mid N \to TM_2 \mid N$ which pulls back Ω_2 to Ω_1. Using tubular neighborhoods, we can construct from this bundle map a diffeomorphism $[g] : [M_1, N] \to [M_1, N]$ which is "tangent" to γ, so that $g \mid N = 1_N$ and $g^*(\Omega_2 \mid N) = \Omega_1 \mid N$. By Theorem 4.1, there is a diffeomorphism $[h] : [M_1, N] \to [M_1, N]$ such that $h \mid N = 1_N$ and $h^*(g\Omega_2) = \Omega_1$. Setting $f = gh : [M_1, N] \to [M_2, N]$ completes our proof. Q.E.D.

COROLLARY 6.2. *Let $[M, N]$ be a local manifold pair with a symplectic structure Ω such that N is a lagrangian submanifold. Then there is a diffeomorphism $[f] : [M, N] \to [T^*N, Z_N]$ such that $f \mid N = 1_N$ modulo the identification of N with Z_N and $f^*\Omega_N = \Omega$, and the lagrangian submanifolds of M "near" N are in $1-1$ correspondence with "small" closed forms on N.*

The local (in N) and finite-dimensional version of Corollary 6.2 is due to Souriau [16].

The Automorphism Group of a Symplectic Manifold

The diffeomorphisms of a manifold M into itself may be identified with their graphs, i.e., with the submanifolds of $M \times M$ which are mapped diffeomorphically onto M by both of the projections π_1 and π_2. If M carries a symplectic structure Ω, $M \times M$ carries the symplectic structure $\Omega^\times = \pi_1{}^*\Omega - \pi_2{}^*\Omega$. It is easy to check that a diffeomorphism $f : M \to M$ is a symplectic automorphism (i.e., $f^*\Omega = \Omega$) if and only if its graph is a lagrangian submanifold of $M \times M$.

Let Δ denote the graph of the identity 1_M. Δ is a lagrangian submanifold of M, and, by Corollary 6.2, $[M \times M, \Delta]$ is symplectically diffeomorphic to $[T^*\Delta, Z_\Delta]$, which is symplectically diffeomorphic in a natural way to $[T^*M, Z_M]$. The diffeomorphisms "near" the identity 1_M are thereby put in $1-1$ correspondence with a "neighborhood" of zero in the space $\mathscr{F}^1(M)$ in such a way that the symplectic diffeomorphisms go onto the subspace of *closed* 1-forms. The correspondence is differentiable in the sense that smooth dependence on parameters is preserved. This gives a "coordinate chart" for the diffeomorphism group around the identity, in which the symplectic automorphism group goes onto a linear subspace, so that the symplectic automorphism group of M is a manifold modeled on the space of closed 1-forms on M. This space is naturally identified by $\tilde{\Omega}^{-1}$ with the space of infinitesimal symplectic automorphisms of M, i.e., the vector fields X on M such that $\mathscr{L}_X \Omega = 0$.

We have been deliberately vague about the topologies and differentiable structures on the spaces of diffeomorphisms and differential forms. The topologies should be strong enough that the sets on which the 1—1 correspondence is defined are really open. The charts are C^∞ if the spaces are considered as differentiable spaces in the sense of [7]. The charts may also be extended to charts for the larger groups of diffeomorphisms determined by various section functors on vector bundles, e.g., H^k, C^k, $C^{k+\alpha}$, etc. (see [13].) In these situations, the charts have as many degrees of differentiability as the mapping $f \mapsto f^{-1}$ in the corresponding diffeomorphism group, as one may verify by following through the identification of diffeomorphisms with their graphs and then with sections of the normal bundle of Δ. Unfortunately, the degree of differentiability is usually zero, except at special points. Ebin and Marsden [5] have shown the existence of charts with more differentiability in the H^k cases, but their method yields no result at all in the C^k cases.

7. Lagrangian Foliations

Let (M, Ω) be a symplectic manifold. A foliation $\mathscr{F}$ of M is *lagrangian* if every leaf of $\mathscr{F}$ is a lagrangian submanifold of M. If we think of a foliation $\mathscr{F}$ as an integrable distribution (subbundle) $E \subseteq TM$, then $\mathscr{F}$ is lagrangian if and only if the fibres of E are lagrangian subspaces of the fibres of TM.

The standard example of a lagrangian foliation is given by the fibres of any cotangent bundle (T^*N, Ω_N).

THEOREM 7.1. *Let $\mathscr{F}$ be a lagrangian foliation of (M, Ω). Let $N \subset M$ be a lagrangian submanifold which is transversal to $\mathscr{F}$ in the sense that $TM \mid N = TN \oplus E \mid N$, where $E \subseteq TM$ is the subbundle corresponding to $\mathscr{F}$. Then there is a diffeomorphism $[f] : [M, N] \to [T^*N, Z_N]$ such that $f \mid N = 1_N$ modulo the identification of N with Z_N, $f^*\Omega_N = \Omega$, and $[f]$ takes the leaves of $\mathscr{F}$ onto the fibres of T^*N.*

Proof. By Corollary 6.2 and the proof of Theorem 6.1, there is a diffeomorphism $[g] : [M, N] \to [T^*N, Z_N]$ satisfying all the requirements of $[f]$ except that the images under $[g]$ of the leaves of $\mathscr{F}$ are only *tangent* along Z_N to the fibres of the cotangent bundle. Denote by $g\mathscr{F}$ the foliation whose leaves are those images. One can certainly find a diffeomorphism $[h] : [T^*N, Z_N] \to [T^*N, Z_N]$ whose derivative is the identity at each point of Z_N and which maps the leaves of $g\mathscr{F}$ onto the fibres of the cotangent bundle. Now $[hg]$ satisfies all the requirements of $[f]$ except that $(gh^{-1})^* \Omega \mid Z_N = \Omega_N \mid Z_N$, whereas we need $(hg^{-1})^* \Omega = \Omega_N$. Now, by Theorem 4.1, there is a diffeomorphism $[k] : [T^*N, Z_N] \to [T^*N, Z_N]$ such that $k \mid Z_N$ is the identity and $k^*\Omega_N = (hg^{-1})^*\Omega$. Then $[f] = [khg]$ is the "identity" on N and $f^*\Omega_N = \Omega$. We will be done if we can show that f takes the leaves of $\mathscr{F}$ onto the fibres of the cotangent bundle, which will be the case if k preserves the fibres. Since k is obtained by integrating a time-dependent vector field $\{Y_t\}_{t\in[0,1]}$, it suffices to show that each Y_t is tangent to the fibres. Now (following the notation of Section 4, with the substitution of $(hg^{-1})^*\Omega$ for Ω_1 and Ω_N for Ω_0), $Y_t \lrcorner \Omega_t = -\phi$, where $\phi = I(\Omega_1 - \Omega_0)$. We suppose I to have been constructed by means of the usual vector bundle structure on T^*N, for which the vector fields X_t of Section 3 are tangent to the fibres. Now if ξ is *any* vector field on T^*N which is tangent to the fibres, we have

$$\Omega_t(Y_t, \xi) = -\phi(\xi) = -\int_0^1 \pi_t^*((\Omega_1 - \Omega_0)(X_t, \xi))\, dt,$$

which is zero because the tangent spaces to the fibres are isotropic for both Ω_0 and Ω_1. Since the fibres are *maximal* isotropic for each Ω_t, it follows that Y_t must be tangent to the fibres. Q.E.D.

COROLLARY 7.2. *Let $\mathscr{F}$ be a lagrangian foliation of (M, Ω), and let $x \in M$ be any point. Then there is a reflexive Banach space V and a diffeomorphism $[f] : [M, x] \to [T^*V, 0]$ such that $f^*\Omega_V = \Omega$ and f takes the leaves of $\mathscr{F}$ onto the fibres of T^*V.*

Proof. By Theorem 7.1, we need only find a lagrangian submanifold through x which is transversal to $\mathscr{F}$. By Corollary 6.2, we may assume that (M, Ω) is a cotangent bundle with the canonical symplectic structure, and that the leaf of $\mathscr{F}$ through x is the zero section. The fibre through x of the cotangent bundle is the required transversal submanifold. Q.E.D.

Functions in Involution

Our theorem on lagrangian foliations is actually a generalization of a result of Carathéodory on systems of functions in involution.[3]

We recall that, if f and g are functions on the symplectic manifold (M, Ω), their *Poisson bracket* (f, g) is the function $X_f g$, where X_f is the vector field $\tilde{\Omega}^{-1}(df)$. It follows immediately that $(f, g) = X_f g = \Omega(X_g, X_f) = -X_g f$. The functions $f_1, \ldots, f_n$ on M are *in involution* if $(f_i, f_j) = 0$ for all i and j.

Carathéodory [3] proved that if M is $2n$-dimensional, $f_1, \ldots, f_n$ are in involution, and $df_1, \ldots, df_n$ are linearly independent, then one can find, locally, functions $g_1, \ldots, g_n$ such that $\Omega = dg_1 \wedge df_1 + \cdots + dg_n \wedge df_n$. The relation between Carathéodory's theorem and ours may be seen from the following result.

PROPOSITION 7.3. *Let $f_1, \ldots, f_n$ be functions on the $2n$-dimensional symplectic manifold (M, Ω) such that $df_1, \ldots, df_n$ are linearly independent. Then $f_1, \ldots, f_n$ are in involution if and only if the foliation $\mathscr{F}$ defined by the equations $f_i = \text{constant}$ is lagrangian.*

Proof. If $f_1, \ldots, f_n$ are in involution, $0 = (f_i, f_j) = X_{f_i} f_j$ implies that each X_{f_i} is tangent to the leaves of $\mathscr{F}$. The linear independence of the df_i implies that X_{f_i} are linearly independent and, for dimension reasons, span the tangent spaces to the leaves. But $\Omega(X_{f_i}, X_{f_j}) = (f_j, f_i) = 0$, so the tangent spaces to the leaves are isotropic and, having dimension n, they are lagrangian.

Conversely, suppose $\mathscr{F}$ is lagrangian, and let X be any vector field tangent to the leaves of $\mathscr{F}$. Then $\Omega(X_{f_j}, X) = (X_{f_j} \lrcorner\, \Omega)(X) = df_j(X) = Xf_j = 0$ for all j, so that X_{f_j} is Ω-orthogonal to the leaves. Since the tangent spaces to the leaves are maximal isotropic, each X_{f_j} must be tangent to the leaves. Since the tangent spaces to the leaves are isotropic, $(f_i, f_j) = \Omega(X_{f_i}, X_{f_j}) = 0$, and $f_1, \ldots, f_n$ are in involution.

Q.E.D.

Structure of Lagrangian-Foliated Manifolds

Corollary 7.2 shows that any lagrangian-foliated symplectic manifold $(M, \Omega, \mathscr{F})$ may be covered by coordinate charts with values in T^*V, for some Banach space V, such that the charts take Ω and $\mathscr{F}$ into Ω_V and the foliation by the fibres. The coordinate changes, therefore, are

[3] I would like to thank J. Roels for bringing this result to my attention.

elements of the pseudogroup $\mathscr{G}$ of local diffeomorphisms of T^*V which preserve Ω_V and the foliation by the fibres.

To describe these diffeomorphisms, we identify T^*V with $V \oplus V^*$. Since the translations of $V \oplus V^*$ are in $\mathscr{G}$, we may restrict our attention to the group $\mathscr{G}_0 \subseteq \mathscr{G}$ of foliation-preserving symplectic diffeomorphisms of $[V \oplus V^*, 0]$. Every element $[f]$ of $\mathscr{G}_0$ induces a diffeomorphism $[f_V]$ of $[V, 0]$ such that the diagram

$$\begin{array}{ccc} V \oplus V^* & \xrightarrow{f} & V \oplus V^* \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ V & \xrightarrow{f_v} & V \end{array}$$

commutes. The correspondence $[f] \to [f_v]$ is a homomorphism from $\mathscr{G}_0$ to the group $\mathscr{D}_0$ of diffeomorphisms of $[V, 0]$. The kernel $\mathscr{K}$ of this homomorphism consists of those elements f such that $f(x, x^*) = (x, g(x, x^*))$ for some $[g] : [V \oplus V^*, 0] \to [V^*, 0]$. We denote by $\partial_1 g : V \oplus V^* \to \hom(V, V^*)$ and $\partial_2 g : V \oplus V^* \to \hom(V^*, V^*)$ the "partial derivatives" of g. The derivative

$$f_* : V \oplus V^* \to \hom(V \oplus V^*, V \oplus V^*)$$

is then given by

$$f_*(x \oplus x^*)(y \oplus y^*) = y \oplus [\partial_1 g(x \oplus x^*)(y) + \partial_2 g(x \oplus x^*) y^*].$$

The fact that f preserves Ω_V is expressed by the equation

$$\begin{aligned} z^*(y) - y^*(z) &= \Omega_V(y \oplus y^*, z \oplus z^*) \\ &= \Omega_V(f_*(x \oplus x^*)(y \oplus y^*), f_*(x \oplus x^*)(z \oplus z^*)) \\ &= \Omega_V(y \oplus [\partial_1 g(x \oplus x^*)(y) + \partial_2 g(x \oplus x^*)(y^*)], \\ &\qquad z \oplus [\partial_1 g(x \oplus x^*)(z) + \partial_2 g(x \oplus x^*)(z^*)]) \\ &= \partial_1 g(x \oplus x^*)(z)(y) + \partial_2 g(x \oplus x^*)(z^*)(y) \\ &\qquad - \partial_1 g(x \oplus x^*)(y)(z) - \partial_2 g(x \oplus x^*)(y^*)(z). \end{aligned} \tag{7.1}$$

Setting y^* and z to zero in (7.1), we find $z^*(y) = \partial_2 g(x \oplus x^*)(z^*)(y)$. Since y is arbitrary, $\partial_2 g(x, x^*)$ is the identity for all $x \oplus x^*$, and we may write $g(x \oplus x^*) = x^* + h(x)$, where $[h] : [V, 0] \to [V^*, 0]$. Now $\partial_1 g(x \oplus x^*) = h_*(x)$, and (7.1) becomes

$$z^*(y) - y^*(z) = h_*(x)(z)(y) + z^*(y) - h_*(x)(y)(z) - y^*(z),$$

which holds if and only if $0 = h_*(x)(y)(z) - h_*(x)(z)(y)$. If h is thought of as a 1-form on V, this equation says that h is a closed 1-form. We have just proved

PROPOSITION 7.4. *$\mathscr{K}$ is naturally isomorphic to the additive group of closed 1-forms on $[V, 0]$ which vanish at 0, acting by "translation" on $[V \oplus V^*, 0]$.*

Next, we show that the map $\mathscr{G}_0 \to \mathscr{D}_0$ has a section, so that there is a split exact sequence $0 \to \mathscr{K} \to \mathscr{G}_0 \to \mathscr{D}_0 \to 0$. Namely, let $[g]$ be any element of $\mathscr{D}_0$. Then $[g]$ lifts in a natural way to a diffeomorphism $[g^*]$ of $[T^*V, 0] = [V \oplus V^*, 0]$ such that $g_V{}^* = g$. Since Ω_V and the foliation by the fibres are natural objects on T^*V, $[g^*]$ is an element of $\mathscr{G}_0$. It is clear that $[g] \mapsto [g^*]$ is a homomorphism. We have proven

THEOREM 7.5. *$\mathscr{G}_0$ is naturally isomorphic to the semidirect product of the additive group $\mathscr{K}$ of closed 1-forms on $[V, 0]$ vanishing at 0 and the group $\mathscr{D}_0$ of diffeomorphisms of $[V, 0]$. The normal subgroup $\mathscr{K}$ acts on T^*V by "translation", and the group $\mathscr{D}_0$ acts by the natural lifting. The action of $\mathscr{D}_0$ as automorphisms of $\mathscr{K}$ is by pullback of forms.*

COROLLARY 7.6. The restriction of an element of $\mathscr{G}$ to any fibre is an affine transformation onto the image.

Combining this corollary with the remarks above about charts, we have

THEOREM 7.7. *Let $(M, \Omega, \mathscr{F})$ be a lagrangian-foliated symplectic manifold. Then the leaves of $\mathscr{F}$ carry a natural affine connection with curvature and torsion zero,* i.e., *affinely equivalent to the natural affine structure on a vector space.*

Theorem 7.7 generalizes the following result of Arnold and Avez [2]. In case the leaf is the set of zeros of functions in involution, the flat affine connection comes from a paralelzation (by the vector fields X_{f_i}). If the leaf is compact, it must be a torus.

The converse of Theorem 7.7 is also true. Namely, let N be an affine manifold with curvature and torsion zero. Then the distribution of horizontal spaces in T^*N defines a lagrangian foliation, as may be readily seen from the local affine isomorphism of N with a vector space. Since N, by identification with the zero section, may be considered as a leaf of this foliation, we have

THEOREM 7.8. *A manifold N is a leaf of a lagrangian foliation of some symplectic manifold if and only if N admits an affine connection with curvature and torsion zero.*

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