ADVANCES IN MATHEMATICS 20, 57–85 (1976)

The Geometry of Algebraic Groups

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The purpose of this paper is to study the geometry of a smooth connected linear algebraic group G defined over an algebraically closed field of any characteristic.

The main tool is the characteristic map for G. Namely, let T be a maximal torus in G and B a Borel subgroup of G containing T. A character χ of T gives rize to a homogeneous linebundle $L(\chi)$ on G/B. This defines the characteristic map for G

$$X(T) \to \mathrm{Pic}(G/B).$$

The characteristic map fits into an exact sequence (Section 1) $0 \to X(G) \to X(T) \to \mathrm{Pic}(G/B) \to \mathrm{Pic}(G) \to 0$, which we use to study central isogenies (Section 2). By a central isogeny we understand an isogeny whose kernel is diagonalizable. We obtain two companion theorems.

There exists a central isogeny $\tilde{G} \to G$ with $\mathrm{Pic}\, \tilde{G} = 0$.

If $f\colon G' \to G$ is a central isogeny, then there is an exact sequence

$$0 \to X(G) \to X(G') \to X(\mathrm{Ker}\, f) \to \mathrm{Pic}\, G \to \mathrm{Pic}\, G' \to 0.$$

These two theorems are used (Section 3) to construct the fundamental group $\pi_1 G$ of G, in case $X(G) = 0$. The fundamental group $\pi_1 G$ appears as a finite diagonalizable group, and we obtain

$$\mathrm{Pic}\, G = X(\pi_1 G).$$

The same techniques allow us to study extensions of G by a diagonalizable group D, and we obtain

$$\mathrm{Ext}(G, D) = \mathrm{Hom}(\pi_1 G, D).$$

From here, we turn to the case where G is reductive, and give first (Section 4) a theory of coroots based on the same geometric ideas, especially the simple connectedness of Sl_2.

In the central part of the paper we return to characteristic map $X(T) \to \operatorname{Pic}(G/B)$ and construct (Section 5, after Chevalley) a basis $(D_\alpha)_{\alpha \in S}$ for the divisor classes on G/B, here, S denotes the simple roots of G. This basis has the following miraculous property.

> The divisor class of $\sum n_\alpha D_\alpha (n_\alpha \in \mathbb{Z})$ contains a positive divisor if and only if $n_\alpha \geqslant 0$ for all $\alpha \in S$.

Next (Sections 5 and 6), we prove the formula, let $\chi \in X(T)$, then

$$c_1(L(\chi)) = \sum_{\alpha \in S} \langle \alpha^\vee, \chi \rangle D_\alpha$$

where $\alpha^\vee$ denotes the coroot associated with α.

A particularly interesting case is when G is semisimple adjoint (Section 7). Then, the formula above tells that the characteristic map for G is given by the Cartan matrix $\langle \alpha^\vee, \beta \rangle_{(\alpha,\beta) \in S \times S}$. Recalling that the cokernel of the characteristic map is isomorphic to Pic G and $X(\pi_1 G) =$ Pic G, this allows us to calculate $\pi_1 G$ from the Cartan matrix.

In Sections 8 and 9, we apply these results to the character theory for linear representations. For a character χ of B, let $E(\chi)$ denote the linear representation induced from $G_a(\chi)$, the one-dimensional representation of B of weight χ. $E(\chi)$ is characterized by Frobenius reciprocity

$$\operatorname{Hom}_G(V, E(\chi)) = \operatorname{Hom}_B(V, G_a(\chi))$$

for all linear representations V of G.

In characteristic zero, the $E(\chi)$'s describe all the simple representations and we derive (Section 8) their character theory and Weyl's character formula (Section 9).

In characteristic p, the $E(\chi)$'s are no longer simple, but at least in the case where G is of type A_n, they satisfy Weyl's character formula

$$\operatorname{tr} E(\chi) = J(e^{\chi+\rho})/J(e^\rho), \qquad (E(\chi) \neq 0).$$

The geometric ideas in this paper have a long history. In case of semisimple complex Lie groups, the characteristic map $X(T) \to \operatorname{Pic}(G/B)$ may be identified with the transgression

$$H^1(B, \mathbb{Z}) \to H^2(G/B, \mathbb{Z}),$$

(Borel's thesis, complexification of H, S where H is a compact Lie group and S a maximal torus). The identification of the simple representations as sections in homogeneous linebundles is due to Borel and Weil [2], and developed further by Bott [3]. The algebraization of these ideas is due to Chevalley [5], Demazure [7, 8], and Kempf [16].

Since I wrote the first version of this paper, various improvements and clarifications have been achieved by means of [12], with which there is an overlap and [19] for a good Lefschetz formula, and through a suggestion by A. Borel for a more geometric proof of (5.3) based on (6.6), a result that can be found in an unpublished paper of Chevalley. It should also be said that many of the results in Sections 3 and 7 can be proved by means of *donnée radicielle* [10, exposé XXI], in connection with [9, Théorème Fundamentale, 3.6.] However, the main aim of this paper is to illustrate how far one can get with geometric means.

1. Homogeneous Linebundles

Throughout this section G denotes a smooth connected linear algebraic group.

Let X be a variety. By a *principal* G_m-*bundle* on X, we shall understand a G_m-bundle that is locally trivial in the Zarisky topology. Let $E \to X$ be a principal G_m-bundle. G_m acts canonically on G_a. Let $E \times {}^{G_m}G_a$ denote the space obtained by taking the quotient of $E \times G_a$ under the G_m-action given by $(e, x)z = (ez, z^{-1}x)$, $e \in E$, $x \in G_a$, $z \in G_m$. There is a canonical projection $E \times {}^{G_m}G_a \to X$, and we obtain in this way a linebundle on X. This construction is well known to give a $1 - 1$ correspondence between principal G_m-bundles and linebundles.

Now, let G act on X (from the left). By a $(G-)$ *homogeneous principal* G_m-*bundle* we understand a pair $(E \to X, \tau)$ where $E \to X$ is a principal G_m-bundle and τ is a (left) action of G on E that

(1) commutes with the G_m-action;

(2) makes $E \to X$ G-equivariant.

In this case, we shall call $E \times {}^{G_m}G_a$ together with the induced action of G, a *homogeneous linebundle*.

Given a principal G_m-bundle (or a line bundle) $E \to X$. An action τ of G on E such that $(E \to X, \tau)$ is a homogeneous G_m-bundle is called a $(G-)$ *homogenization* if $E \to X$. In case there exists a homogenization of $E \to X$, we say that this bundle can be *homogenized*.

Let $L \to X$ be a homogeneous linebundle. G acts on $\Gamma(X, L)$ in virtue of the formula ($g \in G$, $x \in X$, $s \in \Gamma(X, L)$)

$$(gs)(x) = gs(g^{-1}x). \tag{1.1}$$

EXAMPLE 1.2. Let V be a (left) linear representation of G. G_m acts on V in a canonical way. Let V^* denote the linear dual of V with the contragredient actions of G and G_m (*Caution*: Given $z \in G_m$ and $x' \in V$, we have ${}^z x' = z^{-1}x'$). The natural projection $V^* - 0 \to \mathrm{Proj}(V)$ is a homogeneous principal G_m-bundle. The corresponding linebundle is the "universal" line-bundle L_{univ} on Proj V: An element $v \in V$ defines a section of L_{univ}, $x' \mapsto (x', x'(v))$. In fact, this identifies V and $\Gamma(\mathrm{Proj}(V), L_{\mathrm{univ}})$. If we transport the action (2.1) to V we get the original action back.

EXAMPLE 1.3. Let V be a finite-dimensional vector space. $Gl(V)$ acts on $\mathrm{Proj}(V)$ via the contragredient representation; this gives a morphism

$$Gl(V) \to PGl(V).$$

Given a morphism f: $G \to PGl(V)$. This defines a G action on $\mathrm{Proj}(V)$. L_{univ} can be (G—) homogenized if and only if f factors through $Gl(V) \to PGl(V)$, in fact it follows from the discussion in (1.2) that there is a one-to-one correspondence between homogenizations of L_{univ} and such factorizations of f.

In the remaining part of this section B denotes a Borel subgroup of G and p: $G \to G/B$, the canonical projection.

PROPOSITION 1.4. *Let L be a linebundle on G/B such that $V = H^0(G/B, L) \neq 0$. Then, there is a (canonical) morphism s: $G \to PGl(V)$ making the canonical map t: $G/B \to \mathrm{Proj}(V)$ G-equivariant and such that $t^*L_{\mathrm{univ}} \simeq L$ as homogeneous bundles, L_{univ} is homogenized as in* (1.2).

Proof. Since G acts transitively on G/B, the linebundle L is generated by its global sections. This defines t: $G/B \to \mathrm{Proj}(V)$ with $t^*L_{\mathrm{univ}} \simeq L$. The construction of s (on the level of geometric points) is easy to carry out by means of the mapping property of $\mathrm{Proj}(V)$, see [13, II., 4.2, or 15, No. 1], where also a proof is given for the fact that s is morphism of varieties. Q.E.D.

Let L denote a homogeneous bundle on G/B. $p(e)$ is a fixedpoint for the action of B on G/B. Thus, B acts on the fiber of L at $p(e)$. This gives a one-dimensional representation of B; let $\chi_L \in X(B)$ denote the corresponding character.

PROPOSITION 1.5. *$L \mapsto \chi_L$ gives a one-to-one correspondence between isomorphism classes of homogeneous linebundles on G/B and $X(B)$.*

Proof. Let us give the inverse construction and leave the details to the reader. Given $\chi \in X(B)$, let B act from the right on $G \times G_a$ by the formula ($g \in G$, $x \in G_a$, $b \in B$)

$$(g, x)b = (gb, \chi(b^{-1})x.$$

$L(\chi)$ denotes the quotient of $G \times G_a$ for this action. That this defines a linebundle follows from the fact that $G \to G/B$ has a section locally for the Zarisky topology [20]. Q.E.D.

DEFINITION 1.6. The linebundle constructed in the proof of (1.5) will be denoted $L(\chi)$.

DEFINITION 1.7. Let $T \subseteq B$ be a maximal torus. Let us recall that any character $\chi \in X(T)$ extends uniquely to a character of B. The linear map

$$\chi \mapsto L(\chi), \qquad X(T) \to \mathrm{Pic}(G/B)$$

is called *the characteristic map for G.*

EXAMPLE 1.8. $G = Sl_2$, T_2 the subgroup of diagonal matrices in Sl_2, B_2 the upper triangular matrices. Consider the natural action of Sl_2 on $\mathbb{A}^2$. Following the conventions of (1.2) we consider the contragredient action on $\mathbb{P}^1$. $(0, 1) \in \mathbb{P}^1$ has B as stabilizer and this gives rise to an isomorphism $Sl_2/B_2 \simeq \mathbb{P}^1$. The canonical linebundle L_{univ} on $\mathbb{P}^1$ comes equipped with a Sl_2-homogenization and B_2 acts on the fiber of L_{univ} above $(0, 1)$ with weight

$$\begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} \mapsto z^{-1}.$$

Whence, $X(T_2) \to \mathrm{Pic}(Sl_2/B_2)$ is an isomorphism.

PROPOSITION 1.9. *The following sequence is exact*

$$0 \to X(G) \to X(T) \to \operatorname{Pic}(G/B) \to \operatorname{Pic} G \to 0.$$

Proof. See [12, Proposition 3.1].

2. CENTRAL ISOGENIES

DEFINITION 2.1. A morphism $f: G' \to G$ of linear algebraic groups is called a central isogeny if

(a) G' and G are smooth and connected,

(b) $f: G' \to G$ is surjective,

(c) $\operatorname{Ker}(f)$ is finite and diagonalizable.

By rigidity of diagonalizable groups, $\operatorname{Ker}(f)$ is contained in the center of G'.

Let us first make three general remarks on central isogenies.

2.2 If $G'' \to G'$ and $G' \to G$ are central isogenies then the composite $G'' \to G'$ is a central isogeny.

This follows from [11, IV, Sect. 1, No. 46].

2.3 If $f: G' \to G$ is a central isogeny, H a smooth connected linear algebraic group and r, s: $H \rightrightarrows G'$ morphisms such that $fr = fs$, then $r = s$.

This follows from the observation that $\chi \mapsto r(\chi)\, s(\chi)^{-1}$ maps H into $\operatorname{Ker}(f)$.

2.4 Let $f: H \to G$ be a morphism of smooth connected linear algebraic groups. Then, a central isogeny $G' \to G$ "may be pulled back along f."

The precise meaning is this: Let H' denote the reduced connected component of $G' \times_G H$. $H' \to H$ is a central isogeny that we call the pull-back of $G' \to G$ along f. We leave it to the reader to exhibit the universal property of this construction.

LEMMA 2.5. *Let $f: G \to H$ be a surjective morphism of smooth connected linear algebraic groups. Then, the inverse image by f of a maximal torus, respectively, a Borel subgroup is maximal torus, respectively, a Borel subgroup.*

Proof. Let D denote the kernel of f. Let us first prove that any maximal torus T of G contains D:

The centralizer $C = Z_G(T)$ is smooth [13] and connected [1, 11.12]. Let T act on D by inner conjugation. This action is trivial by rigidity of diagonalizable groups. Hence, $D \subseteq C$. C is a product of T and a unipotent group [1, 11.7]. Thus, $D \subseteq T$.

The image of T by f is a maximal torus [1, 11.14]; Conclusion by the fact that maximal tori are conjugated.

Same proof in the case of a Borel subgroup. Q.E.D.

PROPOSITION 2.6. *Let f: $G' \to G$ be a surjective morphism of smooth connected linear algebraic groups whose kernel is diagonalizable. Then, there is an exact sequence*

$$0 \to X(G) \to X(G') \to X(\mathrm{Ker}\, f) \to \mathrm{Pic}(G) \to \mathrm{Pic}(G') \to 0.$$

Proof. Pick a maximal torus T of G and a Borel subgroup B of G containing T. Put $T' = f^{-1}(T)$ and $B' = f^{-1}(B)$, see (2.5). The proposition now follows from the exact (1.9), commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & X(G) & \longrightarrow & X(G') & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X(T) & \longrightarrow & X(T') & \longrightarrow & X(\mathrm{Ker}\, f) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & | & & \\
0 & \longrightarrow & \mathrm{Pic}(G/B) & \longrightarrow & \mathrm{Pic}(G'/B') & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Pic}(G) & \longrightarrow & \mathrm{Pic}(G') & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and the snake lemma. Q.E.D.

THEOREM 2.7. *Let G be a smooth connected linear algebraic group. Then, there exists a central isogeny $\overline{G} \to G$ with* Pic $\overline{G} = 0$

Proof. Let B be a Borel subgroup of G. $\mathrm{Pic}(G/B)$ is finitely generated, since G/B is a rational variety as it follow from the Bruhat decomposition [1, 14.11]. $\mathrm{Pic}(G/B)$ can be generated by linebundles L, for which $H^0(G/B, L) \neq 0$. By virtue of (1.5) and (1.9), it suffices to prove: Let L be a linebundle on G/B with $H^0(G/B, L) \neq 0$. Then, there exists a central isogeny $f: G' \to G$ (put $B' = f^{-1}(B)$) such that the pull-back of L along $G'/B' \simeq G/B$ can be G'-homogenized.

Put $V = H^0(G/B, L)$. With the notation of (1.4), we have canonical maps $s: G \to PGl(V)$, and $t: G/B \to \mathrm{Proj}(V)$ such that $t^*L_{\mathrm{univ}} = L$. Now, let $f: G' \to G$ denote the pull-back of the central isogeny $Sl(V) \to PGl(V)$ along s. Consider L_{univ} as a homogeneous $Sl(V)$-bundle in the canonical way. This gives the pull-back of L_{univ} along the composite

$$G'/B' \to G/B \xrightarrow{t} \mathrm{Proj}(V)$$

a G'-homogenization. Q.E.D.

Corollary 2.8. (cf. [6, 5-21]). *Let G be a smooth connected linear algebraic group. Then,* Pic G *is a finite group.*

Proof. Let $f: \bar{G} \to G$ be a central isogeny with $\mathrm{Pic}\, \bar{G} = 0$. Then, by (2.6), we have an exact sequence

$$X(\ker f) \to \mathrm{Pic}(G) \to \mathrm{Pic}(\bar{G}).$$

Q.E.D.

Corollary 2.9. *Let the smooth connected linear algebraic group G act on the normal projective variety X, and let L be a linebundle on X. Then, a tensor power of L admits a G-linearization.*

Proof. Using that all components of $\mathrm{Pic}(X)$ are proper in this case and the see-saw principle [18, Corollary 6, p. 54], Mumford [19, Sect. 1.3] proves the following:

"The pull-back L_x of L along $(x \in X)\ g \mapsto gx$, $G \to X$ is independent of $x \in X$. L admits a (G—) homogenization if and only if $L_x \in \mathrm{Pic}(G)$ is trivial."

3. Fundamental Group

Definition 3.1. A smooth connected linear algebraic group G is called simply connected if G does not admit a nontrivial central isogeny, $G' \to G$.

We shall limit the discussion to character-free groups.

PROPOSITION 3.2. *Let G be a smooth, connected linear algebraic group which is character-free. Then, G is simply connected if and only if* Pic $G = 0$.

Proof. Suppose Pic $G = 0$. Let $G' \to G$. From (2.6), we get an exact sequence

$$0 \to X(G) \to X(G') \to X(\ker f) \to \mathrm{Pic}(G).$$

$\mathrm{Pic}(G) = X(G) = 0$ by assumption. $X(G')$ is torsion-free as it follows from (1.9), and $X(\ker f)$ is finite. Whence, $X(G') = X(\ker f) = 0$, and therefore, $\ker f = 0$.

The converse follows immediately from (2.7).

COROLLARY 3.3. *Let G be a smooth connected linear algebraic group. Suppose G is character-free. Then, there exists a central isogeny* $\tilde{G} \to G$ *with* $\tilde{G}$ *simply connected.*

Proof. Follows from (3.2) and (2.7).

DEFINITION 3.4. Let G be a smooth connected linear algebraic group that is character-free. The central isogeny $\tilde{G} \to G$ of (3.3), unique by (2.3), is called the universal covering of G. Its kernel is denoted $\pi_1 G$ and is called the fundamental group of G.

PROPOSITION 3.5. *Let G be a smooth connected linear algebraic group which is character-free. Then,*

$$\mathrm{Pic}\, G = X(\pi_1 G).$$

Proof. Follows immediately from (3.2) and (2.6). Q.E.D.

In the remaining part of this Section, we shall apply the same kind of technique to reductive groups.

PROPOSITION 3.6. *Let* $f: G' \to G$ *be a morphism between smooth connected linear algebraic groups, and let f have a diagonalizable kernel. If G is semisimple and simply connected, then* ker f *is a torus and f admits a section.*

Proof. By (2.6), we have an exact sequence

$$0 \to X(G) \to X(G') \to X(\ker f) \to \operatorname{Pic} G.$$

$\operatorname{Pic} G = X(G) = 0$ since G is simply connected and semisimple. G' is reductive, and hence, $T' = G'/[G', G']$ is a torus by [1, 14.2]. Since the projection of $\ker f$ onto T' induces an isomorphism on the character groups, it follows that $\ker f \to T'$ is an isomorphism. From this, it follows that the inclusion $\ker f \to G'$ has a retraction r, say. The endomorphism $g \mapsto r(g^{-1})g$ of G' is a morphism of groups and factor through $G' \to G$ to give the required section. Q.E.D.

COROLLARY 3.7. *Let G be a reductive linear algebraic group. Then, there exists a semisimple group G', a torus T and a central isogeny*

$$G' \times T \to G.$$

PROPOSITION 3.8. *Let G be a semisimple linear algebraic group, and D a diagonalizable group. Then, there is a canonical isomorphism*

$$\operatorname{Hom}(\pi_1 G, D) \xrightarrow{\sim} \operatorname{Ext}(G, D).$$

Proof. Let us first construct the central extension of G by D associated with a morphism $f\colon \pi_1 G \to D$. Let $\tilde{G} \times^{\pi_1 G} D$ denote the cokernel of the morphism $\pi_1 G \to \tilde{G} \times D$ given by $x \mapsto (x, f(x^{-1}))$. $\tilde{G} \times^{\pi_1 G} D \to G$ is a central extension of G by D. This construction defines a linear map

$$\operatorname{Hom}(\pi_1 G, D) \to \operatorname{Ext}(G, D).$$

We are now going to construct the inverse to this map. Thus, given $h\colon G' \to G$ a central extension of G by D. Let us first show that $\tilde{G} \to G$ factors through h. Consider $\tilde{G} \times_G G' \to \tilde{G}$, $(\tilde{G} \times_G G')^0_{\text{red}} \to \tilde{G}$ has a section by (3.6), and consequently, $\tilde{G} \to G$ factors through h. Such a factorization is unique: Suppose there were two factorizations of h $g_1, g_2 : \tilde{G} \to G'$. Then

$$x \mapsto g_1(x)\, g_2(x^{-1})$$

defines a morphism of groups $\tilde{G} \to G'$, which is trivial on $[\tilde{G}, \tilde{G}]$. But, $[\tilde{G}, \tilde{G}] = \tilde{G}$ since $\tilde{G}$ is semisimple. The remaining details are left to the reader. Q.E.D.

COROLLARY 3.9. *Let G be a semisimple linear algebraic group. Then, the canonical map*

$$\mathrm{Ext}(G, G_m) \to \mathrm{Pic}(G)$$

is an isomorphism.

Proof. Combine (3.5) and (3.8).

4. COROOTS

Our presentation is based on the properties of Sl_2. We shall introduce some notation for that group.

$\alpha_2^\vee$: $G_m \to Sl_2$ is given by

$$\alpha_2^\vee(z) = \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}.$$

$T_2 =$ the image of $\alpha_2^\vee$.
$U_2 =$ the group of matrices of the form $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $z \in G_a$.
$B_2 = TU_2$.
$\alpha_2 \in X(T_2)$ is $\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} \mapsto z^2$.
$s_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

LEMMA 4.1. *Sl_2 is simply connected and all automorphisms of Sl_2 are inner.*

Proof. That Sl_2 is simply connected follows from (1.8) and the easy part of (3.2).

Let u be an automorphism of Sl_2. Composing u with an inner automorphism, we may assume that u stabilizes T_2. If necessary, compose u with conjugation by s_2 to obtain that u leaves T_2 elementwise fixed. This ensures that $u(U_2)$ is T-isomorphic to U_2. Hence, u stabilizes U_2, and therefore, B_2. Composing u with conjugation by an element of T_2, we can obtain that u leaves B_2 elementwise fixed. $x \mapsto u(x)x^{-1}$ factors through Sl_2/B_2, and consequently, is constant.

PROPOSITION 4.2. *Let G_* be a reductive linear algebraic group of semisimple rank 1. Let T_* denote a maximal torus of G_*, and α_* a root of G_* relative to T_*. Then, there exists a morphism $\pi_* : Sl_2 \to G_*$ such that*

(1) *the kernel of* π_* *is central,*

(2) $\pi_*(T_2) \subseteq T_*$,

(3) $\alpha_* \pi_* = \alpha_2$.

π_* *is unique up to inner conjugation by an element of* T_2, *and* $\pi_*(s_2)$ *normalizes* T_*.

Proof. Let Z_* denote the center of G_* (in the scheme sense). $[G_*, G_*] \to G_*/Z_*$ is a central isogeny by [1, p. 325]. The existence of a $\pi_* : Sl_2 \to G_*$ with central kernel is a consequence of (4.1), once we prove that "G_*/Z_* is isomorphic to PGL_2." Proof: Borel [1, p. 309] constructs a surjective morphism $G \to PGl_2$ whose kernel Z_*' is the intersection of all Borel subgroups of G. Let B and B' be the two Borel subgroups of G containing T. By [1, p. 310–311], we have $LB \cap LB' = LT$, and $B \cap B' = T$. Consequently, Z_*' is a diagonalizable normal subgroup of G. The rigidity of diagonalizable groups implies that $Z_*' \subseteq Z_*$. The opposite inclusion follows from (7.3). Now, let $\pi_* : Sl_2 \to G_*$ be a morphism with central kernel. We have $\pi_*(T_2') \subseteq T_*$ for some maximal torus T_2' of Sl_2: Let us first note that $\dim Z^* < \dim T$ by [1, p. 264]. On the other hand, $\dim Z_* = \dim G/[G, G]$, and consequently, $[G, G] \cap T$ contains a maximal torus, S of $[G, G]$. $\pi_*^{-1}(S)$ is a maximal torus in Sl_2 by (2.5). Composing π_* by an inner conjugation, we may assume $\pi_*(P_2) \subseteq T_*$. $\alpha_*\pi_*$ is a root of Sl_2 relative to T_2: The restriction of π_* to U_2 is a closed immersion, and hence, T_2 operates nontrivially on LG and trivially on the kernel of Lie(π_*). This proves the assertation. To make π_* satisfy condition (3), it suffices (if necessary) to compose π_* with conjugation with s_2. This proves the existence. The uniqueness follows from (4.1). The last remark follows from the fact that Z_* and $\pi_2(T_2)$ generate T_*. Q.E.D.

DEFINITION 4.3. Let G be a reductive linear algebraic group, T a maximal torus of G, and α a root of G relative to T. Let T_α denote the reduced connected component of Ker(α), $G_\alpha = Z_G(T_\alpha)$ is reductive of semisimple rank 1. Let $\pi_* : Sl_2 \to G_\alpha$ denote a morphism as in (4.2) relative to the triple (G_α, T, α). The composite of π_* and the inclusion of G_α into G will be denoted $\pi_\alpha : Sl_2 \to G$. $\pi_\alpha \alpha_2^\vee$: $G_m \to T$, which does not depend on the choice of π_*, will be denoted $\alpha^\vee$ and is called the *coroot* associated with α. We put $s_\alpha = \pi_\alpha(s_2)$, viewed as an element of $W = N_G(T)/T$.

Notation 4.4. Let $X_*(T)$ denote the multiplicative 1-parameter subgroups of T, i.e., $\mathrm{Hom}(G_m, T)$. If $\eta \in X_*(T)$ and $\chi \in (T)$, then we let $\langle \eta, \chi \rangle$ denote the integer such that

$$\eta(\chi(z)) = z^{\langle \eta \chi \rangle}, \qquad z \in G_m,$$

$\langle\, ,\, \rangle$: $X_*(T) \times X(T) \to \mathbb{Z}$ is a perfect pairing, [1, p. 205].

Let us make it clear that we always consider the *left* operation of W on $X(T)$ and $X_*(T)$. Thus, we have the formula

$$\langle w(\eta), w(\chi) \rangle = \langle \eta, \chi \rangle, \qquad w \in W, \eta \in X_*, \chi \in X.$$

PROPOSITION 4.5. *Let α be a root of G relative to T. Then*

(1) $\langle \alpha^\vee, \alpha \rangle = 2$,

(2) $(-\alpha)^\vee = -\alpha^\vee$,

(3) $(w(\alpha))^\vee = w(\alpha^\vee)$, $w \in W$.

Proof. Consider the morphism $\pi_\alpha : Sl_2 \to G$ introduced in (4.3).

(1) $\alpha(\alpha^\vee(z)) = \alpha(\pi_\alpha(\alpha_2{}^\vee(z)) = \alpha_2(\alpha_2{}^\vee(z)) = z^2$.

(2) If we compose π_α by inner conjugation by s_2, we obtain π_α, thus, $(-\alpha)^\vee(z) = \pi_\alpha(s_2\alpha_2{}^\vee(z)s_2) = \pi_\alpha(\alpha_2{}^\vee(z^{-1})) = \alpha^\vee(z^{-1})$.

(3) With the notation of (4.3), note that ${}^wG_\alpha = G_{w(\alpha)}$. Thus, w will conjugate π_α into $\pi_{w(\alpha)}$. Q.E.D.

PROPOSITION 4.6. *Let α be a root of G relative to T and $\alpha^\vee$ the corresponding coroot. Then*

$$s_\alpha(\chi) = \chi - \langle \alpha^\vee, \chi \rangle \alpha,$$

for all $\chi \in X(T)$.

Proof. Let us first remark that this means

$$(*) \quad s_\alpha(t) = t\alpha^\vee(\alpha(t^{-1})), \qquad \text{all} \quad t \in T.$$

We may assume that G is of semisimple rank 1. Let S be a maximal torus of $[G, G]$ contained in T, and R the reduced connected component of $\mathrm{Ker}(\alpha)$. Recall from the proof of (4.2) that $R \subseteq ZG$, and that S and R generate T. The two expressions in (*) obviously coincide for $t \in R$.

Consider the morphism $\pi_\alpha :_\gamma Sl_2 \to G$ introduced in (4.3); $S = \mathrm{Im}(\alpha^\vee)$. Substitute $t = \alpha^\vee(z)$, $z \in G_m$ in (*) to get

$$s_\alpha(\alpha^\vee(z)) = \alpha^\vee(z^{-1}),$$

$$\alpha^\vee(z)\,\alpha^\vee(\alpha(\alpha^\vee(z^{-1}))) = \alpha^\vee(z)\,\alpha^\vee(z^{-2}) = \alpha^\vee(z^{-1}).$$

Q.E.D.

Remark 4.7. Put $M = X(T)$, $M^* = X_*(T)$. We have given a perfect pairing

$$\langle\, ,\rangle \colon M^* \times M \to \mathbb{Z},$$

a finite subset (the roots of G relative to T)

$$R \subseteq M,$$

a map

$$\alpha \mapsto \alpha^\vee \colon R \to M^*,$$

with the following properties.

(Define $s_\alpha : M \to M$ by $s_\alpha(\chi) = \chi - \langle \alpha^\vee, \chi\rangle\alpha$)

(i) $(-\alpha)^\vee = -\alpha^\vee, \qquad \alpha \in R$

(ii) $\langle \alpha^\vee, \alpha\rangle = 2, \qquad \alpha \in R$

(iii) $s_\alpha(R) = R$

(iv) $\langle \alpha^\vee, s_\beta(\chi)\rangle = \langle s_\beta(\alpha^\vee), \chi\rangle, \qquad \chi \in M;\ \alpha, \beta \in R$

as it follows from (4.5) and (4.6).

A structure like this

$$(M, M^*, \langle\, ,\rangle, R, \alpha \mapsto \alpha^\vee)$$

satisfying i,..., iv is studied in [10, Exposé XXI] under the name *donnée radicielle* (the axioms there are slightly different, but equivalent).

5. Chern Class of a Homogeneous Linebundle

Throughout the remaining part of this paper, we will fix the following standard notation.

G = a reductive, connected linear algebraic group.

T = *a* maximal torus in G.

$W = W(T, G)$, the Weyl group of G relative to T ($N_G(T)/T$).

$s_\alpha =$ (α a root of G relative to T), the reflection with respect to α.

$\alpha^\vee =$ the coroot associated with α.

$B =$ a Borel subgroup in G containing T.

$U =$ the unipotent radical of B.

$w_0 \in W =$ the symmetry with respect to B, (${}^{w_0}B = B^-$, the opposite Borel group).

$S =$ the basis for the roots of G relative to T with respect to B (if $\beta \in S$, then $-\beta$ is a weight of T in LB).

positive $=$ those elements in $X(T)$, the character group of T, that are of the form $\sum_{\alpha \in S} n_\alpha \alpha$, $n_\alpha \in \mathbb{N}$.

p: $G \to G/B =$ the canonical projection.

Let us recall the cellular decomposition of G/B, namely, that G/B is the disjoint union of the U-orbits of $p(w)$ as w runs through W.

LEMMA 5.1. *The U-orbit of $p(w)$ in G/B has codimension* 1 *if and only if $w = w_0 s_\alpha$ with $\alpha \in S$.*

Proof. The stabilizer of $p(w)$ under the action of U is

$$U \cap wBw^{-1} = U \cap wUw^{-1},$$

which is a T-stable subgroup of U. For a root γ, let U_γ denote the one-dimensional T-stable subgroup of U on which T has weight γ [1, IV, 14.4]. The T stable subgroup above is directly spanned (in any order) of the U_γ's it contains loc. cit. That is spanned by U_γ's as γ runs through the set $E(w) = \{\gamma \text{ root} \mid \gamma < 0 \text{ and } w^{-1}(\gamma) < 0\}$, hence, it suffices to determines the w's for which this set has cardinality 1.

In order to analyze this, let us recall three facts about root system

(1) $\gamma > 0$ implies $w_0(\gamma) < 0$,

(2) $w \in \mapsto$, $w(\gamma) > 0$ for all $\gamma > 0$ implies $w = e$,

(3) $\alpha \in S$, $\beta > 0$ and $\beta \neq \alpha$ implies $s_\alpha(\beta) > 0$.

Returning to our initial problem, let $\alpha \in S$. Then, $E(w_0 s_\alpha) = \{w_0(\alpha)\}$. Suppose, conversely, that Card $E(w) = 1$. Expanding an element of $E(w)$ after the basis S, one sees immediately that $E(w)$ if nonempty always contains an element of S. Put $E(w) = \{w_0(\alpha)\}$. It is easy to verify that $w^{-1} w_0 s_\alpha(\beta) > 0$ for all $\beta > 0$. Hence, $w = w_0 s_\alpha$. Q.E.D.

PROPOSITION 5.2. *$(D_\alpha)_{\alpha \in S}$ is a basis for the group of divisor classes on G/B, where D_α is the closure of the U-orbit of $p(w_0 s_\alpha)$.*

The divisor class of $\sum n_\alpha D_\alpha$, $n_\alpha \in \mathbb{Z}$, contains a positive divisor if and only if $n_\alpha \geqslant 0$, all $\alpha \in S$.

Proof. $Up(w_0)$ is an open subset of G/B, [1, 14.11]. Let us first prove the D_α's are the only subvarieties of G/B of codimension 1 that do not meet $Up(w_0)$. Let F be such a subvariety. The complement of $Up(w_0)$ in G/B is stable under U, and hence, F, which is an irreducible component of that complement, is stable under U. This makes F the union of closures of U-orbits. The number of U-orbits being finite, F being irreducible, we conclude that F is the closure of a U-orbit, hence, $F = D_\alpha$ for some $\alpha \in S$.

Let D be a divisor on G/B, D' its restriction to $Up(w_0)$. Since $Up(w_0) \simeq \mathbb{A}^n$, we can find a rational function f on G/B, whose restriction to $Up(w_0)$ has divisor D'. D-div(f) is supported by the complement of $Up(w_0)$, hence, the divisor class group is generated by the classes of $(D_\alpha)_{\alpha \in S}$. Suppose $\sum n_\alpha D_\alpha \equiv 0$. Let f be a rational function of G/B with divisor $\sum n_\alpha D_\alpha$. Then, the restriction of f to $Up(w_0)$ has no poles and no zero's, hence, f is constant on $Up(w_0)$, and hence, f is constant.

Let us now prove that divisors of the form $\sum n_\alpha D_\alpha$ are the only U-invariant divisors on G/B. Let D be an U-invariant divisor. By the preceding result we can write

$$D = D' + \operatorname{div}(f),$$

where D' is a linear combination of the D_α's and f a rational function. This gives $\operatorname{div}^u f = \operatorname{div} f$ for $u \in U$. This makes f a semiinvariant for U, U being unipotent, f is invariant under the action of U. Consequently, f is constant on the open set $Up(w_0)$, and therefore, f is constant on G/B.

Let us also remark the action of U on the divisors of G/B preserves $\equiv$ as it follows from the fact that G/B is rational [12, 2.5]. See [5, Exposé 15, Proposition 4], for a direct proof in this case.

Let us now return to the proof of the last part of (5.9). Hence, suppose the set V of positive divisors in the divisor class of $\sum n_\alpha D_\alpha$ is nonempty. V is a projective variety, with U acting on it. Hence, by Borel's fixed point theorem, U has a fixed point in V. This means that the divisor class of $\sum n_\alpha D_\alpha$ contains a U-invariant positive divisor, hence, $n_\alpha \geqslant 0$, $\alpha \in S$. Q.E.D.

THEOREM 5.3. *Let χ be a character of T. Then*

$$c_1(L(\chi)) = \sum_{\alpha \in S} \langle \alpha^\vee, \chi \rangle D_\alpha .$$

The proof will be given in Section 6.

Let us next investigate the action (1.1) of B on $H^0(G/B, L(\chi))$. This is done by identifying this space with the space of regular functions on G satisfying the *functional equation* (5.5).

PROPOSITION 5.4. *Let $\chi \in X(T)$, and suppose $H^0(G/B, L(\chi)) \neq 0$. Then*

(i) *The space of B-semiinvariants is one-dimensional. The weight in question is $w_0(\chi)$.*

(ii) *The space of T-semiinvariants of weight $w_0(\chi)$ is one-dimensional*

(iii) *All weights of T are $\geqslant w_0(\chi)$.*

Proof. $\chi \in X(T)$ extends uniquely to a character of B, which we shall still denote χ. Recall that $L(\chi)$ is the quotient of $G \times G_a$ under the right action of B given by

$$(g, x)b = (gb, \chi(b^{-1})x), \qquad g \in G, x \in G_a, b \in B.$$

From this, it follows that $H^0(G/B, L(\chi))$ may be identified with the set of regular functions f of G, which satisfies the functional equation

$$f(xy) = f(x)\, \chi(y^{-1}), \qquad x \in G, y \in B. \tag{5.5}$$

When we transport the action (1.1) of G on $H^0(G/B, L(\chi))$, we get that the result ${}^gf$ of acting out with a $g \in G$ satisfies

$${}^gf(x) = f(g^{-1}x), \qquad x \in G.$$

Let us recall [1, IV, 14.13], that $U \times B \to G$, $(u, b) \to uw_0b$ is an open immersion. Therefore, it is clear that a solution f to (5.5) is known when we know the regular function f_U on U given by $f_U(x) = f(xw_0)$.

Remark the formulas

$$({}^uf)_U(x) = f_U(u^{-1}x), \qquad u \in U, x \in G \tag{5.6}$$

$$({}^tf)_U(x) = f_U(t^{-1}\, xt)\, w_0(\chi)(t), \qquad t \in T, x \in G. \tag{5.7}$$

The first one is clear, the second follows from

$$({}^{t}f)_U(x) = ({}^{t}f)(xw_0) = f(t^{-1}xw_0) = f(t^{-1}xtw_0(w_0t^{-1}w_0)).$$

Now, using the function equation, we get

$$({}^{t}f)_U(x) = f(t^{-1}xtw_0)\, w_0(\chi)(t) = f_U(t^{-1}xt)\, w_0(\chi)(t).$$

Let us now prove (i). By Borel's fixedpoint theorem, $H^0(G/B, L(\chi))$ contains a nontrivial U-invariant. The corresponding solution to the functional equation f must satisfy $f_U =$ constant, as it follows from (5.6). This proves the first half of (i). The second half follows from (5.7).

Let us now investigate the action of T on $\Gamma(U, O_U)$ given by

$$^{t}h(x) = h(t^{-1}xt), \qquad h \in \Gamma(U, O_U),\ x \in U,\ t \in T.$$

U is directly spanned by the U_α's as α runs through all negative roots, [1, p. 328]. Thus, U is T-isomorphic (as a variety) to the product at the U_α's. U_α is T-isomorphic (${}^{t}u = tut^{-1}$, $u \in U_\alpha$, $t \in T$) to G_a when we let T act on G_a through α. In conclusion, $\Gamma(U, O_U)$ T-isomorphic to a polynomial ring in variables T_α being a semiinvariant of weight α. Consequently, all T-semiinvariants in $\Gamma(U, O_U)$ are positive, and the space of T-invariant is one-dimensional.

Parts (ii) and (iii) now follow from this and (5.7). Q.E.D.

6. Parabolic Subgroup Associated with a Simple Root

Let us recall that UwB as w runs through W from a partition of G, the Bruhat decomposition.

Proposition 6.1 (cf. Remark 6.7). *Let α be a simple root of G with respect to B. Then, $Us_\alpha B \cup B$ is a closed connected subgroup of G.*

The proof depends on the following.

Lemma 6.2. *Let α be a root of G. Then*

(i) $U_\alpha \subseteq Us_\alpha B \cup B$,

(ii) *if α is a simple root, then*

$$Us_\alpha B = U_{-\alpha}s_\alpha B.$$

Proof. (i) Let G_α denote the centralizer of the reduced connected component of Ker(α) (4.3). $U_\alpha \subseteq G_\alpha$, and we may consider s_α as an element of G_α. $B_\alpha = B \cap G_\alpha$ is a Borel subgroup of G_α. Bruhat decomposition in G_α gives

$$U_\alpha \subseteq B_\alpha \cup B_\alpha s_\alpha B_\alpha .$$

(ii) Write $U = U_{-\alpha} U^\alpha$, where U^α is the product (in some order) of the $U_{-\beta}$'s, $\beta > 0$, $\beta \neq \alpha$)

$$U s_\alpha B = U_{-\alpha} U^\alpha s_\alpha B = U_{-\alpha} s_\alpha s_\alpha U^\alpha s_\alpha B.$$

Now, $s_\alpha U^\alpha s_\alpha$ is a product of groups of the form $U_{s_\alpha(-\beta)}$, $\beta > 0$, $\beta \neq \alpha$. Since α is simple we have $s_\alpha(-\beta) < 0$. Q.E.D.

Proof of 6.1. By (6.2 (ii)) the U-orbit $p(s_\alpha) \in G/B$ is one-dimensional. The closure of $Up(s_\alpha)$ in G/B is $\{p(\epsilon)\} \cup Up(s_\alpha)$. The inverse image of this set by p: $G \to G/B$ is $Us_\alpha B \cup B$, consequently, this set is closed and connected.

Put $P_\alpha = Us_\alpha B \cup B$. P_α is obviously stable under $g \mapsto g^{-1}$ and contains e. To show that P_α is stable under products, it suffices to see that the product of two elements from $Us_\alpha B$ is contained in P_α. By (6.2 (ii)) we get $Us_\alpha B U s_\alpha B = U s_\alpha U s_\alpha B = U s_\alpha U_{-\alpha} s_\alpha B = U U_\alpha B$; conclusion by (6.2, (i)). Q.E.D.

DEFINITION 6.3. Let α be a simple root of G relative to B. Then, P_α denotes the closed connected subgroup of G defined in (6.1).

Remark 6.4. $P_\alpha/B \to P/B$ identifies the closure C_α of the U-orbit of $p(s_\alpha)$ and P_α/B. Reasoning as in (5.1) it is easy to see that $(C_\alpha)_{\alpha \in S}$ gives all the closures of the U-orbits of dimension 1.

The proof of Theorem (5.4) consists in a conjugation of the following two lemmas.

LEMMA 6.5. *Let α be a simple root of G relative to B. C_α the closure in G/B of the U-orbit of $p(s_\alpha)$. Then, $C_\alpha \xrightarrow{\sim} \mathbb{P}^1$. Moreover, if $\chi \in X(T)$, the restriction of $L(\chi)$ to C_α has degree $\langle \alpha^\vee, \chi \rangle$.*

Proof. Consider a morphism as in (4.3)

$$\pi_\alpha : Sl_2 \to G$$

(unique up to inner conjugation by an element of T_2). π_α induces a closed immersion

$$Sl_2/B_2 \to G/B$$

whose image is C_α, as it follows from the preceding results.

We have the canonical immersion $\alpha_2^\vee$: $G_m \to Sl_2$. Let e_2 denote the projection of the origin of Sl_2 onto Sl_2/B_2. G_m has weight $\langle\alpha^\vee, \chi\rangle$ in the fiber above e_2 of the pull-back of $L(\chi)$ to Sl_2/B_2.

On the other hand, the weight of G_m in the fiber above e_2 of the canonical linebundle is 1 according to (1.8). Q.E.D.

LEMMA 6.6. *For the intersection numbers we have*

$$\begin{aligned}(C_\alpha D_\beta) &= 1 \quad \alpha = \beta \\ &= 0 \quad \alpha \neq \beta.\end{aligned}$$

Proof. Consider the fibration $G/B \to G/P_\alpha$. The fiber of this fibration through $p(s_\alpha)$ is C_α. Hence, $(C_\alpha D_\beta)$ can be computed as the intersection number for D_β and any fiber of $G/B \to G/P_\alpha$. The result now follows once we establish: If $\beta = \alpha$, the restriction of $G/B \to G/P_\alpha$ to the U-orbit of $p(w_0 s_\beta)$ is an open immersion. If $\beta \neq \alpha$, the image by $G/B \to G/P_\alpha$ of the U-orbit of $p(w_0 s_\beta)$ has codimension 1.

The image of the U-orbit of $p(w_0 s_\beta)$ by $G/B \to G/P_\alpha$ is the U-orbit of the image of $w_0 s_\beta$ in G/P_α. The U-stabilizer of the last point is $U \cap w_0 s_\beta P_\alpha s_\beta w_0$. By [1, IV, 14.1] this group is directly spanned by the U_γ's it contains. The γ's in question are those for which either (i) or (ii) below

(i) $\gamma < 0$, and $s_\beta(w_0(\gamma)) < 0$,

(ii) $\gamma < 0$, and $s_\beta(w_0(\gamma)) = \alpha$.

Since β is simple, the complete solution to (i) is $\gamma = w_0(\beta)$. As to (ii), let us distinguish between two cases.

$\beta \neq \alpha$. The complete solution to (ii) is $\gamma = w_0(s_\beta(\alpha))$. Hence, the stabilizer in this case is two-dimensional. This proves the second case of the statement above.

$\beta = \alpha$. (ii) means $\gamma < 0$, and $\gamma = w_0(-\alpha)$, which is impossible. Hence, the stabilizer in this case is $U_{w_0(\beta)}$. A simple consideration of the Lie-algebras now concludes the first case of the statement. Q.E.D.

Proof of Theorem 5.4. Write $c_1(L(\chi)) = \sum n_\beta D_\beta$. Restricting this to c_α and counting degree, we get

$$\text{left-hand side } \langle \alpha^\vee, \chi \rangle, \qquad \text{by (6.5)},$$

$$\text{right-hand side } n_\alpha \qquad \text{by (6.6)}.$$

Q.E.D.

Remark 6.7. The considerations made in Lemma 6.2 have the following important generalization.

(i′) If B and B' are Borel subgroups containing T, and α is a root, then

$$U_{-\alpha} s_\alpha \subseteq B'B \cup B' s_\alpha B,$$

which follows by applying (4.3) $\pi_\alpha : Sl_2 \to G$ to the identity

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 1 & x^{-1} \end{pmatrix} \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix}.$$

From (i′) and (6.2 (ii)), one gets if α is a simple root with respect to B,

$$B s_\alpha B \subseteq B'B \cup B' s_\alpha B,$$

or with $B' = w^{-1} B w$

$$w B s_\alpha \subseteq BwB \cup B w s_\alpha B, \qquad w \in W,$$

which is the crucial axiom for a Tits system [4, IV, Sect. 2.1]. The theory of Tits systems gives a complete description of the parabolic subgroups containing B, [4, IV, Sect. 2.5].

7. Cartan Matrix

Let G be a reductive linear algebraic group. The matrix

$$\langle \alpha^\vee, \beta \rangle_{(\alpha,\beta) \in S \times S}$$

is called the Cartan matrix of G and is denoted Cartan (G). It depends up to a simultaneous permutation of the rows and columns only on G.

By the cokernel of an $m \times m$ matrix with $\mathbb{Z}$-coefficients, we understand the cokernel of

$$\mathbb{Z}^m \to \mathbb{Z}^m, \qquad V \mapsto CV.$$

For the simple linear algebraic group, the cokernel of the Cartan matrix takes the following value depending on the type of the root system.

A_n	B_n	C_n	D_{2m}	D_{2m+1}
$\mathbb{Z}/(n+1)$	$\mathbb{Z}/(2)$	$\mathbb{Z}/(2)$	$\mathbb{Z}/(2) \times \mathbb{Z}(2)$	$\mathbb{Z}/(4)$
E_6	E_7	E_8	F_4	G_2
$\mathbb{Z}/(3)$	$\mathbb{Z}/(2)$	0	0	0

Proof. Work through the list of Cartan matrices given at the end of [4]. See also [24].

THEOREM 7.1. *Let G be a semisimple simply connected linear algebraic group. Then, the center ZG of G is diagonalizable and*

$$X(ZG) = \text{coker}\,(\text{Cartan}(G)).$$

Proof. Let $X_r(T)$ denote the subgroup of $X(T)$ generated by the roots of G. The characteristic map of G

$$X(T) \to \text{Pic}(G/B)$$

is an isomorphism by (3.2) and (1.9). The restriction of the characteristic map to $X_r(T)$ is given by the Cartan matrix of G according to (5.3), hence, it suffices to prove.

LEMMA 7.2. *Let G be reductive and let $X_r(T)$ denote the subgroup of $X(T)$ generated by the roots of G. Then, the sequence*

$$0 \to X_r(T) \to X(T) \to X(ZG) \to 0$$

is exact.

Proof. T is its own centralizer, hence, $ZG \subseteq T$. Let us first prove ZG is the intersection of $\ker(\alpha\colon T \to G_m)$ as α runs through the roots of G. Now, U^-B is dense in G, when the center of G is the same as the centralizer of U^-B, and therefore, equals the intersection of the centralizer of the U_α's as α runs through all roots. U_α is T-isomorphic to G_a when we let T act on G_a through α. This proves the assertation.

This means we have an exact sequence

$$0 \to ZG \to T \xrightarrow{(\alpha)} \prod_{\alpha \in S} G_m \to 0.$$

Passing from this exact sequence to characters proves the lemma. Q.E.D.

Remark 7.3. Let G be a reductive group. Then, we can characterize ZG as the largest diagonalizable normal subgroup of G, or alternatively, as the intersection of all maximal tori of G.

Proof. This follows immediately from the conjugacy theorem for tori, rigidity of diagonalizable group, and the fact that a maximal torus in a reductive group is self-centralizing. Q.E.D.

PROPOSITION 7.4. *Let G be a semisimple linear algebraic group. Then, there is an exact sequence*

$$0 \to X(ZG) \to \text{Coker Cartan }(G) \to X(\pi_1 G) \to 0.$$

Proof. Let $\tilde{G} \to G$ denote the universal covering space of G. By (7.3)

$$0 \to \pi_1 G \to Z\tilde{G} \to ZG \to 0$$

is exact, and $\tilde{G}$ and G have the same Cartan matrix. Conclusion by (7.1). Q.E.D.

8. INDUCED REPRESENTATIONS

All linear representations considered are of finite rank.

PROPOSITION 8.1. *Let F denote a linear representation of B. Then, there exists a linear representation E of G satisfying Frobenius reciprocity*

$$\mathrm{Hom}_G(V, E) = \mathrm{Hom}_B(V, F)$$

for all linear representations V of G.

Proof. Let B act on $G \times F$ via the formula

$$(g, f)b = (gb, b^{-1}f), \qquad g \in G, f \in F, b \in B.$$

$G \times {}^B F$ comes equipped with a projection onto G/B, and a left action of G making $G \times {}^B F$ into a G-homogeneous bundle on G/B. Let $\mathscr{F}$ denote the sheaf of sections of $G \times {}^B F$ over G/B. $E = \Gamma(G/B, \mathscr{F})$ is a finite-dimensional vector space since G/B is a projective variety. Moreover, E is in the canonical way a linear representation of G. We are going to prove that for a G-representation V

$$\mathrm{Hom}_G(V, E) = \mathrm{Hom}_B(V, F).$$

Let V_c denote the constant G-homogeneous bundle with fiber V. We have

$$\mathrm{Hom}_G(V, E) = \mathrm{Hom}_G(V, \Gamma(G/B, \mathscr{F})) = \mathrm{Hom}_G(V_c, G \times {}^B F).$$

For a G-homogenized bundle K on G/B, let $K(e)$ denote the fiber of K at the marked point of G/B. $K(e)$ will be viewed as a linear representation of B. Recall that $K \mapsto K(e)$, induces an equivalence between the category of G-homogenized vector bundles on G/B and linear representations of B. Thus,

$$\mathrm{Hom}_G(V_c, G \times {}^B F) = \mathrm{Hom}_B(V, F).$$

Q.E.D.

DEFINITION 8.2. Let χ be a character of B, $G_a(\chi)$ the one-dimensional B-representation of weight χ. Then, $E(\chi)$ denotes the G-representation induced by $G_a(\chi)$, i.e., we have for all G-representations V

$$\mathrm{Hom}_G(V, E(\chi)) = \mathrm{Hom}_B(V, G_a(\chi)).$$

THEOREM 8.3. *Let χ be a character of B. Then, $E(\chi) \neq 0$ if and only if $\langle \alpha^\vee, \chi \rangle \geqslant 0$ for all positive roots α. If $E(\chi) \neq 0$, then:*

(i) *$E(\chi)$ contains precisely one B-stable line, the weight in question is $w_0(\chi)$.*

(ii) *The space of T semiinvariants in $E(\chi)$ of weight $w_0(\chi)$ is one-dimensional.*

(iii) *$\chi' \geqslant w_0(\chi)$ for all weights χ' of T in $E(\chi)$.*

Proof. By the proof of (8.1) we have

$$E(\chi) = H^0(G/B, L(\chi)).$$

The first part of the theorem now follows from (5.2) and (5.3). The second part from (5.4). Q.E.D.

We are now going to study the representations $E(\chi)$ by means of Borel's fixedpoint theorem. Let us first remark that if we are in characteristic zero, then, it follows from Weyl's complete reducibility theorem and (8.3(i)) that $E(\chi)$ is a simple G-representation.

PROPOSITION 8.4. *Let χ, χ' be characters of B. Then,*

(i) *If $E(\chi) \neq 0$, then $E(\chi)$ contains precisely one simple subrepresentation (which we denote $S(\chi)$).*

(ii) *If $S(\chi)$ is isomorphic to $S(\chi')$, then $\chi = \chi'$.*

(iii) *Any simple G-representation is isomorphic to a representation of the form $S(\chi)$.*

Proof. (i) Follows from Borel's fixedpoint theorem and (8.3(i)). By Borel's fixedpoint theorem, $S(\chi)$ contains a B-stable line. The weight of B in such a line is $w_0(\chi)$ by (8.3(i)). Thus, $S(\chi) \simeq S(\chi')$ implies $\chi = \chi'$.

Now, let S be a simple G-representation, then, by Borel's fixedpoint theorem, $S^\vee$ contains a B-stable line. Let B have weight χ in this line. This defines a nontrivial B-linear map $G_a(\chi) \to S^\vee$, and hence, a nontrivial B-linear map $S \to G_a(-\chi)$, and therefore, by Frobenius reciprocity, a nontrivial G-linear map $S \to E(-\chi)$, hence, $S \approx S(-\chi)$. Q.E.D.

COROLLARY 8.5. (cf. [20, Section 12]). *Let S be a simple G-representation. Then, the set of weights of T in S' contains a largest element (the highest weight of T in S).*

Two simple G-representations are isomorphic if and only if the highest weight of T in the two representations are the same.

There exists a simple G-representation with highest weight χ ($\chi \in X(T)$) if and only if

$$\langle \alpha^\vee, \chi \rangle \geqslant 0 \textit{ for all positive roots } \alpha.$$

If S is a simple representation of G with highest weight χ, then the space of T-semiinvariants in S of weight χ is one-dimensional.

Proof. By the proof of (8.4), we have $S(\chi) \simeq S(-w_0(\chi))^\vee$. Q.E.D.

COROLLARY 8.6. *Let $R(G)$, respectively, $R(T)$ denote the representation ring of G, respectively T, then, the restriction map $T \to G$ induces an isomorphism*

$$R(G) \simeq R(T)^W.$$

Proof. Follows from (8.5) and a general lemma on root systems [22, Lemma 6]. Q.E.D.

9. WEYL'S CHARACTER FORMULA

For notational convenience, see (9.2) below, we shall in this paragraph assume that our reductive group G has vanishing Picard group. In case G is semisimple, this means that G is simply connected.

Let X denote the character group of T and put $R = \mathbb{Z}[X]$ and let $\chi \mapsto e^{\chi}$ denote the canonical embedding $X \to R$. A linear representation E of T can be decomposed $E = \bigoplus_{\chi \in X} E_{\chi}$, where T has weight χ in E_{χ}. Define tr $E \in R$ by

$$\operatorname{tr} E = \sum_{\chi \in X} (\operatorname{rank} E_{\chi}) e^{\chi}.$$

We are now going to "polarize" R in two ways. $W = W_G(T)$ acts on X from the left. For $w \in W$ we let $\epsilon(w)$ denote the determinant for the action of w on X, and define the antisymmetry operator

$$J\colon R \to R,\ J(e^{\chi}) = \sum_{w \in W} \epsilon(w) e^{w(\chi)}. \tag{9.1}$$

The half sum of the positive roots

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \tag{9.2}$$

belongs to X since Pic $G = 0$, as it follows from (1.9), (5.3) and [4, Chap. VI, No. 3.3].

Let us recall the formula,

$$\prod_{\alpha < 0} (1 - e^{\alpha}) = e^{-\rho} J(e^{\rho}). \tag{9.3}$$

THEOREM 9.4. *Let χ be a character of T. Then,*

$$\sum_i (-1)^i \operatorname{tr} H^i(G/B, L(\chi)) = J(e^{\chi+\rho})/J(e^\rho).$$

Proof. Let us first quote the general Lefschetz formula for action of a torus T on a smooth projective variety V. Suppose T acts with isolated fixed points on 0 and let E be a homogenized vector bundle on V. Then, in the fraction field of R, one has

$$\sum_i (-1)^i \operatorname{tr} H^i(V, E) = \sum_x \frac{\operatorname{tr} E_x}{\sum_i (-1)^i \operatorname{tr} \wedge^i T_x(V)^\vee}.$$

The last sum being over all fixed point χ of T on V, [19].

In the present case, we let e denote the image of the origin of G by $G \to G/B$. We have an exact sequence

$$0 \to LB \to LG \to T_e(G/B) \to 0,$$

which shows that

$$\operatorname{tr} T_e(G/B)^\vee = \sum_{\alpha<0} e^\alpha,$$

and hence, using (9.3),

$$\sum_i (-1)^i \operatorname{tr} \Lambda^i T_e(G/B)^\vee = \prod_{\alpha<0} (1 - e^\alpha) = e^{-\rho} J(e^\rho).$$

The T-fixed point e, thus contributes with $e^{\chi+\rho}/J(e^\rho)$ to the above sum. Recalling that the fixed point of T in G/B is parametrized by W, we get that the fixed point corresponding to $w \in W$ contributes with

$$w(e^{\chi+\rho})/w(J(e^\rho)) = \epsilon(w)\, w(e^{\chi+\rho})/J(e^\rho).$$

Q.E.D.

COROLLARY 9.5 (Weyl). *Suppose the ground field has characteristic zero. Let S be a simple linear representation of G, and χ the highest weight of T in S. Then*

$$\operatorname{tr} S = J(e^{\chi+\rho})/J(e^\rho).$$

Proof. It follows from Section 8 that S is isomorphic to $H^0(G/B, L(\chi))$ and from [7] that $H^i(G/B, L(\chi)) = 0$ for $i \geqslant 1$. Conclusion by 9.4.
Q.E.D.

COROLLARY 9.6. *Let the ground field have any characteristic, and suppose $G \not\simeq Sl_n$. Then, for a character χ of T for which the induced representation $E(\chi) \neq 0$,*

$$\operatorname{tr} E(\chi) = J(e^{\chi+\rho})/J(e^{\rho}).$$

Proof. We have $E(\chi) = H^0(G/B, L(\chi))$ by the proof of (8.1). On the other hand, we have

$$H^i(G/B, L(\chi)) = 0, \quad \text{for } i \geqslant 1,$$

by a result of Kempf [16]. Q.E.D.

Note added in proof. For an extensive study of the geometry of G/B, see: M. Demazure, Désingularization des varietés de Schubert Généralisées, *Ann. Scien. École Norm. Sup.* **7** (1974), 53–88.

The vanishing theorem aluded to in 9.6 has now been proved incomplete generality by G. Kempf. For partial results, see: L. Bai, C. Musili, and C. S. Seshadri, Cohomology of linebundles on G/B, *Ann. Scien. École Norm. Sup.* **7** (1974), 89–138.

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