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Lie Superalgebras

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DEDICATED TO THE MEMORY OF MY FATHER, G. M. KAC
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Contents.

INTRODUCTION

"Graded Lie algebras have recently become a topic of interest in physics in the context of 'supersymmetries' relating particles of different statistics" (see the survey [22], from which this quotation is taken and which contains an extensive bibliography).

In this paper, we attempt to construct a theory of Lie superalgebras or, as the physicists call them, $\mathbf{Z}_2$-graded Lie algebras. We prefer the term "superalgebra," which is also inspired by physicists, because speaking generally, Lie superalgebras are not Lie algebras.

A *superalgebra* is a $\mathbf{Z}_2$-graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ (that is, if $a \in A_\alpha$, $b \in A_\beta$, $\alpha, \beta \in \mathbf{Z}_2 = \{\bar{0}, \bar{1}\}$, then $ab \in A_{\alpha+\beta}$). A *Lie superalgebra* is a superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ with an operation [,] satisfying the following axioms:

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$$[a, b] = -(-1)^{\alpha\beta}[b, a] \qquad \text{for} \quad a \in G_\alpha, \quad b \in G_\beta,$$
$$[a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]] \qquad \text{for} \quad a \in G_\alpha, \quad b \in G_\beta.$$

We mention that the Whitehead operation in homotopy groups satisfies these axioms; Lie superalgebras also occur in several cohomology theories, for example, in deformation theory (see [22, 24]).

Lie superalgebras appear in [4] as Lie algebras of certain generalized groups, nowadays called Lie supergroups, whose function algebras are algebras with commuting and anticommuting variables. Recently, a satisfactory theory, similar to Lie's theory, has been developed on the connection between Lie supergroups and Lie superalgebras [5].

We now give a brief account of the main features of the theory of finite-dimensional Lie superalgebras. Let G be a finite-dimensional Lie superalgebra. Then G contains a unique maximal solvable ideal R (the solvable radical). The Lie superalgebra G/R is semisimple (that is, has no solvable ideals). Therefore, the theory of finite-dimensional Lie superalgebras is reduced in a certain sense to the theories of semisimple and of solvable Lie superalgebras. (But note that Levi's theorem on G being a semidirect sum of R and G/R is not true, in general, for Lie superalgebras.)

The main fact in the theory of solvable Lie algebras is Lie's theorem, which asserts that every finite-dimensional irreducible representation of a solvable Lie algebra over $\mathbf{C}$ is one-dimensional. For Lie superalgebras this is not true, in general. In the paper we obtain a classification of finite-dimensional irreducible representations of solvable Lie superalgebras (Section 5.2.2, Theorem 7). In particular, we derive a necessary and sufficient condition for any finite-dimensional irreducible representation to be one-dimensional (Section 5.2.2, Proposition 5.2.4).

Next, it is well known that a semisimple Lie algebra is a direct sum of simple ones. This is by no means true for Lie superalgebras. However, there is a construction that allows us to describe finite-dimensional semisimple Lie superalgebras in terms of simple ones (Section 5.1.3, Theorem 6). It is similar to the construction in [21].

So we come to the fundamental problem of classifying the finite-dimensional simple Lie superalgebras. A solution of this problem in the case of an algebraically closed field of characteristic 0 is the main aim of the paper and occupies the major part of it (Chapters 2–4). The principal difficulty lies in the fact that the Killing form (see the definition in Section 2.3.1) may be degenerate, which cannot happen in the case of simple Lie algebras. Therefore, the classical technique Killing–Cartan is not applicable here. The classification is divided into two main parts (presented in Chapters 2 and 4, respectively).

In the first part we give a classification of the classical Lie superalgebras. A Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is called *classical* if it is simple and the representation of the Lie algebra $G_{\bar{0}}$ on $G_{\bar{1}}$ is completely reducible. This classification is divided into two parts, corresponding to the cases of a nondegenerate and a zero Killing form. In Section 2.3 we give a classification of all finite dimensional Lie superalgebras with a nondegenerate Killing form (Theorem 1). This is the first key point of the classification. Here the usual technique is applicable. In Section 2.4 we consider the second key point: the case of a zero Killing form (Proposition 2.4.1). The fact that the Killing form is zero is used to obtain severe restrictions on the index of the representations of $G_{\bar{0}}$ on $G_{\bar{1}}$ (for the definition of the index see Section 1.4.3). Each of the parts corresponding to Sections 2.3 and 2.4 is, in its turn, divided into two parts according to whether or not the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is irreducible (see Section 2.2). The resulting classification of the classical Lie superalgebras that are not Lie algebras is as follows (Theorem 2): (a) four series $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, and $\mathbf{D}(m, n)$, in many respects similar to the Cartan series $\mathbf{A}_n$, $\mathbf{B}_n$, $\mathbf{C}_n$, and $\mathbf{D}_n$; (b) two exceptional Lie superalgebras: a 40-dimensional $\mathbf{F}(4)$ and a 31-dimensional $\mathbf{G}(3)$, and a family of 17 dimensional exceptional Lie superalgebras $\mathbf{D}(2, 1; \alpha)$, which are deformations of $\mathbf{D}(2, 1)$; (c) two "strange" series $\mathbf{P}(n)$ and $\mathbf{Q}(n)$. The construction of all these classical Lie superalgebras is carried out in Section 2.1.

In the second part we give a classification of the nonclassical simple Lie superalgebras. For this purpose we construct a filtration $G = L_{-1} \supset L_0 \supset L_1 \supset \cdots$, where L_0 is a maximal subalgebra containing $G_{\bar{0}}$, and $L_i = \{a \in L_{i-1} \mid [a, L] \subset L_{i-1}\}$ for $i > 0$. Then we classify $\mathbf{Z}$-graded Lie superalgebras with the properties that the associated graded Lie superalgebra $\operatorname{Gr} G = \bigoplus_{i \geq -1} \operatorname{Gr}_i G$ necessarily has (Section 4.1.1, Theorem 4). This is the third key point. In the proof we make essential use of the method developed in our paper [11] for the classification of infinite-dimensional Lie algebras. After this it only remains to reconstruct the Lie superalgebra G with filtration from the $\mathbf{Z}$-graded Lie superalgebra $\operatorname{Gr} G$.

The final classification of simple finite-dimensional Lie superalgebras is as follows (Section 4.2.1, Theorem 5): (a) the classical Lie superalgebras (listed above); (b) the Lie superalgebras of Cartan type $\mathbf{W}(n)$, $\mathbf{S}(n)$, $\mathbf{H}(n)$, $\tilde{\mathbf{S}}(n)$, where the first three series are analogous to the corresponding series of simple infinite-dimensional Lie algebras of Cartan type and $\tilde{\mathbf{S}}(n)$ is a deformation of $\mathbf{S}(n)$. The construction of the Lie superalgebras of Cartan type is carried out in Sections 3.1 and 3.3.

The finite-dimensional irreducible representations of the simple Lie algebras are described by the theorem on the highest weight. A similar result holds for simple Lie superalgebras (Section 5.2.3, Theorem 8). Full reducibility of finite-dimensional representations is lacking, in general.

It is not hard to reduce the classification of simple Lie superalgebras over nonclosed fields for the classical Lie superalgebras to the same problem for simple Lie algebras and for Lie superalgebras of the Cartan type a complete list can be made. This is done in Section 5.3, where we also list all finite-dimensional simple real Lie superalgebras (Theorem 9).

Finally, in Section 5.4 we attempt to extend Cartan's results on the classification of complete infinite-dimensional primitive Lie algebras to Lie superalgebras. In this direction we have only obtained a partial result (Theorem 10). This also makes clear the reason for the appearance of finite-dimensional Lie superalgebras of Cartan type: Lie superalgebras of Cartan type are Lie superalgebras of vector fields in commuting and anticommuting variables, and also their subalgebras defined by the action on the volume, Hamiltonian, and contact forms. If there are no commuting variables, then the superalgebra is finite-dimensional, and so there is no finite-dimensional analog for the contact Lie algebra.

Here is a brief account of the contents of the paper.

Chapter 1 is introductory. In it we give the basic definitions (Section 1.1), establish the simplest properties of gradings and filtrations (Sections 1.2 and 1.3), and quote the necessary information on finite-dimensional representations of semisimple Lie algebras (Section 1.4).

Chapter 2 is devoted to a description (Section 2.1) and classification (Sections 2.2–2.4) of Lie superalgebras with a nondegenerate Killing form (Theorem 1) and of the classical Lie superalgebras (Theorem 2). In Section 2.5 we describe the root systems of the classical Lie superalgebras and find all up to equivalence systems of simple roots. We classify the simple finite-dimensional contragredient Lie superalgebras (Theorem 3); their properties are very close to those of simple Lie algebras.

In Chapter 3 we introduce and study two algebras of differential forms (Section 3.2) with anticommuting and commuting differentials; it is curious that the second algebra has all the properties that one would naturally expect of an algebra of differential forms. In Sections 3.1 and 3.3 we construct the finite-dimensional Lie superalgebras of Cartan type and study their properties.

In Chapter 4 we classify $\mathbf{Z}$-graded Lie superalgebras that arise in the construction of filtrations in simple Lie superalgebras for which the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is reducible (Section 4.1, Theorem 4), and then, on the basis of this classification, we complete the classification of simple Lie superalgebras (Section 4.2, Theorem 5).

Theorems 1, 2, 4, and 5, and also partially Theorems 6 and 7, were announced by the author in the note [16] (Theorem 4 even earlier in [13]).

In Chapter 5 we discuss the following problems. In Section 5.1 we give a description of the finite-dimensional semisimple Lie superalgebras in terms of the simple ones (Theorem 6) and we find the Lie superalgebras of derivations of all simple Lie superalgebras. As in [21], Theorem 6 is a consequence of a general result on differentially simple superalgebras (Proposition 5.1.1). Section 5.2 is concerned with the theory of finite-dimensional irreducible representations of solvable and simple Lie superalgebras (Theorem 7 and 8). In Section 5.3 we treat the classification of simple finite-dimensional Lie superalgebras over nonclosed fields (Propositions 5.3.1–5.3.3). We also give a classification of the simple real Lie superalgebras (Theorem 9). In Section 5.4 we introduce infinite-dimensional Lie superalgebras of Cartan type and formulate the theorem on $\mathbf{Z}$-graded Lie superalgebras that arise in the classification of infinite-dimensional complete primitive Lie superalgebras (Theorem 10). Finally, in Section 5.5 we discuss some unsolved problems.

All spaces and algebras are regarded over a ground field k, which is assumed to be algebraically closed and of characteristic 0 unless the contrary is stated. The symbol $\langle M \rangle$ denotes the linear span over k of a subset M of a linear space, the symbol $\oplus$ the direct sum of k-spaces, and $\otimes$ the tensor product of k-spaces.

Here, I would also like to express my deep indebtedness to F. A. Berezin, E. B. Vinberg, and D. A. Leites for numerous conversations and constructive help. I also thank Professor I. Kaplansky for his interest in my work; having become acquainted with his preprint on root systems of simple Lie superalgebras with a nondegenerate invariant form I could remove some errors that had slipped into the original version of the article.

Remark. The history of this article began in 1969 when, impressed by Stavraky's example of a simple Lie superalgebra $\mathbf{A}(1, 0)$ [19], the author was led to employ the technique of [11] to prove the present Theorem 4. Two years later, having read [4], I decided to publish this result [13]. The classification of classical Lie superalgebras (Theorem 2) was obtained in 1974 under the stimulation of the physicists' interest in the subject. At the beginning of 1975 the key to the complete solution of the classification problem of simple Lie superalgebras was found (filtration!) and Theorem 5 was proved. The results were announced in [16]. Then the results of the Chapter 5 were obtained and by September 1975 the work was completed. In October 1975 the manuscript was submitted to the Soviet journal *Uspehi Matematičeskih Nauk*, but later was withdrawn and resubmitted to the present journal. In the beginning of 1976 the paper was translated into English. I am grateful to Professor Sternberg for his genuine interest in my work and for making the translation of it possible. I am obliged to Professor Hirsch who translated the text.

In the English version some remarks on further results in the field have been added.

1. Basic Definitions and Preliminary Remarks

1.1. *Superalgebras and Lie Superalgebras—Supertrace*

1.1.1. *Superalgebras.* We recall that if A is an algebra and M an Abelian group, then an *M-grading* of A is a decomposition of A into a direct sum of subspaces $A = \bigoplus_{\alpha \in M} A_\alpha$ for which $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$. An algebra A equipped with an M-grading is called M-graded. If $a \in A_\alpha$, then we say that a is *homogeneous of degree* α and we write $\deg a = \alpha$. A subspace B of an M-graded algebra A is called *M-graded* if $B = \bigoplus_{\alpha \in M} (B \cap A_\alpha)$. A *subalgebra* (*or ideal*) of an M-graded algebra is an M-graded subalgebra (or ideal). A *homomorphism* $\Phi: A \to A'$ of M-graded algebras preserves the grading in the sense that $\Phi(A_\alpha) \subseteq A'_{\varphi(\alpha)}$, where φ is an automorphism of M.

Now let $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$ be the residue class ring mod 2, with the elements $\bar{0}$ and $\bar{1}$. A *superalgebra* is a $\mathbf{Z}_2$-graded algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$. The elements of $A_{\bar{0}}$ are called *even*, those of $A_{\bar{1}}$ *odd*. Throughout what follows, if $\deg a$ occurs in an expression, then it is assumed that a is homogeneous, and that the expression extends to the other elements by linearity.

The direct and semidirect sum of superalgebras are defined in the usual way. With the definition of the tensor product things are different. Let A and B be superalgebras. Their *tensor product* $A \otimes B$ is the superalgebra whose space is the tensor product of the spaces of A and B, with the induced $\mathbf{Z}_2$-grading and the operation defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{(\deg a_2)(\deg b_1)} a_1 a_2 \otimes b_1 b_1, \qquad a_i \in A, \quad b_i \in B.$$

There is a natural way of defining *a bracket* $[\ ,\ ]$ in a superalgebra A, i.e., by the equality,

$$[a, b] = ab - (-1)^{(\deg a)(\deg b)} ba. \tag{1.1.1}$$

A superalgebra is called *commutative* if $[a, b] = 0$ for all $a, b \in A$. Quite generally, permutability in a superalgebra is understood in the sense of the bracket (1.1.1). Associativity of superalgebras is defined as for algebras.

For an associative superalgebra A we have the following important identity:

$$[a, bc] = [a, b]c + (-1)^{(\deg a)(\deg b)} b[a, c]. \tag{1.1.2}$$

EXAMPLE 1. Let M be an Abelian group and $V = \bigoplus_{\alpha \in M} V_\alpha$ an M-graded space. Then the associative algebra $\operatorname{End} V$ is equipped with the induced M-grading $\operatorname{End} V = \bigoplus_{\alpha \in M} \operatorname{End}_\alpha V$, where

$$\operatorname{End}_\alpha V = \{a \in \operatorname{End} V \mid a(V_s) \subseteq V_{s+\alpha}\}.$$

In particular, for $M = \mathbf{Z}_2$ we obtain the associative superalgebra $\operatorname{End} V = \operatorname{End}_{\bar{0}} V \oplus \operatorname{End}_{\bar{1}} V$.

EXAMPLE 2. Let $\Lambda(n)$ be the Grassmann algebra in n variables $\xi_1, \ldots, \xi_n$. Then $\Lambda(n)$ becomes $\mathbf{Z}_2$-graded if we set $\deg \xi_i = \bar{1}$, $i = 1, \ldots, n$. The result is called a *Grassmann superalgebra*. It is commutative and associative. Evidently $\Lambda(m) \otimes \Lambda(n) = \Lambda(m + n)$.

A generalization of this example is the commutative superalgebra $\Lambda(m, n) = k[x_1, \ldots, x_m] \otimes \Lambda(n)$, where the polynomial algebra $k[x_1, \ldots, x_m]$ is regarded as a superalgebra with trivial $\mathbf{Z}_2$-grading.

1.1.2. *Definition of a Lie superalgebra.* A *Lie superalgebra* is a superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ with an operation [,] satisfying the following axiom:

$$[a, b] = -(-1)^{(\deg a)(\deg b)}[b, a] \qquad \text{(anticommutativity)},$$
$$[a, [b, c]] = [[a, b], c] + (-1)^{(\deg a)(\deg b)}[b, [a, c]] \qquad \text{(Jacobi identity)}.$$

Observe that $G_{\bar{0}}$ is an ordinary Lie algebra, that multiplication on the left by elements of $G_{\bar{0}}$ determines a structure of a $G_{\bar{0}}$-module on $G_{\bar{1}}$, and that multiplication of elements of $G_{\bar{1}}$ determines a homomorphism of $G_{\bar{0}}$-modules $\varphi: S^2G_{\bar{1}} \to G_{\bar{0}}$. Thus, every Lie superalgebra can be specified by three objects: the Lie algebra $G_{\bar{0}}$, the $G_{\bar{0}}$-module $G_{\bar{1}}$, and the homomorphism of $G_{\bar{0}}$-modules $\varphi: S^2G_{\bar{1}} \to G_{\bar{0}}$, with the sole condition

$$\varphi(a, b)c + \varphi(b, c)a + \varphi(c, a)b = 0 \qquad \text{for} \quad a, b, c \in G_{\bar{1}}. \tag{1.1.3}$$

EXAMPLE 1. If A is an associative superalgebra, then the bracket (1.1.1) turns A into a Lie superalgebra. (The Jacobi identity follows from (1.1.2).) We denote the resulting Lie superalgebra by A_L.

EXAMPLE 2. Let G be a Lie superalgebra and $\Lambda(n)$ a Grassmann superalgebra. Then $G \otimes \Lambda(n)$ is also a Lie superalgebra.

The definitions of a solvable and a nilpotent Lie superalgebra are the same as for Lie algebras. A Lie superalgebra is called simple (semisimple) if it contains no nontrivial (no solvable) ideals.

1.1.3. *The universal enveloping superalgebra.* Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Lie superalgebra. As usual, a pair $(U(G), i)$, where $U(G)$ is an associative superalgebra and $i: G \to U(G)_L$ is a homomorphism of Lie superalgebras, is called the *universal enveloping superalgebra* of G if for any other pair (U', i') there is a unique homomorphism $\theta: U \to U'$ for which $i' = \theta \circ i$.

The universal enveloping superalgebra of $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is constructed as follows [24]. Let $T(G)$ be the tensor superalgebra over the space G with the induced $\mathbf{Z}_2$-grading, and R the ideal of $T(G)$ generated by the elements of the form:

$$[a, b] - a \otimes b + (-1)^{(\deg a)(\deg b)} b \otimes a.$$

We set $U(G) = T(G)/R$. The natural map $G \to U(G)$ evidently induces a homomorphism $i: G \to U(G)_L$, and the pair $(U(G), i)$ is the required enveloping superalgebra.

In [24] the following theorem is verified.

THE POINCARÉ–BIRKHOFF–WITT THEOREM. *Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Lie superalgebra, $a_1, \ldots, a_m$ be a basis of $G_{\bar{0}}$, and $b_1, \ldots, b_n$ be a basis of $G_{\bar{1}}$. Then the elements of the form*

$$a_1^{k_1} \cdots a_m^{k_m} b_{i_1} \cdots b_{i_s}, \qquad \text{where } k_i \geqslant 0 \text{ and } 1 \leqslant i_1 < \cdots < i_s \leqslant n,$$

form a basis of $U(G)$.

Finally, we define the diagonal homomorphism. As it is easy to see, the map $a \mapsto i(a) \otimes 1 + (-1)^{\deg a} 1 \otimes i(a)$, $a \in G$, is a homomorphism of Lie superalgebras $G \to (U(G) \otimes U(G))_L$ and therefore determines a homomorphism of associative superalgebras:

$$\Delta: U(G) \to U(G) \otimes U(G),$$

which is called the *diagonal homomorphism.*

1.1.4. *Derivations and automorphisms of a superalgebra.* A *derivation of degree s*, $s \in \mathbf{Z}_2$, of a superalgebra A is an endomorphism $D \in \mathrm{End}_s A$ with the property

$$D(ab) = D(a)b + (-1)^{s \deg a} aD(b).$$

We denote by $\mathrm{der}_s A \subset \mathrm{End}_s A$ the space of all derivations of degree s, and we set $\mathrm{der}\, A = \mathrm{der}_{\bar{0}} A \oplus \mathrm{der}_{\bar{1}} A$. The space $\mathrm{der}\, A \subset \mathrm{End}\, A$ is easily seen to be closed under the bracket (1.1.1), in other words, it is a subalgebra of $(\mathrm{End}\, A)_L$; it is called the *superalgebra of derivations of A*. Every element of $\mathrm{der}\, A$ is called a *derivation* of A.

EXAMPLE 1. Let G be a Lie superalgebra. It follows from the Jacobi identity that $\mathrm{ad}\, a$: $b \mapsto [a, b]$ is a derivation of G. These derivations are called inner; they form an ideal inder G of $\mathrm{der}\, G$, because $[D, \mathrm{ad}\, a] = \mathrm{ad}\, Da$ for $D \in \mathrm{der}\, G$.

EXAMPLE 2. Let $\Lambda(n) = \Lambda_{\bar{0}}(n) \oplus \Lambda_{\bar{1}}(n)$ be a Grassmann superalgebra. Let us find $\mathrm{der}\, \Lambda(n)$. For this purpose it is convenient to represent $\Lambda(n)$ in the form $\tilde{\Lambda}(n)/I$, where $\tilde{\Lambda}(n)$ is the free associative superalgebra with the generators $\xi_1, \ldots, \xi_n$ whose $\mathbf{Z}_2$-grading is given by $\deg \xi_i = \bar{1}$, $i = 1, \ldots, n$, and I is the ideal generated by all the elements $\xi_i \xi_j + \xi_j \xi_i$. Note that if P and Q are homogeneous elements of $\tilde{\Lambda}(n)$, then $PQ - (-1)^{(\deg P)(\deg Q)} QP \in I$.

Let D be a derivation of degree s of $\tilde{\Lambda}(n)$. Then

$$\begin{aligned} D(\xi_i \xi_j + \xi_j \xi_i) &= D(\xi_i)\xi_j + (-1)^s \xi_i D(\xi_j) + D(\xi_j)\xi_i + (-1)^s \xi_j D(\xi_i) \\ &= (D(\xi_i)\xi_j + (-1)^s \xi_j D(\xi_i)) + (D(\xi_j)\xi_i + (-1)^s \xi_i D(\xi_j)) \in I, \end{aligned}$$

from which it follows that I is invariant under D. Since, obviously, there is one and only one derivation of $\tilde{\Lambda}(n)$ with prescribed values $D(\xi_i) \in \tilde{\Lambda}(n)$, we see that for any $P_1, ..., P_n \in \Lambda(n)$ there is one and only one derivation $D \in \operatorname{der} \Lambda(n)$ for which $D(\xi_i) = P_i \in \Lambda(n)$.

In particular, the relations $\partial/\partial\xi_i(\xi_j) = \delta_{ij}$ define the derivation $\partial/\partial\xi_i$, $i = 1,..., n$. The derivation $D \in \operatorname{der} \Lambda(n)$ for which $D(\xi_i) = P_i$ can now be written as a linear differential operator:

$$D = \sum_{i=1}^{n} P_i \frac{\partial}{\partial \xi_i}, \qquad P_i \in \Lambda(n).$$

It is equally easy to find the automorphism group of $\Lambda(n)$ [3]. Observe that there is a unique homomorphism $\varphi: \Lambda(n) \to k$; we agree to write $f(0)$ instead of $\varphi(f)$, $f \in \Lambda(n)$. If now Φ is an automorphism of $\Lambda(n)$, then $\deg \Phi(\xi_i) = \bar{1}$ and $\det(\partial/\partial\xi_i(\Phi(\xi_j))(0)) \neq 0$; any map $\xi_i \mapsto \Phi(\xi_i) \in \Lambda(n)$, $i = 1,..., n$, having these two properties extends uniquely to an automorphism of $\Lambda(n)$.

Any automorphism Φ of $\Lambda(n)$ induces an automorphism of der $\Lambda(n)$ according to the formula

$$(\Phi D)f = \Phi(D(\Phi^{-1}f)), \qquad f \in \Lambda(n).$$

Note that if D is an even derivation of a superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$, then $\exp tD$, $t \in k$, is a one-parameter group of automorphisms. In particular, if A is a Lie superalgebra, then $\exp(\operatorname{ad} a)$ for $a \in A_{\bar{0}}$ is an automorphism of A; the group generated by these automorphisms is called the group of inner automorphisms. The preceding remark leads to the following result.

Proposition 1.1.1. *Let $\mathscr{G}$ and $\mathscr{G}_0$ be the connected components of the identity in the automorphism groups of superalgebras A and $A_{\bar{0}}$, and let $\mathscr{H}$ be the subgroup of $\mathscr{G}$ consisting of the automorphisms that act identically on $A_{\bar{0}}$. Then the restriction induces an epimorphism $\mathscr{G} \to \mathscr{G}_0$ with kernel $\mathscr{H}$. In particular, if A is a Lie superalgebra, then every inner automorphism of $A_{\bar{0}}$ extends to an inner automorphism of A.*

1.1.5. *The superalgebra $l(V)$ and the supertrace.* Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbf{Z}_2$-graded space. Then (Section 1.1.1) the algebra End V is endowed with a $\mathbf{Z}_2$-grading and so becomes an associate superalgebra. Now $(\operatorname{End} V)_L$ is a Lie superalgebra (Section 1.1.2) which we denote by $l(V)$ or $l(m, n)$, where $m = \dim V_{\bar{0}}$, $n = \dim V_{\bar{1}}$. In the theory of Lie superalgebras $l(V) = l(V)_{\bar{0}} \oplus l(V)_{\bar{1}}$ plays the same role as the general linear Lie algebra in the theory of Lie algebras. If we regard the same decomposition $V = V_0 \oplus V_1$ as a $\mathbf{Z}$-grading of V, then it corresponds to a $\mathbf{Z}$-grading of $l(V)$, which is compatible with the $\mathbf{Z}_2$-grading: $l(V) = l_{-1} \oplus l(V)_{\bar{0}} \oplus l_1$.

Let $e_1, ..., e_m, e_{m+1}, ..., e_{m+n}$ be a basis of V, formed from bases of $V_{\bar{0}}$ and $V_{\bar{1}}$. It is natural to call such a basis homogeneous. In this basis the matrix of an operator a from $l(V)$ can be written in the form $\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$, where α is an $(m \times m)$-, δ an $(n \times n)$-, β an $(m \times n)$-, and γ an $(n \times m)$-matrix. The matrices of even elements have the form $\left[\begin{smallmatrix}\alpha & 0\\ 0 & \delta\end{smallmatrix}\right]$, and those of odd ones $\left[\begin{smallmatrix}0 & \beta\\ \gamma & 0\end{smallmatrix}\right]$. Here l_1 consists of the matrices of the form $\left[\begin{smallmatrix}0 & \beta\\ 0 & 0\end{smallmatrix}\right]$ and l_{-1} of the form $\left[\begin{smallmatrix}0 & 0\\ \gamma & 0\end{smallmatrix}\right]$. Hence it is clear that the $l_{\bar{0}}$-modules l_1 and l_{-1} are contragredient and the $l_{\bar{0}}$-module l_1 isomorphic to $gl_m \otimes gl_n$.

Now we come to the definition of the *supertrace*. For the matrix $a = \left[\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right] \in l_{(m,n)}$ this is the number

$$\operatorname{str}(a) = \operatorname{tr} \alpha - \operatorname{tr} \delta.$$

Observe that the supertrace of the matrix of an operator $a \in l(V)$ does not depend on the choice of a homogeneous basis. Therefore, we have the right to speak of the supertrace of a, meaning the supertrace of this operator in any homogeneous basis.

To state properties of the supertrace (and for other purposes) it is useful to introduce the following definitions. Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a $\mathbf{Z}_2$-graded space and f be a bilinear form on G. Then f is called *consistent* if $f(a, b) = 0$ for $a \in G_{\bar{0}}$, $b \in G_{\bar{1}}$, and *supersymmetric* if $f(a, b) = (-1)^{(\deg a)(\deg b)} f(b, a)$. If G is a Lie superalgebra, f is called *invariant* if $f([a, b], c) = f(a, [b, c])$.

PROPOSITION 1.1.2. (a) *The bilinear form* $(a, b) = \operatorname{str}(ab)$ *on* $l(V)$ *is consistent, supersymmetric, and invariant.*

(b) $\operatorname{str}([a, b]) = 0$ *for any* $a, b \in l(V)$.

Proof. The consistency follows from the fact that $ab \in l(V)_{\bar{1}}$ for $a \in l(V)_{\bar{0}}$, $b \in l(V)_{\bar{1}}$.

Supersymmetry for $a, b \in l(V)_{\bar{0}}$ follows from the corresponding property of the trace, and for $a \in l(V)_{\bar{0}}$, $b \in l(V)_{\bar{1}}$ from consistency. It remains to consider the case $a, b \in l(V)_{\bar{1}}$. Let $a = \left(\begin{smallmatrix}0 & \alpha\\ \beta & 0\end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix}0 & \gamma\\ \delta & 0\end{smallmatrix}\right)$ be the matrices of a and b in a homogeneous basis. Then $(a, b) = \operatorname{tr} \alpha\delta - \operatorname{tr} \beta\gamma$, $(b, a) = \operatorname{tr} \gamma\beta - \operatorname{tr} \delta\alpha$, from which it follows that $(a, b) = -(b, a)$, as required.

(b) is simply another way of writing down supersymmetry.

We still have to verify invariance. By (1.1.2) we have $[b, ac] = [b, a]c + (-1)^{(\deg a)(\deg b)} a[b, c]$. Therefore, by (b):

$$0 = \operatorname{str}([b, ac]) = ([b, a], c) + (-1)^{(\deg a)(\deg b)}(a, [b, c]),$$

as required.

1.1.6. *Linear representations of Lie superalgebras.* Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbf{Z}_2$-graded linear space. A *linear representation* ρ of a Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ in V is a homomorphism $\rho\colon G \to l(V)$.

For brevity we often say in this case that V is a G-module, and instead of $\rho(g)(v)$ we write $g(v)$, $g \in G$, $v \in V$. Note that, by definition, $G_i(V_j) \subseteq V_{i+j}$, $i, j \in \mathbf{Z}_2$, and $[g_1, g_2](v) = g_1(g_2(v)) - (-1)^{(\deg g_1)(\deg g_2)} g_2(g_1(v))$. Note also that the map ad: $G \to l(G)$ for which $(\operatorname{ad} g)(a) = [g, a]$ is a linear representation of G. It is called the *adjoint representation.*

A submodule of a G-module V is assumed to be $\mathbf{Z}_2$-graded; a G-module V is said to be irreducible if it has no nontrivial submodules. By a homomorphism of G-modules $\Phi\colon V \to V'$ we mean one that preserves the $\mathbf{Z}_2$-grading in the sense that $\Phi(V_i) = V'_{\varphi(i)}$, where φ is a bijection $\mathbf{Z}_2 \to \mathbf{Z}_2$.

SCHUR'S LEMMA. *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$, $\mathscr{M}$ an irreducible family of operators from $l(V)$, and $C(\mathscr{M}) = \{a \in l(V) \mid [a, m] = 0, m \in \mathscr{M}\}$. Then either $C(\mathscr{M}) = \langle 1 \rangle$ or* $\dim V_{\bar{0}} = \dim V_{\bar{1}}$ *and $C(\mathscr{M}) = \langle 1, A \rangle$, where A is a nondegenerate operator in V permuting $V_{\bar{0}}$ and $V_{\bar{1}}$, and $A^2 = 1$.*

EXAMPLE. We consider the Lie superalgebra $N = N_{\bar{0}} \oplus N_{\bar{1}}$, where $N_{\bar{0}} = \langle e \rangle$, $N_{\bar{1}} = \langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle$ and $[a_i, b_i] = e$, $i = 1, \ldots, n$, the remaining brackets being zero. We construct a family of representations ρ_α, $\alpha \in k^*$, of N in $\Lambda(n)$ by setting: $\rho_\alpha(a_i)u = \partial u/\partial \xi_i$, $\rho_\alpha(b_i)u = \alpha \xi_i u$, $\rho_\alpha(e)u = \alpha u$. Clearly, ρ_α is a 2^n-dimensional irreducible representation of N.

We now consider the Lie superalgebra $N' = N \oplus \langle c \rangle$, where $[N, c] = 0$, $[c, c] = e$, and the superalgebra $\Lambda'(n) = \Lambda(n) \otimes k[\epsilon]$, where $\deg \epsilon = \bar{1}$, $\epsilon^2 = \alpha/2$, $\alpha \in k$. We define a representation ρ_α' of N' in $\Lambda'(n)$ by setting $\rho_\alpha'(h)(u \otimes v) = \rho_\alpha(h)u \otimes v$, $\rho_\alpha'(c)(u \otimes v) = (1 \otimes \epsilon)(u \otimes v)$, $u \otimes v \in \Lambda'(n)$. Clearly, ρ_α' is a 2^{n+1}-dimensional irreducible representation of N'.

Both N and N' are nilpotent. They are called Heisenberg superalgebras. Note that ρ_α and ρ_α' fall under the two cases of Schur's lemma.

This example shows that Lie's theorem need not be true for Lie superalgebras. However, Engel's theorem remains valid, and the proof is the same as for Lie algebras [10].

ENGEL'S THEOREM. *Let G be a subalgebra of $l(V)$ and suppose that all the operators of G are nilpotent. Then there is a vector $v \in V$, $v \neq 0$, that is annihilated by all the operators of G.*

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbf{Z}_2$-graded space. By the symmetric (respectively, exterior) algebra over V we mean the $\mathbf{Z}$-graded superalgebra $S(V) = S(V_{\bar{0}}) \otimes \Lambda(V_{\bar{1}}) = \bigoplus S^k(V)$ (respectively, $\Lambda(V) = \Lambda(V_{\bar{0}}) \otimes S(V_{\bar{1}}) = \bigoplus \Lambda^k(V)$).

If V is a module of the Lie superalgebra G, then we have homomorphisms $G \to \operatorname{der} S(V)$ and $G \to \operatorname{der} \Lambda(V)$, so $S(V)$ and $\Lambda(V)$ become G-modules. The submodules $S^k(V)$ (respectively, $\Lambda^k(V)$) are called the *symmetric* (respectively, *exterior*) *powers* of the G-module V.

1.2. **Z**-*Graded Lie Superalgebras*

1.2.1. **Z**-*gradings*. A **Z**-*grading* of a superalgebra G is a decomposition of it into a direct sum of finite-dimensional $\mathbf{Z}_2$-graded subspaces $G = \oplus_{i\in\mathbf{Z}} G_i$ for which $G_iG_j \subseteq G_{i+j}$. A **Z**-grading is said to be *consistent* if $G_{\bar{0}} = \oplus G_{2i}$, $G_{\bar{1}} = \oplus G_{2i+1}$.

By definition, if G is a **Z**-graded Lie superalgebra, then G_0 is a subalgebra and $[G_0, G_i] \subseteq G_i$; therefore, the restriction of the adjoint representation to G_0 induces linear representations of it on the subspaces G_i.

A **Z**-graded Lie superalgebra $G = \oplus_{i\in\mathbf{Z}} G_i$ is called *irreducible* if the representation of G_0 on G_{-1} is irreducible.

A **Z**-graded Lie superalgebra $G = \bar{G}_{i\in\mathbf{Z}} G_i$ is called *transitive* if for $a \in G_i$, $i \geqslant 0$, it follows from $[a, G_{-1}] = 0$ that $a = 0$, and *bitransitive* if in addition for $a \in G_i$, $i \leqslant 0$, it follows from $[a, G_1] = 0$ that $a = 0$.

These properties are closely connected with G being simple, as is shown by the following proposition.

PROPOSITION 1.2.1. *If in a simple* **Z**-*graded Lie superalgebra* $G = \oplus_{i\in\mathbf{Z}} G_i$ *the subspace* $G_{-1} \oplus G_0 \oplus G_1$ *generates* G, *then it is bitransitive.*

The proof is the same as that of [11, Proposition 1].

1.2.2. *Local Lie superalgebras.* Let $\hat{G}$ be a $\mathbf{Z}_2$-graded space, decomposed into a direct sum of $\mathbf{Z}_2$-graded subspaces, $\hat{G} = G_{-1} \oplus G_0 \oplus G_1$. Suppose that whenever $|i+j| \leqslant 1$ a bilinear operation is defined $G_i \times G_j \to G_{i+j}((x, y) \mapsto [x, y])$, satisfying the axiom of anticommutativity and the Jacobi identity for Lie superalgebras, provided that all the commutators in this identity are defined. Then $\hat{G}$ is called a *local Lie superalgebra.*

To a **Z**-graded Lie superalgebra $G = \oplus G_i$ there corresponds a local Lie superalgebra $G_{-1} \oplus G_0 \oplus G_1$, which we call the *local part* of G.

Homomorphisms, transitivity, bitransitivity, etc., for local Lie superalgebras are defined as for **Z**-graded Lie superalgebras.

In this subsection we consider only **Z**-graded Lie superalgebras $G = \oplus G_i$ for which the subspace $G_{-1} \oplus G_0 \oplus G_1$ generates G.

A **Z**-graded Lie superalgebra $G = \oplus G_i$ with local part $\hat{G}$ is said to be *maximal* (respectively, *minimal*) if for any other **Z**-graded superalgebra G' an isomorphism of the local parts $\hat{G}$ and $\hat{G}'$ extends to an epimorphism of G onto G' (respectively, G' onto G).

PROPOSITION 1.2.2. *Let* $\hat{G} = G_{-1} \oplus G_0 \oplus G_1$ *be a local Lie superalgebra. Then there is a maximal and a minimal* **Z**-*graded Lie superalgebra whose local parts are isomorphic to* $\hat{G}$.

PROPOSITION 1.2.3. (a) *A bitransitive* $\mathbf{Z}$*-graded Lie superalgebra is minimal.*

(b) *A minimal* $\mathbf{Z}$*-graded Lie superalgebra with bitransitive local part is bitransitive.*

(c) *Two bitransitive* $\mathbf{Z}$*-graded Lie superalgebras are isomorphic if and only if their local parts are isomorphic.*

These two propositions are proved just as the corresponding assertions for Lie algebras (see [11, Propositions 4 and 5]).

1.2.3. *Invariant bilinear forms.* The following proposition is proved in the same manner as [11, Proposition 7].

PROPOSITION 1.2.4. *Suppose that on the local part of a* $\mathbf{Z}$*-graded Lie superalgebra* $G = \bigoplus G_i$ *a consistent supersymmetric invariant bilinear form* $(\; , \;)$ *is given (see Section* 1.1.5) *for which* $(G_i, G_j) = 0$ *when* $i + j \neq 0$. *If* $G_{-1} \oplus G_0 \oplus G_1$ *generates* G, *then the form can be extended uniquely to a consistent supersymmetric invariant bilinear form with the same property on the whole* G.

The following assertion is proved in the same manner as [7, Corollary 2 to Theorem 4].

PROPOSITION 1.2.5. *Let* $G = \bigoplus_{i=-d}^{t} G_i$ *be a simple finite-dimensional* $\mathbf{Z}$*-graded Lie superalgebra, with* $G_{\pm k} = G_{\pm 1}^{k}$, $k \geqslant 0$, $G_{-d} \neq 0$, $G_t \neq 0$. *On* G *there exists a nondegenerate consistent supersymmetric invariant bilinear form* $(\; , \;)$ *if and only if the representations of* G_0 *on* G_{-d} *and* $(G_t)^*$ *are equivalent. For this form* $(G_i, G_j) = 0$ *when* $i + j \neq t - d$.

Clearly, the kernel of an invariant form is an ideal. Therefore, we have the following result.

PROPOSITION 1.2.6. *If* G *is a simple Lie superalgebra, then an invariant form on it is either nondegenerate or identically zero, and any two invariant forms on* G *are proportional.*

1.2.4. *Conditions for simplicity.* In this subsection we state some conditions for the simplicity of Lie superalgebras. The proofs are standard.

PROPOSITION 1.2.7. *The following conditions are necessary for a Lie superalgebra* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *to be simple:*

(1) *The representation of* $G_{\bar{0}}$ *on* $G_{\bar{1}}$ *is faithful and* $[G_{\bar{1}}, G_{\bar{1}}] = G_{\bar{0}}$.

If, in addition,

(2) *the representation of* $G_{\bar{0}}$ *on* $G_{\bar{1}}$ *is irreducible,*

then G *is simple.*

PROPOSITION 1.2.8. *The following conditions are necessary for a $\mathbb{Z}$-graded Lie superalgebra $G = \bigoplus_{i \geq -1} G_i$ to be simple:*

(1) *G is transitive and irreducible; $[G_{-1}, G_1] = G_0$.*

If, in addition,

(2) *the kernel of the G_0-module G_1 is 0 and $G_i = G_1{}^i$ for $i > 0$,*

then G is simple.

1.2.5. *Some properties of $\mathbb{Z}$-graded Lie superalgebras of the form $\bigoplus_{i \geq -1} G_i$.* The following assertion facilitates the work with $\mathbb{Z}$-graded Lie superalgebras.

PROPOSITION 1.2.9 (cf. [23]). *Let $G = \bigoplus_{i \geq -1} G_i$ be a transitive irreducible Lie superalgebra with a consistent $\mathbb{Z}$-grading, and $G_1 \neq 0$. Then $[G_0, G_0] \subseteq [G_{-1}, G_1]$.*

Proof. Observe, first of all, that $[G_{-1}, [G_{-1}, G_1]] = G_{-1}$, because $[G_{-1}, [G_{-1}, G_1]]$ is a nontrivial G_0-submodule of G_{-1}. Let C be the centralizer in G_0 of $[G_{-1}, G_1]$. Since the Lie algebra G_0 is reductive, it is sufficient to show that the Lie algebra C of linear transformations of the space $[G_{-1}, [G_{-1}, G_1]]$ is Abelian. To do this we have to verify that for $x, y \in G_{-1}$, $a, b \in C$ the expression $d = [[[[t, x], y], a], b]$ is symmetric in a and b. Now $d = [[[[[t, x], [y, a]], x], a], b] = -[[[t, y], [x, a]], b] = [[t, [x, a]], [y, b]] = -[[t, [y, b]], [x, a]]$, which proves the assertion.

PROPOSITION 1.2.10. *Let $G = \bigoplus_{i \geq -1} G_i$ be a $\mathbb{Z}$-graded Lie superalgebra satisfying the conditions of Proposition 1.2.9 and suppose that, in addition, the representation of G_0 on G_1 is irreducible. Let H be a Cartan subalgebra of G_0, F_Λ the highest weight vector of the representation of G_0 on G_{-1}, and E_M the lowest weight vector of the representation of G_0 on G_1.*

(a) *If the representations of G_0 on G_{-1} and G_1 are contragredient, then*

(1) $M = -\Lambda$,

(2) $[F_\Lambda, E_M] = h \neq 0$, *where* $h \in H$,

(3) $[G_1, G_1] = 0$,

(4) *the Lie superalgebra $G_{-1} \oplus [G_{-1}, G_1] \oplus G_1$ is simple.*

(b) *If the representations of G_0 on G_{-1} and G_1 are not contragredient, then*

(1) $[F_\Lambda, E_M] = e_\alpha \neq 0$, *where $\alpha = \Lambda + M$ is a nonzero root of the Lie algebra $[G_0, G_0]$,*

(2) $[G_{-1}, G_1] = [G_0, G_0]$,

(3) $[G_0, G_0]$ *is simple.*

Proof. Since $G_{-1} = \langle[\cdots[F_\Lambda, e_{-\gamma_1}],..., e_{-\gamma_k}]\rangle$ and $G_1 = \langle[\cdots[E_M, e_{\gamma_1}],..., e_{\gamma_k}]\rangle$, where $\gamma_1,..., \gamma_k > 0$, we evidently have

$$[G_{-1}, G_1] = \langle[\cdots [F_\Lambda, E_M], e_{\gamma_1},..., e_{\gamma_k}]\rangle. \tag{1.2.1}$$

Since, by transitivity, $[G_{-1}, G_1] \neq 0$, we obtain from (1.2.1) that $[F_\Lambda, E_M] \neq 0$. But $[t, [F_\Lambda, E_M]] = (\Lambda + M)(t)[F_\Lambda, E_M]$ for any $t \in H$. Since contragredience of the representations of G_0 on G_{-1} and G_1 means that $\Lambda + M = 0$, we have now established (1) and (2) in (a) and (1) in (b).

Let us prove (3) in (a). We consider the graded subalgebra $\bar{G}$ of G generated by the subspace $G_{-1} \oplus G_0 \oplus G_1$. Clearly $\bar{G}$ is bitransitive. There is an obvious automorphism φ of its local part carrying the positive roots of G_0 into the negative ones and interchanging G_{-1} with G_1. Since, according to Proposition 1.2.3, $\bar{G}$ is minimal, φ extends to an automorphism of $\bar{G}$. Therefore, it follows from $[G_{-1}, G_{-1}] = 0$ that $[G_1, G_1] = 0$, as required.

(4) in (a) follows from Proposition 1.2.8.

Let us now prove (2) and (3) in (b). From (1.2.1) we see that $[G_{-1}, G_1] = \langle[\cdots[e_\alpha, e_{\gamma_1}],..., e_{\gamma_k}]\rangle \subseteq \tilde{H}$, where $\tilde{H}$ is the simple component of the semisimple Lie algebra $[G_0, G_0]$ the root of which is α. From Proposition 1.2.9 we see that $[G_{-1}, G_{-1}] = \tilde{H} = [G_0, G_0]$, which proves (2) and (3).

PROPOSITION 1.2.11. *Let $G = G_{-1} \oplus G_0 \oplus G_1$ be a transitive $\mathbf{Z}$-graded Lie superalgebra satisfying the conditions of Proposition* 1.2.9. *Then either the representation of G_0 on G_1 is faithful and irreducible, or* $\dim G_1 = 1$.

Proof. Let $G_1 = G_1' \oplus G_1''$ be some nontrivial decomposition of G_1 into a direct sum of G_0-submodules. Applying Proposition 1.2.9 to $G_{-1} \oplus G_0 \oplus G_1'$ we see that $[G_0, G_0] \subseteq [G_{-1}, G_1']$. On the other hand, clearly $[[G_{-1}, G_1'], G_{-1}] \oplus [G_{-1}, G_1'] \oplus G_1'$ is an ideal in G. Hence it follows, in particular, that $[[G_0, G_0], G_1''] = 0$. So we find that if the G_0-module G_1 is reducible, then $[G_0, G_0]$ acts trivially on G_1. But in that case it clearly follows from transitivity that $\dim G_1 = 1$.

It remains to show that the following situation is impossible: $H \subset G_0$, $H \neq [G_0, G_0]$ is a simple subalgebra of G_0 whose representation on G_1 is trivial. In that case $[x, G_{-1}] \subseteq H$ for every $x \in G_1$, which contradicts Proposition 1.2.9. This completes the proof of Proposition 1.2.11.

PROPOSITION 1.2.12. *Let $G = \oplus_{i\geqslant -1} G_i$ be a $\mathbf{Z}$-graded irreducible transitive Lie superalgebra. If the even part of the center C of G_0 is nontrivial, then $C_{\bar{0}} = \langle z\rangle$, and $[z, g] = sg$ for $g \in G_s$.*

Proof. By Schur's lemma, $C_{\bar{0}} = \langle z \rangle$, where $[z, g] = -g$ for $g \in G_{-1}$. Now let $g_{k+1} \in G_{k+1}$, $x \in G_{-1}$. Then we have by induction:

$$[z, [x, g_{k+1}]] = -[x, g_{k+1}] + [x, [z, g_{k+1}]] = k[x, g_{k+1}],$$

from which we see that $[x, [z, g_{k+1}] - (k+1)g_{k+1}] = 0$. By the transitivity of G, what we need now follows.

PROPOSITION 1.2.13. *Let $G = \bigoplus_{i \geqslant -1} G_i$ be a $\mathbf{Z}$-graded irreducible transitive Lie superalgebra for which the representation of G_0 on G_1 is faithful. Then G is bitransitive.*

Proof. Clearly, $V = \{a \in G_{-1} \mid [a, G_1] = 0\}$ is a submodule of the G_0-module G_{-1}. By the transitivity of G we have $[G_{-1}, G_1] \neq 0$; therefore, $V \neq G_{-1}$; consequently, $V = 0$.

1.3. *Lie Superalgebras with Filtrations*

1.3.1. *Filtrations.* A sequence of embedded $\mathbf{Z}_2$-graded subspaces in a superalgebra $L: L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ is called a *filtration* if

$$L_i L_j \subseteq L_{i+j} \qquad \text{and} \qquad \bigcap L_i = 0, \qquad i, j \in \mathbf{Z}.$$

A Lie algebra L with a filtration is called *transitive* if for any $a \in L_i \backslash L_{i+1}$, $i \geqslant 0$, there is an element $b \in L$ for which $[a, b] \notin L_i$. This condition can also be written as follows:

$$L_i = \{a \in L_{i-1} \mid [a, L] \subseteq L_{i-1}\}, \qquad i > 0. \tag{1.3.1}$$

Let L be a Lie superalgebra and L_0 be a subalgebra of L that contains no nonzero ideals of the whole algebra L. Then (1.3.1) defines a filtration in L.

The first property of a filtration is easily verified by induction, using the Jacobi identity, and the second follows from the fact that $\bigcap L_i$, clearly, is an ideal of L and so, by hypothesis, $\bigcap L_i = 0$.

The filtration constructed in this way is called the *transitive filtration* of the pair (L, L_0).

With a Lie superalgebra L we can associate, in the usual way, the $\mathbf{Z}$-graded Lie superalgebra

$$\operatorname{Gr} L = \bigoplus_{i \geqslant -1} \operatorname{Gr}_i L, \qquad \text{where} \quad \operatorname{Gr}_i L = L_i / L_{i+1}.$$

Owing to the grading of the subspace L_i, the algebra $\operatorname{Gr} L$ is equipped with a natural $\mathbf{Z}_2$-grading; however, the $\mathbf{Z}$-grading of $\operatorname{Gr} L$ is not, in general, consistent.

A $\mathbf{Z}$-graded superalgebra $G = \bigoplus_{i \geqslant -1} G_i$ is canonically equipped with a filtration: $L_i = \bigoplus_{s \geqslant i} G_s$.

A superalgebra L with filtration is transitive if and only if $\mathrm{Gr}\,L$ is transitive. If $\mathrm{Gr}\,L$ is simple, then so is L.

1.3.2. *Connection between* L *and* $\mathrm{Gr}\,L$. If in a Lie superalgebra with filtration $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ subspaces G_i are given such that $L_s = G_s \oplus L_{s+1}$ and $[G_i, G_j] \subseteq G_{i+j}$, then we say that L is equipped with a grading that is consistent with its filtration. In that case, clearly, $L \simeq \mathrm{Gr}\,L$, provided that L is finite-dimensional.

PROPOSITION 1.3.1. *Let* $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ *be a transitive finite-dimensional Lie superalgebra with a filtration for which the representation of* $\mathrm{Gr}_0 L$ *on* $\mathrm{Gr}_{-1} L$ *is irreducible and even part of the center of* $\mathrm{Gr}_0 L$ *is nontrivial. Then* $L \simeq \mathrm{Gr}\,L$.

Proof. According to Proposition 1.2.12 there is an element $z \in \mathrm{Gr}_0 L$ such that $[z, g] = sg$ for $g \in \mathrm{Gr}_s L$. Let $\tilde{z}$ be some inverse image of z under the map $L_0 \to \mathrm{Gr}_0 L = L_0/L_1$. As is easy to see, $\tilde{z}$ is diagonalizable in L, so that $L_s = G_s \oplus L_{s+1}$, where G_s is the eigenspace of $\tilde{z}$ for the eigenvalue s. This gives us the required grading, consistent with the filtration of L.

1.3.3. *Properties of filtrations.* Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra and L_0 be a maximal proper subalgebra containing $L_{\bar{0}}$. Suppose that $L_{\bar{0}}$ does not contain nonzero ideals of L. Let us construct the transitive filtration of the pair (L, L_0) (see 1.1.1):

$$L_i = \{a \in L_{i-1} \mid [a, L] \subseteq L_{i-1}\}, \qquad i > 0.$$

Let $\mathrm{Gr}\,L = \bigoplus_{i \geqslant -1} \mathrm{Gr}_i L$ be the associated $\mathbf{Z}$-graded Lie superalgebra.

PROPOSITION 1.3.2. $\mathrm{Gr}\,L$ *has the following properties*:

(a) $\mathrm{Gr}\,L$ *is transitive*;

(b) *the* $\mathbf{Z}$*-grading of* $\mathrm{Gr}\,L$ *is consistent with the* $\mathbf{Z}_2$*-grading*;

(c) $\mathrm{Gr}\,L$ *is irreducible*;

(d) *if the representation of* $L_{\bar{0}}$ *on* $L_{\bar{1}}$ *is reducible, then* $\mathrm{Gr}_1 L \neq 0$.

Proof. (a) follows from the transitivity of L. The fact that L_0 contains $L_{\bar{0}}$ implies that $\mathrm{Gr}_{-1} L \subseteq (\mathrm{Gr}\,L)_{\bar{1}}$. By the transitivity of $\mathrm{Gr}\,L$, we obtain (b) by induction.

Let us now prove (c). Suppose the contrary; then there exists a $\mathbf{Z}_2$-graded subspace $\tilde{L}$ of L containing L_0 but different from L and L_0 for which $[L_0, \tilde{L}] \subseteq \tilde{L}$. Then $\tilde{L} = L_0 \oplus V$, where $V \subseteq L_{\bar{1}}$ and $[V, V] \subseteq L_0$ because $L_0 \supseteq L_{\bar{0}}$. Therefore, $[\tilde{L}, \tilde{L}] = [L_0 \oplus V, L_0 \oplus V] = [L_0, L_0] + [L_0, V] + [V, V] \subseteq \tilde{L}$. But this contradicts the maximality of L_0.

Now we prove (d). If $\mathrm{Gr}_1 L = 0$; then clearly $\mathrm{Gr}_0 L = L_{\bar{0}}$. Consequently, by (c), the representation of $L_{\bar{0}}$ on $L_{\bar{1}}$ is then irreducible.

PROPOSITION 1.3.3. *Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is solvable iff Lie algebra $G_{\bar{0}}$ is solvable.*

Proof. Let $G_{\bar{0}}$ be solvable, L_0 be a maximal proper subalgebra containing $G_{\bar{0}}$, and J be a maximal ideal among ideals of G, containing in L_0. We have a filtered Lie superalgebra $G/J = \bar{L} \supset \bar{L}_0 \supset \bar{L}_1 \supset \cdots$, which satisfies all the conditions of Proposition 1.3.2. In particular, $\mathrm{Gr}\, L$ is irreducible. But $\mathrm{Gr}_0 L$ is solvable since $G_{\bar{0}}$ is solvable and so $\dim \mathrm{Gr}_0 L = 0$ or 1. Therefore $\dim G/J = 1$ or 2 and G/J is a solvable Lie superalgebra. By induction, J is a solvable superalgebra too. So G is a solvable Lie superalgebra.

1.4. *Information from the Theory of Representations of Semisimple Lie Algebras*

1.4.1. *The theorem on the highest weight.* Let G be a semisimple Lie algebra and H be a Cartan subalgebra of it. We consider a representation ρ of G in a finite-dimensional space V (or, as we usually say, a G-module V). For $\lambda \in H^*$ we set $V_\lambda = \{v \in V \mid h(v) = \lambda(h)v\}$. If $V_\lambda \neq 0$, then λ is called a *weight of* ρ and a nonzero vector v_λ in V_λ is called a *weight vector*. Let $\mathscr{L}_\rho$ be the set of all weights; then $V = \oplus_{\lambda \in \mathscr{L}_\rho} V_\lambda$.

A weight α of the adjoint representation of G is called a *root* of G. Then $G = \oplus_\alpha G_\alpha$, where $G_0 = H$ and $\dim G_\alpha = 1$ for $\alpha \neq 0$. A nonzero vector e_α in G_α is called a *root vector*. $G_\alpha V_\lambda \neq 0$ is contained in $V_{\lambda+\alpha}$ if $\lambda + \alpha \in \mathscr{L}_\rho$, and $G_\alpha V_\lambda = 0$ if $\lambda + \alpha \notin \mathscr{L}_\rho$.

Let $(a, b) = \mathrm{tr}(\mathrm{ad}\, a)(\mathrm{ad}\, b)$ be the Killing form on G. Both the Killing form and its restriction to H are nondegenerate; therefore, it induces on H^* a nondegenerate form. If $\alpha \neq 0$, then $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha \neq 0$, where the vector $h_\alpha \in H$ is determined by the relation $\alpha(h) = (h_\alpha, h)$.

Let Δ' be the set of all nonzero roots, Δ^+ be the set of positive roots (in some fixed lexicographical ordering), and $\Sigma = \{\alpha_1, \ldots, \alpha_s\}$ be the set of simple roots of G. Then $\Delta' = \Delta^+ \cup -\Delta^+$, the system Σ forms a basis of the space H^*, and every root $\alpha \in \Delta^+$ is of the form $\alpha = \sum k_i \alpha_i$, where the k_i are nonnegative integers.

Let H_0^* be the linear span of Δ' over $\mathbf{Q}$. The Killing form is positive definite on H_0^*, and $\mathscr{L}_\rho \subseteq H_0^*$. Let $\alpha \in \Delta'$, $\lambda \in \mathscr{L}_\rho$; then the set of weights of the form $\lambda + s\alpha$ forms a progression: $\lambda - p\alpha, \lambda - (p-1)\alpha, \ldots, \lambda - \alpha, \lambda, \lambda + \alpha, \ldots, \lambda + q\alpha$, where p and q are nonnegative integers and $p - q = 2(\lambda, \alpha)/(\alpha, \alpha)$.

The numbers $2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ are called the *numerical marks* of the linear function $\lambda \in H^*$.

If $\lambda \in \mathscr{L}_\rho$, then its numerical marks are integers. A function $\lambda \in H_0^*$ is said to be dominant if its numerical marks are nonnegative integers.

Now let ρ be an irreducible finite-dimensional representation of a Lie algebra G. A *highest* (respectively, *lowest*) *weight* of ρ is a weight $\Lambda \in \mathscr{L}_\rho$ (respectively, $M \in \mathscr{L}_\rho$) for which $\Lambda + \alpha \notin \mathscr{L}_\rho$ (respectively, $M - \alpha \notin \mathscr{L}_\rho$) for $\alpha \in \Delta^+$. The highest and lowest weights are unique, and $\dim V_\Lambda = \dim V_M = 1$. Every nonzero vector in V_Λ (respectively, V_M) is called a highest (respectively, lowest) vector of ρ. The theorem on the highest weight asserts that the function Λ is dominant and that for any dominant linear function Λ there is a unique irreducible finite-dimensional representation with the highest weight Λ.

A representation ρ of a Lie algebra G in a space V induces a representation ρ^* of it in the dual space V^*; ρ and ρ^* are said to be contragredient. $\lambda \in \mathscr{L}_\rho$ if and only if $-\lambda \in \mathscr{L}_{\rho^*}$. In particular, if Λ is the highest weight of ρ, then $(-\Lambda)$ is the lowest weight of ρ^*.

If a Lie algebra G has a faithful irreducible finite-dimensional representation ρ, then $G = G' \oplus C$, where G' is a semisimple Lie algebra, C is the center of G, and $\rho(C)$ are scalar operators. The restriction of ρ to G' is also irreducible and its highest weight (vector) is defined as the highest weight (vector) of ρ.

1.4.2. *Diagrams of highest and dominant roots.* A semisimple Lie algebra can be represented by a *Dynkin diagram.* Let $\Sigma = \{\alpha_1, ..., \alpha_r\}$ by the system of simple roots; then $a_{ij} = -2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_j)$ are nonnegative integers. The Dynkin diagram of G consists of r circles correspondnig to the simple roots, and the ith circle is joined to the jth by an $a_{ij}a_{ji}$ segment with arrows pointing to the ith circle when $a_{ij} < a_{ji}$. An irreducible representation of G is represented by a Dynkin diagram equipped with the nonzero numerical marks $2(\Lambda, \alpha_i)/(\alpha_i, \alpha_i)$ of the highest weight Λ standing against the corresponding circles.

If G is simple, then its adjoint representation is irreducible; its highest weight θ is the highest root of G. The diagrams of the highest roots of all simple Lie algebras are given in Table I. Apart from the highest roots, the simple Lie algebras also have the dominant roots given in Table II.

1.4.3. *The index.* Let ρ be a finite-dimensional faithful linear representation of a semisimple Lie algebra G in a space V. Then the bilinear form $(a, b)_V =$ tr $\rho(a)\,\rho(b)$ on G is nondegenerate. If G is simple, then

$$(a, b)_V = l_V(a, b), \qquad a, b \in G,$$

TABLE I

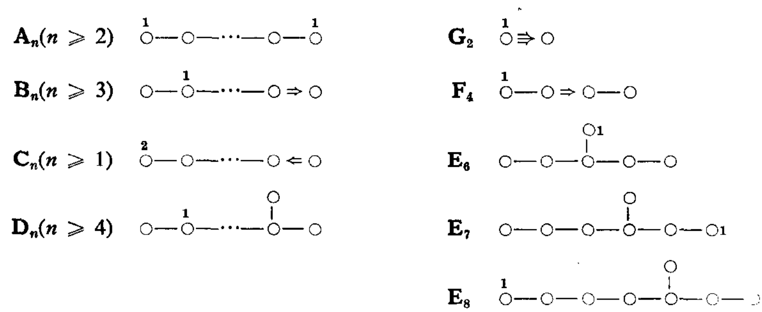


TABLE II

$\mathbf{B}_n (n \geqslant 3)$	$\overset{1}{\circ}-\circ-\cdots-\circ\Rightarrow\circ$	$\mathbf{G}_2$	$\circ\Rrightarrow\overset{1}{\circ}$
$\mathbf{C}_n (n \geqslant 2)$	$\overset{1}{\circ}-\circ-\cdots-\circ\Leftarrow\circ$	$\mathbf{F}_4$	$\circ-\circ\Rightarrow\circ-\overset{1}{\circ}$

TABLE III

Type		r	Diagram	dim $\overset{\circ}{V}$	l_V
A	sl_n	$n-1$	$\overset{1}{\circ}-\circ-\cdots-\circ-\circ$	n	$\frac{1}{2n}$
	$\Lambda^2 sl_n$	$n-1$	$\circ-\overset{1}{\circ}-\cdots-\circ-\circ$	$\frac{n(n-1)}{2}$	$\frac{n-2}{2n}$
	$S^2 sl_n$	$n-1$	$\overset{2}{\circ}-\circ-\cdots-\circ-\circ$	$\frac{n(n+1)}{2}$	$\frac{n+2}{2n}$
	$\Lambda^3 sl_n$, $n = 6, 7, 8$	5, 6, 7	$\circ-\circ-\overset{1}{\circ}-\cdots-\circ$	20, 35, 56	$\frac{1}{2}, \frac{5}{7}, \frac{15}{16}$
B	so_n	$\frac{n-1}{2}$	$\overset{1}{\circ}-\circ-\cdots-\circ\Rightarrow\circ$	n	$\frac{1}{n-2}$
	spin$_n$, $n = 7, 9, 11, 13$	3, 4, 5, 6	$\circ-\circ-\cdots-\circ\Rightarrow\overset{1}{\circ}$	8, 16, 32, 64	$\frac{1}{5}, \frac{2}{7}, \frac{4}{9}, \frac{8}{11}$
C	sp_n	$\frac{n}{2}$	$\overset{1}{\circ}-\circ-\cdots-\circ\Leftarrow\circ$	n	$\frac{1}{n+2}$
	$\Lambda_0^2 sp_n$	$\frac{n}{2}$	$\circ-\overset{1}{\circ}-\cdots-\circ\Leftarrow\circ$	$\frac{n(n-1)}{2}-1$	$\frac{n-2}{n+2}$
	$\Lambda_0^3 sp_6$	3	$\circ-\circ\Leftarrow\overset{1}{\circ}$	14	$\frac{5}{8}$
D	so_n	$\frac{n}{2}$	$\overset{1}{\circ}-\circ-\cdots-\overset{\overset{\circ}{\mid}}{\circ}-\circ$	n	$\frac{1}{n-2}$
	spin$_n$, $n = 10, 12, 14$	5, 6, 7	$\circ-\circ-\cdots-\overset{\overset{\circ}{\mid}}{\circ}-\overset{1}{\circ}$	16, 32, 64	$\frac{1}{4}, \frac{2}{5}, \frac{2}{3}$
E	E_6	6	$\overset{1}{\circ}-\circ-\overset{\overset{\circ}{\mid}}{\circ}-\circ-\circ$	27	$\frac{1}{4}$
	E_7	7	$\overset{1}{\circ}-\circ-\circ-\overset{\overset{\circ}{\mid}}{\circ}-\circ-\circ$	56	$\frac{1}{3}$
F	F_4	4	$\overset{1}{\circ}-\circ\Leftarrow\circ-\circ$	26	$\frac{1}{3}$
G	G_2	2	$\overset{1}{\circ}\Lleftarrow\circ$	7	$\frac{1}{4}$

where l_V is a positive rational number, independent of a and b. It is called the *index* of ρ. The index of a direct sum of representations is the sum of the indices of the representations.

The index of a one-dimensional representation of a simple Lie algebra G is 0, and that for irreducible representation is 1 only of the adjoint representation. A list of the irreducible representations of the simple Lie algebras for which $0 < l_V < 1$ is given, to within transition to a contragredient representation, in Table III, which is taken from [1].

Here sl_n, sp_n, and so_n stand for standard representations of these Lie algebras; S^k and Λ^k denote kth symmetrical and exterior degrees, respectively, $S_0{}^k$ and $\Lambda_0{}^k$ their highest component, spin_k stands for irreducible spinor representation of so_k, G_2, F_4, E_6, E_7 denote also the simplest representations of the corresponding Lie algebras.

1.4.4. *A technical lemma.* We now prove a lemma on representations that is used in an essential way in the classification of the classical Lie superalgebras.

LEMMA 1.4.1. *Let ρ be a faithful irreducible finite-dimensional representation of a semisimple Lie algebra G in a space V. Let Δ be the system of all roots of G, $\mathscr{L}$ the system of weights of ρ, and Λ the highest weight.*

(a) *If $2\Lambda \in \Delta$, then the G-module V is isomorphic to sp_n;*

(b) *if $\Lambda - \mu \in \Delta$ for any $\mu \in \mathscr{L}$, then G-module V is isomorphic to sl_n or sp_n;*

(c) *if $\Lambda - \mu \in \Delta$ for any $\mu \in \mathscr{L}$, $\mu \neq -\Lambda$, then G-module V is isomorphic to sl_n, sp_n, so_n, spin_7, or G_2.*

Proof. Let M be the lowest weight of ρ. Observe that 2Λ and $\Lambda - M$ can lie in Δ only when G is simple. This is, therefore, the case in (a) and (b).

(a) By hypothesis, $2\Lambda \in \Delta$. However, from Tables I and II it is clear that only $\mathbf{C}_r$ has a dominant root with mark 2. Half this root is the highest weight of the $\mathbf{C}_r$-module sp_{2r}. This proves (a).

(b) By hypothesis, $\Lambda - M \in \Delta$; this root is dominant, and the sum of its marks is not less than 2. From Tables I and II it is clear that $\Lambda - M = \theta$ is the highest root of one of the Lie algebras $\mathbf{A}_r$ or $\mathbf{C}_r$; therefore, the G-module V is isomorphic to sl_{r+1} or sp_{2r}, respectively.

(c) If $\Lambda \neq -M$, then by hypothesis $\Lambda - M \in \Delta$, and from the proof of (b) it is clear that V is isomorphic to sl_n. Now let

$$\Lambda = -M \tag{1.4.1}$$

If θ is the highest root of one of the simple components of G, then clearly $\Lambda - \theta \in \mathscr{L}$. If $\Lambda - \theta = M$, then from (1.4.1) and the proof of (b) it follows that V is isomorphic to sp_n. But if $\Lambda - \theta \neq M$, then there is a simple root α

for which $\Lambda - \theta - \alpha \in \mathscr{L}$. Therefore, $\Lambda - \theta - \alpha = M$, because otherwise, by hypothesis and (1.4.1), $\Lambda - (\Lambda - \theta - \alpha) = \theta + \alpha \in \Delta$, which is impossible. Hence,

$$2\Lambda - \theta = \alpha \text{ is a simple root.} \tag{1.4.2}$$

From (1.4.2) it follows that if G is not simple, then the number of simple components is 2 and the highest root of each simple component is a simple root. Therefore, $G = \mathbf{A}_1 \oplus \mathbf{A}_1$, and V is isomorphic to $so_4 = sl_2 \otimes sl_2$.

Now suppose that G is simple. Now (1.4.2) means that $\Lambda = \frac{1}{2}(\theta + \alpha)$ is a dominant linear function. If the circle corresponding to α is not at an end of the diagram, then it has at least two negative marks, so that θ has at least two positive marks. From Table I we see that in this case G is of type $\mathbf{A}_r$, moreover, that $r = 3$, $\alpha = \alpha_r$, and V is isomorphic to $\Lambda^2 sl_4 = so_6$. But if the circle is at an end of the diagram and $r > 2$, then the mark of α is negative, therefore, the positive mark of θ is not at an end. From Table I we see that in this case G is of type $\mathbf{B}_r$ or $\mathbf{D}_r$, $\alpha = \alpha_1$, and V is isomorphic to so_n with $n > 6$, or G is of type $\mathbf{B}_3$, $\alpha = \alpha_3$, and V is isomorphic to spin_7. Finally, the case $r \leqslant 2$, as is easy to see, gives the G-modules so_n with $n = 3, 5$ and G_2.

The proof of the lemma is now complete.

2. Classical Lie Superalgebras

A finite-dimensional Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is called *classical* if it is simple and the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is completely reducible.

The aim of this chapter is the description and classification of the classical Lie superalgebras.

2.1. *Examples of Classical Lie Superalgebras*

2.1.1. *The Lie superalgebras* $\mathbf{A}(m, n)$. We recall some facts from Section 1.1. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbf{Z}_2$-graded space, $\dim V_{\bar{0}} = m$, $\dim V_{\bar{1}} = n$. The associative algebra End V becomes an associative superalgebra if we let

$$\mathrm{End}_i\, V = \{a \in \mathrm{End}\, V \mid aV_s \subseteq V_{i+s}\}, \qquad i, s \in \mathbf{Z}_2 .$$

The bracket $[a, b] = ab - (-1)^{(\deg a)(\deg b)} ba$ makes End V into a Lie superalgebra, denoted by $l(V)$ or $l(m, n)$. If we regard the same decomposition $V = V_0 \oplus V_1$ as a $\mathbf{Z}$-grading of V, then the same construction gives a consistent $\mathbf{Z}$-grading: $l(V) = G_{-1} \oplus l(V)_{\bar{0}} \oplus G_1$. On $l(V)$ we define the supertrace, a linear function $\mathrm{str}\colon l(V) \to k$. Its basic property is $\mathrm{str}([a, b]) = 0$, $a, b \in l(V)$.

From this it follows that the subspace

$$sl(m, n) = \{a \in l(m, n) \mid \mathrm{str}\, a = 0\}$$

is an ideal in $l(m, n)$ of codimension 1. The resulting $\mathbf{Z}$-grading $sl(m, n) = G_{-1} \oplus sl(m, n)_{\bar{0}} \oplus G_1$ looks in some homogeneous basis of V as follows: $sl(m, n)_{\bar{0}}$ is the set of matrices of the form $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \delta\end{smallmatrix}\right)$, where $\operatorname{tr}\alpha = \operatorname{tr}\delta$, G_1 is the set of matrices of the form $\left(\begin{smallmatrix}0 & \beta\\ 0 & 0\end{smallmatrix}\right)$ and G_{-1} of the form $\left(\begin{smallmatrix}0 & 0\\ \gamma & 0\end{smallmatrix}\right)$ (where α is an $(m \times m)$-, δ an $(n \times n)$-, β an $(m \times n)$-, and γ an $(n \times m)$-matrix).

Now $sl(n, n)$ contains the one-dimensional ideal consisting of the scalar matrices $\lambda 1_{2n}$. The Lie superalgebra $sl(1, 1)$ is three-dimensional and nilpotent.

We set

$$\begin{aligned} \mathbf{A}(m, n) &= sl(m+1, n+1) \qquad \text{for} \quad m \neq n, \qquad m, n \geqslant 0,\\ \mathbf{A}(n, n) &= sl(n+1, n+1)/\langle 1_{2n+2}\rangle, \qquad\qquad\qquad n > 0. \end{aligned}$$

The $\mathbf{Z}$-grading of $sl(m+1, n+1)$ induces a $\mathbf{Z}$-grading of $\mathbf{A}(m, n)$ of the form $\mathbf{A}(m, n) = G_{-1} \oplus G_0 \oplus G_1$.

2.1.2. *The Lie superalgebras* $\mathbf{B}(m, n)$, $\mathbf{D}(m, n)$ *and* $\mathbf{C}(n)$. Again, let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbf{Z}_2$-graded space, $\dim V_{\bar{0}} = m$, $\dim V_{\bar{1}} = n$. Let F be a non-degenerate consistent supersymmetric bilinear form on V, so that $V_{\bar{0}}$ and $V_{\bar{1}}$ are orthogonal and the restriction of F to $V_{\bar{0}}$ is a symmetric and to $V_{\bar{1}}$ a skew-symmetric form (in particular, $n = 2r$ is even).

We define in $l(m, n)$ the subalgebra $\operatorname{osp}(m, n) = \operatorname{osp}(m, n)_{\bar{0}} \oplus \operatorname{osp}(m, n)_{\bar{1}}$ by setting

$$\operatorname{osp}(m, n)_s = \{a \in l(m, n)_s \mid F(a(x), y) = -(-1)^{s(\deg x)} F(x, a(y))\}, \qquad s \in \mathbf{Z}_2.$$

We call $\operatorname{osp}(m, n)$ an *orthogonal-symplectic* superalgebra (for $n = 0$ or $m = 0$ it turns into an orthogonal or symplectic Lie algebra, respectively).

Let us find the explicit matrix form of the elements of $\operatorname{osp}(m, n)$. We treat two cases separately.

$m = 2l + 1$. In some basis the matrix of the form F can be written as

$$\left[\begin{array}{ccc|cc} 0 & 1_l & 0 & & \\ 1_l & 0 & 0 & & \\ 0 & 0 & 1 & & \\ \hline & & & 0 & 1_r \\ & & & -1_r & 0 \end{array}\right],$$

from which we see that a matrix in $\operatorname{osp}(m, n)$ is of the form

$$\left[\begin{array}{ccc|cc} a & b & u & x & x_1 \\ c & -a^T & v & y & y_1 \\ -v^T & -u^T & 0 & z & z_1 \\ \hline y_1{}^T & x_1{}^T & z_1{}^T & d & e \\ -y^T & -x^T & -z^T & f & -d^T \end{array}\right];$$

here a is any $(l \times l)$-matrix; b and c are skew-symmetric $(l \times l)$-matrices; d is any $(r \times r)$-matrix; e and f are symmetric $(r \times r)$-matrices; u and v are $(l \times 1)$-matrices; x and y are $(l \times r)$-matrices, and z is an $(r \times 1)$-matrix.

In particular, we see that $\operatorname{osp}(m, n)_{\bar 0}$ is a Lie algebra of type $\mathbf{B}_l \oplus \mathbf{C}_r$, and the $\operatorname{osp}(m, n)_{\bar 0}$-module $\operatorname{osp}(m, n)_{\bar 1}$ is isomorphic to $so_m \otimes sp_n$.

$m = 2l$. In some basis the matrix of F can be written as

$$\left[\begin{array}{cc|cc} 0 & 1_l & & \\ 1_l & 0 & & \\ \hline & & 0 & 1_r \\ & & -1_r & 0 \end{array}\right],$$

from which we see that a matrix in $\operatorname{osp}(m, n)$ has the same form as in the first case, with the middle row and column deleted.

In particular, we find that $\operatorname{osp}(m, n)_{\bar 0}$ for $l \geqslant 2$ is a Lie algebra of type $\mathbf{D}_l \oplus \mathbf{C}_r$, and that the $\operatorname{osp}(m, n)_{\bar 0}$-module $\operatorname{osp}(m, n)_{\bar 1}$ is isomorphic to $so_m \otimes sp_n$.

By analogy with Cartan's notation we set:

$$\begin{aligned} \mathbf{B}(m, n) &= \operatorname{osp}(2m + 1, 2n), & m \geqslant 0, \quad n > 0; \\ \mathbf{D}(m, n) &= \operatorname{osp}(2m, 2n), & m \geqslant 2, \quad n > 0; \\ \mathbf{C}(n) &= \operatorname{osp}(2, 2n - 2), & n \geqslant 2. \end{aligned}$$

We now examine the Lie superalgebra $\mathbf{C}(n)$. Subalgebra $\mathbf{C}(n)_{\bar 0}$ consists of matrices of the form

$$\left[\begin{array}{cc|cc} \alpha & 0 & & \\ 0 & -\alpha & & \\ \hline & & a & b \\ & & c & -a^T \end{array}\right],$$

where a, b, and c are $(n - 1 \times n - 1)$-matrices, b and c being symmetric, and $\alpha \in k$. Furthermore, $\mathbf{C}(n)$ has the consistent $\mathbf{Z}$-grading: $\mathbf{C}(n) = G_{-1} \oplus \mathbf{C}(n)_{\bar 0} \oplus G_1$, where G_{-1} and G_1 consist of the matrices of the form (respectively):

$$\left[\begin{array}{cc|cc} & & 0 & 0 \\ & & y & y_1 \\ \hline y_1^T & 0 & & \\ -y^T & 0 & & \end{array}\right] \quad \left[\begin{array}{cc|cc} & & x & x_1 \\ & & 0 & 0 \\ \hline 0 & x_1^T & & \\ 0 & -x^T & & \end{array}\right].$$

The representations of $\mathbf{C}(n)_{\bar 0}$ on G_{-1} and G_1 are contragradient, and the $\mathbf{C}(n)_{\bar 0}$-module G_1 is isomorphic to $\operatorname{csp}_{2n-2}$.

Supplement. We consider another realization of osp(m, n). Let $V_{\bar{0}}$ be an m-dimensional space with a nondegenerate symmetric bilinear form $(\, , \,)_0$ and $V_{\bar{1}}$ an n-dimensional space with a nondegenerate skew-symmetric bilinear form $(\, , \,)_1$, $n = 2r$. Then osp(m, n) can be realized as follows:

$$\mathrm{osp}(m, n)_{\bar{0}} = \Lambda^2 V_{\bar{0}} \oplus S^2 V_{\bar{1}}, \qquad \mathrm{osp}(m, n)_{\bar{1}} = V_{\bar{0}} \otimes V_{\bar{1}}.$$

The definitions of the operations are

$$\begin{aligned} [a \wedge b, c] &= (a, c)_0 b - (b, c)_0 a, \qquad & a \wedge b \in \Lambda^2 V_0, \quad c \in V_0; \\ [a \circ b, c] &= (a, c)_1 b + (b, c)_1 a, \qquad & a \circ b \in S^2 V_1, \quad c \in V_1. \end{aligned}$$

These brackets define brackets on $\Lambda^2 V_0$ and $S^2 V_1$ in the usual way:

$$[ab, cd] = [ab, c]d + c[ab, d].$$

Finally, for $a \otimes c, b \otimes d \in V_0 \otimes V_1$ we set

$$[a \otimes c, b \otimes d] = (a, b)_0 c \circ d + (c, d)_1 a \wedge b.$$

In this realization there is a natural way of defining an interesting $\mathbf{Z}$-grading of osp(m, n):

$$\mathrm{osp}(m, n) = G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2.$$

To obtain this, we represent $V_{\bar{1}}$ as a direct sum of isotropic subspaces $V_{\bar{1}} = V_{\bar{1}}' \oplus V_{\bar{1}}''$. The following decomposition is then a $\mathbf{Z}$-grading:

$$\mathrm{osp}(m, n) = S^2 V_{\bar{1}}' \oplus (V_{\bar{0}} \otimes V_{\bar{1}}') \oplus (V_{\bar{1}}' \otimes V_{\bar{1}}'' \oplus \Lambda^2 V_{\bar{0}}) \oplus (V_{\bar{0}} \otimes V_{\bar{1}}'') \oplus S^2 V_{\bar{1}}''.$$

Clearly, $G_0 \simeq gl_r \oplus so_n$, the representations of G_0 on G_i and G_{-i} are contragredient, the G_0-module G_1 is isomorphic to $gl_r \otimes so_n$, and G_2 to $S^2 gl_r$.

2.1.3. *The Lie superalgebra* $\mathbf{P}(n)$, $n \geqslant 2$. This is a subalgebra of $sl(n+1, n+1)$, consisted of the matrices of the form:

$$\left(\begin{array}{c|c} a & b \\ \hline c & -a^T \end{array}\right)$$

where tr $a = 0$, b is a symmetric matrix, and c is a skew-symmetric matrix.

2.1.4. *The Lie superalgebra* $\mathbf{Q}(n)$, $n \geqslant 2$. First we denote $\tilde{\mathbf{Q}}(n)$ a subalgebra of $sl(n + 1, n + 1)$, consisting of the matrices of the form

$$\left(\begin{array}{c|c} a & b \\ \hline b & a \end{array}\right),$$

where tr $b = 0$. Lie superalgebra $\tilde{\mathbf{Q}}(n)$ has one-dimensional center $C = \langle 1_{2n+2} \rangle$. We put $\mathbf{Q}(n) = \tilde{\mathbf{Q}}(n)/C$.

2.1.5. *The Lie superalgebras* $\mathbf{F}(4)$, $\mathbf{G}(3)$, *and* $\mathbf{D}(2, 1; \alpha)$.

PROPOSITION 2.1.1. (a) *There is one and only one* 40-*dimensional classical Lie superalgebra* $\mathbf{F}(4)$ *for which* $\mathbf{F}(4)_{\bar 0}$ *is a Lie algebra of type* $\mathbf{B}_3 \oplus \mathbf{A}_1$ *and its representation on* $\mathbf{F}(4)_{\bar 1}$ *is* $spin_7 \otimes sl_2$.

(b) *There is one and only one* 31-*dimensional classical Lie superalgebra* $\mathbf{G}(3)$ *for which* $\mathbf{G}(3)_{\bar 0}$ *is a Lie algebra of type* $\mathbf{G}_2 \oplus \mathbf{A}_1$ *and its representation on* $\mathbf{G}(3)_{\bar 1}$ *is* $G_2 \otimes sl_2$.

(c) *There is a one-parameter family of* 17-*dimensional Lie superalgebras* $\mathbf{D}(2, 1; \alpha)$, $\alpha \in k^*\backslash\{0, -1\}$, *consisting of all simple Lie superalgebras for which* $\mathbf{D}(2, 1; \alpha)_{\bar 0}$ *is a Lie algebra of type* $\mathbf{A}_1 \oplus \mathbf{A}_1 \oplus \mathbf{A}_1$ *and its representation on* $\mathbf{D}(2, 1; \alpha)_{\bar 1}$ *is* $sl_2 \otimes sl_2 \otimes sl_2$.

The proof can be obtained by a direct construction of epimorphisms of G_0-modules $S^2 G_{\bar 1} \to G_{\bar 0}$ satisfying (1.1.3). However, in Section 2.5 (Proposition 2.5.4) we give an alternative proof, by means of contragredient Lie superalgebras (cf. [8, 11]).

2.1.6. *Properties and uniqueness.* From the description of the classical Lie superalgebras in Sections 2.1.1–2.1.5 above and Propositions 1.2.7 and 1.2.8, we derive the following result.

PROPOSITION 2.1.2. (a) *All the Lie superalgebras* $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, $\mathbf{P}(n)$, $\mathbf{Q}(n)$ *are classical.*

(b) *For the Lie superalgebras* $\mathbf{B}(m, n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, *and* $\mathbf{Q}(n)$ *the* $G_{\bar 0}$-*module* $G_{\bar 1}$ *is irreducible and isomorphic to the modules in the following list:*

G	$G_{\bar 0}$	$G_{\bar 0} \mid G_{\bar 1}$	G	$G_{\bar 0}$	$G_{\bar 0} \mid G_{\bar 1}$
$\mathbf{B}(m, n)$	$\mathbf{B}_m \oplus \mathbf{C}_n$	$so_{2m+1} \otimes sp_{2n}$	$\mathbf{F}(4)$	$\mathbf{B}_3 \oplus \mathbf{A}_1$	$spin_7 \otimes sl_2$
$\mathbf{D}(m, n)$	$\mathbf{D}_m \oplus \mathbf{C}_n$	$so_{2m} \otimes sp_{2n}$	$\mathbf{G}(3)$	$\mathbf{G}_2 \oplus \mathbf{A}_1$	$G_2 \otimes sl_2$
$\mathbf{D}(2, 1; \alpha)$	$\mathbf{A}_1 \oplus \mathbf{A}_1 \oplus \mathbf{A}_1$	$sl_2 \otimes sl_2 \otimes sl_2$	$\mathbf{Q}(n)$	$\mathbf{A}_n$	$ad\, sl_{n+1}$

(c) *The Lie superalgebras* $\mathbf{A}(m, n)$, $\mathbf{C}(n)$, *and* $\mathbf{P}(n)$ *admit a unique consistent* $\mathbf{Z}$-*grading of the form* $G_{-1} \oplus G_0 \oplus G_1$. *Here the* G_0-*modules* G_1 *and* G_{-1} *are irreducible and for* $\mathbf{A}(m, n)$ *and* $\mathbf{C}(n)$ *contragredient; they are isomorphic to the modules in the following list:*

G	G_0	$G_0 \mid G_{-1}$	G	G_0	$G_0 \mid G_{-1}$	$G_0 \mid G_1$
$\mathbf{A}(m, n)$	$\mathbf{A}_m \oplus \mathbf{A}_n \oplus k$	$sl_{m+1} \otimes sl_{n+1} \otimes k$	$\mathbf{C}(n)$	$\mathbf{C}_{n-1} \oplus k$	csp_{2n-2}	csp_{2n-2}^*
$\mathbf{A}(n, n)$	$\mathbf{A}_n \oplus \mathbf{A}_n$	$sl_{n+1} \otimes sl_{n+1}$	$\mathbf{P}(n)$	$\mathbf{A}_n$	$\Lambda^2 sl_{n+1}^*$	$S^2 sl_{n+1}$

PROPOSITION 2.1.3. *Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be one of the Lie superalgebras* $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, $\mathbf{P}(n)$, *or* $\mathbf{Q}(n)$. *Then the $G_{\bar{0}}$-module $S^2G_{\bar{1}}$ contains $G_{\bar{0}}$ with multiplicity* 1.

This is not hard to prove, by using the table in [9]. Here we can also exploit the fact that in the tensor product of two irreducible $G_{\bar{0}}$-modules, of which one has a simple spectrum, the multiplicity of any simple submodule is at most 1.

PROPOSITION 2.1.4. *Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a simple Lie superalgebra for which the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is the same as for one of the Lie superalgebras* $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$ *where* $(m, n) \neq (2, 1)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, $\mathbf{P}(n)$, *or* $\mathbf{Q}(n)$. *Then G is isomorphic to this algebra.*

Proof. Let $\Phi: S^2G_{\bar{1}} \to G_{\bar{0}}$ be the homomorphism of $G_{\bar{0}}$-modules defined by the bracket on $G_{\bar{1}}$, and let Φ' be the same map for the corresponding superalgebra, as listed in the proposition. On account of simplicity, Φ and Φ' are epimorphisms. By Proposition 2.1.3, Φ and Φ' are projections of $S^2G_{\bar{1}}$ onto the same subspace. Since the homomorphism Φ, to within a constant factor, determines the superalgebra uniquely, we have to show that Φ and Φ' are proportional projections onto $G_{\bar{0}} \subseteq S^2G_{\bar{1}}$. If $G_{\bar{0}}$ is simple, this is clear; therefore, we assume that $G_{\bar{0}}$ is not simple. The projections ϕ and ϕ' can be decomposed: $\Phi = \Phi_0 + \Phi_1 + \cdots$, $\Phi' = \Phi_0' + \Phi_1' + \cdots$, where Φ_0 and Φ_0' are projections onto the center, and Φ_i and Φ_i', $i > 0$, are projections onto the simple components; $\Phi_i, \Phi_i' \neq 0$; by Schur's lemma, $\Phi_i = c_i\Phi_i'$, $c_i \in k^*$.

Now we observe that the kernel of each bilinear map Φ_i (similarly, Φ_i'): $S^2G_{\bar{1}} \to G_{\bar{0}}$ is trivial. For $\ker \Phi_i \subseteq G_{\bar{1}}$ is a $G_{\bar{0}}$-submodule; therefore, if $G_{\bar{0}}$ is irreducible on $G_{\bar{1}}$, then $\ker \Phi_i = 0$; but if $G_{\bar{0}}$ is reducible on $G_{\bar{1}}$, then $G_{\bar{1}} = G_{-1} \oplus G_1$ is a direct sum of irreducible $G_{\bar{0}}$-modules, and if $G_{-1} \subseteq \ker \Phi_i$; then we see again that $\ker \Phi_i = 0$ because $\Phi_i(G_1, G_1) = 0$.

Now suppose that Φ and Φ' are not proportional. Taking $\Phi'' = \Phi + c\Phi'$ for a suitable $c \in k$, we may assume that $\Phi''(G_{\bar{1}}, G_{\bar{1}}) = H_{\bar{0}}$ is a nonzero ideal in $G_{\bar{0}}$ that does not contain simple component $G_{\bar{0}}^{(s)}$ of $G_{\bar{0}}$ of maximal dimension. From what we have said above it follows that $H = H_{\bar{0}} \oplus G_{\bar{1}}$ is a simple Lie superalgebra. Evidently, in all cases listed in the proposition the $H_{\bar{0}}$-module $G_{\bar{1}}$ contains more than two irreducible components. This contradicts Proposition 2.2.2 below, and Proposition 2.1.4 is proved.

2.2. *Splitting of the Classification of Classical Lie Superalgebras into Two Cases*

Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a classical Lie superalgebra. Then $G_{\bar{0}} = G_{\bar{0}}' \oplus C$, where $G_{\bar{0}}'$ is a semisimple Lie algebra and C is the center of G. We treat two cases separately:

Case I. The representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is irreducible. Then $G_{\bar{0}}$ is a semisimple Lie algebra. If this is not so, then there exists a center element $z \in G_{\bar{0}}$ for which $[z, g] = 2g$ for a $g \in G_{\bar{0}}$, a contradiction.

Case II. The representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is reducible. Then we consider in G a proper maximal subalgebra L_0 containing $G_{\bar{0}}$. We construct the appropriate transitive filtration (see Section 1.3.1) $G = L_{-1} \supset L_0 \supset L_1 \supset \cdots$, where

$$L_i = \{a \in L_{i-1} \mid [a, L] \subset L_{i-1}\}, \qquad i > 0.$$

Let $\operatorname{Gr} L = \bigoplus_{i \geqslant -1} \operatorname{Gr}_i L$, where $\operatorname{Gr}_i L = L_i/L_{i+1}$ is an associated $\mathbf{Z}$-graded Lie superalgebra. This filtration induces one on $G_{\bar{0}}$: $G_{\bar{0}} = (L_0 \cap G_{\bar{0}}) \supset (L_1 \cap G_{\bar{0}}) \supset \cdots$. Since $G_{\bar{0}}$ is a reductive Lie algebra, we see that $L_1 \cap G_{\bar{0}} = 0$. Since the $\mathbf{Z}$-grading of $\operatorname{Gr} L$ is consistent with the $\mathbf{Z}_2$-grading (Proposition 1.3.2), we have $\operatorname{Gr} G_{\bar{0}} = \bigoplus_{i \geqslant 0} \operatorname{Gr}_{2i} L$, from which it follows that $\operatorname{Gr}_2 L = 0$, because $L_1 \cap G_{\bar{0}} = 0$.

Thus, $\operatorname{Gr} L = \operatorname{Gr}_{-1} L \oplus \operatorname{Gr}_0 L \oplus \operatorname{Gr}_1 L$. From Proposition 1.3.2 and Proposition 1.2.11, we now obtain:

LEMMA 2.2.1. *Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a classical Lie superalgebra for which the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is reducible. Then G has a filtration $G = L_{-1} \supset L_0 \supset L_1$ with the following properties*:

(a) *transitivity*;

(b) $L_0 = G_{\bar{0}} \oplus L_1$;

(c) *the representations of $G_{\bar{0}}$ on L_{-1}/L_0 and on L_1 are irreducible*;

(d) *either the representation of $G_{\bar{0}}$ on L_1 is faithful or* $\dim L_1 = 1$.

We now derive from Lemma 2.2.1 the following result:

PROPOSITION 2.2.2. *Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a classical Lie superalgebra for which the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is reducible. Then G has a filtration $G = L_{-1} \supset L_0 \supset L_1$ for which $\operatorname{Gr} L = \operatorname{Gr}_{-1} L \oplus \operatorname{Gr}_0 L \oplus \operatorname{Gr}_1 L$ is a simple $\mathbf{Z}$-graded Lie superalgebra*; *the representations of $\operatorname{Gr}_0 L$ on $\operatorname{Gr}_1 L$ and $\operatorname{Gr}_{-1} L$ are faithful and irreducible, $(\operatorname{Gr} L)_{\bar{0}} = \operatorname{Gr}_0 L \simeq G_{\bar{0}}$, and the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is equivalent to that of $\operatorname{Gr}_0 L$ on $\operatorname{Gr}_{-1} L \oplus \operatorname{Gr}_1 L$.*

Proof. If the center of $\operatorname{Gr}_0 L$ is nontrivial, then $L \simeq \operatorname{Gr} L$, according to Proposition 1.3.1, and there is nothing to prove. But if it is trivial, then $[\operatorname{Gr}_{-1} L, \operatorname{Gr}_1 L] = \operatorname{Gr}_0 L$, by Proposition 1.2.9. If, in addition, the representation of $\mathrm{Gr}_0 L$ on $\mathrm{Gr}_1 L$ is faithful, then all the conditions of Proposition 1.2.8 are satisfied, and therefore, $\mathrm{Gr}\, L$ is simple.

According to Lemma 2.2.1 it only remains to show that the case $\dim L_1 = 1$ is impossible. Now $G = V \oplus G_{\bar{0}} \oplus L_1$ is a decomposition into the direct sum of $G_{\bar{0}}$-invariant subspaces, $[V, L_1] = G_{\bar{0}}$ and $[V, V] \subsetneq G_{\bar{0}}$, from parity arguments, and $[G_{\bar{0}}, L_1] = 0$. Hence, it follows that $V \oplus G_{\bar{0}}$ is an ideal in G, which contradicts simplicity.

Thus, Case II leads us to a classification of the simple $\mathbf{Z}$-graded Lie superalgebras $G = G_{-1} \oplus G_0 \oplus G_1$, with $G_0 = G_{\bar{0}}$, where the representations of G_0 on G_1 and G_{-1} are faithful and irreducible.

2.3. *Classification of Lie Superalgebras with Nondegenerate Killing Form*

2.3.1. *Definition and properties of the Killing form.* The *Killing form* on a Lie superalgebra G is the bilinear form

$$(a, b) = \mathrm{str}((\mathrm{ad}\, a)(\mathrm{ad}\, b)).$$

From the properties of the supertrace (see Proposition 1.1.2) we obtain corresponding properties of the Killing form.

PROPOSITION 2.3.1. *The Killing form on a Lie superalgebra $G_{\bar{0}} \oplus G_{\bar{1}}$ has the following properties*:

$$\begin{aligned} (a, b) &= 0 \quad \text{for} \quad a \in G_{\bar{0}},\ b \in G_{\bar{1}} && (\textit{consistency}), \\ (a, b) &= (-1)^{(\deg a)(\deg b)}(b, a) && (\textit{supersymmetry}), \\ ([a, b], c) &= (a, [b, c]) && (\textit{invariance}). \end{aligned}$$

From Proposition 2.3.1 we derive, in particular, the next result.

PROPOSITION 2.3.2. *If the Killing form on $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is nondegenerate, then its restriction to $G_{\bar{0}}$ is nondegenerate, and its restriction to $G_{\bar{1}}$ gives a nondegenerate bilinear skew-symmetric form that is invariant under the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$.*

Just as for Lie algebras (see [10], for example), we can prove the following two propositions.

PROPOSITION 2.3.3. *A Lie superalgebra with a nondegenerate Killing form splits into an orthogonal direct sum of Lie superalgebras (with nondegenerate Killing forms).*

PROPOSITION 2.3.4. *Every derivation of a Lie superalgebra with nondegenerate Killing form is inner.*

PROPOSITION 2.3.5. *Let* $G = \bigoplus G_i$ *be a* $\mathbb{Z}$*-graded Lie superalgebra with nondegenerate Killing form. Then*

(a) $(G_i, G_j) = 0$ *for* $i \neq -j$,

(b) $(a, G_i) \neq 0$ *for* $a \in G_{-i}$, $a \neq 0$,

(c) *the representations of* G_0 *on* G_i *and* G_{-i} *are contragredient,*

(d) *there is an element* $z \in G_0$ *for which* $[z, g] = sg$ *for* $g \in G_s$.

Proof. If $a \in G_i$, $b \in G_j$, then $(\mathrm{ad}\, a)(\mathrm{ad}\, b)$ for $i + j \neq 0$ is clearly a nilpotent operator on G; therefore, $(a, b) = 0$, which gives (a). Now (b) follows from (a) and the fact that the Killing form is nondegenerate; (c) follows from (b).

Let us prove (d). The endomorphism D for which $D(g) = sg$ for $g \in G_s$ evidently gives rise to a derivation of degree 0. According to Proposition 2.3.4, this derivation is inner.

Now let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Lie superalgebra. On $G_{\bar{0}}$ we can define two bilinear forms:

$$(a, b)_0 = \mathrm{tr}(\mathrm{ad}\, a)(\mathrm{ad}\, b)|_{G_{\bar{0}}} \quad \text{and} \quad (a, b)_1 = \mathrm{tr}(\mathrm{ad}\, a)(\mathrm{ad}\, b)|_{G_{\bar{1}}}. \tag{2.3.1}$$

By the definition of the Killing form:

$$(a, b) = (a, b)_0 - (a, b)_1 \quad \text{for} \quad a, b \in G_{\bar{0}}. \tag{2.3.2}$$

From (2.3.1) and (2.3.2) it follows that if $G_{\bar{0}} = G_{\bar{0}}' \oplus G_{\bar{0}}''$ is a direct sum of Lie algebras and G_0' is simple, then

$$(a, b) = (1 - l)(a, b)_0 \quad \text{for} \quad a, b \in G_{\bar{0}}', \tag{2.3.3}$$

where l is the index of the representation of $G_{\bar{0}}'$ on $G_{\bar{1}}$ (see Section 1.4.3).

PROPOSITION 2.3.6. *A simple Lie superalgebra* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *with nondegenerate Killing form is classical.*

Proof. The unipotent radical N of $G_{\bar{0}}$ is known (see [9], for example) to lie in a kernel of the form $(\ ,\ )_V$ of the representation of $G_{\bar{0}}$ in V. Therefore, if $a \in N$, $b \in G_{\bar{0}}$, then by (2.3.2): $(a, b)_0 = (a, b)_1 = 0$. It follows that a lies in the kernel of the Killing form of G. Hence $N = 0$ and G is a classical Lie superalgebra, because of Proposition 1.3.2.

By means of the Killing form we can write the Jacobi identity for three odd elements in a very convenient way. Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Lie superalgebra with nondegenerate Killing form. We choose in $G_{\bar{0}}$ any basis u_i and the dual basis v_i relative to the restriction of the Killing form to $G_{\bar{0}}$. Let $a, b, c \in G_{\bar{1}}$. Then: $[a, b] = \sum_s \alpha_s v_s$. Multiplying both sides scalarly by u_i, we obtain $\alpha_i = ([a, b], u_i)$. Making use of the invariance of the Killing form, we have: $\alpha_i = (a, [b, u_i]) = -(a, [u_i, b])$. Thus, $[a, b] = -\sum_i (a, [u_i, b])v_i$.

Therefore, the Jacobi identity $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ gives us:

$$\sum_i ((a, [u_i, b])[v_i, c] + (b, [u_i, c])[v_i, a] + (c, [u_i, a])[v_i, b]) = 0. \qquad (*)$$

2.3.2. *Classification of the simple Lie superalgebras with nondegenerate Killing form in Case* I. In this section we prove the following proposition:

PROPOSITION 2.3.7. *Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a simple Lie superalgebra with nondegenerate Killing form for which the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is irreducible and $G_{\bar{1}} \neq 0$. Then G is isomorphic to* $\mathbf{B}(m, n)$, $\mathbf{D}(m, n)$ *with $m - n \neq 1$,* $\mathbf{F}(4)$, *or* $\mathbf{G}(3)$.

Before the proof we give a lemma.

As we have shown in Section 2.2, $G_{\bar{0}}$ is semisimple. Let H be a Cartan subalgebra of $G_{\bar{0}}$, Δ be the system of all roots, and Δ' be that of nonzero roots. Let $\mathscr{L}$ be the system of weights of the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$, and $G_{\bar{1}} = \oplus V_\lambda$ the weight decomposition. From Proposition 2.3.2 it follows that

$$(V_\lambda, V_\mu) = 0 \qquad \text{for} \quad \lambda \neq -\mu; \qquad (2.3.4)$$

$$\text{if } \lambda \in \mathscr{L}, \text{ then } -\lambda \in \mathscr{L} \text{ and } (v_\lambda, V_{-\lambda}) \neq 0. \qquad (2.3.5)$$

Let $G_{\bar{0}} = \oplus G_{\bar{0}}^{(s)}$ be the decomposition of $G_{\bar{0}}$ into a direct sum of simple components. Evidently, this decomposition is orthogonal relative to the bilinear forms $(\ ,\ )$ and $(\ ,\ )_0$. We denote by $(\ ,\ )_0^{(s)}$ the restriction of $(\ ,\ )_0$ to $G_{\bar{0}}^{(s)}$.

Let l_s be the index of the representation of $G_{\bar{0}}^{(s)}$ in $G_{\bar{1}}$. Let $h_1, ..., h_r$ be a basis of H formed from bases of the Cartan subalgebras $H \cap G_{\bar{0}}^{(s)}$ of $G_{\bar{0}}^{(s)}$. Let $\hat{h}_1, ..., \hat{h}_r$ be the dual basis with respect to $(\ ,\ )$ and $\tilde{h}_1, ..., \tilde{h}_r$ with respect to $(\ ,\ )_0$. From (2.3.3) it clearly follows that

$$\tilde{h}_i = (1 - l_s)\, \hat{h}_i \qquad \text{for} \quad h_i \in G_{\bar{0}}^{(s)}. \qquad (2.3.6)$$

In particular,

$$\text{Killing form is nondegenerate iff } l_s \neq 1 \text{ for some } s. \qquad (2.3.7)$$

LEMMA 2.3.8. (a) *If $\lambda \in \mathscr{L}$, $2\lambda \notin \Delta$, then*

$$(\lambda, \lambda) = \sum_s \frac{(\lambda, \lambda)_0^{(s)}}{1 - l_s} = 0.$$

(b) *If $\lambda, \mu \in \mathscr{L}$ and $\lambda \pm \mu \notin \Delta$, then*

$$(\lambda, \mu) = \sum_s \frac{(\lambda, \mu)_0^{(s)}}{1 - l_s} = 0.$$

Proof. We consider the following basis of $G_{\bar{0}}$:

$$\{u_i\} = \{e_\alpha, \alpha \in \Delta'; h_i, i = 1,..., r\}.$$

The dual basis with respect to (,) is

$$\{v_i\} = \{e_{-\alpha}, \alpha \in \Delta'; \mathring{h}_i, i = 1,..., r\}.$$

Let us prove (a). Let $\lambda \in \mathscr{L}$. We set $a = c = v_\lambda$, $b = v_{-\lambda}$, where $(v_\lambda, v_{-\lambda}) = 1$ (by (2.3.5) such a vector exists). Now we write the identity $(*)$ for the chosen bases $\{u_i\}$ and $\{v_i\}$ and the vectors a, b, c. Taking (2.3.4) into account, we have

$$(v_\lambda, [e_{-2\lambda}, v_\lambda])[e_{2\lambda}, v_{-\lambda}] = 2v_\lambda \sum_i \lambda(h_i)\,\lambda(\mathring{h}_i).$$

Therefore, if $2\lambda \notin \Delta$, then

$$\sum_i \lambda(h_i)\,\lambda(\mathring{h}_i) = 0. \tag{2.3.8}$$

Since $(\lambda, \mu) = \sum_i \lambda(h_i)\,\mu(\mathring{h}_i)$ and $(\lambda, \mu)_0 = \sum \lambda(h_i)\,\mu(\tilde{h}_i)$ for any $\lambda, \mu \in H^*$, when (2.3.6) is taken into account, it can be rewritten in the form

$$(\lambda, \lambda) = \sum_s \frac{(\lambda, \lambda)_0^{(s)}}{1 - l_s} = 0,$$

as required.

(b) is proved similarly, but in $(*)$ we must put $a = v_\lambda$, $b = v_{-\lambda}$, $c = v_\mu$. This proves Lemma 2.3.8.

Proof of Proposition 2.3.7. If $G_{\bar{0}}$ is simple, and Λ is the highest weight, then

$$\sum_s \frac{(\Lambda, \Lambda)_0^{(s)}}{1 - l_s} = \frac{(\Lambda, \Lambda)_0}{1 - l_1} \neq 0,$$

and by Lemma 2.3.8(a) it follows that $2\Lambda \in \Delta$. From Lemma 1.4.1(a) it therefore follows that $G_{\bar{0}}$-module $G_{\bar{1}}$ is isomorphic to sp_n. By Proposition 2.1.4, we now see that G is isomorphic to $\mathbf{B}(0, n/2)$. Since $l_1 = 1/(n + 2)$ (see Table III), by virtue of (2.3.7) the Killing form on $\mathbf{B}(0, n/2)$ is nondegenerate.

Suppose now that $G_{\bar{0}}$ is semisimple, but not simple. We represent $G_{\bar{0}}$ in the form $G_{\bar{0}} = G_{\bar{0}}^{\mathrm{I}} \oplus G_{\bar{0}}^{\mathrm{II}}$, where $G_{\bar{0}}^{\mathrm{I}}$ and $G_{\bar{0}}^{\mathrm{II}}$ consist of all simple components of $G_{\bar{0}}$ for which $1 - l_i$ is positive and negative, respectively. As is clear from Lemma 2.3.8(a), both these subalgebras are nontrivial. Let $\Lambda = \Lambda^{\mathrm{I}} + \Lambda^{\mathrm{II}}$ be the highest weight of the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ (where I and II indicate that the weight is restricted to the relevant direct summand). We consider a weight of the form $\mu = \mu^{\mathrm{I}} + \Lambda^{\mathrm{II}}$, where $\mu^{\mathrm{I}} \neq \pm\Lambda^{\mathrm{I}}$. Observe that, clearly,

$$\Lambda + \mu \notin \Delta. \tag{2.3.9}$$

Next, $(\Lambda, \mu) = (\Lambda^{\mathrm{I}}, \mu^{\mathrm{I}}) + (\Lambda^{\mathrm{II}}, \Lambda^{\mathrm{II}}) = (\Lambda^{\mathrm{I}}, \mu^{\mathrm{I}}) + (\Lambda, \Lambda) - (\Lambda^{\mathrm{I}}, \Lambda^{\mathrm{I}})$. Since $2\Lambda \notin \Delta$, by Lemma 2.3.8(a),

$$(\Lambda, \mu) = (\Lambda^{\mathrm{I}}, \mu^{\mathrm{I}}) - (\Lambda^{\mathrm{I}}, \Lambda^{\mathrm{I}}) \tag{2.3.10}$$

This relation can be rewritten (see the proof of Lemma 2.3.8) in the form

$$(\Lambda, \mu) = \sum_s \frac{(\Lambda, \mu)_{\bar{0}}^{(s)} - (\Lambda, \Lambda)_{\bar{0}}^{(s)}}{1 - \lambda_s}, \tag{2.3.11}$$

where the summand is over the simple components occurring in $G_{\bar{0}}^{\mathrm{I}}$. Since Λ is the highest weight, $(\Lambda, \Lambda)_{\bar{0}}^{(s)} \geqslant (\mu, \mu)_{\bar{0}}^{(s)}$ for all s. Therefore, all the terms in (2.3.11) are negative, by the Cauchy–Bunjakowskii inequality. Consequently,

$$(\Lambda, \mu) \neq 0. \tag{2.3.12}$$

Now we can use Lemma 2.3.8(b), according to which it follows from (2.3.9) and (2.3.12) that $\Lambda - \mu \in \Delta^{\mathrm{I}}$. Thus, if $\mu^{\mathrm{I}} \neq -\Lambda^{\mathrm{I}}$, then $\Lambda^{\mathrm{I}} - \mu^{\mathrm{I}} \in \Delta^{\mathrm{I}}$. Of course, the same is true for $G_{\bar{0}}^{\mathrm{II}}$. Therefore, we find from Lemma 1.4.1(c) that the linear representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ can only be equivalent to the tensor product of two of the following linear Lie algebras: sp_n, $n \geqslant 2$; sl_n, $n \geqslant 3$; so_n, $n \geqslant 3$; spin_7; G_2.

We recall now (Proposition 2.3.2) that the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ admits a nondegenerate skew-symmetric invariant bilinear form. This can only be the case when one of the factors of the tensor product has a skew-symmetric invariant and bilinear form and the other an invariant symmetric form. Therefore, only the following possibilities remain:

(1) $so_m \otimes sp_n$,

(2) $sp_n \otimes \mathrm{spin}_7$,

(3) $sp_n \otimes G_2$.

In case (1) we obtain from Proposition 2.1.4 that G is isomorphic to $\mathbf{B}(m - 1/2, n/2)$ for odd $m > 1$, or $\mathbf{D}(m/2, n/2)$ for even $m > 2$. Since $l_1 = n/m - 2$ and $l_2 = m/n + 2$, (Table III), by (2.3.7) the Killing form is nondegenerate on $\mathbf{B}(m, n)$ and also on $\mathbf{D}(m, n)$ when $m - n \neq 1$.

In cases (2) and (3) we use Lemma 2.3.8(a) again:

$$\frac{(\Lambda, \Lambda)_{\bar{0}}^{(1)}}{1 - l_1} + \frac{(\Lambda, \Lambda)_{\bar{0}}^{(2)}}{1 - l_2} = 0. \tag{2.3.13}$$

In cases (2) and (3), (2.3.13) yields that $n = 2$ and therefore we see from Proposition 2.1.4 that G is isomorphic to $\mathbf{F}(4)$ and $\mathbf{G}(3)$, respectively.

This completes the proof of Proposition 2.3.7.

2.3.3. *Classification of the simple Lie superalgebras with nondegenerate Killing form in Case* II. In this section we prove the following proposition.

PROPOSITION 2.3.9. *Let $G = G_{-1} \oplus G_0 \oplus G_1$ be a simple Lie superalgebra with a consistent $\mathbf{Z}$-grading for which the representations of G_0 on G_1 and G_{-1} are faithful and irreducible and the Killing form is nondegenerate. Then G is isomorphic (even as a $\mathbf{Z}$-graded superalgebra) to one of $\mathrm{A}(m, n)$, $m \neq n$, or $\mathrm{C}(n)$.*

The proof of this proposition is based on the same arguments as that of Proposition 2.3.7.

It follows from Proposition 2.3.5 that G_0 is the direct sum of the one-dimensional center C and the semisimple Lie algebra G_0'. Here $C = \langle z \rangle$, where $[z, g] = \pm g$ for $g \in G_{\pm 1}$, and the representations of G_0 on G_{-1} and G_1 are contragredient. Let H be a Cartan subalgebra of G_0', Δ its root system, $\mathscr{L}_{\pm 1}$ the systems of weights of the representations of G_0' on $G_{\pm 1}$. From Proposition 2.3.5 it follows that

$$\mathscr{L}_{-1} = -\mathscr{L}_1, \tag{2.3.14}$$

$$(G_1, G_1) = (G_{-1}, G_{-1}) = 0, \tag{2.3.15}$$

$$(v_\lambda, V_{-\lambda}) \neq 0 \quad \text{for} \quad \lambda \in \mathscr{L}_1, \quad -\lambda \in \mathscr{L}_{-1}. \tag{2.3.16}$$

Let $G_0' = \oplus G_0^{(s)}$ be the decomposition of G_0' into the direct sum of simple components. We denote by $(\ ,\ )_0^s$ the restriction of $(\ ,\ )_0$ to $G_0^{(s)}$ and by l_s the index of the representation of $G_0^{(s)}$ in G_1. Note that it is also the index of the representation of $G_0^{(s)}$ in G_{-1}. Just as in Section 2.2, we choose a basis $h_1, \ldots, h_r$ of H, its dual basis $\check{h}_1, \ldots, \check{h}_r$ with respect to $(\ ,\ )$ and $\tilde{h}_1, \ldots, \tilde{h}_r$ with respect to $(\ ,\ )_0$.

From (2.3.3) it follows that

$$\tilde{h}_i = (1 - 2l_s)\, \check{h}_i \quad \text{for} \quad h_i \in G_0^{(s)}. \tag{2.3.17}$$

In particular,

$$\text{Killing form is nondegenerate, iff } l_s \neq \tfrac{1}{2} \text{ for some } s. \tag{2.3.18}$$

LEMMA 2.3.10. (a) *If $\lambda \in \mathscr{L}_1$, then*

$$(\lambda, \lambda) = \sum_s \frac{(\lambda, \lambda)_0^{(s)}}{1 - 2l_s} - \frac{1}{2 \dim G_1} = 0.$$

(b) *If $\lambda, \mu \in \mathscr{L}_1$ and $\lambda - \mu \notin \Delta$, then*

$$(\lambda, \mu) = \sum_s \frac{(\lambda, \mu)_0^{(s)}}{1 - 2l_s} - \frac{1}{2 \dim G_1} = 0.$$

Proof. We consider the following basis of G_0:

$$\{u_i\} = \{e_\alpha, \alpha \in \Delta'; h_i, i = 1, ..., r; z\}.$$

The dual basis with respect to (,) is

$$\{v_i\} = \{e_{-\alpha}, \alpha \in \Delta'; \hat{h}_i, i = 1, ..., r; -1/(2 \dim G_1)z\}.$$

Let us prove (a). Let $\lambda \in \mathscr{L}_1$. We set $a = c = v_{-\lambda} \in G_{-1}$, $b = v_\lambda \in G_1$, where $(v_\lambda, v_{-\lambda}) = 1$. (According to (2.3.16) such vectors exist.) We now write down the identity (*) in the chosen bases $\{u_i\}$ and $\{v_j\}$ and the vectors a, b, c. Taking (2.3.15) and (2.3.4) into account, we have

$$(\lambda, \lambda) = \sum_i \lambda(h_i)\, \lambda(\hat{h}_i) - \frac{1}{2 \dim G_1} = 0,$$

from which, using (2.3.17), we obtain (a).

(b) is proved similarly, only in (*) we must put $a = v_{-\lambda} \in G_{-1}$, $b = v_\lambda \in G_1$, $c = v_\mu \in G_1$.

Proof of Proposition 2.3.9. We represent G_0' in the form $G_0' = G_0'^{\mathrm{I}} \oplus G_0'^{\mathrm{II}}$, where $G_0'^{\mathrm{I}}$ and $G_0'^{\mathrm{II}}$ consist of those simple components for which $1 - 2l_i$ is positive and negative, respectively. For definiteness, let $G_0'^{\mathrm{I}} \neq 0$. Let $\Lambda = \Lambda^{\mathrm{I}} + \Lambda^{\mathrm{II}}$ be the highest weight of the representation of G_0' in G_1. We consider a weight of the form $\mu = \mu^{\mathrm{I}} + \Lambda^{\mathrm{II}}$. Just as in the proof of Proposition 2.3.7 we find that

$$(\Lambda, \mu) \neq 0.$$

From Lemma 2.3.10(b) it now follows that $\Lambda^{\mathrm{I}} - \mu^{\mathrm{I}} \in \Delta^{\mathrm{I}}$. So we see that if $\mu \in \mathscr{L}^{\mathrm{I}}$, then $\Lambda^{\mathrm{I}} - \mu \in \Delta^{\mathrm{I}}$. Therefore, we obtain from Lemma 1.4.1(b) that the G_0'-module G_1 can only be isomorphic to a linear Lie algebra sl_n or sp_n or to any tensor product of them. So we have the following possibilities for the representation of G_0 on G_1:

(1) $gl_m \otimes sl_n$,

(2) csp_n,

(3) $gl_m \otimes sp_n$, $m \geqslant 2$, $n \geqslant 4$,

(4) $csp_m \otimes sp_n$, $m \geqslant 2$, $n \geqslant 4$.

In case (1), $l_1 = n/(2m)$, $l_2 = m/(2n)$ (Table III); hence, by (2.3.18), $m \neq n$. But then we see from Proposition 2.1.4 that G is isomorphic to $\mathbf{A}(m-1, n-1)$. In case (2), we derive from the same Proposition 2.1.4 that G is isomorphic to $\mathbf{C}((n/2)+1)$. From (2.3.18) it is clear that the Killing form for these Lie superalgebras is nondegenerate.

That cases (3) and (4) are impossible we deduce from Lemma 2.3.10(a):

$$\frac{(\Lambda, \Lambda)_0^{(1)}}{1 - 2l_1} + \frac{(\Lambda, \Lambda)_0^{(2)}}{1 - 2l_2} = \frac{1}{2 \dim G_1}. \tag{2.3.19}$$

In case (3), (2.3.19) yields

$$\frac{m-1}{m(2m-2n)} + \frac{1}{2(n+2-2m)} = \frac{1}{2mn},$$

so that either $m = 1$ or $n = 2$.

In case (4), (2.3.19) yields

$$\frac{1}{2(n+2-2m)} + \frac{1}{2(m+2-2n)} = \frac{1}{2mn},$$

which is impossible.

Thus, cases (3) and (4) cannot occur. This completes the proof of Proposition 2.3.9.

2.3.4. *Conclusion of the classification of simple Lie superalgebras with nondegenerate Killing form.*

THEOREM 1. *A simple finite-dimensional Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ with nondegenerate Killing form is isomorphic to one of the simple Lie algebras or to one of the following classical Lie superalgebras:*

$\mathbf{A}(m, n)$ *with* $m \neq n$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$ *with* $m - n \neq 1$, $\mathbf{F}(4)$, *or* $\mathbf{G}(3)$.

Proof. From Proposition 2.3.6 it follows that G is classical. In accordance with Section 2.2 we have to discuss two cases.

Case I. The representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is irreducible. Then Theorem 1 follows from Proposition 2.3.7.

Case II. The representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is reducible. In that case, according to Lemma 2.2.1, G has a filtration $G = L_{-1} \supset L_0 \supset L_1$ for which $\operatorname{Gr} G$ is a $\mathbf{Z}$-graded Lie superalgebra satisfying all the conditions of Proposition 2.3.9, $(\operatorname{Gr} G)_{\bar{0}} \simeq G_{\bar{0}}$, and the $(\operatorname{Gr} G)_{\bar{0}}$-module $(\operatorname{Gr} G)_{\bar{1}}$ is isomorphic to the $G_{\bar{0}}$-module $G_{\bar{1}}$. By Proposition 2.3.9, $\operatorname{Gr} G$ is one of $\mathbf{A}(m, n)$, $m \neq n$, or $\mathbf{C}(n)$. From Proposition 2.1.4 it now follows that $G \simeq \operatorname{Gr} G$, and the theorem is proved.

2.4. *Completion of the Classification of the Classical Lie Superalgebras*

The classification of the classical Lie superalgebras is given by the following theorem.

THEOREM 2. *A classical Lie superalgebra is isomorphic either to one of the simple Lie algebras* $\mathbf{A}_n$, $\mathbf{B}_n$,..., $\mathbf{E}_8$, *or to one of* $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, $\mathbf{P}(n)$, *or* $\mathbf{Q}(n)$.

By virtue of Theorem 1 and of Proposition 1.2.6, what remains to be proved is the following proposition.

PROPOSITION 2.4.1. *A classical Lie superalgebra* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *with zero Killing form is isomorphic to one of* $\mathbf{A}(n, n)$, $\mathbf{D}(n+1, n)$, $\mathbf{P}(n)$, $\mathbf{Q}(n)$, *or* $\mathbf{D}(2, 1; \alpha)$.

2.4.1. *Beginning of the proof of Proposition* 4.1.

LEMMA 2.4.2. *Let* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *be a Lie superalgebra with zero Killing form for which the representation of* $G_{\bar{0}}$ *on* $G_{\bar{1}}$ *is faithful and completely reducible. Then the index* l_i *of the representation of any simple component* $G_{\bar{0}}^{(i)}$ *of* $G_{\bar{0}}$ *on* $G_{\bar{1}}$ *is* 1. *In particular, the index of the representation of* $G_{\bar{0}}^{(i)}$ *on any irreducible component of* $G_{\bar{1}}$ *does not exceed* 1.

Proof. By (2.3.3), $(a, b) = (1 - l_i)(a, b)_0$ for $a, b \in G_{\bar{0}}^{(i)}$. Since $(a, b) = 0$ and $(a, b)_0$ is nondegenerate on $G_{\bar{0}}^{(i)}$, we see that $l_i = 1$.

2.4.2. *Classification of the classical Lie superalgebras with zero Killing form in Case* I.

PROPOSITION 2.4.3. *Let* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *be a simple Lie superalgebra with zero Killing form for which the representation of* $G_{\bar{0}}$ *on* $G_{\bar{1}}$ *is irreducible. Then* G *is isomorphic to one of* $\mathbf{Q}(n)$, $\mathbf{D}(n+1, n)$, *or* $\mathbf{D}(2, 1; \alpha)$.

Proof. As we remarked in Section 2.2, $G_{\bar{0}}$ is semisimple. Therefore, the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is equivalent to the tensor product of some simple irreducible linear Lie algebras:

$$G_{\bar{0}}(G_{\bar{1}}) = G_{\bar{0}}^{(1)}(V_1) \otimes \cdots \otimes G_{\bar{0}}^{(k)}(V_k). \tag{2.4.1}$$

Let $\tilde{l}_1$,..., $\tilde{l}_k$ be their indices. Then, clearly, the index l_i of the representation of $G_{\bar{0}}^{(i)}$ on $G_{\bar{1}}$ is equal to

$$l_i = \tilde{l}_i \prod_{s \neq i} \dim V_s = 1, \tag{2.4.2}$$

according to Lemma 2.4.2.

If $G_{\bar{0}}$ is simple, then $l_1 = \tilde{l}_1 = 1$, and so (see Section 1.4.3) the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is the adjoint one, and an epimorphism of $G_{\bar{0}}$-modules, $S^2 G_{\bar{1}} \to G_{\bar{0}}$ exists only when $G_{\bar{0}}$ is of type $\mathbf{A}_n$ (see [9], for example). From Proposition 2.1.4, it follows that this case leads to $\mathbf{Q}(n)$.

Now suppose that $G_{\bar{0}}$ is not simple. Then it follows from (2.4.2) that $\tilde{l}_i = t_i^{-1}$, where $t_i \geqslant 2$ is an integer. If $t_i \leqslant 4$, then $\prod_{s\neq i} \dim V_s \leqslant 4$ by (2.4.2); hence, $\dim V_s \leqslant 4$ for $s \neq i$. From Table III (see Section 1.4.3) it is clear that if $\dim V_s \leqslant 4$, then $t_s \leqslant 8$, and by (2.4.2), then $\dim V_i \leqslant 8$. Thus, from $t_i \leqslant 4$ it follows that $\dim V_i \leqslant 8$. From Table III we can now see that only sl_n, $sp_n so_n$, spin_7, and G_2 can occur in (2.4.1). We claim that the last two cases are impossible. If $G_{\bar{0}}^{(1)}(V_1) = \mathrm{spin}_7$ in (2.4.1), then $s = 2$ and $\dim V_2 = 5$. But then $G_{\bar{0}}^{(2)}(V_2) = sl_5$ or so_5. In the first case $l_2 = \frac{8}{10}$, and in the second $l_2 = \frac{8}{3}$. If $G_{\bar{0}}^{(1)}(V_1) = G_2$, then $G_{\bar{0}}(G_{\bar{1}}) = G_2 \otimes sl_4$ or $G_2 \otimes sl_2 \otimes sl_2$ or $G_2 \otimes sp_4$. In the first case $l_2 = \frac{7}{8}$, in the second $l_2 = \frac{7}{4}$, and in the third $l_2 = \frac{7}{6}$. Hence, only sl_n, sp_n, and so_n can occur in (2.4.1):

$$G_{\bar{0}}(G_{\bar{1}}) = sl_{n_1} \otimes \cdots \otimes sl_{n_\alpha} \otimes sp_{r_1} \otimes \cdots \otimes sp_{r_\beta} \otimes so_{m_1} \otimes \cdots \otimes so_{m_\gamma},$$

where $2 \leqslant n_1 \leqslant \cdots \leqslant n_\alpha$, $4 \leqslant r_1 \leqslant \cdots \leqslant r_\beta$, $5 \leqslant m_1 \leqslant \cdots \leqslant m_\gamma$.

Relations (2.4.2) can be rewritten in the form

$$\Big(\prod_{s\neq i} n_s\Big)\Big(\prod_{s} r_s\Big)\Big(\prod_{s} m_s\Big) = 2n_i, \tag{2.4.3}$$

$$\Big(\prod_{s} n_s\Big)\Big(\prod_{s\neq i} r_s\Big)\Big(\prod_{s} m_s\Big) = r_i + 2, \tag{2.4.4}$$

$$\Big(\prod_{s} n_s\Big)\Big(\prod_{s} r_s\Big)\Big(\prod_{s\neq i} m_s\Big) = m_i - 2. \tag{2.4.5}$$

From these relations it is evident that $\alpha \leqslant 3$, $\beta \leqslant 2$, $\gamma \leqslant 1$. If $\alpha = 3$, then it is clear from (2.4.3) that $G_{\bar{0}}(G_{\bar{1}}) = sl_2 \otimes sl_2 \otimes sl_2$, and we have $\mathbf{D}(2, 1; \delta)$. If $\alpha = 2$, the only possibility is $G_{\bar{0}}(G_{\bar{1}}) = sl_{n_1} \otimes sl_{n_2}$; but then $n_1 = 2n_2$, $n_2 = 2n_1$, which is impossible. If $\beta = 2$, then by (2.4.4) the only possibility is $G_{\bar{0}}(G_{\bar{1}}) = sp_r \otimes sp_{r+2}$, which clearly cannot be realized. If $\alpha = \beta = \gamma = 1$, then we have by multiplying (2.4.3), (2.4.4), and (2.4.5), $n_1 r_1^2 m_1^2 = 2(r_1 + 2)(m_1 - 2)$, which is impossible. The cases $\alpha = \beta = 1$, $\gamma = 0$ and $\alpha = \gamma = 1$, $\beta = 0$, are also impossible. There remains the case $G_{\bar{0}}(G_{\bar{1}}) = sp_r \otimes so_n$. Then $n = r + 2$ (see Table III).

Thus, the only remaining possibility for the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$ is $so_{n+2} \otimes sp_n$, $n > 2$. This is, in fact, realized for $\mathbf{D}(n/2 + 1, n/2)$. By Proposition 2.1.4, there can only be one superalgebra with this representation of $G_{\bar{0}}$ on $G_{\bar{1}}$. This proves the proposition.

2.4.3. *Classification of the classical Lie superalgebras with zero Killing form in Case* II.

PROPOSITION 2.4.4. *Let $G = G_{-1} \oplus G_0 \oplus G_1$ be a simple Lie superalgebra with a consistent $\mathbf{Z}$-grading for which the representations of G_0 on G_1 and G_{-1}*

are faithful and irreducible, and the Killing form is zero. Then G is isomorphic (even as a $\mathbf{Z}$-graded superalgebra) to one of $\mathbf{A}(n, n)$ *or* $\mathbf{P}(n)$.

Before the proof we give two lemmas.

LEMMA 2.4.5. *Under the conditions of Proposition* 2.4.4, *G_0 is semisimple.*

Proof. If the center C of G_0 is nontrivial, then by Proposition 1.2.12, there exists a $z \in C$ such that $[z, g] = \pm g$ for $g \in G_{\pm 1}$. But then, clearly, $(z, z) = -\dim G_{-1} - \dim G_1$, which is impossible, because the Killing form is zero.

LEMMA 2.4.6. *If under the conditions of Proposition* 2.4.4 *the representations of G_0 on G_1 and G_{-1} are contragradient, then the highest weight of the representation of G_0 on G_1 must have more than one nonzero numerical mark.*

Proof. Let H be a Cartan subalgebra of G_0, $\alpha_1, \ldots, \alpha_r$ its system of simple roots, $h_1, \ldots, h_r$ a basis of H, where $h_i = [e_{\alpha_i}, e_{-\alpha_i}]$, $\alpha_i(h_i) = 2$. If E_{α_0} is the lowest weight vector of the representation of G_0 on G_{-1}, and $F_{-\alpha_0}$ the highest weight vector of that of G_0 on G_1, $[E_{\alpha_0}, F_{-\alpha_0}] = h_0 \in H$, $h_0 \neq 0$ (see Proposition 1.2.10(a)), then

$$\alpha_0(h_0) = 0, \tag{2.4.6}$$

$$\alpha_i(h_0) = 0 \quad \text{for} \quad \alpha_0(h_i) = 0, \tag{2.4.7}$$

$$\det(\alpha_i(h_j))_{i,j=0}^{r} = 0. \tag{2.4.8}$$

(2.4.6) follows from $0 = [E_{\alpha_0}, [E_{\alpha_0}, F_{-\alpha_0}]] = -2\alpha_0(h_0)F_{\alpha_0}$, (2.4.7) is obtained by multiplying both sides of $[E_{\alpha_0}, F_{-\alpha_0}] = h_0$ by e_{α_i}, and (2.4.8) follows from the linear dependence of the vectors $h_0, h_1, \ldots, h_r$.

Suppose now that $\alpha_0(h_s) \neq 0$ for one s only. Then $\alpha_s(h_0) \neq 0$; otherwise, $h_0 = 0$ by (2.4.6) and (2.4.7), and the remaining elements of the first column are zeros. By hypothesis, $\alpha_0(h_s) \neq 0$, but the remaining elements of the first row are zeros. It then follows that $\det(\alpha_i(h_j)) = \alpha_0(h_s)\, \alpha_s(h_0) \det A$, where A is the Cartan matrix of the Dynkin diagram of G_0, with the sth circle omitted. Since $\det A \neq 0$, we have reached a contradiction to (2.4.8).

We divide the proof of Proposition 2.4.4 into two cases corresponding to the following two lemmas.

LEMMA 2.4.7. *If under the conditions of Proposition* 2.4.4 *the representations of G_0 on G_{-1} are contragradient, then G is isomorphic to* $\mathbf{A}(n, n)$.

Proof. By Lemma 2.4.5, the representation of G_0 on G_1 is equivalent to the tensor product of some simple irreducible linear Lie algebras:

$$G_0(G_1) = G_0^{(1)}(V_1) \otimes \cdots \otimes G_0^{(k)}(V_k).$$

Let $l_1, \ldots, l_k$ be their indices. Since contragredient representations have equal indices, the index of the representation of $G_0^{(i)}$ on G_1 is $\frac{1}{2}$ (by Lemma 2.4.2). Therefore,

$$l_i \prod_{s \neq i} \dim V_s = \tfrac{1}{2}. \tag{2.4.9}$$

If G_0 is simple, then we see from (2.4.9) that $l_1 = \frac{1}{2}$. From Table III it is clear that there are only the two possibilities for $G_0(G_1)$: $\Lambda^3 sl_6$ and $\Lambda_0{}^3 sp_6$. Both cannot occur according to Lemma 2.4.6.

Suppose now that G_0 is not simple. From Table III it is clear that $2(\dim V_i) l_i \geqslant 1$ and that equality holds for sl_n only. Therefore, we see from (2.4.9) at once that the only possibility for $G_0(G_1)$ is $sl_n \otimes sl_n$. By Proposition 2.1.4, G is then isomorphic to $\mathbf{A}(n-1, n-1)$, and the lemma is proved.

LEMMA 2.4.8. *Under the conditions of Proposition* 2.4.4, *if the representations of G_0 on G_{-1} and G_1 are not contragredient, then G is isomorphic to* $\mathbf{P}(n)$.

Proof. It follows from Lemma 2.4.5 that G_0 is semisimple, and from Proposition 1.2.10 (and Proposition 1.2.1) that G_0 is simple. Thus, the relevant pair of representations of G_0 on G_{-1} and G_1 can only be one from Table III, for which the sum of the indices is 1. This leads to the following cases:

(1) sl_8 and $\Lambda^3 sl_8$;
(2) $sl_8{}^*$ and $\Lambda^3 sl_8$;
(3) $\Lambda^2 sl_n$ and $S^2 sl_n$, $n > 4$;
(4) $\Lambda^2 sl_n{}^*$ and $S^2 sl_n$.

Proposition 1.2.10(b) imposes yet another restriction: If Λ is the highest weight of the representation of G_0 on G_{-1} and M is the lowest weight of that on G_1, then $\Lambda + M$ is a root of G_0. This rules out cases (1)–(3) at once. The fourth case corresponds (on the basis of Proposition 2.1.4) only to $\mathbf{P}(n)$, and the lemma is proved.

The conclusion of the proof of Proposition 2.4.1 proceeds verbatim on the same lines as that of Theorem 1 (see Section 2.3.4), on the basis of Propositions 2.4.3 and 2.4.4.

This completes the proof of Theorem 2.

Additional Remark. The following authors independently obtained classification results on classical Lie superalgebras under the following restrictions:

A. Pais and V. Rittenberg, *J. Math. Phys.* **16** (1975), 2062–2073, $G_{\bar{0}}$ is simple and the Killing form is nondegenerate.

D. Z. Djokovic, *J. Pure Appl. Algebra* **7** (1976), 217–230, $G_{\bar{0}}$ is simple.

P. G. O. Freund and I. Kaplansky, *J. Math. Phys.* **17** (1976), 228–231, $G_{\bar{0}}$ is reductive and G admits bilinear invariant form.

W. Nahm, V. Rittenberg, and M. Scheunert, *Phys. Lett. B* **61** (1976), 383–385, Killing form is nondegenerate (and later all classical ones in *Journal of Math. Phys.* **17** (1976), 1626–1640).

2.5. *Contragredient Lie Superalgebras*

2.5.1. *Definition of the superalgebras* $G(A, \tau)$. Let $A = (a_{ij})$ be an $(r \times r)$-matrix with elements from a field k and τ be a subset of $I = \{1, 2,..., r\}$. Let G_{-1}, G_0, and G_1 be vector spaces over k with bases $\{f_i\}$, $\{h_i\}$, and $\{e_i\}$, $i \in I$, respectively. As is easy to see, the following relations determine the structure of a local Lie superalgebra $\hat{G}(A, \tau)$ on the space $G_{-1} \oplus G_0 \oplus G_1$:

$$\begin{gathered}
[e_i, f_j] = \delta_{ij} h_i, \qquad [h_i, h_j] = 0, \\
[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j, \\
\deg h_i = \bar{0}, \qquad \deg e_i = \deg f_i = \bar{0} \quad \text{for} \quad i \notin \tau, \\
\deg e_i = \deg f_i = \bar{1} \quad \text{for} \quad i \in \tau.
\end{gathered}$$

According to Proposition 1.2.2, there exists a minimal $\mathbf{Z}$-graded Lie superalgebra $G(A, \tau)$ with local part $\hat{G}(A, \tau)$. We call $G(A, \tau)$ *contragredient Lie superalgebra*, A its *Cartan matrix*, and r its *rank*. Note that when $\tau = \phi$, we have contragredient Lie algebras whose theory is developed in [11].

When h_i is replaced by ch_i and f_i by cf_i, $c \in k^*$, then the ith row of A is multiplied by c. Therefore, we may (and will) assume that if $a_{ii} \neq 0$, then $a_{ii} = 2$. If we can obtain the pair $(\tilde{A}, \tilde{\tau})$ from (A, τ) by multiplying several rows by nonzero constants and renumbering the indices, then we regard (A, τ) and $(\tilde{A}, \tilde{\tau})$ as equivalent; the corresponding contragredient Lie superalgebras are isomorphic.

Observe that if $I_1 \subset I$, A_1 is the corresponding principal minor of A, and $\tau_1 = I_1 \cap \tau$, then the subalgebra of $G(A, \tau)$ generated by the elements e_i, f_i, and h_i, $i \in I_1$, is isomorphic to $G(A_1, \tau_1)$. Note also that if A is decomposable,

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix};$$

then $G(A, \tau)$ splits into the direct sum of the algebras with the Cartan matrices A_1 and A_2.

Let $H = \langle h_1, ..., h_r \rangle$, let $\alpha_1, ..., \alpha_r$ be the linear functions on H defined by the relations $\alpha_i(h_j) = a_{ji}$, $j = 1,..., r$, and let M be the free Abelian group with the generators $\alpha_1, ..., \alpha_r$. We set $G_\alpha = \langle [\cdots[e_{i_1}, e_{i_2}],..., e_{i_t}] \rangle$, $G_{-\alpha} = \langle [\cdots[f_{i_1}, f_{i_2}],..., f_{i_t}] \rangle$, $\alpha = \sum \alpha_{i_s}$.

As in [11], it is easy to show that $G(A, \tau) = H \oplus (\oplus G_\alpha)$.

Many assertions about contragredient Lie algebras in [8, 11] remain valid for Lie superalgebras (with the same proofs). Here we state only those that are needed in what follows.

PROPOSITION 2.5.1. *The center C of $G(A, \tau)$ consists of the elements of the form* $\sum \gamma_i h_i$, *where* $\sum a_{ij}\gamma_i = 0$.

PROPOSITION 2.5.2. *Let $G(A, \tau)$ be finite-dimensional and C be its center. Then $G(A, \tau)/C$ is simple if and only if*

$$\begin{aligned}&\text{for any } i, j \in I \text{ there exists a sequence } i_1, \dots, i_t \in I\\ &\text{for which } a_{ii_1}a_{i_1i_2}, \dots, a_{i_tj} \neq 0.\end{aligned} \tag{m}$$

PROPOSITION 2.5.3. *Suppose that $G(A, \tau)$ is finite-dimensional and satisfies* (m). *Then on $G(A, \tau)/C$ there is a nondegenerate consistent supersymmetric invariant bilinear form. This induces a form $(\ ,\ )$ on $G(A, \tau)$ having the following properties*:

(1) *The kernel of the form $(\ ,\ )$ is C;*

(2) $(G_\alpha, G_\beta) = 0$ *when* $\alpha \neq -\beta$;

(3) *the form $(\ ,\ )$ determines a nondegenerate pairing of $G_{-\alpha}$ with G_α;*

(4) $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$, *where h_α is a nonzero vector in H for which* $(h_\alpha, h) = \alpha(h)$, $h \in H$.

Proof. The mapping θ: $e_i \mapsto -f_i$, $h_i \mapsto -h_i$, $f_i \mapsto -(-1)^{\deg \alpha_i} e_i$ evidently induces an automorphism of $G(A, \tau)/C$. Hence, all the conditions for Proposition 1.2.5 are satisfied and the required form exists. The remaining properties are proved as in [11].

2.5.2. *Existence of the exceptional Lie superalgebras* $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, *and* $\mathbf{G}(3)$.

PROPOSITION 2.5.4. *We consider the following matrices* ($\alpha \in k/\{0, -1\}$):

$$D_\alpha = \begin{bmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix}.$$

(a) *For* $G(D_\alpha, \{1\})$, $G(F_4, \{1\})$, *and* $G(G_3, \{1\})$ *the $G_{\bar 0}$-module $G_{\bar 1}$ is isomorphic to* $sl_2 \otimes sl_2 \otimes sl_2$, $\mathrm{spin}_7 \otimes sl_2$, *and* $G_2 \otimes sl_2$, *respectively.*

(b) *The $G(D_\alpha, \{1\})$ exhaust all simple Lie superalgebras $G = G_{\bar 0} \oplus G_{\bar 1}$ for which the $G_{\bar 0}$-module $G_{\bar 1}$ is isomorphic to $sl_2 \otimes sl_2 \otimes sl_2$. Two members* $\mathbf{D}(2, 1; \alpha)$ *and* $\mathbf{D}(2, 1; \beta)$ *of this family are isomorphic if and only if α and β lie in the same orbit of the group V of order 6 generated by* $\alpha \mapsto -1 - \alpha$, $\alpha \mapsto 1/\alpha$.

Proof. (a) and the condition for isomorphy of members of the family $\mathbf{D}(2, 1; \alpha)$ are established exactly as for [8, Proposition 3.6].

Now let $G = G_{\bar 0} \oplus G_{\bar 1}$ be a simple Lie superalgebra for which the $G_{\bar 0}$-

module $G_{\bar{1}}$ is isomorphic to $sl_2 \otimes sl_2 \otimes sl_2$. Then $G_{\bar{0}} = \mathbf{A}_1^{(1)} \oplus \mathbf{A}_1^{(2)} \oplus \mathbf{A}^{(3)}$, where $\mathbf{A}_1^{(i)} = \langle e_{\alpha_i}, h_{\alpha_i}, f_{-\alpha_i} \rangle$. Let e_1 and f_1 be the lowest and highest weight vectors. Since G is simple, $[e_1, f_1] = h_1 \neq 0$. We define a $\mathbf{Z}$-grading $G = \oplus G_i$ on G by setting $G_{-1} = \langle f_1, f_{\alpha_2}, f_{\alpha_3} \rangle$, $G_0 = \langle h_1, h_{\alpha_2}, h_{\alpha_2} \rangle$, and $G_1 = \langle e_1, e_{\alpha_2}, e_{\alpha_3} \rangle$. Then G is equipped with the structure of a contragredient Lie superalgebra $G(A, \tau)$, where

$$A = \begin{bmatrix} 0 & \alpha_2(h_1) & \alpha_3(h_1) \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}, \qquad \tau = \{1\}.$$

Since G is simple, $\alpha_2(h_1), \alpha_3(h_1) \neq 0$. So we have $G \simeq \mathbf{D}(2, 1; \alpha_3(h_1)\,\alpha_2(h_1)^{-1})$.

Remark. The family $\mathbf{D}(2, 1; \alpha)$ becomes a family of Lie algebras if the characteristic of the field is 2. This family of Lie algebras is studied in [8]. Note that group V has three exceptional orbits: $\{0, -1, \infty\}$, $\{1, -2, -\frac{1}{2}\}$, and $\{-\frac{1}{2} \pm i3^{1/2}/2\}$. $\mathbf{D}(2, 1)$ corresponds to the second orbit. Superalgebra, corresponded to the third orbit, admits an outer automorphism of order 3.

2.5.3. *The root decomposition of the classical Lie superalgebras.* Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Lie superalgebra and H be a Cartan subalgebra of $G_{\bar{0}}$. We call H a *Cartan subalgebra* of G. Since every inner automorphism of $G_{\bar{0}}$ extends to one of G (see Proposition 1.1.1) and Cartan subalgebras of a Lie algebra are conjugate, so are Cartan subalgebras of a Lie superalgebra.

A Cartan subalgebra of a classical Lie superalgebra is diagonalizable. Therefore, we have the root decomposition:

$$G = \bigoplus_{\alpha \in H^*} G_\alpha, \qquad \text{where} \quad G_\alpha = \{a \in G \mid [h, a] = \alpha(h)\, a \text{ for } h \in H\}. \tag{2.5.1}$$

The set $\Delta = \{\alpha \in H^* \mid G_\alpha \neq 0\}$ is called the root system. Clearly, $\Delta = \Delta_0 \cup \Delta_1$, where Δ_0 is the root system of $G_{\bar{0}}$ and Δ_1 is the system of weights of the representation of $G_{\bar{0}}$ on $G_{\bar{1}}$; Δ_0 is called the *system of even* and Δ_1 that of *odd roots.*

A straightforward inspection of examples of classical Lie superalgebras together with standard arguments from the theory of Lie algebras yields the following information on their root decompositions.

Proposition 2.5.5. *Let G be a classical Lie superalgebra and let $G = \oplus G_\alpha$ be its root decomposition relative to a Cartan subalgebra H.*

(a) *$G_0 = H$ in all cases except* $\mathbf{Q}(n)$;

(b) $\dim G_\alpha = 1$ *for* $\alpha \neq 0$, *except for* $\mathbf{A}(1, 1)$, $\mathbf{P}(2)$, $\mathbf{P}(3)$, *and* $\mathbf{Q}(n)$;

(c) *On G there is one and, up to a constant factor, only one nondegenerate invariant supersymmetric bilinear form* $(\ ,\ )$, *except for* $\mathbf{P}(n)$ *and* $\mathbf{Q}(n)$.

(d) *Δ_0 and Δ_1 are invariant under the Weil group W of $G_{\bar{0}}$.*

(e) *If G is* $\mathbf{A}(m, n)$, $(m, n) \neq (1, 1)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, *or* $\mathbf{G}(3)$, *then the following properties hold:*

(1) $[G_\alpha, G_\beta] \neq 0$ *if and only if* $\alpha, \beta, \alpha + \beta \in \Delta$;

(2) $(G_\alpha, G_\beta) = 0$ *for* $\alpha \neq -\beta$;

(3) *the form* (,) *determines a nondegenerate pairing of* G_α *with* $G_{-\alpha}$;

(4) $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$, *where* h_α *is the nonzero vector determined by* $(h_\alpha, h) = \alpha(h)$, $h \in H$;

(5) *if* α *is in* Δ (*or* Δ_0, *or* Δ_1), *then so is* $-\alpha$;

(6) $k\alpha \in \Delta$ *for* $\alpha \neq 0$, $k \neq \pm 1$, *if and only if* $\alpha \in \Delta_1$ *and* $(\alpha, \alpha) \neq 0$; *here* $k = \pm 2$.

2.5.4. *Explicit description of the system of roots and of simple roots.* A system of roots $\Pi = \{\alpha_1, ..., \alpha_r\} \subseteq \Delta$ is said to be simple if there are vectors $e_i \in G_{\alpha_i}$, $f_i \in G_{-\alpha_i}$, for which $[e_i, f_j] = \delta_{ij} h_i \in H$, the vectors e_i and f_i, $i = 1, ..., r$, generate G, and Π is minimal with these properties. Below we describe for $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, and $\mathbf{G}(3)$ the systems of even nonzero roots Δ_0' and of odd roots Δ_1, and all systems of simple roots, up to W-equivalence.

In all the examples the Cartan subalgebra H is a subspace of the space D of diagonal matrices; the roots are expressed in terms of the standard basis ϵ_i of D^* (more accurately, the restrictions of the ϵ_i to H).

$\mathbf{A}(m, n)$. The roots are expressed in terms of linear functions $\epsilon_1, ..., \epsilon_{m+1}$, $\delta_1 = \epsilon_{m+2}, ..., \delta_{n+1} = \epsilon_{m+n+2}$.

$$\Delta_0' = \{\epsilon_i - \epsilon_j ; \delta_i - \delta_j\}, \qquad i \neq j; \qquad \Delta_1 = \{\pm(\epsilon_i - \delta_j)\}.$$

Up to W-equivalence, all the systems of simple roots are determined by two increasing sequences $S = \{s_1 < s_2 < \cdots\}$ and $T = \{t_1 < t_2 < \cdots\}$ and a sign:

$$\Pi_{S,T} = \pm\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, ..., \epsilon_{s_1} - \delta_1, \delta_1 - \delta_2, ..., \delta_{t_1} - \epsilon_{s_1+1}, ...\}.$$

The simplest such system is

$$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, ..., \epsilon_{m+1} - \delta_1, \delta_1 - \delta_2, ..., \delta_n - \delta_{n+1}\}.$$

$\mathbf{B}(m, n)$. The roots are expressed in terms of linear functions $\epsilon_1, ..., \epsilon_m$, $\delta_1 = \epsilon_{2m+1}, ..., \delta_n = \epsilon_{2m+n}$.

$$\Delta_0' = \{\pm\epsilon_i \pm \epsilon_j ; \pm 2\delta_i ; \pm\epsilon_i ; \pm\delta_i \pm \delta_j\}, \qquad i \neq j; \qquad \Delta_1 = \{\pm\delta_i ; \pm\epsilon_i \pm \delta_j\}.$$

Up to W-equivalence, all the systems of simple roots are determined by two increasing sequences S and T:

$$\Pi_{S,T} = \{\epsilon_1 - \epsilon_2, ..., \epsilon_{s_1} - \delta_1, \delta_1 - \delta_2, ..., \delta_{t_1} - \epsilon_{s_1+1}, ..., \pm\delta_n \text{ (or } \pm\epsilon_m)\}.$$

The simplest such system is

$$\{\delta_1 - \delta_2, \ldots, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_m\}$$

if $m > 0$, and

$$\{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n\}$$

if $m = 0$.

C(n). The roots are expressed in terms of linear functions ϵ_1, $\delta_1 = \epsilon_3, \ldots, \delta_{n-1} = \epsilon_{n+1}$.

$$\Delta_0' = \{\pm 2\delta_i\,;\ \pm\delta_i \pm \delta_j\}; \qquad \Delta_1 = \{\pm\epsilon_1 \pm \delta_i\}.$$

Up to W-equivalence there are the following systems of simple roots:

$$\pm\{\epsilon_1 - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-2} - \delta_{n-1}, 2\delta_{n-1}\};$$
$$\pm\{\delta_1 - \delta_2, \ldots, \delta_i - \epsilon_1, \epsilon_1 - \delta_{i+1}, \ldots, \delta_{n-2} - \delta_{n-1}, 2\delta_{n-1}\};$$
$$\pm\{\delta_1 - \delta_2, \ldots, \delta_{n-2} - \delta_{n-1}, \delta_{n-1} - \epsilon_1, \delta_{n-1} + \epsilon_1\}.$$

D(m, n). The roots are expressed in terms of linear functions $\epsilon_1, \ldots, \epsilon_m$, $\delta_1 = \epsilon_{2m+1}, \ldots, \delta_n = \epsilon_{2m+n}$.

$$\Delta_0' = \{\pm\epsilon_i \pm \epsilon_j\,;\ \pm 2\delta_i\,;\ \pm\delta_i \pm \delta_j\}, \qquad i \neq j, \qquad \Delta_1 = \{\pm\epsilon_i \pm \delta_j\}.$$

Up to W-equivalence, all the systems of simple roots are determined by two increasing sequences S and T, and a number:

$$\Pi^1_{S,T} = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{s_1} - \delta_1, \delta\ - \delta_2, \ldots, \delta_{t_1} - \epsilon_{s_1+1}, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_{m-1} + \epsilon_m$$
$$(\text{or } \delta_n - \epsilon_m, \delta_n + \epsilon_m)\};$$

$$\Pi^2_{S,T} = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{s_1} - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{t_1} - \epsilon_{s_1+1}, \ldots, \delta_{n-1} - \delta_n, 2\delta_n\}.$$

The simplest such systems are

$$\{\delta_1 - \delta_2, \ldots, \delta_n - \epsilon_1, \epsilon_1 - \epsilon_2, \ldots, \epsilon_{m-1} - \epsilon_m, \epsilon_{m-1} + \epsilon_m\};$$
$$\{\epsilon_1 - \epsilon_2, \ldots, \epsilon_m - \delta_1, \delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, 2\delta_n\}.$$

D(2, 1; α). The roots are expressed in terms of linear functions ϵ_1, ϵ_2, and ϵ_3.

$$\Delta_0' = \{\pm 2\epsilon_i\}; \qquad \Delta_1 = \{\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3\}.$$

Up to W-equivalence there are four systems of simple roots:

$$\{\epsilon_1 + \epsilon_2 + \epsilon_3, -2\epsilon_i, -2\epsilon_j\}, \qquad i \neq j, \qquad i, j = 1, 2,3;$$
$$\{\epsilon_1 + \epsilon_2 + \epsilon_3, \epsilon_1 - \epsilon_2 - \epsilon_3, -\epsilon_1 - \epsilon_2 + \epsilon_3\}.$$

F(4). The roots are expressed in terms of linear functions $\epsilon_1, \epsilon_2, \epsilon_3$, corresponding to $\mathbf{B}_3$, and δ, corresponding to $\mathbf{A}_1$.

$$\Delta_0' = \{\pm\epsilon_i \pm \epsilon_j\,;\ \pm\epsilon_i\,;\ \pm\delta\}, \qquad i \neq j; \qquad \Delta_1 = \{\tfrac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \delta)\}.$$

Up to W-equivalence there are four systems of simple roots:

$$\{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta), -\epsilon_1, \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\};$$
$$\{-\delta, \tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta), -\epsilon_1, \epsilon_1 - \epsilon_2\};$$
$$\{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta), \tfrac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3 - \delta), \tfrac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3 + \delta), \epsilon_1 - \epsilon_2\};$$
$$\{\tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta), \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \delta), \epsilon_2 - \epsilon_1, \epsilon_3 - \epsilon_2\}.$$

G(3). The roots are expressed in terms of linear functions $\epsilon_1, \epsilon_2, \epsilon_3$, corresponding to $\mathbf{G}_2$, $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$, and δ, corresponding to $\mathbf{A}_1$.

$$\Delta_0' = \{\epsilon_i - \epsilon_j\,;\ \pm\epsilon_i\,;\ \pm 2\delta\}; \qquad \Delta_1 = \{\pm\epsilon_i \pm \delta;\ \pm\delta\}.$$

Up to W-equivalence there is a unique system of simple roots:

$$\{\delta + \epsilon_1, \epsilon_2, \epsilon_3 - \epsilon_2\}.$$

2.5.5. *Examples of finite-dimensional contragredient Lie superalgebras.* Examples can be obtained in the following manner. Let G be one of the Lie superalgebras $sl(m+1, n+1)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, or $\mathbf{G}(3)$. Let H be a Cartan subalgebra, Π be one of its systems of simple roots (listed in Section 2.5.4 above), and e_i and f_i be the corresponding nonzero vectors in G_{α_i} and $G_{-\alpha_i}$, respectively, $\alpha_i \in \Pi$. Then the vectors $[e_i, f_i] = h_i$ form a basis of H. Setting $\deg e_i = -\deg f_i = 1$, $\deg h_i = 0$, we define a $\mathbf{Z}$-grading on G. Since G is simple modulo its center, G is the minimal $\mathbf{Z}$-graded Lie superalgebra with the local part $G_{-1} \oplus G_0 \oplus G_1$. In this way the structure of a contragredient Lie superalgebra is introduced in G. Its Cartan matrix is $A = (\alpha_j(h_i))$, and $\tau = \{i \in I \mid \alpha_i \in \Delta_1\}$.

In the next section we show that these examples exhaust all simple modulo center finite-dimensional contragredient Lie superalgebras.

We now list all the resulting pairs (A, τ), up to equivalence. For this it is convenient, as usual, to introduce *Dynkin diagrams*. To begin with we extract from our examples in Tables IV and V all the contragredient Lie superalgebras of rank 1 and 2 with indecomposable Cartan matrices, the corresponding pairs (A, τ), and Dynkin diagrams. The circles ○, ⊗, and ● are called, respectively, *white*, *gray*, and *black*. Contragredient Lie superalgebras of rank r are depicted by a diagram consisting of r white, gray, or black circles; the ith circle is white if $i \notin \tau$ and gray or black if $i \in \tau$ and $a_{ii} = 0$ or 2, respectively. The ith and the jth circles are not joined if $a_{ij} = a_{ji} = 0$; otherwise, they are joined as shown in Table V (note, that isomorphic Lie superalgebras may correspond to different Dynkin diagrams).

TABLE IV

$G(A, \tau)$	A	τ	Diagram	dim $G(A, \tau)$
$\mathbf{A}_1$	(2)	ϕ	○	3
sl(1, 1)	(0)	{1}	⊗	3
B(0, 1)	(2)	{1}	●	5

TABLE V

$G(A, \tau)$	A	τ	Diagram	dim $G(l, \tau)$
$\mathbf{A}_2$	$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$	ϕ	○—○	8
$\mathbf{B}_2$	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	ϕ	○⇒○	10
$\mathbf{G}_2$	$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$	ϕ	○⇛○	14
A(1, 0)	$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$	{1}	⊗—○	8
A(1, 0)	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	{1, 2}	⊗—⊗	8
B(1, 1)	$\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$	{1}	⊗⇒○	12
B(1, 1)	$\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$	{1, 2}	⊗⇒●	12
B(0, 2)	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	{2}	○⇒●	14

Matrix A also satisfies the following restrictions. If $a_{ii} = 0$, then in every submatrix of order 3 with a_{ii} in the center, the diagram of which is not a cycle and does not contain two arrows, the sum of the elements of the second row is 0. For the diagram $\otimes \leftarrow \otimes \rightarrow \bigcirc$, it is always $a_{21} = -2a_{23}$, and for the cycle,

$$\begin{array}{ccc} \otimes & & \\ | & \diagdown & \\ \bigcirc & - & \otimes, \end{array}$$

it is always $a_{23} = -2a_{21}$. We introduce matrices

$$\mathbf{D}_\alpha = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & \alpha \\ 0 & -1 & 2 \end{pmatrix}, \qquad \mathbf{D}_\alpha{}' = \begin{pmatrix} 0 & 1 & -1-\alpha \\ 1 & 0 & \alpha \\ -1-\alpha & \alpha & 0 \end{pmatrix}.$$

The diagrams

$$\bigcirc \leftarrow \otimes \rightarrow \bigcirc \qquad \text{and} \qquad \begin{array}{ccc} \otimes & & \\ | & \diagdown & \\ \otimes & - & \otimes \end{array}$$

always correspond to matrices $\mathbf{D}_\alpha$ and $\mathbf{D}_\alpha{}'$, respectively, and $\alpha = 1$ unless the contrary is stated.

The following proposition is a consequence of the results in Section 2.4.

PROPOSITION 2.5.6. *Let G be one of* $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, *or* $\mathbf{G}(3)$ *and let $G \simeq G(A, \tau)/C$, where C is the center. Then $C \neq 0$ only for* $\mathbf{A}(n, n)$ *and in this case* $\dim C = 1$. *The diagrams of the pairs (A, τ) can be described as follows (each point can be a white or a gray circle):*

A $\quad \cdot - \cdot - \cdot - \cdots - \cdot - \cdot$

B $\quad \cdot - \cdot - \cdots - \cdot - \cdot \Rightarrow \bigcirc, \qquad \cdot - \cdot - \cdots - \cdot - \cdot \Rightarrow \bullet$

C, D

$$\begin{array}{c} \bigcirc \\ \uparrow \\ \cdot - \cdot - \cdots - \cdot - \cdot \rightarrow \bigcirc, \end{array}$$

$$\cdot - \cdot - \cdots - \cdot - \bigcirc \Leftarrow \bigcirc, \qquad \cdot - \cdot - \cdots - \cdot \leftarrow \otimes \rightarrow \bigcirc,$$

$$\begin{array}{ccc} & \otimes & \\ & | & \diagdown \\ \cdot - \cdot - \cdots - \cdot - & \cdot & - \otimes \end{array}$$

$\mathbf{D}(2, 1; \alpha) \qquad \bigcirc \leftarrow \otimes \rightarrow \bigcirc, \qquad A = \mathbf{D}_\beta, \qquad \beta = \alpha, -(1+\alpha), -\alpha(1+\alpha)^{-1}$

$$\begin{array}{ccc} & \otimes & \\ \diagup & & \diagdown \\ \otimes & \text{———} & \otimes, \end{array} \qquad A = \mathbf{D}_\alpha{}'$$

$\mathbf{F}(4) \qquad \otimes - \bigcirc \Leftarrow \bigcirc - \bigcirc, \qquad \bigcirc - \bigcirc \leftarrow \otimes \rightarrow \otimes,$

$$\bigcirc \Rightarrow \bigcirc \leftarrow \otimes \rightarrow \bigcirc, \qquad \begin{array}{ccccc} & & \otimes & & \\ & & | & \diagdown & \\ \bigcirc & \leftarrow & \otimes & \rightarrow & \otimes, \end{array}$$

where for the last two diagrams, subdiagrams without the first circle correspond to matrices $\mathbf{D}_{-3}$ *and* $\mathbf{D}'_{-3}$, *respectively.*

$\mathbf{G}(3) \qquad \otimes - \bigcirc \Lleftarrow \bigcirc.$

Below (Table VI) we list the "simplest" diagrams, the coefficients of the decomposition of the highest root into simple roots, the index s of the only nonwhite circle, and the number r of the circles.

2.5.6. *The classification of finite-dimensional contragredient Lie superalgebras.*

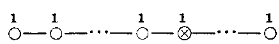
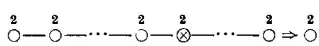
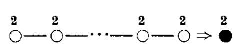
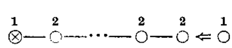
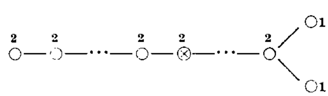
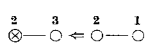
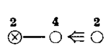
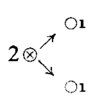
THEOREM 3. *Let $G(A, \tau)$ be a finite-dimensional contragredient Lie superalgebra whose Cartan matrix satisfies condition* (m) *of Proposition* 2.5.2, *and let C be its center. Then $G' = G(A, \tau)/C$ is classical, and (A, τ) is equivalent either to one of the pairs listed in Proposition* 2.5.6 *or to (A, ϕ), where A is the Cartan matrix of a simple Lie algebra.*

Proof. According to Propositions 2.5.1 and 2.5.2, $C \subseteq H$ and G' is simple. Therefore, by Theorem 2 and Proposition 2.5.6, it is sufficient to show that the linear Lie algebra $G_{\bar{0}}'$ acting on $G_{\bar{1}}'$ is reductive.

According to Proposition 2.5.3, on G' there exists a nondegenerate invariant bilinear form (,). We consider a new form on $G': f(x, y) = (x, \theta y)$, where θ is the automorphism from the proof of Proposition 2.5.3 ($\theta(G_\alpha) = G_{-\alpha}$). Clearly, the restriction of f to G_α is nondegenerate, and the operators $\operatorname{ad} e_\alpha$ and $-\operatorname{ad} \theta e_\alpha$ are dual with respect to this form. Hence, it follows that $[e_\alpha, \theta e_\alpha] = h$ is a nonzero element of H. For otherwise we would have two dual nonzero commuting nilpotent operators, which is impossible.

Now let R be the radical of $G_{\bar{0}}'$. Evidently, R is graded relative to the root

TABLE VI

G	Diagram	s	r
$\mathbf{A}(m, n)$		$m+1$	$m+n+1$
$\mathbf{B}(m, n)$, $m > 0$		n	$m+n$
$\mathbf{B}(0, n)$		n	n
$\mathbf{C}(n)$, $n > 2$		1	n
$\mathbf{D}(m, n)$		n	$m+n$
$\mathbf{F}(4)$		1	4
$\mathbf{G}(3)$		1	3
$\mathbf{D}(2, 1; \alpha)$		1	3

decomposition. We claim that $R \subseteq H$, which will prove the theorem. If this were not so, then $e_\alpha \in R$ for some $\alpha \neq 0$, and $[e_\alpha, \theta e_\alpha] = h \neq 0$, $h \in R$. Since R is solvable, $[h, e_\alpha] = 0$; hence $\alpha(h) = 0$. Now we look at the adjoint representation of the subalgebra $\langle \theta e_\alpha, h, e_\alpha \rangle$. By the Lie theorem, in some basis the matrices of $\operatorname{ad} \theta e_\alpha$ and $\operatorname{ad} e_\alpha$ are triangular. But then the matrix of $\operatorname{ad} h = [\operatorname{ad} e_\alpha, \operatorname{ad} \theta e_\alpha]$ is also triangular with zeros along the main diagonal. Since $\operatorname{ad} h$ is diagonalizable, we see that $h = 0$. This is a contradiction and proves the theorem.

2.5.7. *$\mathbf{Z}$-gradings.* It is not hard to show, just as in [12], that the relations $\deg e_i = -\deg f_i = k_i$, $\deg h_i = 0$, $k_i \in \mathbf{Z}$, $i = 1, \dots, r$, determine all possible $\mathbf{Z}$-gradings of finite-dimensional contragredient Lie superalgebras. In particular, if (A, τ) is a pair from Table VI, then for $k_i = 0$, $i \neq s$, $k_s = 1$ we obtain the $\mathbf{Z}$-gradings of $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, and $\mathbf{D}(m, n)$, as described in Section 2.1, and for $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, and $\mathbf{G}(3)$ we obtain consistent $\mathbf{Z}$-gradings of the form $G_{-2} \oplus G_{-1} \oplus G_0 \oplus G_1 \oplus G_2$, where $\dim G_{\pm 2} = 1$ and the G_0-modules $G_{\pm 1}$ are isomorphic, respectively, to $so_4 \otimes k$, $\mathrm{spin}_7 \otimes k$, and $G_2 \otimes k$. In the same way, the $\mathbf{Z}$-gradings are defined for $\mathbf{Q}(n)$ (as $\mathbf{Q}(n)_{\bar 0} = \mathbf{A}_n$ and we can naturally identify spaces $\mathbf{Q}(n)_{\bar 1}$ and $\mathbf{Q}(n)_{\bar 0} : \bar e_\alpha \leftrightarrow e_\alpha$, $\bar h_i \leftrightarrow h_i$).

3. Cartan Lie Superalgebras

3.1. *The Lie Superalgebras* $\mathbf{W}(n)$

3.1.1. *Definition of* $\mathbf{W}(n)$. Let $\Lambda(n)$ be the Grassmann superalgebra with the generators $\xi_1, \dots, \xi_n$. We denote $\operatorname{der} \Lambda(n)$ by $\mathbf{W}(n)$. We recall (see Section 1.1.4) that every derivation $D \in \mathbf{W}(n)$ can be written in the form

$$D = \sum_i P_i \frac{\partial}{\partial \xi_i}, \qquad P_i \in \Lambda(n),$$

where $\partial/\partial\xi_i$ is the derivation defined by

$$\partial/\partial\xi_i(\xi_j) = \delta_{ij}.$$

Letting $\deg \xi_i = 1$, $i = 1, \dots, n$, we obtain a consistent $\mathbf{Z}$-grading of $\Lambda(n)$, which induces one of $\mathbf{W}(n) = \oplus_{k \geq -1} \mathbf{W}(n)_k$, where

$$\begin{aligned} \mathbf{W}(n)_k &= \{D \in \mathbf{W}(n) \mid D(\Lambda(n)_s) \subset \Lambda(n)_{s+k}\} \\ &= \left\{ \sum P_i \frac{\partial}{\partial \xi_i} \,\middle|\, \deg P_i = k + 1, i = 1, \dots, n \right\}. \end{aligned}$$

In particular, $\mathbf{W}(n)_{-1} = \langle \partial/\partial\xi_i ,..., \partial/\partial\xi_n \rangle$. Hence it follows that $[\partial/\partial\xi_i , \partial/\partial\xi_j] = 0$, that is,

$$\frac{\partial}{\partial\xi_i}\frac{\partial}{\partial\xi_j} = -\frac{\partial}{\partial\xi_j}\frac{\partial}{\partial\xi_i}. \tag{3.1.1}$$

Formula (3.1.1) is one of the standard facts of analysis on a Grassmann algebra, as developed in [2].

We now list some properties of $\mathbf{W}(n)$.

PROPOSITION 3.1.1.

(a) $\mathbf{W}(n) = \bigoplus_{i=-1}^{n-1} \mathbf{W}(n)_i$ *is transitive.*

(b) *The* $\mathbf{W}(n)_0$*-module* $\mathbf{W}(n)_{-1}$ *is isomorphic to* gl_n.

(c) $\mathbf{W}(n)_k = \mathbf{W}(n)_k^1$, $k \geqslant 1$.

(d) $\mathbf{W}(n)$ *is simple for* $n \geqslant 2$.

(e) *If* $G = \bigoplus_{i\geqslant -1} G_i$ *is a transitive* $\mathbf{Z}$*-graded Lie superalgebra for which the* G_0*-modules* $G_{\pm 1}$ *are isomorphic to the* $\mathbf{W}(n)_0$*-modules* $\mathbf{W}(n)_{\pm 1}$, *then* $G \simeq \mathbf{W}(n)$.

Proof. Properties (a)–(c) are easily verified directly. (d) follows from Proposition 1.2.8 and (e) follows from (c) and Proposition 3.1.2 below.

3.1.2. *The universality of* $\mathbf{W}(n)$ *as a* $\mathbf{Z}$*-graded Lie superalgebra.* Let $G = \bigoplus_{i\geqslant -1} G_i$ be a transitive Lie superalgebra with a consistent $\mathbf{Z}$-grading, and $\dim G_{-1} = n$. Then the map $\psi_k\colon G_k \to \mathrm{Hom}(\Lambda^{k+1}(G_{-1}), G_{-1})$, defined by

$$\psi_k(g)(a_1 ,..., a_{k+1}) = [\cdots[[g, a_1], a_2],..., a_{k+1}],$$

determines a monomorphism of G_0-modules $\psi_k\colon G_k \to \Lambda^{k+1}(G_{-1}^*) \otimes G_{-1}$. The ψ_k's yield a canonical monomorphism of G_0-modules

$$\psi\colon G \to \bigoplus_{k\geqslant -1} (\Lambda^{k+1}(G_{-1}^*) \otimes G_{-1}).$$

From this we see by dimension arguments that if $G = \mathbf{W}(n)$, then ψ is an isomorphism of $\mathbf{W}(n)_0$-modules.

If we are now given a monomorphism of the G_0-module G_{-1} into $\mathbf{W}(n)_0$-module $\mathbf{W}(n)_{-1}$, we obtain a chain of maps:

$$G \to \bigoplus_k (\Lambda^{k+1}(G_{-1}^*) \otimes G_{-1}) \to \mathbf{W}(n).$$

It is easy to verify that the composite map is a monomorphism of $\mathbf{Z}$-graded Lie superalgebras. So we have the following result.

PROPOSITION 3.1.2. *Let $G = \oplus_{i \geqslant -1} G_i$ be a transitive superalgebra with a consistent $\mathbf{Z}$-grading, and* $\dim G_{-1} = n$. *Then there is an embedding of G in $\mathbf{W}(n)$ preserving the $\mathbf{Z}$-grading.*

3.1.3. *The universality of $\mathbf{W}(n)$ as a Lie superalgebra with a filtration.* $\mathbf{W}(n)$ is canonically equipped with a filtration. Now it turns out that there is also an embedding theorem for Lie superalgebras with filtration.

PROPOSITION 3.1.3. *Let $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ be a transitive Lie superalgebra with a filtration,* $\dim L/L_0 = n$, *and suppose that L_0 contains $L_{\bar{0}}$. Then there is an embedding $\alpha: L \to \mathbf{W}(n)$ preserving the filtration. If β is any other such embedding, then there is one and only one automorphism Φ of $\mathbf{W}(n)$ that is induced by an automorphism of $\Lambda(n)$ for which $\alpha = \Phi \circ \beta$.*

The proof carries over almost verbatim from [20], with the definitions replaced by the relevant definitions in Section 1.1.3. True, the proof in [20] only gives the existence of Φ under the assumption that $(\alpha - \beta)(L) \subseteq L_0$. However, this assumption is easily dispensed with, by modifying β to an automorphism of $\mathbf{W}(n)$ induced by a linear automorphism of $\Lambda(n)$.

As a corollary to Proposition 3.1.3 we have the next result.

PROPOSITION 3.1.4. *Let $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ be a subalgebra of $\mathbf{W}(n)$ with the induced filtration, and* $\dim L/L_0 = n$. *Then every automorphism of L preserving the filtration is induced by an automorphism of $\Lambda(n)$.*

Clearly, in $\mathbf{W}(n)$ with $n \geqslant 3$ there is a unique subalgebra containing $\mathbf{W}(n)_{\bar{0}}$, namely, $\oplus_{k \geqslant 0} \mathbf{W}(n)_k$. Hence for $n \geqslant 3$ this subalgebra, and therefore, the filtration in $\mathbf{W}(n)$, are invariant under all automorphisms. So we obtain the next result from Proposition 3.1.4.

PROPOSITION 3.1.5. *Every automorphism of $\mathbf{W}(n)$ with $n \geqslant 3$ is induced by an automorphism of $\Lambda(n)$.*

3.2. *Two Algebras of Differential Forms*

3.2.1. *The superalgebra $\Omega(n)$.* Let $\Lambda(n)$ be the Grassmann superalgebra on $\xi_1, \ldots, \xi_n$. We denote by $\Omega(n)$ the associative superalgebra over $\Lambda(n)$ with the generators $d\xi_1, \ldots, d\xi_n$ and the defining relations $d\xi_i \circ d\xi_j = d\xi_j \circ d\xi_i$, $\deg d\xi_i = \bar{0}$, $i, j = 1, \ldots, n$. Note that $\Omega(n)$ is commutative (in the sense of the bracket); in particular, $\xi_i d\xi_j = (d\xi_j)\xi_i$.

Every element $\psi \in \Omega(n)$ can be written uniquely as a sum of elements of the form

$$\psi_k = \sum_{i_1 \leqslant \cdots \leqslant i_k} a_{i_1 \ldots i_k} d\xi_{i_1} \circ \cdots \circ d\xi_{i_k}, \qquad \text{where} \quad a_{i_1, \ldots, i_k} \in \Lambda(n).$$

We define on $\Omega(n)$ the differential d as the derivation of degree $\bar{1}$ for which

$$d(\xi_i) = d\xi_i, \qquad d(d\xi_i) = 0, \qquad i = 1,\dots, n.$$

It is easy to verify, as in Section 1.1.4, that this derivation exists and is unique.

$\Omega(n)$ is called *the superalgebra of differential forms with commuting differentials.*

PROPOSITION 3.2.1. *The differential d has the following properties*:

(a) $d(\varphi \circ \psi) = d\varphi \circ \psi + (-1)^{\deg \varphi}\varphi \circ d\psi$, $\varphi, \psi \in \Omega(n)$.

(b) $df = \sum_i \partial f/\partial \xi_i \, d\xi_i$, $f \in \Lambda(n)$.

(c) $d^2 = 0$.

(d) *Every derivation D of $\Lambda(n)$ extends uniquely to a derivation $\tilde{D}$ of $\Omega(n)$ for which $[\tilde{D}, d] = 0$.*

(e) *Every automorphism of $\Lambda(n)$ extends uniquely to an automorphism of $\Omega(n)$ commuting with d.*

Proof. (a) is true, by definition.

(b) is proved by induction on the $\mathbf{Z}$-grading of $\Lambda(n) = \bigoplus \Lambda_i$. Suppose that $f \in \Lambda_k$; it is enough to prove (b) for $f = f_1\xi_j$, where $f_1 \in \Lambda_{k-1}$. By the inductive hypothesis we have

$$df = (df_1)\,\xi_j + (-1)^{k-1} f_1\, d\xi_j = \sum_i \left(\frac{\partial f_1}{\partial \xi_i}\right) d\xi_i + (-1)^{k-1} f_1\, d\xi_j$$

$$= \sum_i \frac{\partial}{\partial \xi_i}(f_1\xi_j)\, d\xi_i,$$

as required.

(c) We now define a height h on $\Omega(n)$ by putting $h(d\xi_i) = 1$, $h(\xi_i) = 0$, $i = 1,\dots, n$. We conduct the proof of (c) by induction on h.

If $h(f) = 0$, then $f \in \Lambda(n)$ and

$$d(df) = d\left(\sum_i \frac{\partial f}{\partial \xi_i}\, d\xi_i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial \xi_i\, \partial \xi_j}\, d\xi_i \circ d\xi_j = 0,$$

by (3.1.1). Suppose, next, that $h(\varphi) = k$; it is enough to prove (c) for $\varphi = \varphi_1 \circ d\xi_i$. By induction we have $d^2(\varphi_1 \circ d\xi_i) = d(d\varphi_1 \circ d\xi_i) = 0$.

(d) Let $D = \sum P_i(\partial/\partial \xi_i)$ be a derivation of $\Lambda(n)$ of degree k. It is easy to verify, as in Section 1.1.4, that there is one (and only one) extension of D

to a derivation $\tilde{D}$ of $\Omega(n)$ for which $\tilde{D}(d\xi_i) = (-1)^k d(D\xi_i)$, $i = 1,..., n$. We claim that $\tilde{D}d = (-1)^k d\tilde{D}$. Let $f \in \Lambda_s$; then we have

$$D(df) = D\left(\sum \frac{\partial f}{\partial \xi_i} d\xi_i\right) = \sum_{i,j} P_j \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} d\xi_i + i(-1)^{ks} \sum_i \frac{\partial f}{\partial \xi_i} dP_i$$

$$= \sum_i \left(\sum_j P_j \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} + (-1)^{ks} \frac{\partial f}{\partial \xi_j} \frac{\partial P_j}{\partial \xi_i}\right) d\xi_i$$

$$= (-1)^k \sum_{i,j} \frac{\partial}{\partial \xi_i} \left(P_j \frac{\partial f}{\partial \xi_j}\right) d\xi_i = (-1)^k d(Df).$$

Now clearly we have

$$Dd(f\, d\xi_{i_1} \circ \cdots \circ d\xi_{i_s}) = (-1)^k dD(f\, d\xi_{i_1} \circ \cdots \circ d\xi_{i_s})$$

as required.

(e) is proved rather like (d) (besides, as is easy to see, it follows from (d)).

The proposition is now proved.

Note that (d) gives us an action of $\mathbf{W}(n)$ on $\Omega(n)$, commuting with d in the sense of the bracket. There is also an analog to Poincaré's lemma.

PROPOSITION 3.2.2. *If a differential form $\psi \in \Omega(n)$ is closed, that is, $d\psi = 0$, then $\psi = d\psi_1$ for some $\psi_1 \in \Omega(n)$.*

Proof. We consider the linear map (homotopy operator) $K: \Omega(n) \to \Omega(n)$ and the homomorphism $\epsilon: \Omega(n) \to \Omega(n)$, defined by the formulas,

$$K(f\, d\xi_{i_1} \circ \cdots \circ d\xi_{i_{s-1}} \circ d\xi_n) = \xi_n f\, d\xi_{i_1} \circ \cdots \circ d\xi_{i_{s-1}},$$

$$K(f\, d\xi_{i_1} \circ \cdots \circ d\xi_{i_s}) = 0 \quad \text{when} \quad i_t \neq n, \quad t = 1,..., s, \quad f \in \Lambda(n),$$

$$\epsilon(\xi_i) = \xi_i \ \text{ and } \ \epsilon(d\xi_i) = d\xi_i \quad \text{if} \quad i \neq n, \quad \epsilon(\xi_n) = \epsilon(d\xi_n) = 0.$$

It is easy to check that $Kd + dK = 1 + \epsilon$. Therefore, if $d\psi = 0$, then $\psi = d(K(\psi)) - \epsilon(\psi)$, where $\epsilon(\psi)$ is closed and does not depend on ξ_n. Now we can use the induction by n.

3.2.2. *The superalgebra $\Theta(n)$.* We denote by $\Theta(n)$ the associative superalgebra over $\Lambda(n)$ with the generators $\theta\xi_1, ..., \theta\xi_n$ and defining relations

$$\theta\xi_i \wedge \theta\xi_j = -\theta\xi_j \wedge \theta\xi_i, \qquad \deg \theta\xi_i = \bar{1}, \qquad i, j = 1,..., n.$$

Note that $\Theta(n)$ is commutative (in the sense of the bracket); in particular, $\xi_i\theta\xi_j = -(\theta\xi_j)\xi_i$.

Every element $\omega \in \Theta(n)$ can be written uniquely as a sum of elements of the form

$$\omega_k = \sum_{i_1<\cdots<i_k} a_{i_1,\ldots,i_k}\theta\xi_{i_1} \wedge \cdots \wedge \theta\xi_{i_k}.$$

We define a differential θ on $\Theta(n)$ as the derivation of degree $\bar{0}$ for which

$$\theta(\xi_i) = \theta\xi_i, \qquad \theta(\theta\xi_i) = 0, \qquad i = 1,\ldots, n.$$

It is easy to verify that this derivation exists and is unique.

PROPOSITION 3.2.3. *The differential θ has the following properties*:

(a) $\theta(\omega_1 \wedge \omega_2) = \theta(\omega_1) \wedge \omega_2 + \omega_1 \wedge \theta(\omega_2)$.

(b) $\theta(f) = \sum_i (\theta\xi_i)(\partial f/\partial\xi_i)$, $f \in \Lambda(n)$.

(c) *Every derivation D of $\Lambda(n)$ extends uniquely to a derivation $\tilde{D}$ of $\Theta(n)$ for which $\tilde{D}\theta f = \theta\tilde{D}f$, $f \in \Lambda(n)$; if $\theta^2(D(\xi_i)) = 0$, $i = 1,\ldots, n$, then $\tilde{D}\theta = \theta\tilde{D}$.*

(d) *Every automorphism Φ of $\Lambda(n)$ extends uniquely to an automorphism $\tilde{\Phi}$ of $\Theta(n)$ for which $\tilde{\Phi}\theta f = \theta\tilde{\Phi}f$, $f \in \Lambda(n)$.*

The proof is similar to that of Proposition 3.2.1. Observe that $\theta^2 \neq 0$. For example, $\theta^2(\xi_1\xi_2) = 2\theta\xi_1 \wedge \theta\xi_2$. Also, it is not true that $[D, \theta] = 0$ for $D \in \mathbf{W}(n)$. Nevertheless, (c) provides us with some action of $\mathbf{W}(n)$ on $\Theta(n)$.

3.3. *Special and Hamiltonian Lie Superalgebras*

3.3.1. *Volume forms and the Lie superalgebras* $\mathbf{S}(n)$ *and* $\tilde{\mathbf{S}}(n)$. A *volume form* is a differential form in $\Theta(n)$ like

$$\omega = f\theta\xi_1 \wedge \cdots \wedge \theta\xi_n, \qquad f \in \Lambda(n)_{\bar{0}}, \qquad f(0) \neq 0.$$

To a volume form ω there corresponds in $\mathbf{W}(n)$ the subalgebra

$$\mathbf{S}(\omega) = \{D \in \mathbf{W}(n) \mid D\omega = 0\}.$$

Among these subalgebras we single out two: $\mathbf{S}(n) = \mathbf{S}(\theta\xi_1 \wedge \cdots \wedge \theta\xi_n)$ and $\tilde{\mathbf{S}}(n) = \mathbf{S}((1 + \xi_1 \cdots \xi_n)\,\theta\xi_1 \wedge \cdots \wedge \theta\xi_n)$ for $n = 2k$.

The condition for an operator $\sum P_i(\partial/\partial\xi_i)$ to belong to $\mathbf{S}(\omega)$ can be written as follows:

$$\sum_i \frac{\partial}{\partial\xi_i}(fP_i) = 0.$$

Hence it is easy to see that $\mathbf{S}(\omega)$ is the linear span of the elements like

$$f^{-1}\frac{\partial a}{\partial\xi_i}\frac{\partial}{\partial\xi_j} + f^{-1}\frac{\partial a}{\partial\xi_j}\frac{\partial}{\partial\xi_i}, \qquad a \in \Lambda(n).$$

On $\mathbf{S}(\omega)$ a filtration is induced from $\mathbf{W}(n)$, and on $\mathbf{S}(n)$ clearly even a $\mathbf{Z}$-grading.

PROPOSITION 3.3.1.

(a) $\mathrm{Gr}\,\mathbf{S}(\omega) \simeq \mathbf{S}(n) = \oplus_{k=-1}^{n-2} \mathbf{S}(n)_k$.

(b) *The semisimple part of* $\mathbf{S}(\omega)_{\bar{0}}$ *is isomorphic to* sl_n.

(c) *The* $\mathbf{S}(n)_0$*-module* $\mathbf{S}(n)_{-1}$ *is isomorphic to* sl_n.

(d) *The* $\mathbf{S}(n)_0$*-modules* $\mathbf{S}(n)_k$ *are irreducible and isomorphic to the highest component of the module* $sl_n \otimes \Lambda^{n-k-1} sl_n$.

(e) $\mathbf{S}(n)_k = \mathbf{S}(n)_1^k$, $k \geqslant 1$.

(f) *The* $\mathbf{Z}$*-graded Lie superalgebra* $\mathbf{S}(n)$ *is transitive. The* $\mathbf{S}(\omega)$ *are simple for* $n \geqslant 3$.

(g) *Every automorphism of* $\mathbf{S}(\omega)$, $n \geqslant 3$, *is induced by an automorphism of* $\Lambda(n)$ *under which the differention form* ω *is multiplied by an element of* k.

(h) *If* $G = \oplus_{i \geqslant -1} G_i$ *is a transitive* $\mathbf{Z}$*-graded Lie superalgebra for which the* G_0*-module* G_{-1} *is isomorphic to* sl_n, *then* $G \simeq \mathbf{S}(n)$.

Proof. Properties (a)–(e) are easily derived from the description of the elements of $\mathbf{S}(\omega)$. The fact that $\mathbf{S}(\omega)$ is simple now follows from Proposition 1.2.8; therefore, (f) is true by (a).

To prove (g), we note (as was done in [15]) that if ω_1 and ω_2 are not proportional, then $\mathbf{S}(\omega_1) \neq \mathbf{S}(\omega_2)$ and that the filtration in $\mathbf{S}(\omega)$ for $n \geqslant 3$ is invariant under automorphisms. (For $n > 3$ this is proved as in Section 3.1.3, and for $n = 3$ it is obvious.)

Finally, (h) is obtained by embedding G in $\mathbf{W}(n)$; clearly, then $G_i = \mathbf{S}(n)_i$ for $i = -1, 0$, and we can then use Proposition 3.3.2 below.

We set $D_0 = \sum \xi_i(\partial/\partial\xi_i)$, $T_k(n) = \{fD_0, f \in \Lambda(n)_k\} \subseteq \mathbf{W}(n)_k$, $k = 0,..., n-1$. The next proposition is easy to obtain, for example, by dimension arguments.

PROPOSITION 3.3.2. $\mathbf{W}(n)_k = \mathbf{S}(n)_k \oplus T_k(n)$, $k \geqslant 0$, *is a direct sum of irreducible* $\mathbf{W}(n)_0$*-modules, and the* $\mathbf{W}(n)_0$*-module* $T_k(n)$ *is isomorphic to* $\Lambda^{n-k} gl_n$.

We now obtain a classification of volume forms.

PROPOSITION 3.3.3. *Every volume form* $\omega = f\theta\xi_1 \wedge \cdots \wedge \theta\xi_n$ *can be reduced by an automorphisms of* $\Lambda(n)$ *to the shape*

$$(\alpha + \beta\xi_1 \cdots \xi_n)\,\theta\xi_1 \wedge \cdots \wedge \theta\xi_n, \qquad \alpha \neq 0; \quad \beta = 0 \quad \textit{when } n \textit{ is odd.}$$

Proof. We may assume that $n \geqslant 3$. The semisimple part of the Lie algebra $\mathbf{S}(\omega)_{\bar{0}}$ (which exists by Levi's theorem) can be carried, by Maltsev's theorem, into $\mathbf{S}(n)_0 \subset \mathbf{W}(n)_0$ by an inner automorphism Φ of $\mathbf{W}(n)_{\bar{0}}$. According to Proposition 1.1.1, Φ extends to an automorphism of $\mathbf{W}(n)$ and is, therefore, induced by an automorphism φ of $\Lambda(n)$ (see Proposition 3.1.5). Replacing ω

by $\varphi(\omega)$, we may assume $D\omega = 0$ for $D \in \mathbf{S}(n)_0$. But $\mathbf{S}(n)_0 \supset \{\xi_i(\partial/\partial\xi_j), i \neq j\}$; therefore, $\xi_i(\partial f/\partial \xi_j) = 0$ for $i \neq j$, and hence, $f = \alpha + \beta\xi_1 \cdots \xi_n$, and $\beta = 0$ for odd n.

Taking Proposition 3.1.3 into account, we can derive the following result from Proposition 3.3.3.

PROPOSITION 3.3.4. *Every superalgebra* $\mathbf{S}(\omega)$ *is isomorphic to one of* $\mathbf{S}(n)$ *or* $\tilde{\mathbf{S}}(n)$. *These two Lie superalgebras are not isomorphic.*

PROPOSITION 3.3.5. *Let* $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ *be a Lie superalgebra with a filtration, and* $\operatorname{Gr} L \simeq \mathbf{S}(n)$. *Then* $L \simeq \operatorname{Gr} L \simeq \mathbf{S}(n)$ *for odd n, and* $L \simeq \mathbf{S}(n)$ *or* $\tilde{\mathbf{S}}(n)$ *for even n.*

Proof. Using Proposition 3.1.3, we embed L in $\mathbf{W}(n)$. Then $L_{\bar{0}} \subseteq \mathbf{W}(n)_{\bar{0}}$, and by applying Maltsev's theorem to this pair, we may assume that the semi-simple part of $L_{\bar{0}}$ (which is isomorphic to sl_n) lies in $\mathbf{W}(n)_0$, and hence, that $L \supseteq \mathbf{S}(n)_0$.

We now observe that $\mathbf{W}(n)$ and L are $\mathbf{S}(n)_0$-modules, that the $\mathbf{S}(n)_0$-modules L and $\mathbf{S}(n)$ are isomorphic and, by Proposition 3.3.2, that $\mathbf{W}(n) = \mathbf{S}(n) \oplus T(n)$ is a direct sum of $\mathbf{S}(n)_0$-modules, where $T(n) = \oplus_k T_k(n)$.

From Propositions 3.3.1(d) and 3.3.2 it is clear that the $\mathbf{S}(n)_0$-modules $T(n)$ and L can only contain a unique common simple component:

$$\operatorname{Gr}_{-1} L \simeq T_{n-1}(n).$$

Therefore, $L = V \oplus (\oplus_{k \geqslant 0} \mathbf{S}(n)_k)$, where $V \subset \mathbf{S}(n)_{-1} \oplus T_{n-1}(n)$. If $V = \mathbf{S}(n)_{-1}$, then $L = \mathbf{S}(n)$. For odd n there is no other possibility because $V \subset L_{\bar{1}}$. But if $V \neq \mathbf{S}(n)_{-1}$ for even n, then $L \simeq \tilde{\mathbf{S}}(n)$, as is easy to see.

3.3.2. *Hamiltonian forms and the Lie superalgebras* $\mathbf{H}(n)$ *and* $\tilde{\mathbf{H}}(n)$. A *Hamiltonian form* is a closed differential form in $\Omega(n)$ of the kind

$$\omega = \sum_{i,j=1}^{n} \omega_{ij}\, d\xi_i \circ d\xi_j, \qquad \omega_{ij} \in \Lambda(n)^{\cdot}, \qquad \omega_{ij} = \omega_{ji}, \qquad \det(\omega_{ij}(0)) \neq 0.$$

To a Hamiltonian form ω there corresponds a subalgebra of $\mathbf{W}(n)$:

$$\tilde{\mathbf{H}}(\omega) = \{D \in \mathbf{W}(n) \mid D\omega = 0\}.$$

We set $\mathbf{H}(\omega) = [\tilde{\mathbf{H}}(\omega), \tilde{\mathbf{H}}(\omega)]$, $\tilde{\mathbf{H}}(n) = \tilde{\mathbf{H}}((d\xi_1)^2 + \cdots + (d\xi_n)^2)$, and $\mathbf{H}(n) = [\tilde{\mathbf{H}}(n), \tilde{\mathbf{H}}(n)]$.

It is not difficult to see that the condition for $\sum P_i(\partial/\partial\xi_i)$ to belong to $\tilde{\mathbf{H}}(\omega)$ can be written as follows:

$$\frac{\partial}{\partial\xi_j}\left(\sum_t \omega_{it} P_t\right) + \frac{\partial}{\partial\xi_i}\left(\sum_t \omega_{jt} P_t\right) = 0.$$

By the analog to Poincaré's lemma, this condition shows that there is an element $f \in \Lambda(n)$ (depending on D) for which $\sum_t \omega_{it} P_t = \partial f / \partial \xi_i$. Therefore, if $(\bar{\omega}_{ij})$ is the inverse matrix to (ω_{ij}), we see that $\tilde{\mathbf{H}}(\omega)$ consists of all the elements of the form

$$D_f = \sum_{i,j} \left(\bar{\omega}_{ij} \frac{\partial f}{\partial \xi_i} \right) \frac{\partial}{\partial \xi_j}, \qquad f \in \Lambda(n), \;\; f(0) = 0,$$

and that $[D_f, D_g] = D_{\{f,g\}}$, where

$$\{f, g\} = (-1)^{\deg f} \sum_{i,j} \bar{\omega}_{ij} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_j}.$$

In particular, $\tilde{\mathbf{H}}(n)$ consists of the elements of the form

$$D_f = \sum_i \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial \xi_i}, \qquad f \in \Lambda(n), \;\; f(0) = 0,$$

and the bracket looks as follows:

$$\{f, g\} = (-1)^{\deg f} \sum \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i}.$$

A filtration is induced on $\tilde{\mathbf{H}}(\omega)$ from $\mathbf{W}(n)$, and on $\tilde{\mathbf{H}}(n)$ and $\mathbf{H}(n)$ even $\mathbf{Z}$-gradings.

PROPOSITION 3.3.6.

(a) $\mathrm{Gr}\, \tilde{\mathbf{H}}(\omega) \simeq \tilde{\mathbf{H}}(n) = \oplus_{k=-1}^{n-2} \tilde{\mathbf{H}}(n)_k$.

(b) *The semisimple part of* $\tilde{\mathbf{H}}(\omega)_{\bar{0}}$ *is isomorphic to* so_n.

(c) $\mathbf{H}(n) = \oplus_{k=-1}^{n-3} \tilde{\mathbf{H}}(n)_k$, *that is* $\tilde{\mathbf{H}}(n) = \mathbf{H}(n) \oplus \langle D_{\xi_1 \cdots \xi_n} \rangle$.

(d) $\mathbf{H}(n)_k = \mathbf{H}(n)_1^k$, $k \geqslant 1$.

(e) $[\tilde{\mathbf{H}}(n)_k, \tilde{\mathbf{H}}(n)_{-1}] = \mathbf{H}(n)_{k-1}$ *for* $k \leqslant n-2$; $[\mathbf{H}(n)_k, \mathbf{H}(n)_{-1}] = \mathbf{H}(n)_{k-1}$, $k \leqslant n$.

(f) *The* $\tilde{\mathbf{H}}(n)_0$*-modules* $\tilde{\mathbf{H}}(n)_k$ *are isomorphic to* $\Lambda^{k+2} so_n$, $-1 \leqslant k \leqslant n-2$.

(g) *The* $\mathbf{Z}$*-graded Lie superalgebras* $\mathbf{H}(n)$ *and* $\tilde{\mathbf{H}}(n)$ *are transitive.*

(h) $\mathbf{H}(n)$ *is simple for* $n \geqslant 4$.

(i) *Every automorphism of* $\mathbf{H}(\omega)$ *for* $n \geqslant 4$ *and of* $\tilde{\mathbf{H}}(\omega)$ *for* $n \geqslant 3$ *is induced by an automorphism of* $\Lambda(n)$ *under which* ω *is multiplied by an element of* k.

(j) *If* $G = \oplus_{i \geqslant -1} G_i$ *is a transitive* $\mathbf{Z}$*-graded Lie superalgebra for which the* G_0*-module* G_{-1} *is isomorphic to* so_n, *then* $G \simeq \mathbf{H}(n)$ *or* $\tilde{\mathbf{H}}(n)$ *or* $\mathbf{P}(3)$.

Proof. Properties (a)–(i) are established just like the corresponding assertions in Proposition 3.3.1.

(j) is now a consequence of the following assertion: If $G = \oplus_{i \geqslant -1} G_i$ is a transitive Lie superalgebra with a consistent $\mathbf{Z}$-grading and if the G_0-module G_{-1} has an invariant symmetric bilinear form (,), then there is an embedding of G in $\tilde{\mathbf{H}}(n)$ preserving the $\mathbf{Z}$-grading. To prove this we construct embeddings $\psi_k\colon G_k \to \Lambda^{k+2}(G_{-1}^*)$ by the formula $\psi_k(g)(a_1,..., a_{k+2}) = ([\cdots[g, a_1],..., a_{k+1}], a_{k+2})$. The subsequent arguments are the same as in Section 3.1.2.

Remark (cf. [22]). Let $\mathscr{C}_n$ be a Clifford superalgebra with the natural $\mathbf{Z}_2$-grading. The bracket turns this into a Lie superalgebra $(\mathscr{C}_n)_{\mathrm{L}}$. The factor algebra $(\mathscr{C}_n)_{\mathrm{L}}/\langle 1 \rangle$ is isomorphic to $\tilde{\mathbf{H}}(n)$.

PROPOSITION 3.3.7. *Let $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ be a Lie superalgebra with a filtration, and* $\operatorname{Gr} L \simeq \mathbf{H}(n)$ *or* $\tilde{\mathbf{H}}(n)$. *Then* $L \simeq \operatorname{Gr} L$.

Proof. By Levi's theorem, $L_{\bar{0}}$ contains a subalgebra G_0 isomorphic to so_n (see Proposition 3.3.6(b)). Now L splits into a $\mathbf{Z}_2$-graded direct sum of irreducible G_0-submodules, and the G_0-module L is isomorphic to the $\operatorname{Gr}_0 L$-module $\operatorname{Gr} L$. Hence we see that $L = G_{-1} \oplus L_0$ is a direct sum of G_0-modules, and that the G_0-module G_{-1} is isomorphic to so_n. In particular, $[G_{-1}, G_{-1}] \subset S^2 so_n$; hence, and from Proposition 3.3.6(f), it is clear that $[G_{-1}, G_{-1}] \subset L_{n-2}$. It follows that $[G_{-1}, G_{-1}]$ can differ from zero only when n is even and $\operatorname{Gr} L \simeq \tilde{\mathbf{H}}(n)$. However, in that case $[G_{-1}, G_{-1}] = L_{n-2} = \langle D_{\xi_1 \cdots \xi_n} \rangle$, and for $a \in G_{-1}$ we have $[[a, a], a] = (a, a) \cdot [D_{\xi_1 \cdots \xi_n}, a] = 0$ (by the Jacobi identity); this contradicts the transitivity of $\tilde{\mathbf{H}}(n)$.

Thus, $[G_{-1}, G_{-1}] = 0$. We now embed L in $\mathbf{W}(n)$ with preservation of the filtration $\alpha\colon L \to \mathbf{W}(n)$, using Proposition 3.1.3. Since $[G_{-1}, G_{-1}] = 0$, there is an embedding $\beta\colon G_{-1} \to \mathbf{W}(n)$ for which $\beta(G_{-1}) = \mathbf{W}(n)_{-1}$. Therefore, α can be modified (by Proposition 3.1.3) to an automorphism of $\mathbf{W}(n)$ for which $\alpha(G_{-1}) = \langle \partial/\partial\xi_i ,..., \partial/\partial\xi_n \rangle$.

Now let $\operatorname{Gr} L = \mathbf{H}(n)$. Then $\mathbf{W}(n) \supset L = L_{-1} \supset L_0 \supset \cdots \supset L_{n-3} \supset 0$, where $G_{-1} = \mathbf{W}(n)_{-1}$ and the G_0-module L_{n-3} is isomorphic to so_n, and $L_{n-3} \subset \mathbf{W}(n)_{n-3} \oplus \mathbf{W}(n)_{n-1}$; hence, it follows evidently that $L_{n-3} \subset \mathbf{W}(n)_{n-3}$. Therefore, $(\operatorname{ad} G_{-1})^{s-3} L_{n-3} \subset \mathbf{W}(n)_{n-s}$.

In this way, in accordance with Proposition 3.3.6(e), a $\mathbf{Z}$-grading is introduced on L that is consistent with the filtration; therefore, $L \simeq \operatorname{Gr} L$.

The arguments for the case $\operatorname{Gr} L \simeq \tilde{\mathbf{H}}(n)$ are similar.

It is now easy to prove the analog to Darboux' lemma.

PROPOSITION 3.3.8. *Every Hamiltonian form* $\omega = \sum \omega_{ij}\, d\xi_i \circ d\xi_j$ *can be brought by an automorphism of* $\Lambda(n)$ *to the shape*

$$\sum_{i=1}^{n} (d\xi_i)^2.$$

Proof. We consider $\bar{\mathbf{H}}(\omega)$. According to Proposition 3.3.6(a), $\operatorname{Gr}\bar{\mathbf{H}}(\omega) \simeq \bar{\mathbf{H}}(n)$. Therefore, by Proposition 3.3.7, $\bar{\mathbf{H}}(\omega) \simeq \bar{\mathbf{H}}(n)$, and from Proposition 3.1.3 it follows that there is an automorphism Φ of $\Lambda(n)$ carrying $\bar{\mathbf{H}}(\omega)$ into $\bar{\mathbf{H}}(n)$. Therefore $\partial/\partial\xi_i(\Phi(\omega)) = 0$ (because $\partial/\partial\xi_i \in \bar{\mathbf{H}}(n)$) and $\Phi(\omega) = \sum c_{ij}\, d\xi_i \circ d\xi_j$, $c_{ij} \in k$. It remains to apply a linear automorphism.

3.3.3. Definition. The Lie superalgebras $\mathbf{W}(n)$ for $n \geqslant 3$, $\mathbf{S}(n)$ for $n \geqslant 4$, $\tilde{\mathbf{S}}(n)$ for even $n \geqslant 4$, and $\mathbf{H}(n)$ for $n \geqslant 5$ are called *Cartan Lie superalgebras.* (For other values of n they are either not simple or isomorphic to classical Lie superalgebras: $\mathbf{W}(2) \simeq \mathbf{A}(1, 0) \simeq \mathbf{C}(2)$, $\mathbf{S}(3) \simeq \mathbf{P}(2)$, $\mathbf{H}(4) \simeq \mathbf{A}(1, 1)$.)

4. The Classification Theorem

In this chapter we complete the classification of the simple Lie superalgebras over an algebraically closed field of characteristic 0.

4.1. *Classification of Certain $\mathbf{Z}$-Graded Lie Superalgebras*

4.1.1. *The noncontragredient case.* Here we prove the following important proposition, which is the most complicated technically.

Proposition 4.1.1. *Let $G = \bigoplus_{i=-d}^{t} G_i$ be a bitransitive Lie superalgebra with a consistent $\mathbf{Z}$-grading, for which d or $t = 1$, and suppose that*

(a) *G_0 is semisimple;*
(b) *the representations of G_0 on G_{-1} and G_1 are irreducible;*
(c) *the representations of G_0 on G_{-1} and G_1 are not contragredient;*
(d) *$G_{-1} \oplus G_0 \oplus G_1$ generates G.*

Then G is isomorphic as a $\mathbf{Z}$-graded superalgebra to one of $\mathbf{S}(n)$, $\mathbf{H}(n)$ with $n > 4$, or $\mathbf{P}(n)$.

The proof of the proposition is based on an analysis of the relations between the highest vector F_Λ of the representation of G_0 on G_{-1} and the lowest vector E_M of the representation of G_0 on G_1. We know (see the proof of Proposition 1.2.10) that

$$[F_\Lambda, E_M] = e_{-\alpha}, \tag{4.1.1}$$

where $\alpha = -(\Lambda + M)$ is a nonzero root of G_0. Interchanging if necessary G_k with G_{-k}, we may assume that $\alpha > 0$, and hence, in particular, that

$$[F_\Lambda, e_\alpha] = [E_M, e_{-\alpha}] = 0. \tag{4.1.2}$$

In the proof of the proposition we need a number of lemmas.

LEMMA 4.1.2. *Let G be a finite-dimensional $\mathbf{Z}$-graded Lie superalgebra. Let $E_i, F_i \in G$, $i = 1, 2$, be $\mathbf{Z}$-homogeneous of nonzero degree and $\mathbf{Z}_2$-homogeneous elements and $H \in G$ a nonzero element such that*

$$[H, E_i] = a_i E_i, \qquad [E_i, F_j] = \delta_{ij} H, \qquad [H, F_i] = -a_i F_i. \tag{4.1.3}$$

Then $a_1 = a_2 = 0$.

Proof. The subalgebra P of G generated by the E_i and F_i is clearly endowed with a $\mathbf{Z}$-grading when we set $\deg E_i = 1$, $\deg F_i = -1$, $\deg H = 0$. Suppose that one of the a_i is not zero. We consider the matrix

$$A = \begin{bmatrix} a_1 & a_2 \\ a_1 & a_2 \end{bmatrix}$$

and set $\tau = \{i \in \{1, 2\} \mid E_i \text{ is odd}\}$. Then the contragredient Lie superalgebra $G(A, \tau)$ is infinite-dimensional. For if both $a_i \neq 0$, this follows, for example, from Theorem 3 (all the matrices of Table V are nondegenerate). But if $a_1 \neq 0$, $a_2 = 0$, then we replace E_2 by $E_2' = [E_1, E_2]$ and F_2 by $F_2' = [F_1, F_2]$ and arrive at the preceding case. The factor algebra $P_1 = G(A, \tau)/C$, where C is the (one-dimensional) center, is also infinite-dimensional.

Now, evidently, the map $E_i \to e_i$, $F_i \to f_i$ induces an epimorphism of $\mathbf{Z}$-graded Lie superalgebras $P \to P_1$. Therefore, $\dim P = \infty$, which contradicts the fact that G is finite-dimensional.

LEMMA 4.1.3. *Let $G = \bigoplus G_i$ be a finite-dimensional $\mathbf{Z}$-graded Lie superalgebra, with G_0 semisimple. Suppose that there exist odd elements x_λ and x_μ, $\mathbf{Z}$-homogeneous of nonzero degree, that are weight vectors of the adjoint representation of G_0 on G, and a root vector $e_{-\delta}$ of G_0, linked by the relations*

$$[x_\lambda, x_\mu] = e_{-\delta}, \tag{4.1.4}$$

$$[x_\lambda, e_\delta] = [x_\mu, e_{-\delta}] = 0. \tag{4.1.5}$$

Then $(\lambda, \delta) = 0$.

Proof. Suppose that $(\lambda, \delta) \neq 0$. The same argument as in the proof of Lemma 3.1 in [12] gives that $2(\lambda, \delta) = (\delta, \delta)$. We choose a root vector $e_{-\delta}$ such that $[e_{-\delta}, e_\delta] = h_\delta$. We consider the elements

$$E_1 = x_\lambda^2, \qquad F_1 = -(4(\lambda, \delta))^{-1}[x_\mu, e_\delta]^2, \qquad H = h_\delta,$$

$$E_2 = [[x_\lambda^2, e_{-\delta}], e_{-\delta}], \qquad F_2 = (8(\lambda, \delta)(\delta - \lambda, \delta)(4\lambda - \delta, \delta))^{-1}[[[x_\mu, e_\delta]^2, e_\delta], e_\delta].$$

A direct calculation shows that these elements satisfy (4.1.3). Therefore, we find from Lemma 4.1.2 that $(\lambda, \delta) = 0$, and we have a contradiction.

Henceforth we assume that all the conditions of Proposition 4.1.1 are satisfied and, using Lemma 4.1.3, we look for restrictions on the weight Λ and the root α. From (4.1.1), (4.1.2), and the lemma we deduce at once the next result.

LEMMA 4.1.4. $(\Lambda, \alpha) = 0$.

LEMMA 4.1.5. *Let β and γ be positive roots of G_0.*

(a) *If $\alpha + \beta$ is a root, then $(\Lambda - \beta, \alpha + \beta) = 0$.*

(b) *If $\alpha + \beta$ is a root and $\alpha - \beta$ is not, then*

$$(\Lambda + \alpha, \beta) = 0 \qquad \textit{and} \qquad \frac{2(\Lambda, \beta)}{(\beta, \beta)} = -\frac{2(\alpha, \beta)}{(\beta, \beta)} = 1.$$

(c) *If $\alpha + \beta$ is a root and $\alpha - \beta$, $\alpha - \gamma$, $\beta - \gamma$ are not, then $(\Lambda, \gamma) = 0$.*

Proof. (a) We set $x_\lambda = [e_{-\beta}, F_\Lambda]$, $x_\mu = E_M$, $\delta = \alpha + \beta$. Then it follows from (4.1.1) and (4.1.2) that relations (4.1.4) and (4.1.5) hold. By Lemma 4.1.3 we now see that $(\Lambda - \beta, \alpha + \beta) = 0$.

(b) Suppose that $(\Lambda + \alpha, \beta) \neq 0$. We set $x_\lambda = [e_{-\beta}, F_\Lambda]$, $x_\mu = [e_\beta, E_M]$, $\delta = \alpha$. Then it follows from (4.1.1) and (4.1.2) that relations (4.1.4) and (4.1.5) hold. By Lemmas 4.1.3 and 4.1.4, $(\Lambda - \beta, \alpha) = -(\alpha, \beta) = 0$. However, by hypothesis, $(\alpha, \beta) < 0$, which is a contradiction.

We have yet to show that $2(\Lambda, \beta)/(\beta, \beta) = 1$. We know that $(\Lambda, \beta) - (\alpha, \beta) - (\beta, \beta) = 0$ and $(\Lambda, \beta) + (\alpha, \beta) = 0$. Adding, we see that $2(\Lambda, \beta) = (\beta, \beta)$, as required.

(c) Suppose that $(\Lambda, \gamma) \neq 0$. Then it is easy to see that $[[[[E_M, e_\alpha], e_\beta], e_\gamma], [E_M, e_\alpha]] \neq 0$ and $[[[F_\Lambda, e_{-\beta}], e_{-\gamma}], F_\Lambda] \neq 0$, which proves the lemma.

LEMMA 4.1.6. (a) *Only one numerical mark of Λ is different from* 0, *and that is equal to* 1; *in particular, G_0 is simple.*

(b) *α is the highest root of one of the parts of the Dynkin diagram of G_0 into which it is divided by the numerical mark of Λ.*

Proof. (a) Since $(\Lambda, \alpha) = 0$, clearly there is a simple root β for which $\alpha + \beta$ is a root, but $\alpha - \beta$ is not. If there is a simple root $\gamma \neq \beta$ for which $(\Lambda, \gamma) \neq 0$, then $\alpha - \gamma$ is not a root and by applying Lemma 4.1.5(c) we arrive at a contradiction.

Thus, the only nonzero numerical mark of Λ corresponds to the simple root β. It is equal to 1 by Lemma 4.1.5(b).

(b) Suppose the contrary. Then there is a simple root β for which $\alpha + \beta$ is a root and $(\Lambda, \beta) = 0$. Multiplying both sides of (4.1.1) by $e_{-\beta}$ we have $[[F_\Lambda, e_{-\beta}], E_M] = e_{-\alpha-\beta}$, from which it follows that $[F_\Lambda, e_{-\beta}] \neq 0$; therefore $(\Lambda, \beta) \neq 0$, which is a contradiction.

We denote by s the number of the circle in the Dynkin diagram of G_0 against which the only nonzero numerical mark of Λ is placed.

LEMMA 4.1.7. *Either the sth circle of the Dynkin diagram of G_0 is at an end, or it is joined to an end circle with the number t, and then $\alpha = \alpha_t$ is a simple root.*

Proof. Suppose that the sth circle is joined both to the $(s-1)$th and $(s+1)$th. Applying Lemma 4.1.5(c) to $\beta = \alpha_s$ and $\gamma = \alpha_{s-1} + \alpha_s + \alpha_{s+1}$ (where α_i is a simple root corresponding to the ith circle), we see that α is a simple root. Lemma 4.1.7 now follows from Lemma 4.1.6(b).

Conclusion of the proof of Proposition 4.1.1. Unfortunately, I have not succeeded in avoiding case distinctions.

Let $\beta = \alpha_s$ be the unique simple root for which $(\Lambda, \alpha_s) \neq 0$.

By Lemma 4.1.5(b):

$$-2(\alpha, \beta)/(\beta, \beta) = 1. \tag{4.1.6}$$

In accordance with Lemma 4.1.7, we consider two cases separately.

Case I. The sth circle of the Dynkin diagram is at an end. If the G_0-module G_{-1} is isomorphic to one of the linear Lie algebras sl_n with $n > 2$ or so_n with $n > 4$, $n \neq 6$, then we see evidently, by Lemma 4.1.6(b), that the local Lie superalgebra $G_{-1} \oplus G_0 \oplus G_1$ is isomorphic to the local part of $\mathbf{S}(n)$ and $\mathbf{H}(n)$, respectively.

We claim that all other cases are impossible. Let t be the number of the circle in the Dynkin diagram that is determined by the following properties: $t \neq s$, t is an end circle and belongs to the longest of the possible "tails" of the diagram. We denote by γ the largest root for which in the decomposition into simple roots the coefficient of α_t is zero. It is not hard to check (using Table I) that in all cases satisfying (4.1.6), except the adjoint representation of $\mathbf{G}_2$, neither $\alpha - \gamma$ nor $\beta - \gamma$ is a root and that $(\Lambda, \gamma) \neq 0$. Therefore, by Lemma 4.1.5(c), this case cannot occur. For $\mathbf{G}_2$ we set $\beta' = \alpha + \beta$. Then $\alpha + \beta'$ is a root; however, $(\Lambda - \beta', \alpha + \beta') \neq 0$, as is easy to see, and this contradicts Lemma 4.1.5(a).

Case II. The sth circle of the Dynkin diagram is not at an end, but is joined by an edge to an end circle with the number t, and $\alpha = \alpha_t$. If the G_0-module G_{-1} is isomorphic to $\Lambda^2 sl_n$, $n > 3$, then we see clearly that the local Lie superalgebra $G_{-1} \oplus G_0 \oplus G_1$ is isomorphic to $\mathbf{P}(n-1)$.

We claim that all other cases are impossible. Let θ be the highest root of G_0.

If $(\theta, \alpha_s) = (\theta, \alpha_t) = 0$, then by setting $\gamma = \theta$ we arrive at a contradiction to Lemma 4.1.5(c). In the remaining cases satisfying (4.1.6), except the representation of $\mathbf{C}_n$ with highest weight $\theta - \alpha_t$, we denote by γ the largest root for which in the decomposition into simple roots the coefficient of α_t is zero, and we again use Lemma 4.1.5(c). In the case of $\mathbf{C}_n$ we set $\beta' = \theta - 2\alpha_t$ and arrive at a contradiction to Lemma 4.1.5(a).

Thus, the local part of G is isomorphic to the local part of one of $\mathbf{S}(n)$, $\mathbf{H}(n)$, or $\mathbf{P}(n)$. The isomorphism of the $\mathbf{Z}$-graded Lie superalgebras themselves now follows from Propositions 1.2.3(c), 3.3.1(e), and 3.3.6(d).

This completes the proof of Proposition 4.1.1.

4.1.2. *Classification of* $\mathbf{Z}$*-graded Lie superalgebras of depth* 1.

We now describe two constructions of transitive $\mathbf{Z}$-graded Lie superalgebras.

Every $\mathbf{Z}$-graded Lie superalgebra $G = \bigoplus G_i$ can be extended by means of an even derivation z defined by

$$[z, x] = kx \qquad \text{for} \quad x \in G_k .$$

So we obtain a $\mathbf{Z}$-graded Lie superalgebra, which we denote by $G^z = \bigoplus G_i^z$, where $G_i^z = G_i$ for $i \neq 0$ and $G_0^z = G_0 \oplus \langle z \rangle$. If G_0 is transitive and the center of G_0 is trivial, then clearly G^z is also transitive.

The other construction goes as follows. Let H be a Lie algebra without center. On it we construct a Lie superalgebra $H^\xi = G_{-1} \oplus G_0 \oplus G_1$ with a consistent $\mathbf{Z}$-grading, by setting $G_{-1} = \xi H$, $G_0 = H$, $G_1 = \langle d/d\xi \rangle$, where the commutators are defined as follows: $[d/d\xi, \xi h] = h$, $[\xi h_1, h_2] = \xi[h_1, h_2]$, $[d/d\xi, h] = 0$. Evidently, H^ξ is transitive.

Now we are in a position to state the main theorem of this section.

THEOREM 4. *A transitive irreducible Lie superalgebra* $G = \bigoplus_{i \geqslant -1} G_i$ *with a consistent* $\mathbf{Z}$*-grading and* $G_1 \neq 0$ *is isomorphic as* $\mathbf{Z}$*-graded superalgebra to one of the following list*:

I. $\mathbf{A}(m, n)$, $\mathbf{C}(n)$, $\mathbf{P}(n)$;

II. $\mathbf{W}(n)$, $\mathbf{S}(n)$, $\mathbf{H}(n)$, $\tilde{\mathbf{H}}(n)$;

III. H^ξ, *where* H *is a simple Lie algebra*;

IV. G^z, *where* G *is of type* I, II, *or* III *and the center of* G_0 *is trivial.*

Proof. Since the representation of G_0 on G_{-1} is faithful and irreducible, $G_0 = G_0' \oplus C$, where G_0 is semisimple, C is the center of G_0, $\dim C \leqslant 1$, and if $\dim C = 1$, then $C = \langle z \rangle$, with $[z, g] = kg$ for $g \in G_k$ (see Proposition 1.2.12). Therefore, the representation of G_0 on G_1 is completely reducible; let

$$G_1 = \bigoplus_i G_1^{(i)} \tag{4.1.7}$$

be the decomposition of G_1 into G_0-irreducible components. We denote by $G^{(s)}$ the $\mathbf{Z}$-graded subalgebra of G:

$$G^{(s)} = G_{-1} \oplus [G_{-1}, G_1^{(s)}] \oplus G_1^{(s)} \oplus (G_1^{(s)})^2 \oplus \cdots .$$

If we consider in G the $\mathbf{Z}$-graded subalgebra $G_{-1} \oplus G_0 \oplus G_1^{(s)} \oplus (G_1^{(s)})^2 \oplus \cdots$, we can infer from Proposition 1.2.9 that $[G_{-1}, G_1^{(s)}] \subseteq G_0'$. Therefore, $G^{(s)}$ is a $\mathbf{Z}$-graded Lie superalgebra satisfying all the conditions of Theorem 4.

There are two possibilities.

(1) The representations of $[G_{-1}, G_1^{(s)}]$ on G_{-1} and $G_1^{(s)}$ are contragredient. According to Proposition 1.2.10(a), $G^{(s)} = G_{-1} \oplus [G_{-1}, G_1^{(s)}] \oplus G_1^{(s)}$ is classical. Propositions 2.3.9 and 2.4.4 now show that $G^{(s)}$ is isomorphic as $\mathbf{Z}$-graded algebra to one of $\mathbf{A}(m, n)$ or $\mathbf{C}(n)$.

(2) The representations of $[G_{-1}, G_1^{(s)}]$ on G_{-1} and $G_1^{(s)}$ are not contragredient. According to Proposition 1.2.10(b), $[G_{-1}, G_1^{(s)}] = G_0'$ is then simple. If the representation of $[G_{-1}, G_1^{(s)}]$ on $G_1^{(s)}$ is not faithful, that is, $\dim G_1^{(s)} = 1$, then, as is easy to see, $G^{(s)} = G_{-1} \oplus [G_{-1}, G_1^{(s)}] \oplus G_1^{(s)}$ is isomorphic as $\mathbf{Z}$-graded algebra to $(G_0')^{\xi}$. If the representation of G_0' on $G_1^{(s)}$ is faithful, then according to Proposition 4.1.1 ($G^{(s)}$ is bitransitive, by Proposition 1.2.13), $G^{(s)}$ is isomorphic as $\mathbf{Z}$-graded Lie superalgebra to one of $\mathbf{S}(n)$, $\mathbf{H}(n)$, or $\mathbf{P}(n)$.

Thus, when $G_1 = G_1^{(s)}$ is an irreducible G_0-module, then Lemma 4.1.8 below and Propositions 3.3.1 and 3.3.6 show that G is one of $\mathbf{A}(m, n)$, $\mathbf{C}(n)$, $\mathbf{S}(n)$, $\mathbf{H}(n)$, $\tilde{\mathbf{H}}(n)$, $\mathbf{P}(n)$, H^{ξ}, or of type IV. (In Lemma 4.1.8 the case $\mathbf{A}(1, 1)$ is excluded; however, $\mathbf{A}(1, 1) \simeq \mathbf{H}(4)$.)

We claim that in (4.1.7) all the G_0-modules are pairwise inequivalent. Let F_Λ be the highest weight vector of the G_0-module G_{-1}, and E_{M_s} the lowest weight vector of $G_1^{(s)}$. Suppose that in (4.1.7) there are two isomorphic G_0-modules, say $G_1^{(1)}$ and $G_1^{(2)}$. If they are contragredient to G_{-1}, then by Proposition 2.1.6, $G^{(1)}$ and $G^{(2)}$ are isomorphic. Therefore, the vectors $[F_\Lambda, E_{M_1}]$ and $[F_\Lambda, E_{M_2}]$ are proportional; consequently, $[F_\Lambda, E_{M_1} - cE_{M_2}] = 0$ for some $c \in k$. But then, clearly, $[G_{-1}, E_{M_1} - cE_{M_2}] = 0$, which contradicts the fact that G is transitive. If $G_1^{(1)}$ and $G_1^{(2)}$ are not contragredient to G_{-1}, then $[F_\Lambda, E_{M_i}]$ are root vectors of G_0' corresponding to one and the same root and are, therefore, proportional. Again, this contradicts the transitivity of G.

When we now compare the possibilities for $G^{(s)}$ obtained above, we see that the G_0-module G_1 can be reducible only if the G_0-module G_{-1} is isomorphic to gl_n or the G_0'-module G_{-1} to $\Lambda^2 sl_4 \simeq so_6$. In the first case, Propositions 3.3.2 and 3.1.1(e) show that G is isomorphic to $\mathbf{W}(n)$. In the second case it follows from Proposition 3.3.6(j) that G is isomorphic to one of $\mathbf{H}(6)$, $\tilde{\mathbf{H}}(6)$, $\mathbf{H}(6)^z$, or $\tilde{\mathbf{H}}(6)^z$.

The proof of Theorem 4 is now complete.

4.1.3. *On extensions of some Lie superalgebras.*

LEMMA 4.1.8. *Let* $G = \bigoplus_{i \geqslant -1} G_i$ *be a transitive* $\mathbf{Z}$*-graded Lie superalgebra and* $G' = G_{-1} \oplus G_0 \oplus G_1$ *isomorphic to one of*

(a) H^ξ;

(b) $\mathbf{A}(m, n)$ *for* $(m, n) \neq (1, 1)$ *or* $\mathbf{C}(n)$;

(c) $\mathbf{P}(n)$.

Then $G = G'$.

Proof. In all three cases we have to show that $G_2 = 0$.

(a) We recall that $H^\xi = \xi H \oplus H \oplus \langle d/d\xi \rangle$. Suppose that there is a $t \in G_2$, $t \neq 0$. Then $[t, \xi a] = \alpha(a)(d/d\xi)$, where α is a nonzero linear function on H (because of transitivity). Therefore, $0 = [[t, \xi a], \xi a] = \alpha(a)$ for every $a \in H$, which is impossible.

(b) Suppose the contrary. Then there is a nonzero weight vector t_λ of the representation of G_0 on G_2. By the transitivity of G, there is a root vector $e_{-\beta} \in G_{-1}$ for which

$$[t_\lambda, e_{-\beta}] = e_{\lambda-\beta} \in G_1 . \tag{4.1.8}$$

By Proposition 2.5.5(e) there is a root vector $e_{-\lambda+\beta} \in G_{-1}$ for which $[e_{-\lambda+\beta}, e_{\lambda-\beta}] = h_{\lambda-\beta}$. Multiplying both sides of (4.1.8) by $e_{-\lambda+\beta}$, we have

$$[[e_{-\lambda+\beta}, t_\lambda], e_{-\beta}] = h_{\lambda-\beta} . \tag{4.1.9}$$

Hence it follows that $[e_{-\lambda+\beta}, t_\lambda] = e_\beta$, so that (4.1.9) shows that $h_{\lambda-\beta}$ is proportional to h_β; and then $\lambda = c\beta$ (also by Proposition 2.5.5(e)). Moreover, as we have seen, $\lambda - \beta = (c - 1)\beta$ is a nonzero root of G'. However, as is clear from Section 2.5.4, multiples of β can only be the roots 0 and $-\beta$, that is, $c = 0$ or 2. In the first case we see that $\dim G_\beta \geqslant 2$, which contradicts Proposition 2.5.5(b). Thus, any weight of the representation of G_0 on G_2 is equal to twice a root of G_0. But clearly this is impossible.

(c) We recall that the G_0-module G_{-1} is isomorphic to $\Lambda^2 sl_n$ or $S^2 sl_n$, and the G_0-module G_1 to $S^2 sl_n{}^*$ or $\Lambda^2 sl_n{}^*$. Let $\Lambda_{\pm 1}$ be the highest and $M_{\pm 1}$ the lowest weights of $G_{\pm 1}$. By transitivity, G_2 is a G_0-submodule of $G_1 \otimes G_{-1}^*$. (A mapping $\psi: G_2 \to G_1 \otimes G_{-1}^* = \mathrm{Hom}(G_{-1}, G_1)$ can be constructed in the obvious fashion: $\psi(g)(a) = [g, a]$, $g \in G_2$, $a \in G_{-1}$.) Thus, G_2 is a G_0-submodule of $\Lambda^2 sl_{n+1}^* \otimes S^2 sl_{n+1}^*$. This module splits into two irreducible components with the lowest weights $M^{(1)} = -\Lambda_{-1} + M_1$ and $M^{(2)} = -\Lambda_{-1} + M_1 + \alpha_1 + \alpha_2$, where α_1 and α_2 are the first two simple roots of the Lie algebra $\mathbf{A}_n$.

We now assume that one of the weight vectors $E_{M^{(i)}}$ is contained in G_2. By transitivity we then have, respectively,

$$[F_{\Lambda_{-1}}, E_{M^{(1)}}] = E_{M_1}, \tag{4.1.10}$$

$$[F_{\Lambda_{-1}}, E_{M^{(2)}}] = [E_{M_1}, e_{\alpha_1+\alpha_2}]. \tag{4.1.11}$$

Now (4.1.10) also gives

$$[[F_{\Lambda_{-1}}, e_{-\alpha_1}], E_{M^{(2)}}] = [E_{M_1}, e_{\alpha_2}]. \tag{4.1.12}$$

Next, $[F_{\Lambda_{-1}}, E_{M_1}] = e_{-\alpha_1}$ (or e_{α_1}). Therefore, from (4.1.10):

$$0 = [F^2_{\Lambda_{-1}}, E_{M^{(1)}}] = 2[F_{\Lambda_{-1}}, E_{M_1}] = e_{\pm\alpha_1}.$$

This is a contradiction of $E_{M^{(1)}} \in G_2$. Similarly, we find from (4.1.11) (or (4.1.12)) that

$$0 = [F^2_{\Lambda_{-1}}, E_{M^{(2)}}] = e_{\alpha_2} \qquad (\text{respectively, } 0 = [[F_{\Lambda_{-1}}, e_{-\alpha_1}]^2, E_{M^{(2)}}]$$
$$= 2[[F_{\Lambda_{-1}}, e_{-\alpha_1}], [E_{M_1}, e_{\alpha_2}]] = e_{\alpha_2});$$

hence $E_{M^{(2)}} \notin G_2$. This proves the lemma.

4.2. *The Classification of the Simple Lie Superalgebras*

4.2.1. *The main theorem.* The following theorem is the central result of the paper.

THEOREM 5. *A simple finite-dimensional Lie superalgebra over an algebraically closed field k of characteristic 0 is isomorphic either to one of the simple Lie algebras or to one of the Lie superalgebras* $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, $\mathbf{P}(n)$, $\mathbf{Q}(n)$, $\mathbf{W}(n)$, $\mathbf{S}(n)$, $\tilde{\mathbf{S}}(n)$, *or* $\mathbf{H}(n)$.

Proof. Let $L = L_{\bar 0} \oplus L_{\bar 1}$ be a simple finite-dimensional Lie superalgebra over k. If the representation of $L_{\bar 0}$ on $L_{\bar 1}$ is irreducible, then L is classical. Therefore, by Theorem 2, L is isomorphic to one of $\mathbf{B}(m, n)$, $\mathbf{D}(m, n)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, $\mathbf{Q}(n)$, or $\mathbf{D}(2, 1; \alpha)$.

Suppose now that the representation of $L_{\bar 0}$ on $L_{\bar 1}$ is reducible. Then, by Proposition 1.3.2, L has a filtration: $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ for which the associated $\mathbf{Z}$-graded Lie superalgebra $\operatorname{Gr} L = \oplus \operatorname{Gr}_i L$ satisfies all the conditions of Theorem 4. Therefore, $\operatorname{Gr} L$ can only be isomorphic to one of the Lie superalgebras of type I–IV listed there.

From the proof of Proposition 2.2.2, it is clear that if $\operatorname{Gr} L \simeq H^\xi$, then L is not simple; hence type III does not occur.

Proposition 1.3.1 shows that if the center of $\mathrm{Gr}_0 L$ is nontrivial, then $L \simeq \mathrm{Gr}\, L$. Hence, type IV does not occur either, because clearly no superalgebra of this type is simple; also $L \simeq \mathbf{W}(n)$ if $\mathrm{Gr}\, L \simeq \mathbf{W}(n)$.

If $\mathrm{Gr}\, L \simeq \mathbf{A}(m, n)$, $\mathbf{C}(n)$, or $\mathbf{P}(n)$, then evidently, the representation of $L_{\bar{0}}$ on $L_{\bar{1}}$ is for L the same as for $\mathrm{Gr}\, L$. Therefore, Proposition 2.1.4 shows that $L \simeq \mathrm{Gr}\, L$, so that L is one of $\mathbf{A}(m, n)$, $\mathbf{C}(n)$, or $\mathbf{P}(n)$.

If $\mathrm{Gr}\, L \simeq \mathbf{H}(n)$ or $\tilde{\mathbf{H}}(n)$, then $L \simeq \mathrm{Gr}\, L$, by Proposition 3.3.7. But $\tilde{\mathbf{H}}(n)$ is not simple, so that this case is impossible, and L is one of the $\mathbf{H}(n)$.

Finally, if $\mathrm{Gr}\, L \simeq \mathbf{S}(n)$, then by Proposition 3.3.5, L is isomorphic to an $\mathbf{S}(n)$ or $\tilde{\mathbf{S}}(n)$.

This completes the proof of the theorem.

4.2.2. *Isomorphisms.* It is not hard to list all the isomorphisms between simple Lie superalgebras. They are: $\mathbf{A}(m, n) \simeq \mathbf{A}(n, m)$; $\mathbf{A}(1, 0) \simeq \mathbf{C}(2) \simeq \mathbf{W}(2)$; $\mathbf{A}(1, 1) \simeq \mathbf{H}(4)$; $\mathbf{P}(2) \simeq \mathbf{S}(3)$.

In the remaining cases, except for $\mathbf{D}(2, 1; \alpha)$, $\mathbf{S}(n)$, and $\tilde{\mathbf{S}}(n)$, simple Lie superalgebras are pairwise nonisomorphic, because for them the $L_{\bar{0}}$-modules $L_{\bar{1}}$ are nonisomorphic. $\mathbf{S}(n)$ and $\tilde{\mathbf{S}}(n)$ are also nonisomorphic, according to Proposition 3.3.4. Conditions for isomorphisms of superalgebras in the family $\mathbf{D}(2, 1; \alpha)$ were derived in Proposition 2.5.4(b).

The following is a list of the dimensions of all the simple Lie superalgebras.

$\mathbf{A}(m, n)$	$(m+n+2)^2 - 1 - \delta_{m,n}$	$\mathbf{P}(n)$	$2(n+1)^2 - 1$
$\mathbf{B}(m, n)$	$2(m+n)^2 + m + 3n$	$\mathbf{Q}(n)$	$2(n+1)^2 - 2$
$\mathbf{C}(n)$	$2n^2 + n - 2$	$\mathbf{W}(n)$	$n \cdot 2^n$
$\mathbf{D}(m, n)$	$2(m+n)^2 - m + n$	$\mathbf{S}(n)$	$(n-1)2^n + 1$
$\mathbf{D}(2, 1; \alpha)$	17	$\tilde{\mathbf{S}}(n)$	$(n-1)2^n + 1$
$\mathbf{F}(4)$, $\mathbf{G}(3)$	40, 31	$\mathbf{H}(n)$	$2^n - 2$

4.2.3. *Classification of finite-dimensional primitive Lie superalgebras.* Let L be a Lie superalgebra and L_0 be a distinguished subalgebra. The pair (L, L_0) is called *even primitive* if L_0 is a maximal proper subalgebra, it contains no ideals of L, and $L_0 \supset L_{\bar{0}}$.

The same arguments as in the proof of Theorem 5 give a classification of the primitive even pairs (clearly, they are all finite-dimensional automatically, because $\mathrm{Gr}\, L$ is embedded in $\mathbf{W}(n)$, where $n = \dim L/L_0$). To state and prove the result, we need only make the following remarks.

As it is easy to see, if L is a Lie superalgebra with a filtration for which $\mathrm{Gr}\, L \simeq H^\epsilon$, then either $L \simeq H^\epsilon$ or $L \simeq \operatorname{der} \mathbf{Q}(n) = \mathbf{Q}(n) \oplus D$ is a semidirect sum of Lie superalgebras, D being, up to a constant factor, the only odd outer derivation of $\mathbf{Q}(n)$ (see Proposition 5.1.2(c)).

To an irreducible faithful representation of a Lie algebra H in a space V

there corresponds the primitive Lie superalgebra $H^V = V \oplus H$, where $[V, V] = 0$, $[h, v] = h(v)$ for $h \in H$, $v \in V$, and $H_{\bar{0}}^V = H$, $H_{\bar{1}}^V = V$.

THEOREM 5′. *Let (L, L_0) be a primitive even pair. Then L is isomorphic to a Lie superalgebra in the following list*:

I. $\mathbf{A}(m, n)$, $\mathbf{C}(n)$, $\mathbf{P}(n)$;

II. $\mathbf{B}(m, n)$, $\mathbf{D}(m, n)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, $\mathbf{Q}(n)$, der $\mathbf{Q}(n)$, $\mathbf{D}(2, 1; \alpha)$;

III. $\mathbf{W}(n)$, $\mathbf{S}(n)$, $\tilde{\mathbf{S}}(n)$, $\mathbf{H}(n)$, $\tilde{\mathbf{H}}(n)$;

IV. H^ξ, *where H is a simple Lie algebra*;

V. G^z, *where G is one of the $\mathbf{Z}$-graded Lie superalgebras* $\mathbf{A}(n, n)$, $\mathbf{P}(n)$, $\mathbf{S}(n)$, $\mathbf{H}(n)$, $\tilde{\mathbf{H}}(n)$, *or* H^ξ.

VI. H^V, *where H is a Lie algebra and V a faithful irreducible H-module.*

Each of these Lie superalgebras, except $\mathbf{P}(n)$ *and* $\mathbf{P}^z(n)$, *admits a unique structure of a primitive even pair. There are two such structures for* $\mathbf{P}(n)$ *and* $\mathbf{P}^z(n)$.

Note that Theorem 5′ also gives a classification of the primitive transitive supergroups of transformations of a supermanifold whose stabilizer contains a maximal reduced subgroup.

5. CONTINUATION OF THE THEORY

5.1. *Description of Semisimple Lie Superalgebras in Terms of Simple Ones*

5.1.1. DEFINITION. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a superalgebra, der A the Lie superalgebra of its derivations, and L a subset of der A. Then A is said to be *L-simple* if A contains no nontrivial ideals that are invariant under all the derivations in L. If A is der A-simple and $A^2 \neq 0$, then A is called *differentiably simple.*

We define operators l_s and r_s, $s \in A$, on A by the formulas

$$l_s(a) = sa, \qquad r_s(a) = (-1)^{(\deg a)(\deg s)} as, \qquad a \in A.$$

It is easy to verify that if $D \in \operatorname{der} A$, then

$$[D, l_s] = l_{D(s)} \qquad \text{and} \qquad [D, r_s] = r_{D(s)} .$$

We denote by $T(A)$ the associative subalgebra of $l(A)$ (all the endomorphisms of A) generated by all the l_s and r_s, $s \in A$.

Finally, the *centroid* of A is the associative superalgebra $\Gamma(A) = \{g \in L(A) \mid [g, h] = 0,\ h \in T(A)\}$. A is called *central* if $\Gamma(A) = k$.

5.1.2. *Differentiably simple superalgebras.* A verbatim repetition of the arguments in [21], with the relevant definition replaced by those above, leads to the following result.

PROPOSITION 5.1.1. *Let G be a finite-dimensional differentiably simple (not necessarily Lie) superalgebra. Then $G \simeq S \otimes \Lambda(n)$, where S is a simple and $\Lambda(n)$ is the Grassmann superalgebra.*

5.1.3. *Description of semisimple Lie superalgebras.* We recall that a superalgebra A is said to be semisimple if $A^2 \neq 0$ and A contains no nontrivial solvable ideals. In [21] the description of differentiably simple algebras is used to derive a description of semisimple Lie algebras over any field. The same arguments are suitable for Lie superalgebras.

THEOREM 6. *Let $S_1, \ldots, S_r$ be finite-dimensional Lie superalgebras, $n_1, \ldots, n_r$ be nonnegative integers, and $S = \bigoplus_{i=1}^r S_i \otimes \Lambda(n_i)$. Then*

$$S = \operatorname{inder} S = \bigoplus_{i=1}^{r} (\operatorname{inder} S_i) \otimes \Lambda(n_i) \subseteq \operatorname{der} S$$

$$= \bigoplus_{i=1}^{r} ((\operatorname{der} S_i) \otimes \Lambda(n_i) + 1 \otimes \operatorname{der} \Lambda(n_i)).$$

Let L be a subalgebra of der S *containing S; we denote by L_i the set of components of elements of L in* $1 \otimes \operatorname{der} \Lambda(n_i)$. *Then:*

(a) *L is semisimple if and only if $\Lambda(n_i)$ is L_i-simple for all i.*

(b) *All finite-dimensional semisimple Lie superalgebras arise in the manner indicated.*

(c) der L *is the normalizer of L in* der S, *provided that L is semisimple.*

5.1.4. *Description of* der G *for the simple Lie superalgebras G.*

PROPOSITION 5.1.2. *Let G be a simple Lie superalgebra,* der G *be the Lie superalgebra of its derivations, and* inder G ($\simeq G$) *be the ideal of* der G *consisting of the inner derivations.*

(a) *If G is one of the classical Lie superalgebras* $\mathbf{A}(m, n)$ *with $m \neq n$,* $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, *or one of the Lie superalgebras of Cartan type* $\mathbf{W}(n)$ *or* $\mathbf{S}(n)$, *then* der G = inder G.

(b) *If $G = \bigoplus G_i$ is one of the* $\mathbf{Z}$*-graded Lie superalgebras* $\mathbf{A}(n, n)$ *with*

$n > 1$, $\mathbf{S}(n)$, or $\mathbf{P}(n)$, then der G = inder $G \oplus \langle z \rangle$ *is a semidirect sum, where z is an even derivation of G such that $[z, g] = kg$ for $g \in G_k$.*

(c) *If $G \simeq \mathbf{Q}(n) = \mathbf{Q}(n)_{\bar{0}} \oplus \mathbf{Q}(n)_{\bar{1}}$, then* der G = inder $G \oplus \langle D \rangle$ *is a semidirect sum, where D is the (within proportionality unique) endomorphism of $\mathbf{Q}(n)$ for which $D(\mathbf{Q}(n)_{\bar{0}}) = 0$, $D(\mathbf{Q}(n)_{\bar{1}}) = \mathbf{Q}(n)_{\bar{0}}$, and $D: \mathbf{Q}(n)_{\bar{1}} \to \mathbf{Q}(n)_{\bar{0}}$ is an isomorphism of $\mathbf{Q}(n)_{\bar{0}}$-modules.*

(d) *If $G \simeq \mathbf{H}(n)$, $n \geqslant 5$, then* der G = inder $G \oplus T$, *where*

$$T = \left\langle \sum \xi_i \frac{\partial}{\partial \xi_i}, \sum \frac{\partial}{\partial \xi_i} (\xi_1 \cdots \xi_n) \frac{\partial}{\partial \xi_i} \right\rangle \subset \mathbf{W}(n)$$

is a two-dimensional solvable Lie superalgebra.

(e) *If $G \simeq \mathbf{A}(1, 1) = G_{-1} \oplus G_0 \oplus G_1$, then* der G = inder $G \oplus P$ *is a semidirect sum, where $P = \langle D_{-1}, z, D_1 \rangle$ is a three-dimensional simple Lie algebra $[z, g] = kg$ for $k \in G_k$, $D_{\pm 1}$ are the (up to a constant factor unique) endomorphisms of G for which $D_{\pm}(G_0) = 0$, $D_{\pm 1}(G_{\pm 1}) = 0$, $D_{\pm 1}(G_{\mp 1}) = G_{\pm 1}$, and $D_{\pm 1}: G_{\mp 1} \to G_{\pm 1}$ are isomorphisms of G_0-modules.*

Proof. $G \simeq$ inder $G \subset$ der G is an ideal of der G. Let G_0 be the reductive part of $G_{\bar{0}}$. Since the G_0-module der G is completely reducible, we have that der G = inder $G \oplus T$ is a direct sum of G_0-modules and T is a $\mathbf{Z}_2$-graded subspace; in particular, $[G_0, T] \subset T$. On the other hand, $[G_0, T] \subset$ inder G, because inder G is an ideal of der G. Therefore, $[G_0, T] = 0$ and if $D \in T$, then ad D is an endomorphism of the G_0-module der G. Using this fact it is now easy to compute T in all cases.

5.2. *Irreducible Finite-Dimensional Representations of Solvable and Simple Lie Superalgebras*

5.2.1. *Induced modules.* Let G be a Lie superalgebra, $U(G)$ its universal enveloping superalgebra (see Section 1.1.3), H a subalgebra of G, and V an H-module. V can be extended to a $U(H)$-module. We consider the $\mathbf{Z}_2$-graded space $U(G) \otimes_{U(H)} V$ (this is the factor space of $U(G) \otimes V$ by the linear span of the elements of the form $gh \otimes v - g \otimes h(v)$, $g \in U(G)$, $h \in U(H)$). This space can be endowed with the structure of a G-module as follows: $g(u \otimes v) = gu \otimes v$, $g \in G$, $u \in U(G)$, $v \in V$. The so constructed G-module is said to be *induced from the H-module V* and is denoted by $\mathrm{Ind}_H{}^G V$.

We list some of the simplest properties of induced modules, which follow from the Poincaré–Birkhoff–Witt theorem (see Section 1.1.3).

PROPOSITION 5.2.1. (a) *Let G be a Lie superalgebra, H be a subalgebra, V be a simple G-module, and W be an H-submodule of V considered as an H-module. Then V is a factor module of the G-module* $\mathrm{Ind}_H{}^G W$.

(b) *If $H_2 \subset H_1 \subset G$ are subalgebras of G and W an H_2-module, then* $\mathrm{Ind}_{H_1}^G(\mathrm{Ind}_{H_2}^{H_1} W) \simeq \mathrm{Ind}_{H_2}^G W$.

(c) *Let $H \subset G$ be a subalgebra of G containing $G_{\bar{0}}$, and $g_1, \dots, g_t$ odd elements of G whose projections onto G/H form a basis. Let W be an H-module. Then* $\mathrm{Ind}_H{}^G W = \bigoplus_{1 \le i_1 < \cdots < i_s \le t} g_{i_1} \cdots g_{i_s} W$ *is a direct sum of subspaces; in particular,* $\dim \mathrm{Ind}_H{}^G W = 2^t \dim W$.

The next result follows from Proposition 5.2.1(c) and Ado's theorem for Lie algebras.

Ado's Theorem. *Every finite-dimensional Lie superalgebra has a finite-dimensional faithful representation.*

5.2.2. *Representations of solvable Lie superalgebras.* Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a Lie superalgebra. A linear form $l \in G^*$ is said to be *distinguished* if $l([G_{\bar{0}}, G_{\bar{0}}]) = l(G_{\bar{1}}) = 0$. We denote by $\mathscr{L}$ the space of distinguished linear forms, by $\mathscr{L}_0$ the subspace consisting of those l for which $l([G, G]) = l(G_1) = 0$, and by $\mathscr{L}_1$ the subgroup of $\mathscr{L}_0$ generated by the linear forms given by the one-dimensional factors of the adjoint representation of G.

Let ρ be a representation of G in a space V, $\mathscr{M}$ a subgroup of $\mathscr{L}_0$, and $\lambda \in \mathscr{M}$. We define a representation $\tilde{\rho}$ of G in V by the formula $\tilde{\rho}(g)v = \rho(g)v + \lambda(g)v$ (i.e., $\tilde{\rho}$ is a tensor product of ρ and a one-dimensional representation). The G-modules ρ and $\tilde{\rho}$ are said to be *$\mathscr{M}$-equivalent.*

Lemma 5.2.2. *Let G be a Lie superalgebra, H be a subalgebra of codimension 1 containing $G_{\bar{0}}$, and g be an odd element for which $G = H \oplus \langle g \rangle$ is a direct sum of subspaces.*

(a) *If W is an irreducible H-module, then all the irreducible factors of the H-module $\mathrm{Ind}_H{}^G W = W \oplus gW$ are $\mathscr{L}_0$-equivalent to W ($\mathscr{L}_0 \subset H^*$).*

(b) *If V is an irreducible G-module and W an irreducible H-submodule with $W \neq V$, then $V \simeq \mathrm{Ind}_H{}^G W$.*

Proof. (a) For $h \in H$ we have $[h, g] = \lambda(h)g + h'$, where $h' \in H$, $\lambda \in \mathscr{L}_0$. Therefore, $h(gv) = g(hv + \lambda(h)v) + h'v$; hence, the H-modules W and $\mathrm{Ind}_H{}^G W/W$ are $\mathscr{L}_0$-equivalent.

(b) follows from (a) and Proposition 5.2.1.

Let $l \in \mathscr{L}$ be a distinguished linear form, considered modulo $\mathscr{L}_0$; we set $G_l = \{g \in G \mid l([g, g_1]) = 0 \text{ for } g_1 \in G\}$. Clearly, G_l is a subalgebra of G containing $G_{\bar{0}}$, and $l([G_l, G_l]) = 0$. A subalgebra $P \subset G$ is said to be *subordinate* to l if $l([P, P]) = 0$ and $G_l \subset P$. Clearly, this concept is well defined.

We single out an important class of solvable Lie superalgebras—the completely solvable ones—for which all irreducible factors of the adjoint representation are one-dimensional. By Engel's theorem, a nilpotent Lie superalgebra is completely solvable, and $\mathscr{L}_1 = 0$.

Finally, we denote by $\{H, l\}$ the one-dimensional H-module given by a linear form $l \in \mathscr{L}_0$ according to formula $h(v) = l(h)v$.

Now we are in a position to state a theorem that describes the finite-dimensional irreducible representations of solvable Lie superalgebras.

THEOREM 7. *Let $G = G_{\bar{0}} \oplus G_{\bar{1}}$ be a solvable Lie superalgebra.*

(a) *If V is an irreducible finite-dimensional G-module, then all the irreducible factors of V considered as a $G_{\bar{0}}$-module are one-dimensional, and their corresponding linear forms, extended by zero to $G_{\bar{1}}$, lie in a single coset $l_V \in \mathscr{L}/\mathscr{L}_0$.*

(b) *Let $\mathfrak{l} \in \mathscr{L}/\mathscr{L}_0$, P be a maximal subalgebra subordinate to $\mathfrak{l}$, and $\{P, l\}$ be the one-dimensional P-submodule given by the linear form $l \in \mathfrak{l}$. Then the G-module $V = \operatorname{Ind}_P{}^G\{P, l\}$ is finite-dimensional and irreducible, and $\mathfrak{l} = l_V$. Two such G-modules V_1 and V_2 are $\mathscr{L}_0$-equivalent if and only if $\mathfrak{l}_1 = \mathfrak{l}_2$.*

(c) *Every finite-dimensional irreducible G-module V is isomorphic to one of the modules $\operatorname{Ind}_P{}^G\{G, l\}$, where $l \in l_V$ and P is a maximal subalgebra subordinate to l.*

(d) *If G is completely solvable, then $\mathscr{L}_0$ can be replaced everywhere by $\mathscr{L}_1$. In particular, if G is nilpotent, we obtain a bijective correspondence $V \mapsto l_V$ between the set of classes of isomorphic finite-dimensional irreducible G-modules and $\mathscr{L}$.*

Proof. (a) is proved by induction on $\dim G$. Let V be an irreducible G-module, H be a subalgebra of codimension 1 containing $[G, G]$, $G = H \oplus \langle g \rangle$, where $g \in G_s$, $s \in \mathbf{Z}_2$, and W be an irreducible submodule of an H-module V. Clearly, then $V = \bigoplus_{i=0}^{t} g^i W$ and since H is an ideal of G, the proof of Lemma 5.2.2(a) shows that all the irreducible factors of V are isomorphic to W. Therefore, if $g \in G_{\bar{1}}$, then V and W, as $G_{\bar{0}}$-modules, have the same stock of irreducible factors, and (a) is true by induction. But if $g \in G_{\bar{0}}$, then $G_{\bar{0}} = H_{\bar{0}} \oplus \langle g \rangle$, and only as $H_{\bar{0}}$-modules do V and W have the same stock of irreducible factors. But then it follows that if $l_1, l_2 \in G^*$ are linear forms that are zero on $G_{\bar{1}}$ and give irreducible factors of the $G_{\bar{0}}$-module V; then by the inductive hypothesis, $l_1 - l_2 |_{[H_{\bar{0}}, H_{\bar{0}}]} = 0$, in particular, $l_1 - l_2 |_{[G_{\bar{1}}, G_{\bar{1}}]} = 0$. Since, of course, $l_1 - l_2 |_{[G_{\bar{0}}, G_{\bar{0}}]} = 0$ we see that $l_1 - l_2 \in \mathscr{L}_0$, as required.

(c) is proved by induction on $\dim G_{\bar{1}}$. Let V be a finite-dimensional irreducible G-module and P be a maximal subalgebra subordinate to l_V. If $\dim G_{\bar{1}} = 0$, then (c) is true by Lie's theorem. We may, therefore, assume that $\dim G_{\bar{1}} > 0$. We analyze first the case $G = P$, that is, $l_V = 0$. By induction on $\dim G_{\bar{1}}$ we show that $\dim V = 1$. Let G' be a subalgebra of G, of codimension 1, containing $G_{\bar{0}}$, and $G = G' \oplus \langle g \rangle$. By the inductive hypothesis, the

G'-module V contains a one-dimensional submodule $\langle v\rangle$. Suppose that $\dim V > 1$; then, evidently, $V = \langle v\rangle \oplus \langle gv\rangle$. Now $g(gv) = \frac{1}{2}[g, g]v = \frac{1}{2}l([g, g])v = 0$. Since V is irreducible, there is an element $h \in G_{\bar{1}}$ for which $h(gv) = v$. Replacing h by $h + cg$ for a suitable $c \in k$, we may assume that $h(v) = 0$. But then $[h, g]v = v$ and $l([h, g]) = 1$, which contradicts our assumption.

Suppose now that $G \neq P$. Then G has a subalgebra H of codimension 1 containing P, and $G = H \oplus \langle g\rangle$, $g \in G_{\bar{1}}$. Let W be an irreducible submodule of the H-module V. By the inductive hypothesis, $W = \mathrm{Ind}_P{}^H\{P, l\}$. If $H = P$, then, as we have shown, $W = \langle v\rangle$ is a one-dimensional H-module. In that case $V \neq W$ because $G \neq P$, and so $V = \mathrm{Ind}_P{}^G\{P, l\}$, by Lemma 5.2.2(b) and Proposition 5.2.1(b).

Suppose now that $H \neq P$. Then $l([h, h]) \neq 0$ for some $h \in H_{\bar{1}}$, and the quadratic equation in α,

$$l([g + \alpha h, g + \alpha h]) = l([g, g]) + 2\alpha l([g, h]) + \alpha^2 l([h, h]) = 0,$$

has a root α_0. Replacing g by $g + \alpha_0 h$, we may assume that $l([g, g]) = 0$.

What we have to show (according to Proposition 5.2.1(b)) is that the G-module $\mathrm{Ind}_H{}^G\, W = W + gW$ is irreducible. Suppose the contrary, i.e., that it contains a nonzero irreducible G-submodule W'. By Lemma 5.2.2(a), the G-modules W and W' are $\mathscr{L}_0$-equivalent. Hence, in particular, there is a one-dimensional P-submodule $\{P, l_1\} = \langle v_1 + gv_2\rangle \in W$, where $l - l_1 \in \mathscr{L}_0$. Now $h(v_1 + gv_2) = l_1(h)(v_1 + gv_2) = h(v_1) + [h, g]v_2 + ghv_2$ for $h \in G_{\bar{0}}$. Since $[h, g] = c(h)g + h'$, $h' \in H$, we see that $h(v_2) = (l_1(h) - c(h))v_2$. In particular, $\langle v_2\rangle$ is a one-dimensional submodule of the $G_{\bar{0}}$-module W; hence by (a), $h(v_2) = l_2(h)v_2$ for $h \in G_{\bar{0}}$, where $l_1 - l_2 \in \mathscr{L}_0$. Therefore, in particular, $l_2([g, g]) = 0$. If $v_1 \neq 0$, then $g(v_1 + gv_2) = gv_1 + \frac{1}{2}l([g, g]v_2 = gv_1 \neq 0$; but if $v_1 = 0$, then $gv_2 \neq 0$. Thus, $W' \cap gW \neq 0$. It therefore follows from Lemma 5.2.2(a) that $W' = gW$. But then $h(gv) = l([h, g])v + ghv$ for $h \in H_{\bar{1}}$. Since gW is a G-module, we infer that $l([h, g]) = 0$ for $h \in H_{\bar{1}}$, and since, furthermore, $l([g, g]) = 0$, it follows that $l([G_{\bar{1}}, g]) = 0$, that is, $g \in G_l \subset P$. This contradicts the choice of g.

(b) evidently follows from (a) and (c), and (d) follows from the fact that if G is a completely solvable Lie superalgebra, then all the irreducible factors of the G-module $U(G)$ are one-dimensional and are given by linear forms in $\mathscr{L}_1$.

This completes the proof of the theorem.

The following propositions are consequences of Theorem 7.

PROPOSITION 5.2.3. *For an irreducible finite-dimensional representation of a solvable Lie superalgebra* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *in a space* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ *we have: Either* $\dim V_{\bar{0}} = \dim V_{\bar{1}}$ *and* $\dim V = 2^s$, *where* $s \leqslant \dim G_{\bar{1}}$, *or* $\dim V = 1$.

PROPOSITION 5.2.4. *All the irreducible finite-dimensional representations of a solvable Lie superalgebra* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *are one-dimensional if and only if* $[G_{\bar{1}}, G_{\bar{1}}] \subseteq [G_{\bar{0}}, G_{\bar{0}}]$.

EXAMPLE 1. It follows from Theorem 7 that the families of representations ρ_α and ρ_α' of the Heisenberg superalgebras N and N', which were constructed in Section 1.1.6, contain all their finite-dimensional non-one-dimensional irreducible representations, and each precisely once.

EXAMPLE 2. Let $G = l(1, 1)$ be the completely solvable Lie superalgebra with the basis $z = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, $e = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$, $f = \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$. The set of representations of dimension >1 is parameterized by the numbers $\alpha = l(h)$, $\beta = l(z) \neq 0$:

$$z \mapsto \begin{pmatrix}\beta & 0\\ 0 & \beta\end{pmatrix}, \qquad h \mapsto \begin{pmatrix}\alpha & 0\\ 0 & \alpha - 1\end{pmatrix}, \qquad e \mapsto \begin{pmatrix}0 & \beta\\ 0 & 0\end{pmatrix}, \qquad f \mapsto \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

For $\beta = 0$ we obtain all the one-dimensional representations $h \mapsto \alpha$, $z, e, f \mapsto 0$.

The proof of Theorem 7 also works for infinite-dimensional representations of completely solvable Lie superalgebras. Two representations are called *weakly equivalent* if they have the same kernel in $U(G)$. We set $\tilde{\mathscr{L}} = \{l \in G^* \mid l(G_{\bar{1}}) = 0\}$. Let $\mathscr{G}_0$ be the Zariski closure of $\operatorname{Aut} G_{\bar{0}}$ in $GL(G_{\bar{0}})$.

It is known (Dixmier) that there is a bijection between the set of $\mathscr{G}_{\bar{0}}$-orbits in $G_{\bar{0}}^*$ and the set of classes of weakly equivalent representations of $G_{\bar{0}}$.

THEOREM 7′. (a) *If G is a completely solvable Lie superalgebra and V is an irreducible G-module, then there is an uncondensed Jordan–Holder series relative to $G_{\bar{0}}$ and all its irreducible factors correspond to a single $\mathscr{G}_0$-orbit Ω_V in $\tilde{\mathscr{L}}/\mathscr{L}_1$.*

(b) *Let $l \in \tilde{\mathscr{L}}/\mathscr{L}_1$, P be a maximal subalgebra subordinate to l, and* $\dim P_{\bar{0}} = \frac{1}{2}(\dim G_{\bar{0}} + \dim(G_l)_{\bar{0}})$. *Then the G-module $V_l = \operatorname{Ind}_P{}^G\{P, l\}$ is irreducible. The correspondence $l \mapsto V_l$ induces a bijective correspondence between the set of classes of $\mathscr{L}_1$-weakly equivalent irreducible representations of G and the set of $\mathscr{G}_0$-orbits in $\tilde{\mathscr{L}}/\mathscr{L}_1$. Here $\Omega_{V_l} = \mathscr{G}_0 \cdot l$.*

5.2.3. *Representations of simple Lie superalgebras.*

PROPOSITION 5.2.5. *Let $G = \bigoplus_{i \geqslant -d} G_i$ be a $\mathbf{Z}$-graded Lie superalgebra of depth d, $N^{\pm} = \bigoplus_{i>0} G_{\pm i}$, $B^{\pm} = N^{\pm} \oplus B_0$, where B_0 is a Borel subalgebra of $(G_0)_{\bar{0}}$. Suppose that*

$$[B_{\bar{1}}^{\pm}, B_{\bar{1}}^{\pm}] \subset [B_0^{\pm}, B_0^{\pm}], \qquad [B_0, (G_{\pm 1})_{\bar{0}}] = (G_{\pm 1})_{\bar{0}}, \qquad \text{and } G_{\pm 1} \text{ generates } N^{\pm}.$$

(a) *Let V be a finite-dimensional irreducible G-module; we set $V_0 = \{v \in V \mid N^+(v) = 0\}$. Then V_0 is an irreducible submodule of the G_0-module V.*

(b) *Two finite-dimensional G-modules V and V' are isomorphic if and only if the corresponding G_0-modules V_0 and V_0' are isomorphic.*

(c) *If the depth $d = 1$, then for any finite-dimensional irreducible G_0-module V_0' there is a finite-dimensional irreducible G-module V for which the G_0-module V_0 is isomorphic to V_0'.*

Proof. Since B^+ is a solvable Lie superalgebra, it follows by Proposition 5.2.4 that any irreducible factor of the B^+-module V is one-dimensional. Using the properties of N^+, we hence find that the N^+-module V is nilpotent. In particular, $V_0 \neq 0$. The same is true for N^-.

Now it only remains to go through the arguments in [7] almost verbatim. First, we show that $W = V_0 \cap G_{-1}V$ is equal to 0. Let $U'(G_{-1})$ be the subalgebra of the enveloping superalgebra $U(G)$ generated by G_{-1} and $U(G_{-1}) = U'(G_{-1}) \oplus \langle 1 \rangle$. Since $U'(G_{-1})$ is nilpotent, $U(G_{-1})W \subseteq U'(G_{-1})V \neq V$. $U(G_{-1})W$ is a G-submodule of V, and since V is a simple G-module, $U(G_{-1})W = 0$; in particular, $W = 0$.

We set $V_i = G_{-1}V_{i-1}$ for $i > 0$. Clearly, V is the sum of the subspaces V_i. We show by induction that this sum is direct. Suppose the contrary, i.e., that $v \in V_{m+1} \cap (\oplus_{i=0}^{m} V_i)$, $v \neq 0$. Then $G_1 v \neq 0$ because $W = 0$. But $G_1 v \subseteq V_m \cap (\oplus_{i=0}^{m-1} V_i)$, which is impossible. Thus, $V = \oplus_{i\geqslant 0} V_i$. From this it follows, obviously, that V_0 is an irreducible G_0-module.

(b) Let V be a finite-dimensional irreducible G-module; then $V \simeq \mathrm{Ind}_{G_0}^{G} V_0/I$, where V_0 is an irreducible G_0-submodule and I is a maximal submodule of the G-module $\mathrm{Ind}_{G_0}^{G} V_0$. Since $V = \oplus_{i\geqslant 0} V_i$, this I is uniquely determined as the sum of all graded submodules of the G-module $\mathrm{Ind}_{G_0}^{G} V_0$, and this proves (b).

(c) We define an action of $G_0 \oplus N^+$ on V_0', setting $N^+V_0' = 0$. Since $d = 1$, the induced G-module $\mathrm{Ind}_{G_0}^{G} V_0$ is finite-dimensional (see Proposition 5.2.1(c)). The required G-module is a factor module of this G-module.

We apply Proposition 5.2.5 to the following $\mathbf{Z}$-graded Lie superalgebras $G = \oplus G_i$: (a) $\mathbf{P}(n)$, $\mathbf{W}(n)$, $\mathbf{S}(n)$, $\mathbf{H}(n)$ with the "standard" $\mathbf{Z}$-grading; (b) $\mathbf{Q}(n)$ with the $\mathbf{Z}$-grading in Section 2.5.7 with $k_1 = \cdots = k_n = 1$; (c) the contragredient Lie superalgebras with "standard" $\mathbf{Z}$-grading (from Section 2.5.7 with $k_1 = \cdots = k_r = 1$).

In cases (b) and (c) we set $H = (G_0)_{\bar{0}} = \langle h_1, \dots, h_r \rangle$, $N^+ = \oplus_{i>0} G_i$, $B = H \oplus N^+$. Let $\Lambda \in H^*$, $a_i = \Lambda(h_i) \in k$, $\langle v_\Lambda \rangle$ be a one-dimensional B-module for which $N^+(v_\Lambda) = 0$, $h_i(v_\Lambda) = a_i v_\Lambda$. We set $V_\Lambda = \mathrm{Ind}_B{}^G \langle v_\Lambda \rangle / I_\Lambda$, where I_Λ is the (unique) maximal submodule of the G-module V_Λ. Λ is called the *highest weight* of the G-module V_Λ. It follows from Proposition 5.2.5(b) that the G-modules V_{Λ_1} and V_{Λ_2} are isomorphic if and only if $\Lambda_1 = \Lambda_2$. Numbers $a_i = \Lambda(h_i)$ are called the *numerical marks* of Λ.

We let $\mathbf{Z}_+$ denote the set of nonnegative integers.

THEOREM 8. (a) *Let $G = \bigoplus G_i$ be one of the following* **Z**-*graded Lie superalgebras of depth* 1: $\mathbf{P}(n)$, $\mathbf{W}(n)$, $\mathbf{S}(n)$, $\mathbf{H}(n)$. *Then the correspondence in Proposition* 5.2.5 *between finite-dimensional irreducible G-modules and finite-dimensional irreducible G_0-modules is bijective.*

(b) *For $\mathbf{Q}(n)$ the set of numerical marks of the highest weight of the finite-dimensional module V_Λ is characterized by the following conditions: $a_i \in \mathbf{Z}_+$ and if $a_i = 0$, then $a_1 + 2a_2 + \cdots + (i-1)a_{i-1} = a_n + 2a_{n-1} + \cdots + (n-i)a_{i+1}$.*

(c) *For the contragredient Lie superalgebras in Table* VI (*the sth row of the Cartan matrix is normalized so that $a_{ss+1} = 1$ for $a_{ss} = 0$*), *the set $\{a_i\}$ of numerical marks of the highest weight of the finite-dimensional module V_Λ is characterized by the following properties:*

(1) $a_i \in \mathbf{Z}_+$ *for* $i \neq s$;

(2) $k \in \mathbf{Z}_+$, *where k is given by the following table:*

G	k	b
$\mathbf{B}(0, n)$	$1/2a_n$	0
$\mathbf{B}(m, n)$, $m > 0$	$a_n - a_{n+1} - \cdots - a_{m+n-1} - 1/2a_{m+n}$	m
$\mathbf{D}(m, n)$	$a_n - a_{n+1} - \cdots - a_{m+n-2} - 1/2(a_{m+n-1} + a_{m+n})$	m
$\mathbf{D}(2, 1; \alpha)$	$(1+\alpha)^{-1}(2a_1 - a_2 - \alpha a_3)$	2
$\mathbf{F}(4)$	$1/3(2a_1 - 3a_2 - 4a_3 - 2a_4)$	4
$\mathbf{G}(3)$	$1/2(a_1 - 2a_2 - 3a_3)$	3

(3) *for $k < b$ (in the table) there are the supplementary conditions:*

$\mathbf{B}(m, n)$: $a_{n+k+1} = \cdots = a_{m+n} = 0$.

$\mathbf{D}(m, n)$: $a_{n+k+1} = \cdots = a_{m+n} = 0$, $k \leqslant m-2$; $a_{m+n-1} = a_{m+n}$, $k = m-1$.

$\mathbf{D}(2, 1; \alpha)$: *all $a_i = 0$ if $k = 0$; $(a_3 + 1)\alpha = \pm(a_2 + 1)$ if $k = 1$.*

$\mathbf{F}(4)$: *all $a_i = 0$ if $k = 0$; $k \neq 1$; $a_2 = a_4 = 0$ if $k = 2$; $a_2 = 2a_4 + 1$ if $k = 3$.*

$\mathbf{G}(3)$: *all $a_i = 0$ if $k = 0$; $k \neq 1$; $a_2 = 0$ if $k = 2$.*

Proof. (a) follows from Proposition 5.2.5.

To prove (b) and (c), we let Π_0 denote the system of simple roots of the (reductive) Lie algebra $G_{\bar{0}}$ determined by the induced **Z**-grading. In order for the G-module V_Λ to be finite-dimensional it is necessary and sufficient that

$$e_{-\alpha}^{c+1} v_\Lambda = 0 \quad \text{for} \quad \alpha \in \Pi_0, \quad \text{where} \quad c = 2(\Lambda, \alpha)/(\alpha, \alpha) \qquad (5.2.1)$$

(it is well known that these relations generate the annihilator of v_Λ).

(b) If $G = \mathbf{Q}(n)$, then $\Pi_0 = \{\alpha_1, \ldots, \alpha_n\}$, where $e_{\alpha_i} = e_i$; hence $a_i \in \mathbf{Z}_+$. It is easy to see that the supplementary conditions only arise when $a_i = 0$:

This condition gives the equality $\bar{e}_i f_i v_\Lambda = 0$, which is equivalent to the second equality in (b).

(c) If $G = \mathbf{A}(m, n)$ or $\mathbf{C}(n)$, then $\Pi_0 = \{\alpha_i, i \neq s\}$, and condition (5.2.1) is equivalent to condition (1). For the remaining contragredient Lie superalgebras, $\Pi_0 = \{\alpha_i, i \neq s, \beta\}$, where β is the maximal root among the roots of the form $\sum_{i \geqslant s} k_i \alpha_i$. By the same token, condition (5.2.1) shows that conditions (1) and (2) are necessary. It is also clear that (1) is sufficient for (5.2.1) when $\alpha = \alpha_i$. However, condition (2) turns out not to be sufficient for (5.2.1) when $\alpha = \beta$. When $\alpha = \beta$, using direct computations from (5.2.1), we can show that condition (3) is necessary. It is also not hard to verify that (1) and (2) are sufficient for (5.2.1) when $G = \mathbf{B}(0, n)$.

It remains to show that conditions (2) and (3) are sufficient for (5.2.1) when $\alpha = \beta$. To do this, it suffices to find a set of highest weights Λ of finite-dimensional modules V_Λ with the property $2(\Lambda, \beta)/(\beta, \beta) \leqslant k$ which generates the plane defined by the equations in (3). It is clear that for $\mathbf{B}(m, n)$ and $\mathbf{D}(m, n)$ without loss of generality we may assume that $n = 1$; then for $\mathbf{B}(m, 1)$ and $\mathbf{D}(m, 1)$ the desired set is the exterior powers of the standard representation. For $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, and $\mathbf{G}(3)$, we must take the exterior powers of the adjoint representation; then we need only verify that (3) is sufficient for (5.2.1) when $\alpha = \beta$, $c = 2$.

The theorem is proved.

Let G be a simple finite-dimensional contragradient Lie superalgebra, and let $(\, , \,)$ be an invariant nondegenerate bilinear form on G. We let ρ denote the difference between the half-sums of the positive even roots and the positive odd roots. It is not hard to show that $\rho(h_{\alpha_i}) = (\alpha_i, \alpha_i)/2$. We define the Casimir operator in the center of the enveloping superalgebra by the formula: $\Gamma = \sum (-1)^{\deg u_i} u_i u^i$, where $\{u_i\}$ and $\{u^i\}$ are dual bases of G relative to the form $(\, , \,)$.

Let V be a finite-dimensional irreducible G-module with highest weight Λ. The action of Γ on V can be written in the form: $\Gamma(v_\lambda) = (\lambda, \lambda + 2\rho)v_\lambda + \sum_{\alpha > 0} e_{-\alpha} e_\alpha v_\lambda$. In particular, $\Gamma(v_\Lambda) = (\Lambda, \Lambda + 2\rho)v_\Lambda$, and, by Schur's lemma, Γ is a scalar operator; hence, $\Gamma(v) = (\Lambda, \Lambda + 2\rho)v$, $v \in V$. We define the *supertrace form* in the usual way: $(a, b)_V = \operatorname{str}(ab)$. Since invariant forms on G are proportional, we have $(a, b)_V = l_V(a, b)$, where $l_V \in k$ is the *index* of the representation V. We have: $\operatorname{str}(\Gamma) = \sum (-1)^{\deg u_i} \operatorname{str}(u_i u^i) = l_V(\dim G_{\bar{0}} - \dim G_{\bar{1}})$. On the other hand, $\operatorname{str}(\Gamma) = (\dim V_{\bar{0}} - \dim V_{\bar{1}})(\Lambda, \Lambda + 2\rho)$. Thus, $l_V(\dim G_{\bar{0}} - \dim G_{\bar{1}}) = (\dim V_{\bar{0}} - \dim V_{\bar{1}})(\Lambda, \Lambda + 2\rho)$, from which we obtain

PROPOSITION 5.2.6. *The supertrace form of a finite-dimensional irreducible representation of a simple contragredient Lie superalgebra with* $\dim G_{\bar{0}} \neq \dim G_{\bar{1}}$ *in a space* V *with highest weight* Λ *is nondegenerate if and only if* $(\dim V_{\bar{0}} - \dim V_{\bar{1}})(\Lambda, \Lambda + 2\rho) \neq 0$.

EXAMPLE (compare [22]). We consider the standard representation osp(1, 2) of the dispin superalgebra $\mathbf{B}(0, 1)$. Then $V^k = \Lambda^k \operatorname{osp}(1, 2)$, $k = 0, 1, \ldots$ are all the irreducible finite-dimensional representations of $\mathbf{B}(0, 1)$. The highest weight of V^k is $2k$, $\dim V^k = 2k + 1$, $V_{\bar{0}}^k$ and $V_{\bar{1}}^k$ are $\mathbf{B}(0, 1)_{\bar{0}}$-irreducible, $\dim V_{\bar{0}}^k = k$, $\dim V_{\bar{1}}^k = k + 1$. The supertrace form is always nondegenerate (Proposition 5.2.6).

5.3. *Simple Lie Superalgebras Over Nonclosed Fields*

In this section the ground field k is arbitrary, of characteristic 0.

5.3.1. *Reduction of the classification of simple Lie superalgebras over k to finding of the forms.* Let $\bar{k}$ be the algebraic closure of k. We recall that a k-algebra G is said to be a *form of a $\bar{k}$-algebra $\bar{G}$* if $G \otimes \bar{k} \simeq \bar{G}$. If G is a form of $\bar{G}$ and V a G-module, where V is a vector space over k, then V is called a *form of the $\bar{G}$-module $V \otimes \bar{k}$*. We recall that if G is a semisimple Lie algebra over k and V is an irreducible G-module over k, then the $G \otimes \bar{k}$-module $V \otimes \bar{k}$ splits into the direct sum of irreducible submodules, which are equivalent up to a "twist" under an outer automorphism of $G \otimes \bar{k}$. We also remark that there is at most one irreducible form of a $\bar{G}$-module V for a given form G of a Lie algebra $\bar{G}$.

The next result is proved just as for Lie algebras [10].

PROPOSITION 5.3.1. *A simple finite-dimensional Lie superalgebra over k is isomorphic either to $G \otimes k'$, where k' is a finite extension of k and G is one of the k-algebras* $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n), \ldots$, $\mathbf{S}(n)$, $\tilde{\mathbf{S}}(n)$, $\mathbf{H}(n)$, *or it is a form of one of these $\bar{k}$-algebras.*

If $G = G_{\bar{0}} \oplus G_{\bar{1}}$ is a Lie superalgebra over k, then for an element $\alpha \in k^* \bmod k^{*2}$ we can construct another form G' for $G \otimes \bar{k}$, by setting $[a, b]' = \alpha[a, b]$ for $a, b \in G_{\bar{1}}'$ and $[a, b]' = [a, b]$ otherwise. This form we call *equivalent* to the original one.

5.3.2. *Forms of the classical Lie superalgebras.*

PROPOSITION 5.3.2. (a) *If a Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ over k is a form of $\bar{G} = \bar{G}_{\bar{0}} \oplus \bar{G}_{\bar{1}}$, then $G_{\bar{0}}$ is a form of $\bar{G}_{\bar{0}}$, and the $G_{\bar{0}}$-module $G_{\bar{1}}$ is a form of the $\bar{G}_{\bar{0}}$-module $\bar{G}_{\bar{1}}$.*

(b) *Suppose that $\bar{G} = \bar{G}_{\bar{0}} \oplus \bar{G}_{\bar{1}}$ is a classical Lie superalgebra over $\bar{k}$; suppose also that $G_{\bar{0}}$ is a form of $\bar{G}_{\bar{0}}$ and a $G_{\bar{0}}$-module V is a form of the $\bar{G}_{\bar{0}}$-module $\bar{G}_{\bar{1}}$. Then there is one and only one up to equivalence Lie superalgebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ over k that is a form of $\bar{G}$, with the $G_{\bar{0}}$-module $G_{\bar{1}}$ isomorphic to V.*

(c) *The Lie superalgebras* $\mathbf{B}(m, n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, $\mathbf{P}(n)$, *and* $\mathbf{Q}(n)$, *defined over* $\bar{k}$, *have at most one up to equivalence form* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *with a given subalgebra* $G_{\bar{0}}$.

(d) *The Lie superalgebras* $\mathbf{A}(m, n)$ *and* $\mathbf{C}(n)$ *over* $\bar{k}$ *have at most one form* $G = G_{\bar{0}} \oplus G_{\bar{1}}$ *with a given subalgebra* $G_{\bar{0}}$ *for which the* $G_{\bar{0}}$*-module* $G_{\bar{1}}$ *is irreducible* (*or reducible, respectively*), *up to equivalence.*

Proof. (a) is obvious. (b) follows from the fact mentioned earlier (see Section 2.1.6) that the $\bar{G}_{\bar{0}}$-module $S^2\bar{G}_{\bar{1}}$ contains ad $G_{\bar{0}}$ as a direct summand with multiplicity 1. The condition for the map $S^2G_{\bar{1}} \to \text{ad } G_{\bar{0}}$ to define a Lie superalgebra (see Section 1.1.2, (1.1.3)) is preserved under a change of field.

(c) and (d) follow from remarks made above in Section 5.3.1.

5.3.3. *Forms of Lie superalgebras of Cartan type.* In Sections 3.1 and 3.3 it was shown that for all Lie superalgebras of Cartan type, that is, $\mathbf{W}(n)$ with $n \geqslant 3$, $\mathbf{S}(n)$ and $\tilde{\mathbf{S}}(n)$ with $n \geqslant 4$, and $\mathbf{H}(n)$ with $n \geqslant 5$, the filtration is invariant under automorphisms, and the reductive part of the automorphism group is isomorphic to GL_n in the first two cases, SL_n, and SO_n, respectively. Hence by using the same arguments as in [18], we obtain the following result.

PROPOSITION 5.3.3. *Let G be a Lie superalgebra over k and a form of one of the Lie superalgebras of Cartan type* $\mathbf{W}(n)$, $\mathbf{S}(n)$, $\tilde{\mathbf{S}}(n)$, *or* $\mathbf{H}(n)$, *over* $\bar{k}$. *Then G is isomorphic to one of the following Lie superalgebras over k, respectively* $\mathbf{W}(n)$, $\mathbf{S}(n)$, $\tilde{\mathbf{S}}(n)$, *or* $\mathbf{H}(\sum \alpha_i(d\xi_i)^2)$, *where* $\alpha_i \in k^*$.

5.3.4. *Classification of simple finite-dimensional real Lie superalgebras.* Let us begin by constructing some series of examples of Lie superalgebras over $\mathbf{R}$. We fix the standard embeddings of the fields of real and of complex numbers in the quaternion field: $\mathbf{R} \subset \mathbf{C} \subset \mathbf{H}$. Throughout what follows, the bar denotes the standard conjugation in $\mathbf{C}$ and $\mathbf{H}$.

(a) *The special linear Lie superalgebras* $sl(m, n; k)$, $k = \mathbf{R}$ *or* $\mathbf{C}$ *or* $\mathbf{H}$. We consider the space $l(m, n; k)$ of all square matrices of order $m + n$ over k. In it we single out the subspaces

$$l(m, n; k)_{\bar{0}} = \left\{\left(\begin{array}{c|c} \alpha & 0 \\ \hline 0 & \delta \end{array}\right)\right\}, \qquad l(m, n; k)_{\bar{1}} = \left\{\left(\begin{array}{c|c} 0 & \beta \\ \hline \gamma & 0 \end{array}\right)\right\},$$

where α is an $(m \times m)$-, δ an $(n \times n)$-, β an $(m \times n)$-, and γ an $(n \times m)$-matrix. We define the bracket in the usual way: $[a, b] = ab - (-1)^{st} ba$ if $a \in l(m, n; k)_s$, $b \in l(m, n; k)_t$. This makes $l(m, n; k)$ into a real Lie superalgebra. The *special linear superalgebra* $sl(m, n; k)$ is the real subalgebra of $l(m, n; k)$ distinguished by the conditions:

$$sl(m, n; k) = \{a \in l(m, n; k) \mid \text{str}(a) = 0\} \qquad \text{for} \quad k = \mathbf{R} \text{ or } \mathbf{C};$$
$$sl(m, n; \mathbf{H}) = \{a \in l(m, n; \mathbf{H}) \mid \text{Re str}(a) = 0\}.$$

For $m = n$ these superalgebras contain a one-dimensional center, which has to be factored out.

(b) *Unitary and orthogonal-symplectic Lie superalgebras.* Again let $k = \mathbf{R}$ or $\mathbf{C}$ or $\mathbf{H}$. We consider the $\mathbf{Z}_2$-graded space $k^{m+n} = k^m \oplus k^n$; let $f = f(x, y)$, be a function on k^{m+n} with values in k, which is linear relative to the first variable, superhermitian, i.e., $f(x, y) = (-1)^{(\deg x)(\deg y)}\overline{f(y, x)}$, nondegenerate and consistent, i.e., $f(x, y) = 0$ if $x \in k^m$, $y \in k^n$. We put

$$\begin{aligned} su(m, n; f)_s &= \{a \in sl(m, n;\ C)_s \mid f(a(x), y) = -(-1)^{s \deg x} f(x, a(y))\}, \\ \mathrm{osp}(m, n; f)_s &= \{a \in sl(m, n;\ R)_s \mid f(a(x), y) = -(-1)^{s \deg x} f(x, a(y))\}, \\ h\mathrm{osp}(m, n; f)_s &= \{a \in sl(m, n;\ H)_s \mid f(a(x), y) = -(-1)^{s \deg x} f(x, a(y))\}, \qquad s \in \mathbf{Z}_2 \end{aligned}$$

The Lie superalgebras $su(m, n; f)$, $\mathrm{osp}(m, n; f)$ and $h\mathrm{osp}(m, n; f)$ are called *unitary*, *orthogonal-symplectic*, and *quaternion orthogonal-symplectic*, respectively.

(c) *The Lie superalgebras* $\mathbf{UQ}(n, p)$ *and* $\mathbf{HQ}(n)$. Let

$$\widetilde{\mathbf{UQ}}(n; p) = \left\{\left(\begin{array}{c|c} a & b \\ \hline ib & a \end{array}\right), a \in u(p, n-p), b \in su(p, n-p)\right\} \subset sl(n, n; \mathbf{C})$$

$$\widetilde{\mathbf{HQ}}(n) = \left\{\left(\begin{array}{c|c} a & b \\ \hline b & a \end{array}\right), a, b \in l(n), \operatorname{Re}\operatorname{tr} b = 0\right\} \subset l(m, n; H).$$

We put $\mathbf{UQ}(n, p) = \widetilde{\mathbf{UQ}}(n, p)/\langle 1_{2n}\rangle$, $\mathbf{HQ}(n) = \widetilde{\mathbf{HQ}}(n)/\{\lambda 1_{2n}, \lambda \in \mathbf{R}\}$.

(d) *The Lie superalgebras* $\mathbf{D}(2, 1; \alpha; p)$. For each of the representations of $so(4, 4-p; \mathbf{R}) \otimes sl(2; \mathbf{R})$, $p = 0, 1, 2$, there is a family of real Lie superalgebras $\mathbf{D}(2, 1; \alpha; p)$, $\alpha \in \mathbf{R}\backslash\{0, -1\}$, that are forms of $\mathbf{D}(2, 1; \alpha)$.

(e) *The Lie superalgebras* $\mathbf{F}(4; p)$. Each of the Lie algebras $so(p, 7-p)$, $p = 0, 1, 2, 3$, has a spinor representation $\mathrm{spin}_{p,7-p}$, which is a real form of the $\mathbf{B}_3$-module spin_7. For each of the four linear Lie algebras $\mathrm{spin}_{7,7-p}$ there is, by Proposition 5.3.2(b) one and only one real Lie superalgebra $\mathbf{F}(4; p)$, $p = 0, 1, 2, 3$, which is a form of the complex Lie superalgebra $\mathbf{F}(4)$.

(f) *The Lie superalgebras* $\mathbf{G}(3; p)$. Each of the real forms $\mathbf{G}_{2,p}$, $p = 0, 1$, of the complex Lie algebra $\mathbf{G}_2$ has a 7-dimensional representation $G_{2,p}$. For each of the two linear Lie algebras $G_{2,p} \otimes sl_2$ there is, by Proposition 5.3.2(b), one and only one real Lie superalgebra $\mathbf{G}(3; p)$, $p = 0, 1$, which is a form of the complex Lie superalgebra $\mathbf{G}(3)$.

(g) *The Lie superalgebras* $\mathbf{H}(n; p; \mathbf{R})$:

$$\mathbf{H}(n; p; \mathbf{R}) = \left\{D \in \mathbf{W}(n; \mathbf{R}) \,\middle|\, D\left(\sum_{i=1}^{p} (d\xi_i)^2 - \sum_{i=p+1}^{n} (d\xi_i)^2\right) = 0\right\}.$$

(h) *The Lie superalgebras* $\mathbf{P}(n; \mathbf{R})$, $\mathbf{Q}(n; \mathbf{R})$, $\mathbf{W}(n; \mathbf{R})$, $\mathbf{S}(n; \mathbf{R})$, *and* $\tilde{\mathbf{S}}(n; \mathbf{R})$. These are $\mathbf{P}(n)$, $\mathbf{Q}(n)$,..., defined for $k = \mathbf{R}$.

Real Lie superalgebras obtained from one another by the construction in Section 5.3.1 (for $\alpha = -1$) are called *dual*. From the classification of simple real Lie algebras and from Propositions 5.3.1–5.3.3 we derive the following theorem.

THEOREM 9. *A simple finite-dimensional real Lie superalgebra that is not a Lie algebra is isomorphic either to one of the complex Lie superalgebras* $\mathbf{A}(m, n)$, $\mathbf{B}(m, n)$, $\mathbf{C}(n)$, $\mathbf{D}(m, n)$, $\mathbf{D}(2, 1; \alpha)$, $\mathbf{F}(4)$, $\mathbf{G}(3)$, $\mathbf{P}(n)$, $\mathbf{Q}(n)$, $\mathbf{W}(n)$, $\mathbf{S}(n)$, $\tilde{\mathbf{S}}(n)$, *or* $\mathbf{H}(n)$, *regarded as real superalgebras, or, up to transition to its dual, to one of the forms of these Lie superalgebras listed below:*

- **A** $sl(m, n; \mathbf{R})$, $su(m, n; p, q)$, $m, n \geqslant 1$, $m + n \geqslant 2$; $sl(m, n, \mathbf{H})$, $m, n \geqslant 1$; $\mathbf{H}(4; p; \mathbf{R})$.
- **B** $\mathrm{osp}(m, n; p; \mathbf{R})$, *m odd*, $m \geqslant 1$, $n \geqslant 2$.
- **C** $\mathrm{osp}(2, n; p; \mathbf{R})$, $h\mathrm{osp}(1, n; p)$, $n \geqslant 2$.
- **D** $\mathrm{osp}(m, n; p; \mathbf{R})$, *m even*, $m \geqslant 4$, $n \geqslant 2$; $h\mathrm{osp}(m, n; p)$, $m \geqslant 2$, $\mathbf{D}(2, 1; \alpha; p)$.
- **F** $\mathbf{F}(4, p)$, $p = 0, 1, 2, 3$.
- **G** $\mathbf{G}(3, p)$, $p = 0, 1$.
- **P** $\mathbf{P}(n, \mathbf{R})$, $n \geqslant 3$.
- **Q** $\mathbf{Q}(n, \mathbf{R})$; $\mathbf{UQ}(n; p)$, $n \geqslant 3$; $\mathbf{HQ}(n)$, $n \geqslant 2$.
- **W** $\mathbf{W}(n, \mathbf{R})$, $n \geqslant 3$.
- **S** $\mathbf{S}(n, \mathbf{R})$, $n \geqslant 4$.
- $\tilde{\mathbf{S}}$ $\tilde{\mathbf{S}}(n, \mathbf{R})$, $n \geqslant 4$.
- **H** $\mathbf{H}(n; p; \mathbf{R})$, $n \geqslant 5$.

5.4. *On the Classification of Infinite-Dimensional Primitive Lie Superalgebras*

In Section 4.2.3, we have given a classification of the primitive Lie superalgebras (L, L_0) for which $L_0 \supset L_{\bar{0}}$; they are all finite-dimensional.

In this section we state without proof some partial results on the classification of infinite-dimensional primitive Lie superalgebras. We recall that Lie superalgebra L with a distinguished subalgebra L_0 is called *primitive* if L_0 is a maximal subalgebra and it does not contain nontrivial ideals of L.

5.4.1. *Two algebras of differential forms.* Let $\Omega_0(m)$ be the superalgebra of differential forms with coefficients from the polynomial algebra $k[x_1, ..., x_m]$; in other words, $\Omega_0(m)$ is the associative superalgebra over $k[x_1, ..., x_m]$ (with trivial $\mathbf{Z}_2$-grading) with the generators $dx_1, ..., dx_m$ with the defining relations

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \qquad \deg dx_i = \bar{1}, \qquad i, j = 1, ..., m.$$

On $\Omega_0(m)$ the differential d is defined in the usual way, as a derivation of degree $\bar{1}$ for which $d(x_i) = dx_i$ and $d^2(x_i) = 0$, $i = 1,..., m$, with the standard properties.

We now define the following superalgebras:

$$\Omega(m, n) = \Omega_0(m) \otimes \Omega(n) \qquad \text{and} \qquad \Theta(m, n) = \Omega_0(m) \otimes \Theta(n).$$

The differentials d and θ are extended from $\Omega(n)$ and $\Theta(n)$ to $\Omega(m, n)$ and $\Theta(m, n)$ in the natural manner, namely, $d = d \otimes 1 + 1 \otimes d$, $\theta = d \otimes 1 + 1 \otimes \theta$. It is not hard to establish their properties, which are similar to those in Chapter 3.

We set $\Lambda(m, n) = k[x_1, ..., x_m] \otimes \Lambda(n)$. The relations $\deg \xi_i = \deg x_j = 1$ determine on $\Lambda(m, n)$ a $\mathbf{Z}$-grading (which is not consistent with the $\mathbf{Z}_2$-grading). Every derivation D of degree s of $\Lambda(m, n)$ extends uniquely to a derivation of $\Theta(m, n)$ and $\Omega(m, n)$, subject to the conditions $[D, \theta]f = [D, d]f = 0, f \in \Lambda(m, n)$.

5.4.2. *Six series of infinite-dimensional Lie superalgebras.* We introduce the following differential forms:

$$v = dx_1 \wedge \cdots \wedge dx_m \wedge \theta\xi_1 \wedge \cdots \wedge \theta\xi_n \in \Theta(m, n),$$

$$h = 2 \sum_{i=1}^{k} dx_i \wedge dx_{k+i} + \sum_{i=1}^{n} (d\xi_i)^2 \in \Omega(m, n), \qquad m = 2k,$$

$$k = dx_{2k+1} + \sum_{i=1}^{k} (x_i \, dx_{k+i} - x_{k+i} \, dx_i) + \sum_{i=1}^{k} \xi_i \, d\xi_i \in \Omega(m, n), \qquad m = 2k + 1.$$

We now define six series of infinite-dimensional Lie superalgebras ($m > 0$):

I. $\mathbf{W}(m, n) = \operatorname{der} \Lambda(m, n)$.

The other five series consist of Lie algebras inside $\mathbf{W}(m, n)$, which are characterized by the following action on the differential forms

II. $\mathbf{S}(m, n) = \{D \in \mathbf{W}(m, n) \mid Dv = 0\}$,
II′. $\mathbf{CS}(m, n) = \{D \in \mathbf{W}(m, n) \mid Dv = \lambda v, \lambda \in k\}$,
III. $\mathbf{H}(m, n) = \{D \in \mathbf{W}(m, n) \mid Dh = 0\}$,
III′. $\mathbf{CH}(m, n) = \{D \in \mathbf{W}(m, n) \mid Dh = \lambda h, \lambda \in k\}$,
IV. $\mathbf{K}(m, n) = \{D \in \mathbf{W}(m, n) \mid Dk = uk, u \in \Lambda(m, n)\}$.

Note that for $n = 0$ we obtain the six standard series of infinite-dimensional Lie algebras of Cartan type.

The $\mathbf{Z}$-grading of $\Lambda(m, n)$ induces an (inconsistent) $\mathbf{Z}$-grading in $\mathbf{W}(m, n)$. We write it down in more detail. Every element $D \in \mathbf{W}(m, n)$ can be expressed as a linear differential operator

$$D = \sum_{i=1}^{m} P_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j \frac{\partial}{\partial \xi_j}, \qquad P_i, Q_j \in \Lambda(m, n). \tag{5.4.1}$$

The relations $\deg x_i = \deg \xi_j = 1$, $\deg \partial/\partial x_i = \deg \partial/\partial \xi_j = -1$ also determine on $\mathbf{W}(m, n)$ a $\mathbf{Z}$-grading $\mathbf{W}(m, n) = \bigoplus_{i \geqslant -1} \mathbf{W}(m, n)_i$, which corresponds in the canonical way to a filtration; the appropriate distinguished subalgebra is $\bigoplus_{i \geqslant 0} \mathbf{W}(m, n)_i$. The filtration and the distinguished subalgebra induce on every subalgebra L a filtration and a distinguished subalgebra $L_0 = L \cap \bigoplus_{i \geqslant 0} \mathbf{W}(m, n)_i$.

The Lie superalgebra $\mathbf{S}(m, n)$ consists of the operators of the form (5.4.1) satisfying the condition

$$\operatorname{div} D = \sum_{i=1}^{m} \frac{\partial P_i}{\partial x_i} + (-1)^{\deg D+1} \sum_{j=1}^{n} \frac{\partial Q_j}{\partial \xi_j} = 0.$$

Hence we see that $\mathbf{S}(m, n)$ is the linear span of the elements of the form

$$\frac{\partial a}{\partial \xi_i} \frac{\partial}{\partial \xi_j} + \frac{\partial a}{\partial \xi_j} \frac{\partial}{\partial \xi_i}, \qquad \frac{\partial a}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial}{\partial x_i},$$

$$\frac{\partial a}{\partial \xi_i} \frac{\partial}{\partial x_j} + (-1)^{\deg a} \frac{\partial a}{\partial x_j} \frac{\partial}{\partial \xi_i}, \qquad a \in \Lambda(m, n).$$

The Lie superalgebra $\mathbf{H}(m, n)$ consists of the operators of the form

$$D_a = \sum_{i=1}^{n} \frac{\partial a}{\partial \xi_i} \frac{\partial}{\partial \xi_i} + \sum_{i=1}^{k} \left(\frac{\partial a}{\partial x_i} \frac{\partial}{\partial x_{k+i}} - \frac{\partial a}{\partial x_{k+i}} \frac{\partial}{\partial x_i} \right), \qquad a \in \Lambda(m, n).$$

Here $[D_a, D_b] = D_{\{a,b\}}$, where

$$\{a, b\} = (-1)^{\deg a} \sum_{i=1}^{n} \frac{\partial a}{\partial \xi_i} \frac{\partial b}{\partial \xi_i} + \sum_{i=1}^{k} \left(\frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_{k+i}} - \frac{\partial a}{\partial x_{k+i}} \frac{\partial b}{\partial x_i} \right).$$

Next,

$$\mathbf{CS}(m, n) = \mathbf{S}(m, n) \oplus \left\langle \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i} + \sum_{i=1}^{m} x_i \frac{\partial}{\partial x_i} \right\rangle,$$

$$\mathbf{CH}(m, n) = \mathbf{H}(m, n) \oplus \left\langle \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i} + 2 \sum_{i=1}^{m} x_i \frac{\partial}{\partial x_i} \right\rangle.$$

Finally, the Lie superalgebra $\mathbf{K}(m, n)$ consists of elements of the form

$$\begin{aligned} D_a' = \sum_{i=1}^{n} \left(\frac{\partial a}{\partial x_m} \xi_i - \frac{\partial a}{\partial \xi_i} \right) \frac{\partial}{\partial \xi_i} + \sum_{i=1}^{k} \Bigg(& \left(\frac{\partial a}{\partial x_m} x_i + \frac{\partial a}{\partial x_{k+i}} \right) \frac{\partial}{\partial x_i} \\ & + \left(\frac{\partial a}{\partial x_m} x_{k+i} - \frac{\partial a}{\partial x_i} \right) \frac{\partial}{\partial x_{k+i}} \\ & + \left(2(-1)^{\deg a} + \sum_{i=1}^{n} \frac{\partial a}{\partial \xi_i} \xi_i - \sum_{i=1}^{m-1} \frac{\partial a}{\partial x_i} x_i \right) \frac{\partial}{\partial x_m} \Bigg), \qquad a \in \Lambda(m, n). \end{aligned}$$

The $\mathbf{Z}$-grading of $\mathbf{W}(m, n)$ also induces a $\mathbf{Z}$-grading of the form $G = \bigoplus_{i\geqslant -1} G_i$ on the Lie superalgebras of series II, II′, III, and III′. This is not so in the case of the last series $\mathbf{K}(m, n)$. However, if we set $\deg x_i = \deg \xi_j = 1$, $\deg \partial/\partial x_i = \deg \partial/\partial \xi_j = -1$ for $1 \leqslant i \leqslant m-1$, $1 \leqslant j \leqslant n$ and $\deg x_m = 2$, $\deg \partial/\partial x_m = -2$, then the resulting $\mathbf{Z}$-grading of $\mathbf{W}(m, n)$ also induces a $\mathbf{Z}$-grading of the form $G = \bigoplus_{i\geqslant -2} G_i$ on $\mathbf{K}(m, n)$.

Note that the G_0-modules G_{-1} for these six series are isomorphic to the following Lie superalgebras:

$l(m, n)$ for $\mathbf{W}(m, n)$ and $\mathbf{CS}(m, n)$, $sl(m, n)$ for $\mathbf{S}(m, n)$, $\mathrm{osp}(n, m)$ for $\mathbf{H}(m, n)$, $c\mathrm{osp}(n, m)$ for $\mathbf{CH}(n, m)$ and $c\mathrm{osp}(n, m-1)$ for $\mathbf{K}(m, n)$.

The superalgebras of all six series are transitive and irreducible, and those of series I, II, III, and IV are even simple.

5.4.3. *On the classification of primitive Lie superalgebras.* Let L be an infinite-dimensional primitive Lie superalgebra and L_0 be the distinguished subalgebra. Let L_{-1} be some minimal ($\mathbf{Z}_2$-graded) subspace of L that contains L_0 and is different from L_0 and $\operatorname{ad} L_0$-invariant. We construct a filtration in L of the form

$$L = L_{-d} \supset L_{-d+1} \supset \cdots \supset L_{-1} \supset L_0 \supset L_1 \supset \cdots, \qquad \text{by setting (see [6]):}$$
$$L_{-(s+1)} = [L_{-1}, L_{-s}] + L_{-s}, \qquad L_s = \{a \in L_{s-1} \mid [a, L_{-1}] \subset L_{s-1}\}, \qquad s > 0.$$

The corresponding associate $\mathbf{Z}$-graded superalgebra $\operatorname{Gr} L = \bigoplus_{i\geqslant -d} G_i$ has the following properties:

1°. $\operatorname{Gr} L$ is transitive and irreducible,

2°. $G_{-s} = G_{-1}^s$ for $s > 0$.

3°. $G_1 \neq 0$.

We may also assume that

4°. $\bigoplus_{i<0} G_i$ does not contain nonzero ideals of $\operatorname{Gr} L$ (because we can factor out such an ideal if it exists).

If the $\mathbf{Z}$-grading is consistent, then

5°. $[G_0, G_0]$ is a contragredient Lie superalgebra.

Apparently, 5° holds in general, but I have not been able to prove this.
Now we can state (without proof) the main result of this section.

THEOREM 10. *Let $G = \bigoplus_{i\geqslant -d} G_i$ be an infinite-dimensional $\mathbf{Z}$-graded Lie superalgebra having properties 1°–5°. Then G is isomorphic as $\mathbf{Z}$-graded superalgebra to one of $\mathbf{W}(m, n)$, $\mathbf{S}(m, n)$, $\mathbf{CS}(m, n)$, $\mathbf{H}(m, n)$, $\mathbf{CH}(m, n)$, or $\mathbf{K}(m, n)$ with $m > 0$.*

The proof uses the same methods as in Chapter 4 and relies on Theorem 3.

A primitive Lie superalgebra L with distinguished subalgebra L_0 is called *complete* if it is complete in the topology defined by the subspaces of the transitive filtration of the pair (L, L_0) (see Section 1.3.1). The superalgebra $\bar{\Lambda}(m, n) = k[[x_1, \ldots, x_m]] \otimes \Lambda(n)$ is complete in the topology defined by its natural filtration.

We denote by $\bar{\mathbf{W}}(m, n)$ the Lie superalgebra of all continuous derivations of $\bar{\Lambda}(m, n)$. Then $\bar{\mathbf{W}}(m, n)$ is a complete primitive Lie superalgebra with the natural distinguished subalgebra. Complete and primitive are also $\bar{\mathbf{S}}(m, n), \ldots, \bar{\mathbf{K}}(m, n)$, which are characterized by the same action on the differential forms v, h, k, as for $\mathbf{S}(m, n), \ldots, \mathbf{K}(m, n)$.

A well-known result of Cartan asserts that $\bar{\mathbf{W}}(m, 0), \ldots, \bar{\mathbf{K}}(m, 0)$ are the only infinite-dimensional complete primitive Lie algebras.

CONJECTURE 1. *An infinite-dimensional complete primitive Lie superalgebra is isomorphic to one of* $\bar{\mathbf{W}}(m, n)$, $\bar{\mathbf{S}}(m, n)$, $\overline{\mathbf{CS}}(m, n)$, $\bar{\mathbf{H}}(m, n)$, $\overline{\mathbf{CH}}(m, n)$, *or* $\bar{\mathbf{K}}(m, n)$ *for* $m > 0$.

5.4.4. *Remarks.* (a) In Chapter 4 we have, in fact, proved that if $G = \bigoplus_{i \geqslant -d} G_i$ is an infinite-dimensional Lie superalgebra with a consistent $\mathbf{Z}$-grading having properties 1°–4°, then G is isomorphic to $\mathbf{K}(1, n)$.

(b) There are general embedding theorems, which generalize standard theorems for Lie algebras and Propositions 3.1.2 and 3.1.3.

If $G = \bigoplus_{i \geqslant -1} G_i$ is a transitive $\mathbf{Z}$-graded Lie superalgebra, and $m = \dim(G_{-1})_{\bar{0}}$, $n = \dim(G_{-1})_{\bar{1}}$, then there is an embedding $G \to \mathbf{W}(m, n)$ preserving the $\mathbf{Z}$-grading.

Let $L = L_{-1} \supset L_0 \supset L_1 \supset \cdots$ be a transitive Lie superalgebra with a filtration, $m = \dim(L/L_0)_{\bar{0}}$, $n = \dim(L/L_0)_{\bar{1}}$. Then there is an embedding $\alpha: L \to \bar{\mathbf{W}}(m,n)$ preserving the filtration. If β is another such embedding and $(\alpha - \beta)L \subset L_0$, then there exists one and only one (continuous) automorphism φ of $\bar{\mathbf{W}}(m, n)$ for which $\alpha = \varphi \circ \beta$; φ can be induced by an automorphism of $\bar{\Lambda}(m, n)$.

(c) By the same method as Proposition 3.3.8, it can be proved that every nondegenerate closed differential form from $\bar{\Omega}(m, n)$ of degree 2 is reduced to the form h by an appropriate automorphism of $\bar{\Lambda}(m, n)$.

5.5. *Some Unsolved Problems*

5.5.1. *Classification of infinite-dimensional primitive Lie superalgebras.* Apropos this topic, see Conjecture 1 and Theorem 10.

5.5.2. *Formulas for the characters and dimensions of irreducible representations.* The most urgent task is to prove a formula for the characters in the case of contragredient Lie superalgebras. For contragredient Lie algebras (including these of infinite dimension) this is done in [14]. However, the proof in [14] only works for $\mathbf{B}(0, n)$.

5.5.3. *Cohomology*. For the definition of the cohomology group $H^k(G, V)$ of a Lie superalgebra V with coefficients in a G-module V, see [17]. As usual, it is shown that if V is a finite-dimensional irreducible G-module and Γ is the Casimir operator (the existence of an invariant bilinear form is assumed), then $H^k(G, V) = 0$ for $\Gamma(V) \neq 0$. In the case of contragredient Lie superalgebras, the latter condition is equivalent to $(\Lambda, \Lambda + 2\rho) \neq 0$, where Λ is the highest weight (see Section 5.2.3), and it is not violated in any nontrivial representation only for $\mathbf{B}(0, n)$.

Now some questions arise at once: the cohomology of the simple finite-dimensional Lie superalgebras with trivial coefficients, and the cohomology of the infinite-dimensional complete primitive Lie superalgebras.

Closely connected with the problem of the triviality of $H^1(G, V)$ is the full reducibility of representations and the theorems of Levi and Mal'tsev. A counter-example to Levi's theorem is $sl(n, n)$, and one to full reducibility is the adjoint representation of $\mathbf{A}(n, n)$. As we have already mentioned, full reducibility always holds for $\mathbf{B}(0, n)$. It is not hard to show that if G is a classical Lie superalgebra, then $H^1(G, V) = 0$ for all irreducible representations, with the exception of a finite set S. It would be interesting to find this S and also to classify all indecomposable representations of the classical Lie superalgebras.

5.5.4. *Infinite-dimensional representations*. Undoubtedly, Kirillov's orbits method extends to Lie superalgebras. (In particular, Theorem 7′ on infinite-dimensional representations of solvable Lie algebras points to this.) We mention that Kirillov's differential form $\omega(x, y) = l([x, y])$ on an orbit of the co-adjoint representation of a Lie superalgebra is a form in dx and $d\xi$ (see Section 5.4). On infinite-dimensional representations of the simple Lie superalgebras almost nothing is known. First in line is, of course, the dispin algebra $\mathbf{B}(0, 1)$.

5.5.5. *Generalized Lie superalgebras*. We consider the ring $\mathrm{M} = \mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2$ (s times). An M-graded algebra is called a *generalized superalgebra*. If $\alpha = (\alpha_1, ..., \alpha_s) \in \mathrm{M}$, we set $(-1)^\alpha = (-1)^{\alpha_1} \cdots (-1)^{\alpha_s}$. Now all the definitions and assertions of Section 1.1 carry over to generalized superalgebras, in particular, the definitions of a Lie superalgebra, of the supertrace, and the Killing form. Just as in Section 2.1, we can define series of generalized Lie superalgebras $sl(n_1, ..., n_{2^s})$, $\mathrm{osp}(n_1, ..., n_{2^s})$, $\mathbf{Q}(n)$, and as in Chapter 3, the series $\mathbf{W}$, $\mathbf{S}$, $\tilde{\mathbf{S}}$, $\mathbf{H}$. The same problems arise here as for Lie superalgebras, first and foremost, the problem of classifying the simple generalized Lie superalgebras.

Additional remarks. To Section 5.2. In my recent article, "Characters of Typical Representations of Classical Lie Superalgebras" (*Commun. Algebra* 5, No. 8, 889–897 (1977)), the formulas for the character and supercharacter of finite-dimensional irreducible representations in "general position" (so-called typical representations) are obtained. For example, all the representations of $\mathbf{B}(0, n)$ are typical and all the representations of $\mathbf{A}(1, 0)$ are typical except for $S^k sl(2, 1)$ and its dual.

To Section 5.3. D. Z. Djokovic and G. Hochshild proved, in their article "Semi-simplicity of $\mathbf{Z}$-graded Lie algebras, II" (*Illinois J. Math.* 20 (1976), 134–143), that every finite-dimensional representation of a Lie superalgebra G is fully reducible if and only if G is a direct sum of a semisimple Lie algebra and several copies of $\mathbf{B}(0, n_i)$.

From my article, mentioned in Section 5.2, it follows that $H^1(G, V) = 0$ for a typical representation V. By the way, $H^1(\mathbf{A}(1, 0), V) = 0$ only for one irreducible representation-standard representation.

To Section 5.4. B. Kostant in his recent paper, "Graded manifolds, graded Lie theory and prequantization," gave a more correct definition of a supermanifold than the one in [5]. This definition allowed him to develop the theory of homogeneous supermanifolds, and "orbits method" for supergroups.

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