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Nicola F. Maaser

Decision-Making in Committees

Game-Theoretic Analysis

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Hamburg, *Nicola F. Maaser*
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Introduction

The belief that individuals take advantage of the opportunities afforded by an institution in order to achieve their goals is at the foundation of much of economic theory. It is summarized by Plott (1979, p. 138) in the equation

$$\text{preferences} \oplus \text{institutions} \oplus \text{physical possibilities} = \text{outcomes} \qquad (*)$$

where the notation $\oplus$ refers to some unspecified abstract operation which combines preferences, institutions, and "physical possibilities" or alternatives into actual outcomes. In the context of decision-making in committees, equation $(*)$ may read as follows: using some procedure, at the end of which stands a vote, a collection of individuals has to choose from a set of possible alternatives on which they have differing and potentially conflicting preferences. Everyday examples abound; they include voting on legislative proposals in the Council of Ministers of the European Union (EU), members of a cartel deciding on prices, shareholder meetings voting on corporate matters, as well as the boards of central banks determining interest rates or money supply. The "institution" here amounts to the formal and informal rules that govern the collective decision-making, establishing, for example, who is entitled to make a proposal, whether or not proposals can be amended, and what constitutes agreement. If, following Lasswell's (1936) formulation, politics is "who gets what," then it is of central importance to any serious attempt at understanding or shaping political decision-making (in a broad sense) to provide an answer to the question: *How does the decision rule allocate power and influence over the collective decision?*

From a game-theoretic perspective, two general approaches to this problem exist. First, traditional power indices, which are rooted in cooperative game theory, can be used to quantify the "a priori voting power" of a committee member, defined as the ability to affect the outcome of a vote. For this purpose, they abstract from decision-makers' preferences. Yet, as equation $(*)$ would suggest, looking at these indices may tell us very little about players' capacities to shape actual outcomes: the latter still depend on preferences and the feasible alternatives. Moreover, the voting rule is typically but one part of a real-world institution. Nevertheless, power indices may be useful in designing new committees, or evaluating in advance how the voting power of decision-makers might change when the voting rule is altered. A comprehensive treatment is Felsenthal and Machover (1998). Holler (1982a) and Holler and Owen (2001) contain theoretical and applied contributions to the field, and also give a sense of its development.

Second, the collective decision process may be described as a non-cooperative game. Although, in principle, allowing for the specific analysis of complex decision structures, noncooperative modeling necessitates a detailed description of who can do what and when, which, in many contexts, is simply not available. Thus, studies that analyze individual influence in this vein have been limited so far to purely distributional decisions (e. g., Baron and Ferejohn, 1989; Snyder et al., 2005), or rely on both cooperative and noncooperative solution concepts (e. g., Napel and Widgrén, 2006).

The borderland between these two approaches is exactly the location of this dissertation. Its contribution is twofold: First, it reviews the existing theories that aim to assess the influence conferred upon committee members by the decision rules. Second, it breaks new ground by combining – in an eclectic manner – elements of these theories to obtain fresh insights into committee decision processes.

Before turning to an outlook on individual chapters, a remark on the relation between the work at hand and the social choice theory of committee decision-making seems expedient. That theory dates back to the formal investigations of voting in committees by Borda (1781) and Condorcet (1785). Since the latter's discovery of cyclical majorities in a three-member, three-alternative example, known as Condorcet's paradox, much of subsequent research has dealt with committees as preference aggregation mechanisms under majority rule. In terms of our introductory equation, social choice concentrates on the first component in $(*)$: It essentially deals with a mapping acting on arbitrarily chosen individual preference profiles in a "committee of the whole," and processing them into a collective or social preference ordering of all feasible outcomes. The focus is on characteristics of the aggregation method, in particular its equilibrium properties, and game aspects such as strategy-proofness (see the seminal contributions by Farquharson, 1969; Gibbard, 1973; Satterthwaite, 1975). Apart from special cases, social choice theory following Arrow's (1963[1951]) pioneering work has largely established that voting outcomes under majority rule are inherently unstable. In the absence of sharp predictions regarding the outcome it would be hard, or even impossible, to determine individual influence on the collective decision. Although the present analysis is partly concerned with the same structure as social choice theory, namely a majority rule committee unrestricted by institutional features that favor any member or any particular outcome, it is unaffected by the difficulties raised by the latter. Either the committees that we will consider face a binary choice, e. g., approving or disapproving a proposal or choosing between exactly two candidates, or committee members are assumed to have single-peaked preferences over a one-dimensional continuous policy space. With only two alternatives, majority cycles cannot occur. But the possibilities of abstention or not showing up are also ruled out. Under the single-peakedness restriction, the collective decision outcome is compellingly predicted by Black's (1958) median voter theorem.

Chapter 1 presents applications of game-theory to political science. The focus will be on the effect of the committee decision rule for "who gets what." One relevant distinction between models precisely concerns what committee members get: Do they decide on (and derive utility from) the contents of a public policy, or do they perceive the decision to be primarily about the division of a fixed-size pie? Depending on the nature of the issue to be decided and on whether the committee acts on an external proposal or not, different frameworks are appropriate to evaluate the distribution of players' influence on the collective decision. Aiming at greater clarity than is usual in these regards, the chapter discusses simple games with and without transferable utility, spatial voting games, and non-cooperative bargaining games. From this, it eventually transpires that traditional power indices figure prominently in conceptually diverse approaches to the problem of assessing influence in decision-making bodies as far as it is related to the voting rule.

Chapter 2 makes use of an innovative model to consider a familiar problem in institutional design, namely the choice of an "adequate" voting rule for a democratic committee of representatives who act on behalf of groups of different sizes. It is argued that formal political equality or compliance with the "one-person, one-vote" principle is an essential property of any democratic decision rule, even though it cannot capture the full idea of "democracy" and "equity" in collective decision-making. So far, the solution to ensuring equal representation of individual citizens in a two-tiered voting system has been the one first suggested by Penrose (1946). In order to equalize individual constituents' chances to indirectly determine the outcome of decisions in the committee of representatives, *Penrose's square root rule* recommends to assign weights to representatives such that their power as measured by the Penrose–Banzhaf index (Penrose, 1946; Banzhaf, 1965) is proportional to the square root of the respective constituency's population size. The snag lies in the fact that the rule rests on a binomial model in which individual voters and representatives are assumed to vote "yes" or "no" independently with (expected) probability $1/2$, which does not apply to many real-world decisions. In contrast, the model presented here considers decisions which are elements of a one-dimensional convex policy space and may result from strategic behavior consistent with the median voter theorem. A mathematical investigation suggests that, under limit conditions and simple majority rule, a square root allocation of weights would be optimal – a result, which is then confirmed, by means of extensive Monte-Carlo simulations, for "small" artificial constituency configurations, as well as the EU and the US. In conclusion, Penrose's square root rule appears to extend from its original model to a setting with many finely graded policy alternatives and strategic interactions. The chapter is based on the article "Equal Representation in Two-tier Voting Systems", co-authored with Stefan Napel and published 2007 in Social Choice and Welfare 28(3), pp. 401–420.

Chapter 3 investigates the robustness of the square root rule for equal representation in two-tiered voting systems. It continues and develops the approach and the model of Chap. 2, but moreover, it sheds light on problems that are of general interest in the design of voting rules, namely the "inverse problem" of finding weights which induce a desired power distribution, and the choice of the decision threshold or quota. A review of the literature shows that the impact of the latter on representation of individual citizens has never been properly examined. When supermajority rules are introduced into the model, representation is demonstrated to become less egalitarian. The conclusion from these findings would be that the quota applied in a committee of representatives has important implications for the equity and hence the legitimacy of decision-making. The chapter then proceeds to explore equal representation in the case that voters' preferences within a constituency are somewhat more similar than across constituencies, i. e., it studies the assumption of heterogeneous constituencies. This can be done without forgoing the "veil of ignorance" perspective appropriate to constitutional design. Thinning the "veil of ignorance" slightly, a new rule for equal representation emerges: In order to give individual citizens from different constituencies a priori equal chances to influence the collective decision, the voting weights of representatives need to be such that their power as measured by the Shapley–Shubik index is proportional to constituency size. Chapter 3 draws on recent unpublished work with Stefan Napel.

Finally, Chap. 4 studies decision-making in a legislative committee under lobby influence. Specifically, lobbyists who have similar policy preferences wish to manipulate the collective decision by offering payments to committee members. A model of endogenous coalition formation among lobbyists is developed. Its analysis uncovers a new link between the status quo bias of a legislative committee and the severity of the collective action problem of the lobbyists. Coalition formation among the lobbyists is conceived as a three-stage game: First, lobbyists form coalitions; second, coalitions offer politically valuable resources to the members of the committee; third, the committee decides by voting under closed rule with a fixed agenda-setter. It turns out that small or no bias in favor of the status quo encourages the formation of the grand coalition of lobbyists, whereas, in the case of a large status quo bias, cooperation is partial in the sense that part of the lobbyists will free-ride on others' lobbying efforts. The equilibrium coalition structure then involves under-provision of lobby efforts, resulting in an inefficient outcome for the lobbyists. To the extent that the legislative bargaining game used in the model can be taken as a stylized description of real world procedures, the model may help to understand the lobbying of supranational legislative institutions like the EU Commission and the Council of Ministers.

Today, EU decisions extend to all areas of public activity in its member states, including market regulation, agriculture, consumer affairs, environmental policy, immigration, and education. Over the past two decades the expansion of the EU, both geographically and in terms of policy competencies, has stimulated considerable interest in the design of committee decision-making, which is reflected in a large amount of related literature, both theoretical and applied. In fact, one might be tempted to interpret the tough-minded and painful haggling over the rules for Qualified Majority Voting in the EU Council of Ministers at all of the more recent Intergovernmental Conferences as evidencing the practical relevance of these rules for "who gets what." In any case, it should be pointed out here that Chaps. 2–4 can be read as applications of game-theoretic tools to decision-making processes in the EU.

However, the complexities and subtleties of the political system of the EU are far beyond the scope of the present work. Considering only the basic institutional quartet of the EU – the Commission, the Council, the European Parliament, and the Court of Justice – and the formal relations between these bodies, it is obvious that our analysis, which mostly concentrates on decision-making in an "isolated" committee, can at best be partial. Even a sound investigation of the whole set of formal and informal decision rules might not be sufficient to complete the picture because these rules still structure negotiation processes that may include deliberation, persuasion, threats, and false pretenses. Getting back once again to equation $(*)$, it is these elements of committee voting, beclouding our understanding of how "$\oplus$" and "$=$" work, that ultimately limit our chances to answer in full the question who has how much influence on collective decisions.

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Symbols

$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+$	The set of non-negative real numbers
$\mathbb{N}^*$	The set of natural numbers not including zero
a, b	Scalars
N	Set of players
2^N	Set of subsets of N
n	Cardinality of N
i, j	Players
x_m	Transfer commodity
u_i	Payoff of player i
u	Payoff vector
Σ_i	Strategy set of player i
σ_i	A strategy of player i
S	A subset of players or coalition
s	Cardinality of the set S
u_S	The projection of u on $\mathbb{R}^s$
Σ_S	Set of correlated strategies of players $i \in S$
σ_S	A correlated strategy of players $i \in S$
$\emptyset$	The empty set
$\setminus$	Set subtraction
$\times$	Cartesian product
$v(S)$	Characteristic function in TU-case
$V(S)$	Characteristic set in NTU-case
$V_\alpha(S)$	NTU characteristic set under α-effectivity
$V_\beta(S)$	NTU characteristic set under β-effectivity
(N, v)	Cooperative game in characteristic function form
P	A set of points/pivot position
$compr(P)$	Comprehensive hull of set P
$conv(P)$	Convex hull of set P
$cch(P)$	Comprehensive convex hull of set P

$\mathcal{W}$	A set of winning coalitions defining a simple game or voting rule
$\mathcal{W}^{\min}$	A set of minimum winning coalitions
$\mathcal{W}_i$	The set of winning coalitions to which i belongs
$\mathcal{W}_{-i}$	The set of winning coalitions to which i does not belong
$\mathfrak{W}$	Set of all voting rules $\mathcal{W}$
$(N, \mathcal{W})$	A simple game
G	A simple game
$\mathcal{G}^N$	The set of monotonic simple TU games with player set N
w	Vector of voting weights
q	The decision quota
$[q, w]$	A weighted majority game
M_n	Simple majority game with n players
X	The set of policy proposals or outcomes
x, y, z	Elements of X
U	Set of feasible payoff combinations
Γ	A normal form game/Hart and Kurz' game Γ
$Core(N, \mathcal{W})$	Core of a simple game
$I(v)$	The set of imputations of game v
λ_i	The ideal point of individual i
$X^*(S)$	The set of Pareto-efficient alternatives for coalition S
χ	A proposal
Q	The status quo
g	A game form
μ	A power measure
μ^{NW}	Power measure proposed by Napel and Widgrén (2002, 2004)
Φ	The Shapley value
$<<$	Strict linear ordering of the set of players
$D(i, <<)$	The set of predecessors of i in the ordering $<<$
ϕ	The Shapley–Shubik index
β	The non-normalized Penrose–Banzhaf index or measure
p_i	Acceptance rate of individual i
π	A partition
ξ	A probability distribution
$\mathbf{U}_{[a,b]}$	Uniform distribution on the interval $[a, b]$
$x \succ_i y$	Player i strictly prefers x to y
B	A bargaining problem
d	Disagreement point
$\mathcal{B}$	The set of bargaining problems
B_{TU}	TU bargaining problem
U_{TU}	Set of feasible payoffs in TU case
ς	A permutation

T	A positive linear transformation/a coalition
φ	Weight vector in the asymmetric Nash bargaining solution
$\text{Nash}^{\varphi}(B)$	Asymmetric Nash bargaining solution for B
J	Set of null players
A	Agenda-setter
ϕ^{λ}	The modified or Shapley–Owen power index
x_{SP}	The strong point
ω	State of the world/vector of median ideal points
Ω	Set of states of the world/set of vectors of median ideal points
$x_g^*(\omega)$	Equilibrium outcome
X_{DD}	The set of policy proposals or outcomes in 'divide-the-dollar'
$\mathscr{M}_2$	The bargaining set $\mathscr{M}_2$
$\mathbf{E}$	Expectation operator
$\mathcal{C}_j$	Constituency j
m	Number of constituencies
λ_i^j	Ideal point of voter i in constituency j
λ_j	Median ideal point in constituency j
$\mathcal{R}$	A committee of representatives
$F_j(x)$	Cumulative density function of individual voters' ideal points
$f_j(x)$	probability density function of individual voters' ideal points
$F_{\lambda_j}(x)$	Cumulative density function of λ_j
$f_{\lambda_j}(x)$	Probability density function of λ_j
μ_j	Mean of F_{λ_j}
σ_j	Standard deviation of F_{λ_j}
ϑ	A constant
η	A constant
$\Pr(\cdot)$	Probability of an event
e	Euler's number
$\tilde{w}_j(x)$	Contribution of $\mathcal{C}_j$ to total weight
$\tilde{W}_{-k}(x)$	Weight accumulated up to position x by players other than k
$\tilde{W}_{-k}^{-1}(\cdot)$	Quasi-inverse of $\tilde{W}_{-k}(x)$
π_j	Probability that representative j is pivotal in $\mathcal{R}$
$\hat{\pi}_j$	Approximation of π_j
$\overline{W}$	Aggregate weight
$\mathbf{1}_X$	Indicator function of set X
$\hat{F}_{\lambda_j}(x)$	Approximation of $F_{\lambda_j}(x)$
$y \propto x$	y is (directly) proportional to x
K_j	Random variable in Monte-Carlo simulation
$\bar{k}_j^s$	Empirical average of s draws of K_j
α	Exponent in power law rule for weight allocation
κ	Skewness parameter of a Pareto density

$\underline{x}$	Lower bound of the support of a Pareto density
$\mathbf{P}(\kappa, \underline{x})$	Pareto distribution
$\mathbf{N}(\mu, \sigma)$	Normal distribution with mean μ and standard deviation σ
$Y \sim \mathbf{N}(\mu, \sigma)$	The random variable Y is distributed according to $\mathbf{N}(\mu, \sigma)$
$\tilde{\Delta}_j$	Distribution median in $\mathcal{C}_j$
Δ	A realization of $\tilde{\Delta}_j$/Hart and Kurz' game Δ
H	Distribution of $\tilde{\Delta}_j$
h	Density of H
σ_H	Standard deviation of H or heterogeneity measure
$F_{\lambda_j \mid 0}$	Distribution of λ_j conditional on $\Delta = 0$
$f_{\lambda_j \mid 0}$	Density of λ_j conditional on $\Delta = 0$
$h * f$	Convolution of functions h and f
u_i^{dNB}	Payoff of citizen i in direct Nash bargaining
λ_M	Ideal point of the median legislator
Λ	Profile of legislators' ideal points
R_A	Resources received by the agenda-setter
R_j	Resources received by legislator j
L	The lobbyists' ideal point
C_j	Costs of lobbyist j
x_d	Dividing point
$\gamma(\Lambda)$	Status quo bias of the legislature
$\rho(x)$	Resources necessary to achieve policy x
r	The rightmost legislator that is still lobbied
ℓ	Number of lobbying coalitions in a given coalition structure
$A \rightrightarrows A$	A mapping from a set $A \subseteq \mathbb{R}^n$ into the power set of A
$\tau(x), \hat{\tau}(x)$	Set-valued mappings
$S_L(k, t)$	Lobby coalition with $k + t$ members
$S_{FR}(k, t)$	Coalition that free-rides on $S_L(k, t)$

Chapter 1
Games and Political Decisions

This chapter provides an introduction to some fundamental aspects of decision-making in committees. 'Committee' is used to refer to a decision-making body that comprises a small number of members (as opposed to a referendum situation), and chooses from a set of well-defined policy alternatives (in contrast to the electorate in a general election which usually chooses between candidates or party platforms). Decisions are ultimately reached by putting alternatives to a vote according to some voting rule specifying which subsets of all committee members can pass a proposal. This notion of a committee differs from everyday language where the term also applies to expert panels with advisory function, or organizational subunits that make recommendations or submit proposals to some superordinate organization.

Simple games provide a model for committees where a majority coalition can choose any outcome from a set of alternatives, but that are otherwise institution-free. As such, they are important tools for the analysis of the outcomes of committee decision-making in the present work. One relevant distinction concerns the nature of the issue to be decided by the committee. Suppose, for example, that a committee of representatives each of whom acts on behalf of a constituency has to decide on the allocation of a fixed-size budget to the constituencies. Under the assumptions that each representative solely cares about the share that goes to his own constituents and that representatives' utility functions are linear in money, the decision is purely *distributive*. By contrast, if the committee has the task to establish a regulatory framework for, say, stem cell research, it confronts a *non-distributive* problem, even though the decision – as most policy decisions – is likely to allocate costs and benefits across different groups in a certain way. Yet, these costs and benefits do not primarily accrue to the decision-makers themselves as is the case with budget allocation under the above assumptions. Rather, their utilities derive from the contents of the public policy they eventually adopt (see Barry, 1980a, pp. 189ff).

Another useful distinction concerns the choice environment of the committee: if any alternative within some large set could be enacted, provided that it has the support of a 'winning coalition', the decision is subject to bargaining among prospective coalition members, which is then confirmed by a final vote. In contrast, a committee might only be entitled to accept or reject a given proposal which an external agency submits to it, i.e., it cannot make proposals of its own or amend the external proposal. Following Laruelle and Valenciano (2007, 2008b),

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Table 1.1 Selected models in the analysis of committees. Cooperative game theory models are highlighted in gray

Decision Mechanism / Nature of decision	Voting on exogeneous Take-it-or-leave-it proposal	Pre-vote bargaining
Distributive	TU simple game	TU simple game
	Napel and Widgrén (2001)	Baron and Ferejohn (1989)
		Snyder et al. (2005)
Non-distributive	TU simple game	NTU simple game
	Romer and Rosenthal (1978)	Laruelle and Valenciano (2007)
	Napel & Widgrén (2002; 2004)	Laruelle and Valenciano (2008b)

these two prototypes are referred to as *bargaining committees* and *take-it-or-leave-it committees*, respectively.

So far, only committees governed by a voting rule but unconstrained by any further institutional arrangements have been considered. This setting can be studied by simple games, and, more generally, by *cooperative game theory*. Here, all communication, signaling, and commitment are assumed "to take place outside the formal context of the game" (Shubik, 1982, p. 258). Communication and binding agreements are available at no cost within the game. In the cooperative portrayal of a committee, decision-makers thus have the opportunity to compare all alternatives simultaneously and costlessly with each other. By contrast, models based on *non-cooperative game theory* can – and, in fact, have to – accommodate procedural rules that specify, e.g., which players may make proposals or a particular sequential structure of voting in a multi-chamber legislature. The incorporation of such institutional detail typically reduces the number of alternatives that can be pitted against each other, and thereby leads to conclusions that differ remarkably from those of cooperative models.

Table 1.1 gives a selective overview of models that have been used in order to analyze committees, and classifies them according to the three distinctions drawn above. This chapter has the purpose to discuss these models – at varying length – with respect to the question 'Who can expect to get what?'. The presentation is also intended to serve as a backdrop for the applications in Chaps. 2–4. The first section presents basic tools that will be used throughout subsequent parts, namely characteristic functions, simple games, and spatial games. In Sect. 1.2, 'non-distributive' decision-making is considered in various contexts. Section 1.3 focuses on the case of distributive politics.

1.1 Basic Concepts

When applying game theory to political problems, the question whether utility can be considered as transferable or not deserves special attention. The *transferable* *utility (TU)* assumption requires that some infinitely divisible commodity exists, and *side payments* in units of this medium permit players to transfer utility without loss. Then, players within a coalition can reallocate payoffs among themselves in any manner they see fit. TU is reflected in *quasilinear* individual utility functions $u_i(x_1, \ldots, x_m) = \hat{u}_i(x_1, \ldots, x_{m-1}) + x_m$, that is, utility for the transfer commodity or 'money' x_m is linear. This makes TU seem a more reasonable approximation for market situations than for policy-making. It is true that committee members typically face not a single proposal, but a whole set, which gives them the opportunity to exchange votes. A TU representation may still not be justified if, for example, vote-trading reduces a legislator's chances of reelection so that he cannot be fully compensated for the lost votes by his trading partners. Similarly, treating minister portfolios as a transfer medium, as is sometimes done in models of coalition formation in multi-party systems, may be appropriate if politicians only care about offices and the associated monetary rewards, but rather inappropriate if they are interested in policies.

In *non-transferable utility (NTU)* games, players are still able to communicate and coordinate strategy choices, but they cannot transfer payoffs within coalitions by means of sidepayments. In principle, NTU games form an interesting 'middle class' between non-cooperative games and cooperative TU games. Although game theory under the TU assumption is accurately described as a special case within NTU theory, historically, solution concepts such as the Shapley value often emerged first from the study of TU games, and only later they were generalized to the NTU case. The likely reason behind this is that a NTU representation necessitates the use of set-theoretic manipulations whereas TU games can be analyzed with the help of simple algebra.

1.1.1 The Characteristic Function

Let $N = \{1, 2, \ldots, n\}$ denote the set of players and 2^N the power set of N, i.e., the set of all subsets of N. Each subset $S \in 2^N$ is referred to as a *coalition*. Small letters are used to denote the cardinality, i.e., set $|N| = n$ etc.

A real-valued mapping $v\colon 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$ is called the *characteristic function* v of a cooperative n-person game *with transferable utility*. The value $v(S)$ is interpreted as a single-number summary of 'what the group can obtain' when its members can bindingly commit to some coordinated course of action, that is, the potential *worth* of the coalition S. A cooperative game in characteristic function or coalitional form is an ordered pair (N, v). Often (N, v) is referred to as game v. The game v is *monotonic* if $v(T) \leq v(S)$ for all $T \subset S$.

A restriction on v that was already proposed by Von Neumann and Morgenstern (1944, pp. 241f) is *superadditivity*. A TU-game in characteristic function form is superadditive if

$$v(S) + v(T) \leq v(S \cup T) \qquad \forall\ S,\ T \subseteq N : S \cap T = \emptyset, \tag{1.1}$$

Table 1.2 A three-person normal form game

	$\sigma_1 = L$			$\sigma_1 = R$	
	$\sigma_2 = L$	$\sigma_2 = R$		$\sigma_2 = L$	$\sigma_2 = R$
$\sigma_3 = L$	(7, 5, 3)	(0, 3, 3)	$\sigma_3 = L$	(3, 3, 6)	(6, 7, 3)
$\sigma_3 = R$	(2, 3, 8)	(6, 9, 2)	$\sigma_3 = R$	(5, 3, 2)	(2, 0, 2)

i.e., if any two disjoint coalitions S and T can achieve at least as much by joining their forces as by remaining separate.

Consider the three-person game in Table 1.2, taken from Michener et al. (1987), with strategy set $\Sigma_i = \{L, R\}$ for each player $i \in \{1, 2, 3\}$ is $\Sigma_i = \{L, R\}$. Starting from the normal form, $v(S)$ is the *maximin payoff* of S in the two-person game between S and its complement $N \backslash S$, derived from the hypothetical situation that these two coalitions form.[1] The maximin payoff can be interpreted as the 'safety level' of a coalition – what that coalition could guarantee itself even if non-members took action to minimize its payoff. In constant-sum games, i.e., when $v(S) + v(N\backslash S) = k$, a constant, for all S, this behavioral assumption is fully warranted, because here minimizing the payoff accruing to the complement is equivalent to maximizing own payoff.

Under the TU-assumption, the characteristic function of the game in Table 1.2 is $v(\{1\}) = 2$, $v(\{2\}) = v(\{3\}) = 3$, $v(\{12\}) = 6$, $v(\{13\}) = 9$, $v(\{23\}) = 8$ and $v(\{123\}) = 17$.[2] It is both monotonic and superadditive.

If side payments between coalition members are not possible, the coalition's 'worth' is represented by a set of vectors instead of a single number. A cooperative NTU-game is a pair (N, V), where V is a map that assigns to each $S \in 2^N$, $S \neq \emptyset$, a non-empty set $V(S) \subseteq \mathbb{R}^s$ of attainable payoff vectors, or *utility profiles*. For each player i, an individually rational payoff $v(\{i\}) \in \mathbb{R}$ is assumed to exist such that $V(\{i\}) = \{u \in \mathbb{R} \mid u \leq v(\{i\})\}$, while, for each $S \in 2^N\backslash\emptyset$, $V(S)$ satisfies the following conditions:

(a) $V(S)$ is a non-empty, closed subset of $\mathbb{R}^s$.
(b) $V(S)$ is comprehensive, i.e., if $u \in V(S)$ and $w \in \mathbb{R}^s$ such that $w \leq u$, then $w \in V(S)$.[3]
(c) $V(S) \cap \{u \in \mathbb{R}^s \mid u_i \geq (v(\{i\}))_{i \in S}\}$ is bounded.

Condition (b) reflects *free disposal*, i.e., players could 'burn' their utility if they feel like it, and thus all these reduced utility vectors might also occur. The *comprehensive hull* of a set $P \subset \mathbb{R}^n$ is defined by

$$compr(P) := \{z \in \mathbb{R}^n \mid \text{there is a } p \in P \text{ such that } z \leq p\}.$$

[1] This definition of a characteristic function was put forward by Von Neumann and Morgenstern (1944, pp. 238ff), and is also known as Von Neumann–Morgenstern characteristic function.

[2] Instead of $\{1, 2\}$, $v(\{1, 2\})$ etc. the notation $\{12\}$, $v(\{12\})$ etc. is used throughout as there is no risk of confusion.

[3] For $u, w \in \mathbb{R}^s$, $u \geq w$ denotes $u_i \geq w_i$ for all $i \in S$, and $u > w$ denotes $u_i > w_i$ for all $i \in S$.

Condition (c) just states that the set of feasible utility profiles for coalition S is bounded above.

A NTU-game is *monotonic* if, for all $S \subset T \subset N$, $S \neq \emptyset$, and for all $x \in V(S)$ there exists a $y \in V(T)$ such that $y_S \geq x$. It is *superadditive* iff $V(S) \times V(T) \subseteq V(S \cup T)$ for all $S, T \in 2^N \backslash \emptyset$ with $S \cap T = \emptyset$, where $V(S) \times V(T) = \{(x, y) \mid x \in V(S) \text{ and } y \in V(T)\}$ is the Cartesian product of the sets $V(S)$ and $V(T)$.

Any TU-game v can be generalized to a NTU-game $V(S)$, given by $V(S) = \left\{x \in \mathbb{R}^s \mid \sum_{i \in S} x_i \leq v(S)\right\}$ for each $S \in 2^N \backslash \emptyset$, where $v(S)$ denotes the value of coalition S in a sidepayment game (N, v). Thus, the set of all TU-games is formally a subset of the set of all NTU-games.

In NTU-games, several ways to derive a characteristic function from the normal form game come into consideration, differing in which *effectiveness* criteria are applied.[4] Coalition S is said to be α-*effective* for some set $V(S)$ if and only if its members can combine their strategies in such a way that the outcome of the game is an element of that set, regardless of the behavior of players outside S. If players in S may use joint mixed or *correlated strategies* (see Aumann, 1974), then the set of utility vectors that can be reached by cooperation forms a compact convex polyhedron (provided that the underlying strategic game is finite). Let Σ_S denote the set of correlated strategies of the members of coalition S. Put more precisely, coalition S is α-effective for vector $x \in \mathbb{R}^s$ if its members have a correlated strategy $\sigma_S \in \Sigma_S$ such that, for any strategy combination $\sigma_{N \backslash S} \in \Sigma_{N \backslash S}$ of the complementary coalition, $u_i(\sigma_S, \sigma_{N \backslash S}) \geq x_i$ (in expected value) for all $i \in S$. By contrast, $V(S)$ could also consist of precisely those payoff vectors which players outside S cannot prevent S from getting. This criterion is termed β-*effectiveness* (Aumann and Peleg, 1960).[5] The normal form game in Table 1.2 translates into the α-characteristic function

$$V_\alpha(\{1\}) = (-\infty, 2], \quad V_\alpha(\{2\}) = V_\alpha(\{3\}) = (-\infty, 3],$$
$$V_\alpha(\{12\}) = V_\alpha(\{13\}) = V_\alpha(\{23\}) = \mathit{compr}\{(3, 3)\},$$
$$V_\alpha(\{123\}) = \mathit{cch}\{(7, 5, 3), (2, 3, 8), (6, 9, 2), (3, 3, 6), (6, 7, 3)\},$$

and the β-characteristic function is given by

$$V_\beta(\{1\}) = (-\infty, 5], \quad V_\beta(\{2\}) = V_\beta(\{3\}) = (-\infty, 3],$$
$$V_\beta(\{12\}) = \mathit{compr}\{(6, 7)\}, \quad V_\beta(\{13\}) = \mathit{compr}\{(6, 3)\},$$
$$V_\beta(\{23\}) = \mathit{cch}\{(5, 3), (3, 6)\},$$
$$V_\beta(\{123\}) = \mathit{cch}\{(7, 5, 3), (2, 3, 8), (6, 9, 2), (3, 3, 6), (6, 7, 3)\},$$

[4] Besides the α- and β-characteristic functions dealt with here, characteristic functions can also be derived under the γ- and δ-assumptions in Hart and Kurz (1983, 1984).

[5] S is β-effective for a payoff vector $x \in \mathbb{R}^s$ if, for any strategy combination $\sigma_{N \backslash S} \in \Sigma_{N \backslash S}$, there exists a $\sigma_S \in \Sigma_S$ such that $u_i(\sigma_S, \sigma_{N \backslash S}) \geq x_i$ for all $i \in S$.

where $cch(P)$ refers to the comprehensive convex hull of $P \subset \mathbb{R}^n$.[6] The two forms generally differ because no minimax theorem for coalitions exists in the absence of transferable utility (see Aumann and Peleg, 1960).

For games with transferable utility, the α- and the β-form of the characteristic function coincide. As the example shows, this is not the case in the class of NTU-games, and the β-characteristic set always includes the α-characteristic set. This conforms to the intuition that the α-characteristic function embodies a rather pessimistic or conservative prejudice, whereas β-effectiveness poses less restrictions. Shubik (1971) offers an interpretation of the two forms of $V(S)$ in sequential terms: α-effectiveness is as if coalition S moves before its complement, whilst β-effectiveness could describe the situation where S announces its intention to form, makes $N \backslash S$ move first, and then responds.

An important constraint of coalitional form games is that they are unable to capture the possibility of externalities across coalitions (for a discussion and example see Rosenthal, 1972). Externalities exist if the payoffs to the members of S depend on the actions of the non-members, which implies that no worth can be assigned to S without specifying those actions. In this situation, the α- and β-characteristic functions inadequately assume away the strategic interaction between S and the outside players. One attempt to deal with this problem by extending the characteristic function concept is the *partition function* approach, introduced by Thrall and Lucas (1963).[7] Another is offered by endogenous coalition formation theory (see, e.g., Bloch, 1996, 1997; Yi, 1997; Ray and Vohra, 1997, 1999), using non-cooperative game theory.

1.1.2 Simple Games

An important class of characteristic function form games in modeling group decision-making are *simple games*, introduced by Von Neumann and Morgenstern (1944, pp. 420ff). Their distinctive feature is that each coalition $S \in 2^N$ can be classified as either *winning* or *losing*. A simple game is a pair $G = (N, \mathcal{W})$ in which $\mathcal{W} \subsetneq 2^N$ is a set of winning coalitions which satisfy

$$\text{(a)} \quad \emptyset \notin \mathcal{W}, \qquad \text{(b)} \quad N \in \mathcal{W},$$

and

$$\text{(c)} \quad S \in \mathcal{W} \text{ and } S \subseteq T \;\Rightarrow T \in \mathcal{W}.$$

[6] The *convex hull* of a set $P \subset \mathbb{R}^n$ is the smallest convex set containing the points in P. It is defined by $conv(P) := \left\{\sum_{i=1}^n a_i p_i \mid p_i \in P,\ a_i \in \mathbb{R}_+,\ \sum_{i=1}^n a_i = 1\right\}$. – If randomization is not possible, the comprehensive convex hull must be replaced by the comprehensive hull.

[7] For each coalition structure, a partition function assigns a 'worth' to each coalition (for a discussion of the limitations of partition functions in the analysis of coalition formation see Ray and Vohra, 1997).

The *monotonicity* condition (c) captures the intuition that a superset of a winning coalition must also be winning.[8]

In the presence of transferable utility, $(N, \mathcal{W})$ is equivalent to the game (N, v) where v is a characteristic function taking on only two values, and that, after a suitable normalization, is given by $v : 2^N \to \{0, 1\}$ with

$$v(S) = \begin{cases} 0 & \text{if } S \text{ is losing,} \\ 1 & \text{if } S \text{ is winning.} \end{cases}$$

Let $\mathcal{G}^N$ denote the set of all n-player monotonic simple games with transferable utility. Then $\mathcal{G}^N$ can be interpreted as the collection of all logically possible voting procedures with n voters if the decision is *binary*, such as voting 'yes' or 'no' on some (exogenous) proposal. These games have been found to be well-suited to model economic or political bodies that exercise some kind of *control*, e.g., over the allocation of some budget. As observed by Shapley and Shubik (1954), any winning coalition can determine which proposals pass. Therefore, all winning coalitions can, in a way, be said to be equally 'powerful'.

If the defection of *any* individual member i of a winning coalition S turns the remaining members $S \backslash \{i\}$ into a losing coalition, S is called a *minimum winning coalition*. Put differently, a minimum winning coalition is a winning coalition with no proper subsets that are also winning. A simple game is completely determined by its set of minimum winning coalitions, denoted $\mathcal{W}^{\min}$, because both $\mathcal{W}$ and $\mathcal{W}^{\min}$ are finite sets, and for $\mathcal{W}$ it holds that

$$\mathcal{W} = \{S \in 2^N | S \supseteq R \in \mathcal{W}^{\min}\}.$$

A player $i \in N$ is called a *null player* or *dummy* in G if

$$S \in \mathcal{W} \Rightarrow S \backslash \{i\} \in \mathcal{W} \tag{1.2}$$

for any coalition $S \in 2^N$. If, by contrast, $S \in \mathcal{W} \Rightarrow i \in S$, then that player i is a *veto player* without whom no coalition can be winning. If it even holds that $S \in \mathcal{W} \Leftrightarrow i \in S$, then i is a *dictator* in G. Note that it follows from this definition that the coalition $S = \{i\}$ is also winning (and the other players are of course dummies). Dictatorial games are *inessential* since there is no incentive for anyone to form a coalition.

A simple game is *proper* if

$$S \in \mathcal{W} \Rightarrow N \backslash S \notin \mathcal{W}.$$

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[8] Following common practice, we include monotonicity in the definition of a simple game. See, for example, Von Neumann and Morgenstern (1944, pp. 420ff), or Shapley (1962). Note, however, that a simple game can also be defined more generally in terms of a set $\mathcal{W}$ which satisfies conditions (a) and (b), but not necessarily condition (c) (see, e.g., Ordeshook, 1986, p. 324).

If both S and its complement $N \backslash S$ are winning coalitions, the game is *improper*. This is the case, for instance, when any subset of k players with $k < n \backslash 2$ is a winning coalition. In improper games, confusion may arise as two separate groups of voters can make decisions at the same time. A game $G \in \mathcal{G}^N$ is superadditive [cf. (1.1)] precisely if it is proper.

A simple game is *strong* if

$$S \notin \mathcal{W} \Rightarrow N \backslash S \in \mathcal{W}.$$

In a strong game, the complement of a losing coalition is always winning, that is, no blocking coalitions exist. By contrast, in a *weak* game, at least one veto player exists. A special case of weak games arises under unanimity rule, which makes *every* individual a veto-player or blocking coalition: A *pure bargaining game* results, in which the outcome is indeterminate as long as one does not impose further assumptions to make a bargaining solution applicable. Games that are both strong and proper are called *decisive*, and in the TU-case, this is equivalent to being constant-sum.

Two sub-classes of simple games are particularly relevant in applications to real-world committees. The first is the class of *weighted majority games* $[q; w]$, where $w = (w_1, \ldots, w_n)$ is a vector of *voting weights*, and $q \in (0, 1]$ is the (relative) *decision quota*. The most elementary example is the canonical simple majority game with an odd number n of voters,

$$M_n = (0.5; \underbrace{1, \ldots, 1}_{n \text{ times}}) \qquad n \text{ odd}. \tag{1.3}$$

Majority games are *symmetric* in the sense that groups of the same number of players are undistinguishable. Generally, a coalition $S \subseteq N$ is winning in a weighted majority game iff

$$\sum_{i \in S} w_i \geq q \sum_{i \in N} w_i.$$

The $(n + 1)$-tuple $[q; w_1, \ldots, w_n]$ is called a representation of G. Obviously, one game has many representations, and it is always possible to find one where both quota and weights are all integers (Shapley, 1962, p. 64). Many simple games, however, cannot be represented as weighted voting games, which is equivalent to saying that no single consistent ranking of the players' relative strengths exists.[9]

The second sub-class is obtained by applying some concatenation operation to simple games, e.g., the product or the sum operation. In these *compound simple games*, described by Shapley (1962), simple games serve as building blocks. Let $G_1 = (N_1, \mathcal{W}_1)$, $G_2 = (N_2, \mathcal{W}_2)$ be simple games. The player sets need not be disjoint. The *product* $G_1 \wedge G_2$ is defined as the game $(N_1 \cup N_2, \mathcal{W}_p)$ where $S \in \mathcal{W}_p$ iff $(S \cap N_1 \in \mathcal{W}_1)$ and $(S \cap N_2 \in \mathcal{W}_2)$ for $S \subseteq N_1 \cup N_2$. The *sum* $G_1 \vee G_2$ is the

[9] Taylor and Zwicker (1992, 1993) introduce a general method to decide whether a simple game is weighted, based on trades of players among coalitions.

game $(N_1 \cup N_2, \mathcal{W}_s)$ where $S \in \mathcal{W}_s$ iff $(S \cap N_1 \in \mathcal{W}_1)$ or $(S \cap N_2 \in \mathcal{W}_2)$, i.e., to win in the sum, it is sufficient that a coalition wins in one component, whilst it must win in both components to win in the product game.[10] A general compound simple game is given by

$$H = K[G_1, \ldots, G_m]$$

where K is an arbitrary simple game with simple games G_j, $j = 1, \ldots, m$, as 'players'. A coalition is winning in H if and only if it comprises winning coalitions in enough component games to constitute a winning coalition in K. Consider for illustration the 9-person compound game

$$H = M_3[M_3, M_3, M_3],$$

where a winning coalition must at least consist of four players such that two players from each of two subgames are included. A game of this type can be used to model a two-stage indirect decision procedure (for an application to the EU see Laruelle and Widgrén, 1998). Another structure to which compound games apply are *multi-cameral legislatures* (see Footnote 10). Imagine for example the legislative assembly of a society that is ethnically divided into four groups. The constitution prescribes that the approval of either groups A and B, or the approval of groups A, C, and D is needed for the passage of a bill. This would give rise to the game $A \times (B + (C \times D))$.[11] Interactions between bodies in the various decision-making procedures of the EU have also been modeled by means of compound games (see, for example, Laruelle and Widgrén, 2001).[12] The approach is, however, unable to capture the sequential structure of these rules.

Compound simple games allow to create sophisticated voting games from simpler ones and to model complex interactions between the simpler components. A measure of the complexity of a voting system modeled as simple game $(N, \mathcal{W})$ is its *dimension*, defined by Taylor and Zwicker (1993) as the smallest number k for which k weighted voting systems $(N, \mathcal{W}_1), \ldots, (N, \mathcal{W}_k)$ exist such that a coalition is winning in $(N, \mathcal{W})$ iff it is winning in every $(N, \mathcal{W}_j)$. The compound game $A \times (B + (C \times D))$ can be represented as the (weak) weighted majority game $[5; 3, 2, 1, 1]$, and thus its dimension is 1. Examples of higher-dimensional voting games include the procedure for amending the Canadian Constitution and the US federal legislative system (see Taylor and Zwicker, 1993), as well the current threefold majority requirements to pass a decision in the EU Council of Ministers (see Felsenthal and Machover, 2001).

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[10] Starting with Shapley (1962), the notation $\times$ and $+$, instead of $\wedge$ and $\vee$, is often used in the literature for the special case that the player sets are disjoint. Thus, $G_1 \times G_2$ and $G_1 + G_2$ model *bicameral*, or more generally, *multicameral* decision structures.

[11] For another example of a legislative system, consisting of a president, a senate, and a house, see Shapley and Shubik (1954).

[12] See Braham and Steffen (2003) for an application of compound weighted voting games to insolvency law rules.

The framework of simple TU-games lends itself only to the study of decision bodies (a) that operate under *closed rule*, i.e., when amendments are not possible, and a binary agenda determined by an external agent (when interpreting the coalition value as 'control'), or (b) whose members compete for a fixed purse (when interpreting the coalition value as a private good). In reality, however, committee decisions rarely concern only two prefabricated alternatives or candidates, and preferences are not always diametrically opposed as is the case in the division of spoils. Committee members often can make proposals themselves, or amend proposals put in front of them. Though some political decisions like, e.g., choosing between two alternatives or allocating a budget or costs, fit into the TU scenario, many others do not.

Simple games without transferable utility enable investigation of more complex committee models. These games are still characterized by the set of winning coalitions, but, in addition, a payoff space must be specified. To be precise, let X denote the set of all possible policy alternatives or outcomes. It is assumed that the preferences of each individual over X can be represented by a Von Neumann–Morgenstern utility function,[13] establishing a perfect correspondence between the set of all lotteries over X and the set of available payoff vectors, or utility possibility set, $U \subseteq \mathbb{R}^n$.[14] Note that no distinction is made between alternatives and final outcomes. This is surely a simplifying assumption as it is usually impossible to select outcomes directly, but it should also be observed that individuals usually have preferences over alternatives because they associate them with certain outcomes.

Then, a set-valued *effectivity function* (see Rosenthal, 1972; Moulin and Peleg, 1982) can be used to determine for each coalition S the subset of X that S can realize if it forms. In general, this raises the question which effectivity concept (see Sect. 1.1.1) should be applied, e.g., α-effectivity (what S can enforce regardless of the actions taken by players in $N \setminus S$) or β-effectivity (what $N \setminus S$ cannot prevent S from achieving). Fortunately, the problem does not arise for (proper) simple games.[15] Here, the idea is that a winning coalition is all-powerful in the sense that it can choose any outcome it pleases from X, whereas a losing coalition has no say at all. If coalition S forms and selects some payoff vector $x \in X$, the resulting utilities for all players $i \in N$ are summarized by the payoff vector $u = (u_1, \ldots, u_n) \in U$. Without loss of generality (since the origin of each player's utility scale is arbitrary), let $\min_{x \in X} u_i(x) \geq 0$ for all $i \in N$. For $u \in \mathbb{R}^n_+$, u_S denotes the projection of u to the coordinates corresponding to S. This leads to the following redefinition of $V(S)$ for simple games,

[13] A utility function is of the Von Neumann–Morgenstern type if it satisfies the *expected utility hypothesis*. See, for example, Ordeshook (1986, pp. 37ff).

[14] The setting consisting of the set of voters, the set of alternatives, and a voting rule has also been studied from a social choice perspective. See Peleg (2002) for an overview.

[15] In voting situations, even taking pre-vote negotiations into account, it seems appropriate to ignore the possibility that a coalition can achieve outcomes x and y, but is not effective for outcome z (see Miller, 1982).

$$V(S) = \begin{cases} cch\{u_S(x) \mid x \in X\} \cap \mathbb{R}^s_+ & \text{if } S \text{ is winning,} \\ \mathbb{R}^s_- & \text{if } S \text{ is losing.} \end{cases} \tag{1.4}$$

In (1.4), the whole set of feasible utility vectors is associated to every winning coalition S. It is important to observe that, generally, this model does not belong to the class of constant-sum games. Moreover, payoffs to all players depend on which coalition S forms, and which outcome x its members agree on. In particular, a non-winning coalition might free-ride on or be exploited by a winning coalition. This is illustrated by the following primitive example taken from Shubik (1982, p. 140):

Example 1.1.1. Consider a three-player, simple majority game where the set of available utility vectors is $cch\,\{(0,0,0),(2,2,0),(2,0,2),(1,2,2)\}$. The first player can reasonably expect to get at least a utility of 1, even if he is not included in the winning coalition. Yet, the 'value' assigned to him by the characteristic function is $(-\infty, 0]$.

These characteristics of the NTU simple game (1.4) transcend the framework of coalitional games.[16] Unlike the TU simple game, (1.4) provides an appropriate, albeit abstract, model of collective decision-making in the case that (a) decision-making is concerned with public policies which are 'public goods' in that they bear on all players, but are judged differently by different players (Barry, 1980a, pp. 189ff), and (b) decision-making is a bargaining process under a voting rule, but without any further institutional constraints.

In principle, it should be possible to derive a characteristic function from some underlying game, e.g., in normal form. To conclude this section, an example illustrates how a strategic game and a TU simple game can be linked to each other.

Example 1.1.2 (M_5 'Divide-the-dollar'). Five persons decide by simple majority voting on the division of some budget, normalized to 1.[17] A winning coalition can appropriate the budget, but due to a legal requirement the latter must be divided equally among the coalition members. First, the situation is modeled by the normal form game $\Gamma = [\{\Sigma_i\}, \{u_i(\cdot)\}]$, $i = 1, \ldots, 5$. The *strategy set* Σ_i of player i is

$$\Sigma_i = \{S \subseteq N : i \in S\},$$

i.e., a strategy $\sigma_i \in \Sigma_i$ amounts to the choice of a coalition to which i belongs. The payoff function u_i, which assigns to each strategy profile $\sigma = (\sigma_1, \ldots, \sigma_5) \in \times\, \Sigma_i$ the payoff to i, is given by

$$u_i(\sigma) = \begin{cases} 1/|S_k| & \text{if } i \in S_k \text{ and } |S_k| \geq 3, \\ 0 & \text{otherwise} \end{cases}$$

---

[16] In Sect. 1.2.2, the NTU simple game will be defined to exclude the possibility of both free-riding and exploitation of non-winning players.

[17] The example can be extended to an arbitrary number of players, see Example 1.3.1.

Table 1.3 'Divide-the-dollar' with five players

Coalition structure	u_1	u_2	u_3	u_4	u_5
$\{123\}, \{45\}$	$1/3$	$1/3$	$1/3$	0	0
$\{1234\}, \{5\}$	$1/4$	$1/4$	$1/4$	$1/4$	0
$\{12345\}$	$1/5$	$1/5$	$1/5$	$1/5$	$1/5$

where

$$S_k(\sigma) = \begin{cases} S_k & \text{if and only if } \sigma_j = S_k \quad \forall\ j \in S_k, \\ \{j\} & \text{otherwise.} \end{cases} \tag{1.5}$$

The coalition formation rule (1.5) states that a particular coalition S_k will come into being if and only if all its prospective members choose S_k. In case that players $i \in S_k$ do not unanimously agree on the formation of S_k, those who have selected the strategy $\sigma_j = S_k$ remain singletons.

Moving on to the NTU-game, the information about the coalition formation process is lost, but the 'equal division of payoffs' rule survives. Omitting permutations of players, Table 1.3 summarizes the attainable efficient payoff vectors.

Finally, the normal form game translates into a TU simple game given by the characteristic function

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq 3, \\ 0 & \text{otherwise.} \end{cases}$$

According to this representation, the payoff 1 can be divided among the members of a winning coalition in any way possible.

1.1.3 Spatial Analysis

In political applications, the set of feasible alternatives from which the committee can choose is often adequately described by a (usually nonempty, compact, and convex) subset X of a one- or many-dimensional Euclidean space $\mathbb{R}^m$: the choice of tax-rates, expenditure levels, or a minimum wage are examples for approximately continuous variables. Moreover, the outcomes of interest to the researcher frequently are policies which at least some individuals (legislators or voters) neither entirely accept nor entirely oppose. For example, let any amount of welfare benefit between 0 and the income of the average earner be feasible for the government. A voter may prefer some moderate amount of welfare benefit over both extremes, thus displaying satiable preferences.[18] These preferences can also be given a *spatial conceptualization*. The dimensions of the policy space are usually referred to as 'issues'. Spatial models presume that all players have a common perception as to what the issue dimensions are. They disagree about which is the most desirable point, but not about how policy alternatives are ordered.

[18] By contrast, in most purely economic applications, preferences are assumed to exhibit nonsatiation, i.e., more of some good (perhaps denominated in money) is always better.

As highlighted by Arrow's 1963[1951] famous theorem, non-dictatorial social choice in a society without some basic value consensus or similarity of attitudes may fail to produce a transitive social preference ordering and therefore be unstable.[19] For simple majority rule in particular, a number of results (e.g., Plott, 1967; Slutsky, 1979) indeed indicate that undominated outcomes, or *core* points, very rarely exist.[20] This holds true a fortiori for the existence of a complete transitive social ordering. Important results of subsequent research concern the characteristics of individual preference orderings that, often in combination with the majority threshold or other institutional constraints, allow equilibrium states of the social aggregation mechanism. With respect to the spatial setting,[21] it transpired from this work that extremely severe restrictions must be imposed on the admissible type of individual preferences or on the domain of the latter (i.e., on the dimensionality of the policy space), or that majority sizes greater than simple majority rule must be employed, to ensure the existence of equilibria, and even then, the latter may lack robustness to slight perturbations (see, e.g., McKelvey, 1979).

The first result on the existence of a 50%-majority winner, Black's 1958 *median voter theorem*, combines restrictions on both preference type and domain. It requires a one-dimensional policy space, where it is assumed that each individual i has a single most preferred policy outcome, an *ideal point* (or bliss point), on the issue, denoted λ_i. Moreover, i's utility is assumed to decrease as the outcome moves away from i's ideal point. The latter condition is termed *single-peakedness*. Formally, preferences are single-peaked, if, for each $i \in N$ there exists a $\lambda_i \in X$ such that $u_i(\lambda_i) \geq u_i(x_1) \geq u_i(x_2)$ for all $x_1, x_2 \in X$ that satisfy either $\lambda_i \geq x_1 \geq x_2$ or $\lambda_i \leq x_1 \leq x_2$. If X is a convex set, single-peakedness is equivalent to quasi-concavity of the utility function. Often, a continuous probability density function is used to describe the (infinite) set of individuals' ideal points.

Another assumption that is sometimes made is *symmetry* of preferences which implies that the individual's utility is a decreasing function of the *distance*, measured usually by the Euclidean norm, between his ideal point and the actual policy outcome. In other words, a departure of a given size from the ideal point yields the same decline in utility, independent of the direction of departure.

The median voter theorem states a sufficient condition, one-dimensionality and single-peakedness, for the existence of an undominated outcome – the median ideal point (or a closed interval of median points if the number of voters is even) – under

[19] The absence of value consensus is expressed in Arrow's axiom of 'unrestricted domain'.

[20] The *core* of a game (N, v) in characteristic function form is defined by $Core(N, v) := \{u \in I(v) \mid \sum_{i \in S} u_i \geq v(S) \text{ for all } S \in 2^N \setminus \emptyset\}$ where $I(v)$ is the set of *imputations*, i.e., vectors $u \in \mathbb{R}^n$ such that (a) $u_i \geq v(\{i\})$ for all $i \in N$, and (b) $\sum_{i=1}^{n} u_i = v(N)$. Conditions (a) and (b) express individual rationality and efficiency, respectively.

[21] In Arrow's 1963[1951] model, individuals have binary preference relations over a finite set of alternatives, which are assumed to satisfy the usual consistency conditions such as reflexivity, transitivity, and completeness.

simple majority rule.[22] Multi-dimensional analogues of the median voter result are offered by Plott (1967) and Davis et al. (1972). However, the symmetry conditions on the distribution of voters' ideal points which are necessary to produce a multidimensional 'median' are too strong and knife-edge to be met in practice: the existence of a policy $x^* \in \mathbb{R}^m$ such that *any* hyperplane through x^* divides the committee into three groups such that the members on one side of it and those with ideal points on the hyperplane can form a majority, as can members on the other side with those on the hyperplane.[23] Generally, in multidimensional issue spaces, no alternative will command a simple majority against every other alternative (see Plott, 1967). The 'chaos theorems' by McKelvey (1976) and Schofield (1978) show that in the absence of an undominated outcome intransitivities are generally global, that is, sequences of paired votes exist such that almost every alternative can be reached from every other one, making social choice susceptible to local agenda manipulation.

As a consequence, in order to permit predictions on the outcomes of voting processes, theories using a spatial conceptualization often assume that the issues to be voted upon can be catched by a single policy dimension. A one-dimensional policy space can obviously be interpreted as an ideological left-right dimension, the level of a particular tax, or the amount to be spent on some project. Yet, most decisions are presumably made along more than one dimension, and two dimensions are usually considered the threshold to enter the realm of real-world political attitudes (e.g., Binmore, 1998, p. 164; for empirical findings on US congress see Poole and Rosenthal, 1991). Another way to escape the no-equilibrium results, originating with Shepsle (1979), is to augment the spatial setting with some institutional structure which narrows the set of implementable outcomes and may thereby yield stability.

In Sect. 1.1.2, (proper) simple games without side payments were introduced as a model of majority rule bargaining without any further institutional constraints. In this context, the spatial conceptualization of preferences and alternatives is only a particularly concrete way to 'put flesh on the bones' of the NTU-game (1.4), that is, to obtain the utility possibility set U.[24] Even without its typical restrictions, the spatial approach imposes considerable structure on the set of alternatives and individual preference patterns. Why would we want to use a less general approach compared to (1.4)? One reason is that low-dimensional spatial models have been found empirically to capture actual voting behavior quite well (see, for example, Poole and Rosenthal, 1991). A possible conclusion from these studies is that the spatial conceptualization is consistent with how people experience and describe political attitudes. From an applied perspective, the assumptions of spatial models thus appear less restrictive. Second, the spatial formulation allows a simple

[22] The theorem was proved by Black (1958) for a finite set of alternatives and extended by Arrow (1963[1951]) to arbitrary sets of alternatives.

[23] A hyperplane is a plane in more than two dimensions.

[24] It is not applicable when issues are purely distributive, or interpreted in terms of distribution by the decision-makers. These situations are more appropriately modeled by TU constant-sum games.

depiction of the similarity in the policy goals of decision-makers, and to evaluate the distance of outcomes under different institutions. It gives an intuitive geometric interpretation to proposals and counter-proposals, dominance relationships between alternatives, and solution concepts such as the core or bargaining sets. Consider for example the simple NTU-game $(N, \mathcal{W})$ with one-dimensional policy space X. If $S \in \mathcal{W}$, S is effective for the whole set X. The set of feasible policy outcomes that are Pareto-efficient for coalition S is given by

$$X^*(S) = \{x \in X \mid \nexists\, y \in X \text{ such that } u_i(y) > u_i(x) \text{ for all } i \in S\}. \tag{1.6}$$

Then, the core of the game $(N, \mathcal{W})$ can be defined as

$$Core(N, \mathcal{W}) = \left\{ u(\tilde{x}) \in U \mid \tilde{x} \in \bigcap_{S \in 2^N \setminus \emptyset} X^*(S) \right\}. \tag{1.7}$$

The core consists of all (feasible) utility vectors with the property that no coalition could achieve more for all its members on its own. If $(N, \mathcal{W})$ is strong, it is sufficient to take the intersection in (1.7) over all $S \in \mathcal{W}^{\min}$ rather than all non-empty coalitions (see Ordeshook, 1986, Theorem 8.1). Figure 1.1 illustrates the sets $X^*(S)$ for the minimum winning coalitions in the canonical majority game M_3 under the assumption of single-peaked preferences. The core consists of the utility vector $u(\lambda_2) \in \mathbb{R}^3$ associated with the median voter's ideal point.

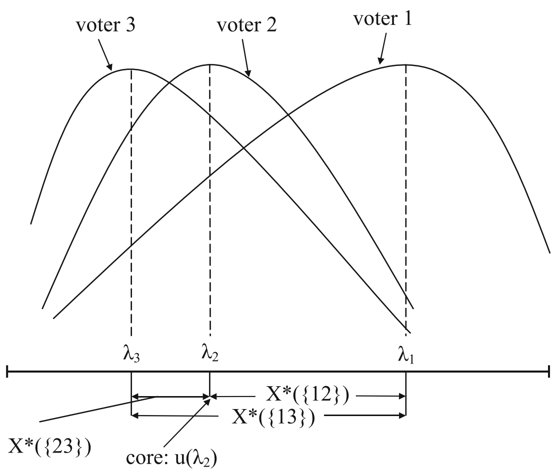

Fig. 1.1 Core with single-peaked preferences. The core is the intersection of the Pareto sets $X^*(S)$

1.2 Voting Rules and Power

Max Weber (1962, p. 117) famously defined power as

> "[..] that opportunity existing within a social relationship which permits one to carry out one's own will even against resistance and regardless of the basis on which this opportunity rests."

and reinforces

> "Every conceivable quality of a person and every combination of circumstances may put someone in a situation where he can demand compliance with his will."

The definition reflects the experience that 'power' is a ubiquitous phenomenon, a potential aspect of any human relationship. There is no doubt that 'every quality' such as patience, better information, persuasiveness, or rhetorical skill can also be a source of power in committees. Yet, this red meat of power largely eludes game-theoretic modeling. In view of the general definition of power, the range of the concepts discussed in the following may thus appear to be regrettably narrow. The analysis focuses on the power as conferred upon committee members by the voting rule taken in isolation. For this purpose, power can be specified as the ability to make a difference to the outcome of a collective decision-making situation.

Power indices, introduced in Sect. 1.2.1, indicate the probability of a committee member to make a decision pass that would not have passed otherwise without making reference to the issue at stake, and hence, without reference to decision-makers' preferences regarding the issue. Instead, committee members are assumed to be in a 'general condition'. Differences in the definition of the latter mainly account for the diversity of power indices.

An approach to evaluate voting rule based influence on the committee outcome in a general NTU framework is put forward by Laruelle and Valenciano (2007, 2008b). Their work is considered in Sect. 1.2.2.

Finally, Sect. 1.2.3 provides a brief survey of attempts to measure power in spatial voting games.

1.2.1 Power Indices

In this section we consider voting power in the context of a committee which can be represented by a weighted majority game or a more general simple game. A simple example serves to introduce some key concepts.

Example 1.2.1. Suppose a three-player take-it-or-leave-it committee $[3;2,1,1]$ is presented with a proposal χ that it can adopt or reject. In the latter case, the status quo Q prevails.

Since only the institutional structure is known, but not players' preferences regarding χ and Q, the natural model for the situation at hand is a *game form* g (Gibbard,

Table 1.4 Relationship between strategy choices and outcome

	$\sigma_1 = 0$			$\sigma_1 = 1$	
	$\sigma_2 = 0$	$\sigma_2 = 1$		$\sigma_2 = 0$	$\sigma_2 = 1$
$\sigma_3 = 0$	Q	Q	$\sigma_3 = 0$	Q	χ
$\sigma_3 = 1$	Q	Q	$\sigma_3 = 1$	χ	χ

1973), i.e., a game where no individual utilities are yet attached to the outcomes. It specifies the set of players N, a set of strategies Σ_i for each player $i \in N$, a set of feasible outcomes X, and an outcome function x that assigns to each n-strategy tuple $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in \Sigma_i$, a single outcome $x \in X$.

In the example, $X = \{Q, \chi\}$, and, ignoring abstention, the strategy set of each committee member i is $\Sigma_i = \{0, 1\}$, where 0 and 1 stand for voting 'no' and 'yes', respectively, on the proposal. The outcome depends on individuals' strategy choices as shown in Table 1.4.

How can we evaluate the voting power of the three players in this game? The most widely used measures of *a priori power* are the Penrose–Banzhaf index, introduced in Penrose (1946) and Banzhaf (1965), and the Shapley–Shubik (1954) index.[25] They are commonly interpreted either in terms of the axioms that characterize them, or as probabilities. A *power measure* or *power index* is a function[26] $\mu\colon \mathcal{G}^N \to \mathbb{R}^n_+$ which assigns to each n-player game v a real-valued vector $\mu(v)$ that is interpreted as the distribution of voting power under a given voting rule modeled by v. The ith component of the vector $\mu(v)$ is seen as indicating player i's voting power as it derives from the formal structure $[q; w_1, \ldots, w_n]$ alone.

The Shapley–Shubik index emerges from the application of the Shapley value (Shapley, 1953) to simple games. The Shapley value $\Phi(v)$ of a (general) n-player-game v is given by

$$\Phi_i(v) = \sum_{\substack{S \in 2^N, \\ i \in S}} \frac{(s-1)!(n-s)!}{n!} \left(v(S) - v(S \backslash \{i\})\right). \tag{1.8}$$

Let $<<$ denote a strict linear ordering of the player set N. Considering a fixed player $i \in N$, let $D(i, <<)$ be the set of players $j \in N$ who precede i in the ordering. From (1.8), it is then clear that $\Phi_i(v)$ is player i's weighted marginal contribution to all coalitions S, where weights are the shares of player orderings $<<$ that include the formation of the coalition $S = D(i, <<) \cup \{i\}$ out of all (equally probable) player orderings. There are exactly $(s-1)!(n-s)!$ orderings or permutations that yield the coalition S.

[25] In the literature, the term 'Banzhaf index' is widespread. Yet, since L.S. Penrose proposed the measure well before Banzhaf, we think it more accurate to call it Penrose–Banzhaf index.

[26] More formally, a power index is a family of functions $\{\mu^n\}_{n=1,2,\ldots}$ because N and n are not fixed.

In his seminal contribution "A Value for n-Person Games", Shapley (1953, p. 316) gives the following informal, 'easy-to-visualize' derivation of his value:

> "The players [..] agree to play the game v in a grand coalition, formed in the following way: (1) Starting with a single member, the coalition adds one player at a time until everyone has been admitted. (2) The order in which the players are to join is determined by chance, with all arrangements equally probable. (3) Each player, on his admission, demands and is promised the amount which his adherence contributes to the value of coalition (as determined by the function v). The grand coalition then plays the game "efficiently" so as to obtain the amount $v(N)$: exactly enough to meet all the promises."

In a simple game, player i's contribution $v(S)-v(S\backslash\{i\})$ to the value of coalition S is either 0 or 1 for a particular ordering $<<$, taking the value 1 if and only if $i \in S$ is in a *pivot position* in the ordering. This is the case if i's entry turns the coalition of the $s-1$ players, who precede i according to the random ordering, from losing into winning, i.e., if $D(i,<<) \notin \mathcal{W}$, but $D(i,<<) \cup \{i\} \in \mathcal{W}$. Let $\phi(v) := \Phi(v)$ if $v \in \mathcal{G}^N$. Then, from (1.8) we obtain

$$\phi_i(v) = \frac{1}{n!} \sum_{S:\ i \text{ is critical in } S} (s-1)!(n-s)!\,, \tag{1.9}$$

where the phrase 'i is critical in S' means that summation takes place over all coalitions S that have the property $S \in \mathcal{W}$, $i \in S$ and $S\backslash\{i\} \notin \mathcal{W}$. The Shapley–Shubik index (1.9) can then be interpreted as the proportion of orderings for which player i is pivotal among all $n!$ logically possible orderings. Shapley and Shubik (1954, p. 788) suggest to regard a permutation as a 'voting order' indicating the players' relative degrees of support for the bill, with those most strongly in favor of it voting first, etc.

The definition of the (non-normalized) Penrose–Banzhaf index for player i in $v \in \mathcal{G}^N$ is

$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{S:\ i \text{ is critical in } S} 1. \tag{1.10}$$

The Penrose–Banzhaf index counts the number of swings player i has, and this number is put in relation to 2^{n-1}, the number of swings that i could maximally have (because it is the number of coalitions containing i in a n-player game). The game $[3;2,1,1]$ in the introductory example yields the Shapley–Shubik index $(\frac{2}{3},\frac{1}{6},\frac{1}{6})$, and the Penrose–Banzhaf index is $(\frac{3}{4},\frac{1}{4},\frac{1}{4})$.[27]

Other power indices are the Deegan–Packel index (Deegan and Packel, 1979), and the Public Good Index (Holler, 1982b; Holler and Packel, 1983); for an overview see Straffin (1994). They differ from the indices introduced above by their focus on minimum winning coalitions. The Deegan–Packel index assumes that coalition members divide the coalition payoff equally, while the Public Good Index

[27] The Penrose–Banzhaf index, or measure, only sums to unity after 'normalizing' the number of swings for each player i with the total number of swings. The *normalized Penrose–Banzhaf index* here is $(\frac{3}{5},\frac{1}{5},\frac{1}{5})$.

treats $v(S)$ as a public good which accrues to each member of the minimum winning coalition.[28] The nucleolus, introduced by Schmeidler (1969), and other cooperative solution concepts that have their origin in the allocation of costs or benefits, have also been proposed as power measures; see, for example, Montero (2005).

The affinity of simple games and, in particular, of power indices, to certain phenomena in non-animate nature has been noticed early (e.g., Shapley, 1962). It provides a perhaps rather rhetorical argument against the use of these indices in the analysis of political or economic institutions (see, for example, Garrett and Tsebelis, 1996, p. 278). The Penrose–Banzhaf index, for example, corresponds in electrical engineering to the *Birnbaum index* (Birnbaum, 1969) which is a measure of structural reliability of components in, for example, an electronic circuit (for more details about the link between such reliability systems and power indices see Freixas and Puente, 2002). In the same context Barlow and Proschan (1975) proposed an 'importance measure' that coincides with the Shapley–Shubik index. The labels 'winning' and 'losing' in a simple game can be interchanged with 'on' and 'off' in switching theory. Other connections exist between weighted majority games and Boolean algebra as well as with the threshold model in neuroscience, where stimuli are accumulated until a given neuron fires in response.

The axiomatic approach explicitly lists a set of properties, usually referred to as *axioms*, that an index is supposed to have, and then proceeds to demonstrate that some function uniquely satisfies them. The Shapley value is characterized by four properties, namely symmetry, linearity, efficiency, and treatment of dummy players (Shapley, 1953). Axiomatizations for the Shapley–Shubik index and the Penrose–Banzhaf index have been proposed by Dubey (1975) and Dubey and Shapley (1979), respectively. Yet, knowledge of the properties that an index has does not help to answer the question what quality it measures. One possibility to interpret the Shapley–Shubik and Penrose–Banzhaf indices is offered by the probabilistic approach under which a player's voting power is regarded as his probability of being crucial in passing a decision.

The probabilistic interpretation of power measures is made operational by the *multilinear extension (MLE)* of the characteristic function of coalitional games. Suppose that each player i independently accepts to cooperate in coalition S (to vote 'yes') with probability p_i, and let $p = (p_1, \ldots, p_n) \in [0, 1]^n$ denote the vector of the individual *acceptance rates*. Then, the 'vote configuration' S forms with probability

$$\prod_{i \in S} p_i \prod_{j \notin S} (1 - p_j),$$

[28] Unlike the Shapley–Shubik and the Penrose–Banzhaf indices, the Deegan–Packel and the Public Good index both violate *local monotonicity*, i.e., for two players i and j with $w_i < w_j$, these indices do not always assign a higher value to player j. In the literature, local monotonicity has been discussed, along with other properties, as a major desideratum for a measure of voting power, and as a criterion for selecting among power indices (see Felsenthal and Machover, 1998, pp. 221ff, and Holler and Napel 2004a, 2004b).

and in this event, yields the worth $v(S)$. Following Owen (1972, 1988), the MLE $f : [0,1]^n \to [0,1]$ of $v \in \mathcal{G}^N$ is defined as

$$f(p_1, \ldots, p_n) := \sum_{S \subseteq N} \prod_{i \in S} p_i \prod_{j \notin S} (1 - p_j)\, v(S) \qquad (1.11)$$
$$= \sum_{S \in \mathcal{W}} \prod_{i \in S} p_i \prod_{j \notin S} (1 - p_j),$$

where $\mathcal{W}$ is the set of winning coalitions in v. From (1.11), the MLE can be seen to be the mathematical expectation of a random variable v, given acceptance rates $(p_1, \ldots, p_n)$. Moreover, as $v(S) = 1$ if $S \in \mathcal{W}(v)$ and $v(S) = 0$ if $S \notin \mathcal{W}(v)$, it gives the probability of formation of a winning coalition in v.

The domain of the MLE is the unit hypercube. Any point $p \in [0,1]^n$ in the cube represents a random coalition, and its 2^n vertices correspond in a natural way to the deterministic coalitions $S \subseteq N$.[29] For example, in Fig. 1.2, illustrating the MLE for a three-person game, the point $(0, 1, 1)$ relates to the vote configuration "Player 1 votes 'no', and players 2 and 3 vote 'yes' with probability 1". In particular, f satisfies $f(p_1, \ldots, p_n) = v(\{i \in N \mid p_i = 1\})$, i.e., it coincides with $v(S)$ on

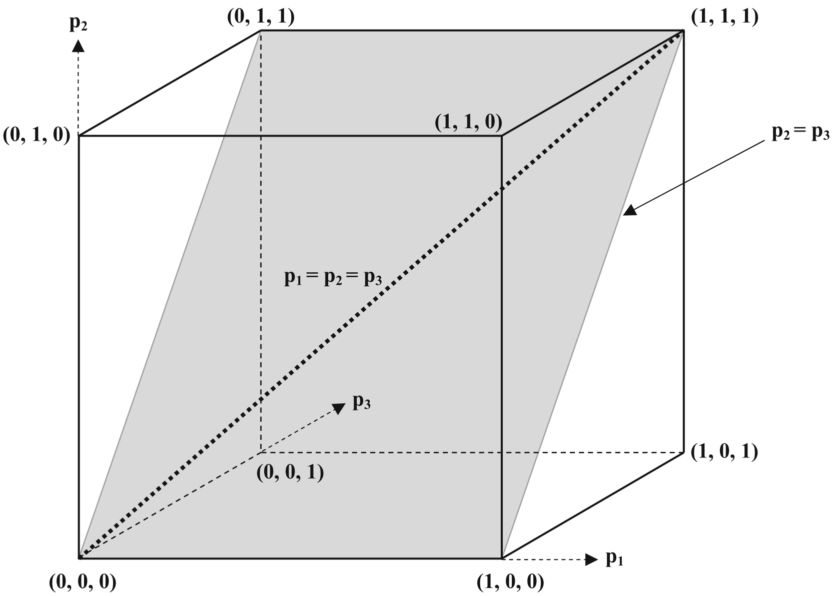

Fig. 1.2 The multilinear extension for a three-person game

[29] The MLE could also be defined for arbitrary real values of p_i. The restriction to the unit cube is due to the interpretation of the p_i as probabilities which are naturally constrained by $0 \leq p_i \leq 1$.

the vertices, and is thus an 'extension' of v. Owen (1972) proves that f is the only multilinear (linear in each p_i) function with this property.

How is (1.11) connected to the power indices (1.9) and (1.10)? For any $i \in N$, the first derivative of $f(p_1, \ldots, p_n)$ with respect to p_i is

$$\frac{\partial f(p_1, \ldots, p_n)}{\partial p_i} = \sum_{\substack{S \subseteq N \\ i \notin S}} \prod_{j \in S} p_j \prod_{\substack{j \notin S \\ j \neq i}} (1 - p_j) \; [v(S \cup \{i\}) - v(S)], \tag{1.12}$$

which is also called the *power polynomial* of player i (Straffin, 1977, 1988). It can be interpreted as a weighted average of player i's (marginal) *contributions* $[v(S \cup \{i\}) - v(S)]$ to all coalitions S that do not include i yet. At the same time, (1.12) is the probability that player i swings in the randomly formed coalition S.

Now consider a partition π of the player set N into m disjoint subsets N_j, $j = 1, \ldots, m$. Let the acceptance rate $p_{i \in N_j}$, common to all voters i in N_j, be a random variable with probability distribution ξ_j on [0, 1]. Then, the probability of voter i having a swing is given by the expected value of (1.12),

$$\Pr(\text{'voter } i \text{ is decisive'} | \pi)$$
$$= \int_0^1 \cdots \int_0^1 \sum_{\substack{S \subseteq N \\ i \notin S}} \prod_{j \in S} p_j \prod_{\substack{j \notin S \\ j \neq i}} (1 - p_j) \; [v(S \cup \{i\}) - v(S)] \, d\xi_1 \ldots d\xi_m. \tag{1.13}$$

In principle, it is possible to use the probability distributions ξ_j to model relationships between voters' preferences, e.g., ideological opposition between two groups A and B by stipulating $p_B = 1 - p_A$, or other determinants of voting behavior, as long as they can be formulated in probabilistic terms.

Two assumptions, introduced by Straffin (1977), concerning the partition and the distributions ξ_j of acceptance rates are particularly prominent: the independence and the homogeneity assumption. Under 'independence' we have $m = n$, i.e., the partition of N is $\pi = \{\{1\}, \{2\}, \ldots \{n\}\}$, and $\xi_j = \mathbf{U}_{[0,1]}$: each individual's acceptance rate p_i is selected independently from the uniform distribution on [0, 1] (or, for that matter, any other distribution on the unit interval with mean $1/2$). The homogeneity assumption, by contrast, requires that some p is realized from the uniform distribution on [0, 1], and then $p_{i \in N_j} = p$ for all j. This corresponds to the case $m = 1$ and, again, $\xi_j = \mathbf{U}_{[0,1]}$.[30] One could think of voters as being homogeneous in the sense that they apply a uniform standard or set of values, reflected by p, to evaluate the acceptability of a proposal, but p varies from issue to issue.

Owen (1972) shows that the Shapley–Shubik index of player i is obtained by integrating the partial derivative $\partial f / \partial p_i$ along the main diagonal of the unit

[30] The term 'independence assumption' may be misleading as voters' decisions to vote 'yes' are independent under the homogeneity assumption as well.

hypercube. Since the main diagonal is characterized by $p_1 = \ldots = p_n$ (see Fig. 1.2), it expresses Straffin's (1977) 'homogeneity' assumption. Similarly, under the 'independence' assumption, (1.13) gives rise to the Penrose–Banzhaf index (1.10), as demonstrated by Straffin (1977). This is easiest to see by noting that 'independence' is equivalent to the assumption that each voter $i \in N$ will vote in favor of any proposal independently with probability $1/2$.

When the partition is neither atomic nor equals the society as a whole, that is, $1 < m < n$, the above model describes a situation with *partial homogeneity*: Voters within the same subset N_j behave homogeneously, but the acceptability levels of different subsets are chosen independently from $\mathbf{U}_{[0,1]}$. In Fig. 1.2, the light gray plane represents the partial homogeneity structure $\pi = \{\{1\}, \{23\}\}$, i.e., players 2 and 3 always vote 'yes' with the same probability $p_2 = p_3$, while player 1's acceptance rate is selected independently. For an application of the partial homogeneity approach to the EU, see Kirman and Widgrén (1995).

Besides permitting (to some extent) preferences to be incorporated into the power index framework, the virtue of the probabilistic approach is that it gives a clear meaning to the term 'voting power': the a priori power of a voter under some voting rule is his probability of casting a decisive vote, given some assumption about voters' behavior.[31] In this view the Penrose–Banzhaf index appears to be based on the assumption – equivalent to the 'independence assumption' – that each voter, independently from the others, randomly votes 'yes' and 'no' exactly with (expected) probability $1/2$.[32] This is obviously tantamount to saying that each coalition forms with equal probability (see also Leech, 1990). Thus reframed, the 'coin toss assumption' looks more palatable, and in the discussion on the 'right' index, the Penrose–Banzhaf index has repeatedly been endorsed on the grounds that it reflects best the basic normative principle that any information beyond the voting rule itself should be ignored. Felsenthal and Machover (1998, p. 38) point to the 'Principle of Insufficient Reason' to justify the independence assumption.[33] While this may be a valid argument in favor of the Penrose–Banzhaf index, it seems that the same principle may be invoked to justify the Shapley–Shubik index: The latter considers all 'attitude orders' equally probable – this is also expressed in the MLE approach where the Shapley–Shubik index emerges as an average over all possible values of the acceptance rate p.[34] Second, even a priori we have good reason to think that

[31] Only in dictatorial games, probabilities can be assessed from the voting rule alone, without a behavioral assumption.

[32] The probability of casting a decisive vote is greatest at $p = 1/2$ and declines rapidly already for small deviations from $p = 1/2$ (see Chamberlain and Rothschild, 1981). This observation translates into a substantial bias of the Penrose–Banzhaf index when applied to voting bodies that do not perfectly satisfy the 'independence assumption' (see Kaniovski, 2008).

[33] The usage the 'Principle of Insufficient Reason' (or 'Principle of Indifference' in Keynes' terminology) is discussed critically and in detail by Keynes (1921, Chaps. IV and VI) who also states conditions under which it is safely applicable.

[34] Leech (1990) favors the Penrose–Banzhaf measure over Shapley–Shubik index arguing that the distributional assumption underpinning the latter is "unduly strong".

the independence assumption is most unlikely to be satisfied: A decision-making body consisting only of indifferent policy-makers seems quite peculiar, a feeling that Barry (1980a, p. 184) put into the catchy comparison that "A committee made up entirely of people who had no interest in pursuing some particular outcome but were fascinated by the process as such would be as frustrating as a brothel all of whose customers were voyeurs".

Before we conclude this section, let us get back to the definition of power as ability to determine the outcome of decision-making, and ask in what sense we can say that power indices measure power in this sense. Consider again the introductory Example 1.2.1 and Table 1.4. Suppose that players have strict preferences over X, i.e., for every $i \in N$ it either holds that $Q \succ_i \chi$ or $\chi \succ_i Q$. Obviously, voting against the preferred alternative is a weakly dominated strategy for any player, and it seems reasonable to expect that players will not choose weakly dominated strategies. In the equilibrium of this 'game', the set of players is partitioned into the set of 'yes'-voters, $\hat{S} = \{i \in N \mid \sigma_i = 1\}$, and its complement $N \backslash \hat{S}$ such that $\hat{S}$ consists exactly of those players who prefer χ to Q. If $\hat{S}$ is a winning (losing) coalition, the outcome is χ (Q), and the members of $\hat{S}$ ($N \backslash \hat{S}$) can be said to be *successful*.[35]

If we take the equilibrium outcome as reference point, a player with strict preferences could at most make a difference to the outcome by choosing a weakly dominated strategy. This sort of behavior amounts to a rather freakish enjoyment of power as an end in itself, and although it certainly occurs, it does not pertain to the 'normal' decision-maker. It would follow that only players who are indifferent between the two alternatives could be credited with power as they can (at times) change the outcome without betraying their actual preferences. The reference point chosen in the voting power literature to assess player i's power is, however, not the equilibrium of the n-player game, but the shadow equilibrium outcome of the game with player set $N \backslash \{i\}$:[36] Following Laruelle and Valenciano (2005), a voter is *decisive* (ex post) if he is successful and his vote is critical to the outcome, i.e., iff

$$\left(i \in \hat{S} \in \mathcal{W} \text{ and } \hat{S} \backslash \{i\} \notin \mathcal{W}\right) \quad \text{or} \quad \left(i \notin \hat{S} \notin \mathcal{W} \text{ and } \hat{S} \cup \{i\} \in \mathcal{W}\right).$$

Power indices make power ascriptions base on decisiveness, i.e., a player's ability to pass a decision that would not have passed had he voted 'no' instead of 'yes'. Call this concept 'decisiveness-power'. Yet, only in the extreme cases of a dictator and a dummy decisiveness translates directly into (zero) a priori power. In intermediate cases, however, player i's being decisive obviously depends on other players' behavior. It requires a particular type of luck for a no-dummy, no-dictator player i to be

[35] The concept of 'success' was introduced in Penrose (1946) and Rae (1969), and elaborated further by Barry (1980a, 1980b). For an analysis of decisiveness and success also see Laruelle and Valenciano (2005).

[36] To see that is a reasonable choice consider a game with a dictator (who faces a dichotomous choice): The equilibrium outcome is always the preferred alternative of the dictator, and he could only make a difference to that by choosing his dominated strategy. His power, however, is that he can impose his will on (the whole of) the *other* players.

decisive, to wit, that players in $N \setminus \{i\}$ produce a configuration that puts him in such a position. If, for example, player 1 endorses the proposal, while player 2 is against it, then player 3's vote will tip the balance in favor of either Q or χ (cf. Table 1.4). The probabilistic approach (indirectly) specifies assumptions about the frequency of such 'lucky' events so as to make an appraisal of i's a priori decisiveness possible. Nevertheless 'decisiveness-power' complies with the intuition that 'power' should include the ability to overcome some resistance, albeit only in a formalistic sense. A winning coalition can make a decision without the votes and irrespective of the actions or desires of the losing players. Player 3, in the configuration just described, can enforce χ or Q 'against the resistance' of the losing coalition (player 2 or 1, respectively) where 'resistance' simply consists in voting on the other side.

It should be stressed that the tight – in fact, proportional – relationship between decisiveness and 'power' is only justifiable for committees that choose from an exogenously given binary agenda. The characteristic function is defined on the collection of coalitions, and only in TU simple games – modeling 0-1 decisions – do 'making a difference to a coalition' and 'making a difference to the outcome' fully coincide. By contrast, if players can make proposals themselves or could amend the exogenous proposal, it will generally be necessary to negotiate a compromise among the desires of prospective members of the winning coalition, and then the fact that a player is usually not unique in being decisive to a coalition would become relevant. Suppose, for example, that players 1 and 2 agree in principle to form the minimum-winning coalition $\{12\}$ and make some still-to-be-specified amendment to the proposal χ. Since both players are decisive here, neither should be able to claim unilateral control over the outcome.

To conclude, the Shapley–Shubik and the Penrose–Banzhaf index provide statistical measures of the a priori voting power relations *within* a take-it-or-leave-it committee. That they do so in a rather crude manner may be seen from the fact that the Shapley–Shubik and the normalized Penrose–Banzhaf index yield $(\frac{1}{n}, \ldots, \frac{1}{n})$ [n elements] for both the n-person unanimity game and the ordinary majority game M_n. Here, the power indices only reflect the symmetry inherent in both decision-making situations, but fail to report their fundamental difference: as any summary, power indices must contain poorer information than the original game.

There is an on-going debate about the adequacy of power indices as measures of (voting) power in general and with respect to their application to EU institutions. One criticism, advanced most prominently by Garrett and Tsebelis (1996, 1999), concerns the fact that power indices disregard players' preferences (for replies see, for example, Holler and Widgrén, 1999; Steunenberg et al. 1999; Napel and Widgrén, 2004). Whilst this point is rather easily countered by pointing to the intended use of the indices for a priori analysis, a second criticism by Garrett and Tsebelis, namely that power indices do not take into account relevant features of the decision-making situation, is far more applicable. Most obviously neglected is the existence of an external agenda-setter, who can, especially in take-it-or-leave-it committees, exercise a great deal of control over the policy outcome *with regard to its content* as distinguished from 'decisiveness-power'.

1.2.2 Bargaining Power

In a simple game with n players and outcome set X where each outcome $x \in X$ is associated with a utility vector $u \in U \subseteq \mathbb{R}^n$ [cf. (1.4)], the players confront the problem of jointly choosing a single feasible utility vector. In the case that this choice has to be made by unanimous consent, the situation corresponds to a classical n-person bargaining problem as studied by Nash (1950) for bilateral bargaining. Its outcome, or solution, is fundamentally indeterminate in the absence of any assumptions about the nature of a 'reasonable agreement' or explicit bargaining rules.

The generalized situation where n players bargain over alternatives $x \in X$ 'in the shadow' of an arbitrary voting rule (with unanimity rule as a special case) is investigated in recent work by Laruelle and Valenciano (2007, 2008b). While Laruelle and Valenciano (2007) obtain a solution to the above problem in the tradition of cooperative game theory, Laruelle and Valenciano (2008b) provide a noncooperative foundation of that solution.[37] Their conceptually very elegant results offer a perspective on power indices which is quite different from that of Sect. 1.2.1.

A *n-player bargaining committee* consists of a classical *bargaining problem* $B = (U, d)$, where $U \subseteq \mathbb{R}^n$ is the *set of feasible payoffs* and $d \in U$ is the *disagreement payoff* or *status quo payoff*, together with a voting rule $\mathcal{W}$.[38] The result of the bargaining process is implemented iff it has the support of a winning coalition as specified by $\mathcal{W}$. U is assumed to be closed, convex and comprehensive. Let $\mathcal{B}$ and $\mathfrak{W}$ denote the set of bargaining problems and the set of voting rules, respectively. In line with the intention to extend the Nash bargaining problem where the bargaining settlement can 'by definition' leave no player worse off than disagreement, it is assumed that no player can be forced upon any alternative that he considers worse than the status quo. Then, any bargaining committee $(B, \mathcal{W})$ corresponds to a NTU simple game $(N, V_{(B,\mathcal{W})})$ defined by

$$V_{(B,\mathcal{W})}(S) = \begin{cases} \{u_S \mid u \in U \text{ and } u_{N\setminus S} = d_{N\setminus S}\} & \text{if } S \text{ is winning,} \\ compr(d_S) & \text{if } S \text{ is losing.} \end{cases} \qquad (1.14)$$

Note that, in contrast to the NTU simple game (1.4), the utility vectors available to coalition S according to (1.14) give to non-members of S their status quo payoffs. Hence, this 'translation' of the bargaining problem eliminates the possibility of free-riding that became apparent in Example 1.1.1 (p. 11): All players $i \in N$ for whom a certain payoff vector is beneficial in the sense that it gives them more than d_i are

[37] Laruelle and Valenciano (2008b) can be considered as a contribution to the *Nash programme* of establishing connections between cooperative solution concepts and non-cooperative game theory. However, the fact that some bargaining protocol implements a given solution does not mean that all relevant protocols do.

[38] As with the NTU game (1.4), it is also assumed here that players have Von Neumann–Morgenstern preferences over X. Thus the bargaining situation can be summarized by (U, d).

generally willing to vote 'yes' on it, that is, to form a coalition S. Similarly, u_j, $j \in N\backslash S$ cannot be arbitrarily small as is the case in the general NTU game.

Using an axiomatic approach, Laruelle and Valenciano (2007) study the question which agreements can reasonably be expected to result in this setting. Following Nash's 1950 pioneering work, an answer is provided by a *bargaining solution*, that is, a function $F : \mathcal{B} \times \mathfrak{W} \to \mathbb{R}^n$ which assigns to every bargaining committee problem $(B, \mathcal{W})$ a single element of $\mathbb{R}^n$, indicating the distribution of payoffs that can be earned by the individual players. As it presupposes unanimous consent, the classical bargaining problem corresponds to the case $\mathcal{W} = \{N\}$.

It is desirable that the solution F selects a payoff vector which is *feasible*, i.e., $F(B, \mathcal{W}) \in U$, and *individually rational*, which requires $F(B, \mathcal{W}) \geq d$. Four more axioms are used to characterize the solution of $(B, \mathcal{W})$:

The first axiom ensures that the names of the players do not matter in determining the solution.

(ANO) Anonymity: For all $(B, \mathcal{W}) \in \mathcal{B} \times \mathfrak{W}$ and any permutation ς of N: $F_{\varsigma(i)}(\varsigma(B, \mathcal{W})) = F_i(B, \mathcal{W})$, where $\varsigma(B, \mathcal{W}) = (\varsigma B, \varsigma \mathcal{W})$ and $\varsigma B = (\varsigma(U), \varsigma(d))$ is defined by $\varsigma(u)_{\varsigma(i)} = u_i$.

The second axiom requires that, if $F(B, \mathcal{W})$ is the solution to the bargaining committee problem $(B, \mathcal{W})$, and if $F(B, \mathcal{W})$ is still feasible for the problem $(B', \mathcal{W})$ with a diminished set of payoff possibilities $U' \subseteq U$, then $F(B, \mathcal{W})$ should still be the agreement in $(B', \mathcal{W})$.

(IIA) Independence of irrelevant alternatives: Let $B, B' \in \mathcal{B}$, with $B = (U, d)$ and $B' = (U', d')$ such that $d = d'$ and $U' \subseteq U$. Then, $F(B, \mathcal{W}) \in U' \Rightarrow F(B', \mathcal{W}) = F(B, \mathcal{W})$ for any $\mathcal{W} \in \mathfrak{W}$.

The third axiom states that two bargaining committee problems $(B, \mathcal{W})$, $(B', \mathcal{W}) \in \mathcal{B} \times \mathfrak{W}$ should be assigned the same solution if B and B' only differ in the units of the scale on which utility is measured.

(IAT) Invariance w.r.t. positive affine transformation: For any positive affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_i(u_i) = a \cdot u_i + b$ with $a, b \in \mathbb{R}$ and $a > 0$, and for all $(B, \mathcal{W}) \in \mathcal{B} \times \mathfrak{W}$:

$$F\left((T(U), T(d)), \mathcal{W}\right) = T\left(F((U, d), \mathcal{W})\right).$$

The fourth axiom requires (a) that status quo payoffs are assigned to null or dummy players (see definition (1.2)) under the voting rule in use ($\Rightarrow$),[39] and (b) that non-null players can expect to receive strictly more than their status quo payoffs ($\Leftarrow$).

[39] As is common practice in the literature, the terms 'null player' and 'dummy player' are used interchangeably here and refer to players whose marginal contribution is null with respect to every coalition. More accurately, a dummy player i is a null player if $v(\{i\}) = 0$ (see Roth, 1988, p. 23).

Part (ii) makes this axiom more demanding than the null player axiom commonly used in the axiomatization of the Shapley (1953) value.[40]

(NP*) Null player*: For all $(B, \mathcal{W}) \in \mathcal{B} \times \mathfrak{W}$: $i \in N$ is a null player in $\mathcal{W} \Leftrightarrow F_i(B, \mathcal{W}) = d_i$.

It turns out that these properties uniquely characterize a solution $F(B, \mathcal{W})$ to the bargaining committee problem (Laruelle and Valenciano, 2007, Theorem 1): $F(B, \mathcal{W})$ satisfies (ANO), (IIA), (IAT), and (NP*) iff

$$F(B, \mathcal{W}) = \text{Nash}^{\varphi(\mathcal{W})}(B). \tag{1.15}$$

for some function $\varphi : \mathcal{W} \to \mathbb{R}^n$ of the voting rule that satisfies (ANO) and (NP*).[41]

In (1.15), $\text{Nash}^{\varphi(\mathcal{W})}(B)$ is the *asymmetric Nash bargaining solution*, introduced by Kalai (1977), with weights determined by φ. Given a vector $\varphi = (\varphi_1, \ldots, \varphi_n)$ of positive weights,[42] the asymmetric Nash bargaining solution for an arbitrary bargaining problem $B = (U, d)$ is defined as

$$\text{Nash}^{\varphi}(B) = \arg \max_{\substack{u \in U, \\ u \geq d}} \prod_{i=1}^{n} (u_i - d_i)^{\varphi_i}. \tag{1.16}$$

It is worth noting that (IIA), (IAT), and (NP*) together implicate efficiency of the solution (for a proof see Roth, 1977), that is, no feasible vector $u \in U$ exists which would make all players better off than F.

Let $B_{\text{TU}} = (U_{\text{TU}}, 0)$ be a normalized TU bargaining problem where $U_{\text{TU}} := \{u \in \mathbb{R}^n_+ \mid \sum_{i \in N} u_i \leq 1\}$ is the TU counterpart of (1.14). Because $\text{Nash}_i^{\varphi}(B_{\text{TU}}) = \varphi_i / \sum_{i \in N} \varphi_i$,[43] and multiplication of the weights with a positive constant does not affect the maximizer of the Nash product, $\varphi(\mathcal{W})$ in (1.15) can, for any $\mathcal{W} \in \mathfrak{W}$, be replaced by $\text{Nash}^{\varphi(\mathcal{W})}(B_{\text{TU}}) = F(B_{\text{TU}}, \mathcal{W})$. Then, the solution (1.15) is fully determined by choosing some specific anonymous function on the domain of simple TU games which satisfies the null player property. Among the many possible choices for $F(B_{\text{TU}}, \cdot) : \mathcal{W} \to \mathbb{R}^n$, e.g., the normalized Penrose–Banzhaf index or the Public Good Index, Laruelle and Valenciano (2007) single out the Shapley–Shubik index

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[40] Shapley's 1953 original axiomatic characterization used another axiom, the carrier axiom, which bundles the efficiency axiom and the null player axiom into one.

[41] Solution (1.15) can alternatively be derived from (ANO), (IIA), (IAT), the $\Rightarrow$-part of (NP*), and the following efficiency axiom: (EFF) For all $(B, \mathcal{W}) \in \mathcal{B} \times \mathfrak{W}$, there is no $u \in U$ such that $u > F(B, \mathcal{W})$. Then, φ must be an efficient anonymous function satisfying the $\Rightarrow$-part of (NP*).

[42] Laruelle and Valenciano (2007) state their result also for the case that some weights are zero. Here, we simplify to the case of positive weights.

[43] $\text{Nash}_i^{\varphi}(B_{\text{TU}})$ is defined as the solution to the maximization problem $\max u_1^{\varphi_1} \cdot \cdots \cdot u_n^{\varphi_n}$ s.t., $u_1 + \ldots + u_n = 1$. The first order conditions for the u_i $(i = 2, \ldots, n)$ can be summarized to $u_i = u_1 \varphi_i / \varphi_1$. Then, the constraint amounts to $u_1 \sum_{i=1}^{n} \varphi_i = \varphi_1$. It follows that $u^* = (u_1, \ldots, u_n)$ with $u_i = \varphi_i / \sum_{i=1}^{n} \varphi_i$ for all $i \in N$ is the solution to the above maximization problem.

(see (1.9)) by adding the following transfer axiom (see Dubey, 1975; Weber, 1988, p. 109) to (ANO), (IIA), (IAT), and (NP*):

(T) Transfer: For all $\mathcal{W}, \mathcal{W}' \in \mathfrak{W}$:

$$F_i(B_{\text{TU}}, \mathcal{W}) + F_i(B_{\text{TU}}, \mathcal{W}') = F_i(B_{\text{TU}}, \mathcal{W} \cup \mathcal{W}') + F_i(B_{\text{TU}}, \mathcal{W} \cap \mathcal{W}') \quad \forall\, i \in N. \tag{1.17}$$

This axiom embodies a linear notion of power: it requires that if $\mathcal{W}$ and $\mathcal{W}'$ are the sets of winning coalitions in any two voting games with player set N, then the sum of any voter i's powers in $\mathcal{W}$ and $\mathcal{W}'$ should be equal to the sum of i's powers in the two games whose sets of winning coalitions are $\mathcal{W} \cup \mathcal{W}'$ and $\mathcal{W} \cap \mathcal{W}'$, respectively. It is well known that (T) is the driving force in the axiomatic characterization of the Shapley–Shubik index (Dubey, 1975).[44] In a few words, every simple game can be written as the composition of unanimity games on its minimum winning coalitions $S_k \in \mathcal{W}^{\min}$, and thus (T) allows to derive the value of player i in a simple game, composed of a number of auxiliary unanimity games, from i's values in the auxiliary unanimity games. Due to the efficiency and symmetry of the Shapley–Shubik index, player i's value in the unanimity game of coalition S_k is $1/|S_k|$ if $i \in S_k$ and 0 otherwise.

Laruelle and Valenciano (2007, Theorem 2) establish that the solution to the bargaining committee problem characterized by (ANO), (IIA), (IAT), (NP*), and (T) is unique and given by

$$F(B, \mathcal{W}) = \text{Nash}^{\phi(\mathcal{W})}(B) \tag{1.18}$$

where ϕ is the Shapley–Shubik index. Note that for the special case $B = B_{\text{TU}}$, the solution proposed in (1.18) is $\text{Nash}^{\phi(\mathcal{W})}(B_{\text{TU}}) = \phi(\mathcal{W})$ (cf. Footnote 43). It is well known that, for TU-games, the n-player Nash solution coincides with the Shapley value when players' (exogenous) status quo payoffs d_i are identified with their security levels ($v(\{i\})$), and agreement is general.

In (1.18), the weight vector in the asymmetric Nash solution is endogenous as it is the Shapley–Shubik index induced by the voting rule. In line with Binmore's (1998, p. 78) interpretation of the weights φ_i in (1.16), the solution (1.18) thus permits construing $\phi(\mathcal{W})$ as the distribution of players' bargaining powers reflecting the strategic advantages bestowed on them by the voting rule.

However, *any* function of $\mathcal{W}$ that is efficient, anonymous and assigns zero to null players is according to (1.15) an equally well-qualified candidate to represent bargaining power. Given $\mathcal{W}$, let $J := \{j \in N \mid j \text{ is a null player in } (N, \mathcal{W})\}$. Whether $\varphi_i(\mathcal{W}) = \phi_i(\mathcal{W})$ is a better representation of bargaining power than, say,

[44] Laruelle and Valenciano (2001) introduce an alternative version of the transfer axiom which states that the effect on any player's power of eliminating a minimal winning coalition from $\mathcal{W}$ is the same in any game in which this coalition is minimal winning.

$$\varphi_i(\mathcal{W}) = \begin{cases} \frac{1}{|N \backslash J|} & \text{if } i \text{ is not a null player in } (N, \mathcal{W}), \\ 0 & \text{otherwise,} \end{cases}$$

depends entirely on the plausibility of the transfer axiom. As pointed out by a number of authors, e.g., Straffin (1994) and Felsenthal and Machover (1998, pp. 193ff), (1.17) lacks an intuitively compelling interpretation in the context of collective decision-making.

Still, (1.18) opens up a much richer setting than the take-it-or-leave-it committee discussed in Sect. 1.2.1 for applying the Shapley–Shubik index as a power measure. The more important criticism concerns the adequacy of the bargaining committee as model of political decision-making. In Nash's bargaining theory, due to its original concern with bilateral exchange, agreements can only be reached by unanimous consent. While the requirement that no player should be imposed upon an outcome which falls short of his status quo utility is entirely plausible for market exchange, it is not very appealing when decisions are by vote. The individual rationality axiom limits the bargaining space to mutually beneficial agreements, and thus ensures the general acceptability of the outcome. However, decisions by vote can usually be expected in situations where consent is lacking. The formulation (1.14) gives every player a rather large degree of veto power, namely with respect to outcomes that are exploitative or otherwise injuring to him, which is then reflected in the agreement. When decisions are eventually made by voting, it is – even on normative grounds – difficult to justify that a winning coalition should not be able or willing to bring about an outcome that is beneficial to its members but encroaches on the utilities of non-members. In direct as well as in representative democracy, majority decisions are binding on everybody, and everybody can eventually be forced to comply with them, including those who see their interests violated. Recognition of the fact that, under majority rule, the interests of the losing minority are inherently unprotected (within constitutional bounds) would make it necessary to endogenize disagreement payoffs. It would also invalidate solutions (1.15) and (1.18) which critically depend on the individual rationality condition.

1.2.3 Power Measurement and Spatial Voting

A still different perspective on power indices is afforded by work on power measurement in spatial settings (cf. Sect. 1.1.3). The earliest contribution in this context, suggested by Owen (1971) and extended by Shapley (1977), is the application of power indices to spatial voting games where fixed positions in an m-dimensional *ideological space* influence players' behavior.[45] In Fig. 1.3, the most-preferred

[45] Owen (1971) introduces a $(n-1)$-dimensional space in order to describe ideological affinities between n voters. A legislative proposal is represented as a point which is chosen randomly from a uniform distribution over the space. The ordering of players is then taken in terms of increasing distances of players' ideal points from that point. Shapley' s (1977) modification, which is followed

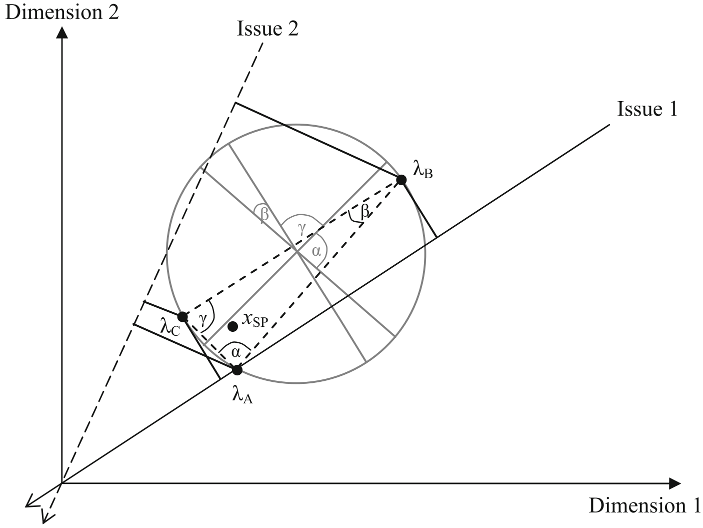

Fig. 1.3 Modified power index in a three-person voting game with spatial preferences

policies of the members of some committee $N = \{A, B, C\}$ are described as points λ_A, λ_B, and λ_C in a two-dimensional space. Each issue determines an ordering of the three players by drawing the perpendiculars from each ideal point: For issue 1 the line-up is given by B, A, C, whereas it is B, C, A for issue 2. Under simple majority voting, these orders imply a spatial swing for player A and C, respectively. Supposing that all issue directions are equally likely, or "that the 'political winds' blow across the ideological space in a perfectly random way" (Shapley, 1977, p. 20), it is obvious that generally not all $n!$ orders are equally probable. In line with common political wisdom, players suffer reduced chances of being pivotal *under simple majority rule* when taking positions at the extremes of the policy space. If the decision rule applied in Fig. 1.3 is $\mathcal{W} = \{AB, AC, BC\}$, the chances of each committee member to be pivotal for a randomly chosen issue vector are proportional to the angles in the triangle. To see this, imagine that the arrow labeled 'Issue 1' is rotated through 360°, and, in any position, is equally likely to be brought to a stop. Then, the *modified power index*, or Shapley–Owen power index, for the game $(N, \mathcal{W}, (\lambda_A, \lambda_B, \lambda_C))$ is given by $\phi^\lambda = (\alpha/\pi,\ \beta/\pi,\ \gamma/\pi)$.[46] In Fig. 1.3, we have $\phi^\lambda = (86/180,\ 17/180,\ 77/180)$. The Shapley–Shubik index (1.9) corresponds to the special case where the players' ideal points $\lambda_1, \ldots, \lambda_n$ are equidistant from each other, i.e., they form the vertices of a regular n-simplex. In Fig. 1.3, this would require the triangle λ_A, λ_B, λ_C to be equilateral.

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in Fig. 1.3, uses an ideological space of arbitrary dimension where dimensions can be interpreted as 'pure' issues. Then, a player ordering is induced by a random vector whose direction indicates the particular ideological 'mixture' of the (exogeneous) proposal faced by the committee.

[46] The calculation of the modified power index for general weighted voting games in two dimensions is explained in Straffin (1994, pp. 1140ff).

As shown by Owen and Shapley (1989), the *strong point* x_{SP}, or Copeland winner, of a (decisive and proper) spatial voting game can be expressed as the weighted average of committee members' ideal points, where the weights are given by their respective Shapley–Owen values. The strong point is the policy position "that defeats or ties the greatest number of alternatives in the space" (Grofman et al., 1987, p. 539). When the core of a decisive spatial voting game is non-empty, the core consists of exactly one element and coincides with the strong point.

Modified power indices provide an analytical tool in situations where it is (empirically) possible to identify decision-makers, e.g., parties in a parliament, in advance with definite positions on the 'political spectrum'. The observation that, once policy positions are taken into account, not all coalitions are equally probable to form, is also at the bottom of the 'lack of preferences' criticism of classical power indices which has been advanced, among others, by Garrett and Tsebelis (1996, 1999). In response to that point, Garrett and Tsebelis (1996, p. 275) propose to build a power measure only on coalitions that are ideologically connected: only coalitions whose members are adjacent to each other in the policy dimension(s) should be expected to arise.[47] Given a realization of ideal points and a *status quo*, denoted by Q, which prevails in case that no decision is passed, player i's 'power' is measured by the number of 'plausible' minimum winning coalitions in which player i participates. Since this is equivalent to the joint assumption that all connected minimum winning coalitions are equally likely, and the coalition value of 1 is a collective good accruing to all members similarly, the measure bears some resemblance with the non-spatial Public Good Index (Holler, 1982b; Holler and Packel, 1983). Blending decision rule-related power with preference-related influence, it has some intuitive appeal if the objective is a *descriptive* assessment or an *ex post* understanding of the voting situation.

If, however, power is to be evaluated *ex ante*, i.e., before a particular preference profile occurs, the assumption of stable ideological affinities is not appropriate. To this purpose, Steunenberg et al. (1999) suggest to consider the *equilibrium outcome* x_g^* of a policy game whose rules are modeled by a game form g (see p. 16). Let $\omega = (\lambda_1, \ldots, \lambda_n;\ Q)$ denote a 'state of the world' consisting of players' ideal points and a status quo, and let Ω be the collection of all states of the world. Then, x_g^* can be thought of as a game-theoretic *solution*, that is, a mapping $x_g^*\colon \Omega \to X$ which assigns to each game defined by g and $\omega \in \Omega$ a policy outcome $x_g^*(\omega)$. The basic idea is that "a player is more powerful than another player if the expected distance between the equilibrium outcome and its ideal point is smaller than the expected distance for the other player" (Steunenberg et al. 1999, p. 348). Given the 'rules of the game' and distributional assumptions about ω, the equilibrium outcome $x_g^*(\omega)$ can be derived as a function of the random variable ω. The expected distance between $x_g^*(\omega)$ and player i's ideal point λ_i is the basic ingredient of a measure, dubbed the *strategic power index* since x_g^* may result from sophisticated strategic interaction.

[47] For one-dimensional policy spaces, (minimum) connected winning coalitions were first suggested by Axelrod (1970) in his "conflict of interest" theory.

In principle, the approach can also address the second main criticism of classical power indices, viz. their neglect of institutional structures. For example, Steunenberg et al. (1999) use their power index to analyze the consultation procedure of EU decision-making whicth includes the EU Commission as a strategic agenda-setter. As pointed out in detail by Napel and Widgrén (2002, pp. 9ff), the drawback of the so-called strategic power index is that it captures *expected success*, but not power, understood as the ability to make a difference to the outcome of the decision process. The reason for this is that the measure exclusively relies on information about x_g^*: coincidence of the latter with player i's preferences implies that i 'gets what he wants'. This success for player i (see p. 23) could come about either because he is decisive, or because he is lucky (see Barry, 1980b).

Instead of comparing $x_g^*(\omega)$ with player i's ideal point, Napel and Widgrén (2002, 2004) base power ascriptions on a comparison between the outcome of the collective decision process modeled by g and a counterfactual or *shadow outcome* which i could bring forth if he liked to. The more sensitive the outcome is to i's behavior, the greater the power attributed to i relative to other players. Given the decision rules g, a solution concept x_g^*, and some particular state of the world $\omega \in \Omega$, sensitivity to a change in player i's preferences is measured *ex post* by $\partial x_g^*(\omega)/\partial\lambda_i$, i.e., by the partial derivative of x_g^* with respect to i's ideal point.[48] It indicates the extent to which the outcome would differ if i's preferences were slightly different, or the *marginal contribution* of player i to the outcome.

Now, power can be measured ex ante (or in the long run) by introducing uncertainty about the state of the world in the form of some probability distribution ξ over profiles $\omega \in \Omega$. The ex ante measure μ^{NW} is then given by the expectation of the above derivative with respect to ξ,

$$\mu_i^{NW} = \int \frac{\partial x_g^*(\omega)}{\partial\lambda_i}\, d\xi\,. \tag{1.19}$$

Just like the MLE expression for voter i's probability of being pivotal (1.13), μ_i^{NW} identifies 'power' with i's expected marginal contribution, but the contribution is to the policy outcome rather than to the 'worth' of coalitions. In line with this analogy, an a priori measure is obtainable by choosing an informationally poor probability measure ξ in (1.19), e.g., a uniform distribution on the state space Ω. For appropriately selected game forms and distributions, the sensitivity approach includes traditional power indices as special cases (see Napel and Widgrén, 2004, pp. 526f). Unlike the latter, (1.19) permits taking into account institutional structures such as the existence of an agenda-setter and the time structure of policy games. In fact, Napel and Widgrén (2006) provide an application to the *co-decision procedure* of

[48] Discrete alternatives to the derivative – corresponding to an infinitesimally small preference change – are discussed in Napel and Widgrén (2004). Sensitivity may also be defined more directly as the reaction of x^* to a change in a given player's actions rather than his preferences. However, as actions are usually thought to originate in preferences, the formulation of the approach in terms of preferences is more elementary.

the EU, modeled as an extensive form game. They find that, although the European Parliament and the Council of Ministers are formally on a par under that procedure, the inter-institutional balance of power is substantially biased in favor of the Council of Ministers as a consequence of the latter's more conservative internal decision rule. In a similar vein, Napel and Widgrén (2008) study the distribution of power between permanent and elected members of the UN Security Council.

As seen in Sect. 1.2.1, traditional power indices capture players' decisiveness with respect to two exogenously given alternatives, e.g., proposal and status quo. The more interesting kind of influence over the outcome, however, seems to rest with the agenda-setter who determines the contents of the proposal.[49] It should be pointed out that agenda-setting is a factor that properly lies outside the voting game. An early model highlighting the power of the agenda-setter (or proposer) vis-à-vis a take-it-or-leave-it committee is provided by Romer and Rosenthal (1978).[50] The agenda-setter is assumed to be fully informed about the policy preferences of committee members, and, rather than making random proposals, he behaves strategically. In the first stage of the game, the agenda-setter makes a proposal $\chi \in X$ to the committee whose members have Euclidean preferences over the set of feasible alternatives X, with an ideal point in the space. Once the proposal is made, the committee votes whether to adopt χ by simple majority rule. In case that χ is not supported by a simple majority, the status quo Q, which is known to all players, prevails. Under the closed rule assumption, the agenda-setter can make a credible ultimatum offer relative to Q. His optimal strategy is to propose the policy he prefers most among those policy alternatives that are preferred to Q by a majority of decision-makers. This set of points is called the *win set* of Q. For the three-member committee illustrated in Fig. 1.4, it corresponds to the shaded area. Suppose that member A is the agenda-setter in this situation. Then, A's equilibrium proposal (and the outcome of the voting game) is his ideal point, i.e., $x_g^*(\omega) = \lambda_A$. Applying the sensitivity approach $\partial x_g^*(\omega)/\partial \lambda_i$ to the committee $N = \{A, B, C\}$, one obtains the ex post power vector $(1, 0, 0)$ for any direction of marginal change in the preferences. Any small change in the agenda-setter's preferences translates into an identical shift of the collective decision, whereas small changes in the ideal points of members B and C have no effect.[51]

To round off the example, consider the case that the committee in Fig. 1.4 operates under an open agenda rule, which permits amendments to proposals. Then, the

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[49] The agenda-setter can also control the outcome under a *sequential* binary agenda, as described by a finite binary voting tree, if he has information about the majority preference relation.

[50] Romer and Rosenthal (1978) analyzed the effect of agenda control for one-dimensional policy choices under closed rule.

[51] In the closed rule game, players B and C could be characterized as *spatially inferior* (compared to A). The concept of *inferior players*, referring to players who are subject to credible ultimatum threats, is introduced by Napel and Widgrén (2001) for TU simple games, and by Napel and Widgrén (2002) and Widgrén ans Napel (2002) for (one-dimensional) spatial voting games. Napel and Widgrén propose power measures, called the *Strict Power Index* and the *Strict Strategic Power Index*, respectively, which regard inferior/spatially inferior players as powerless.

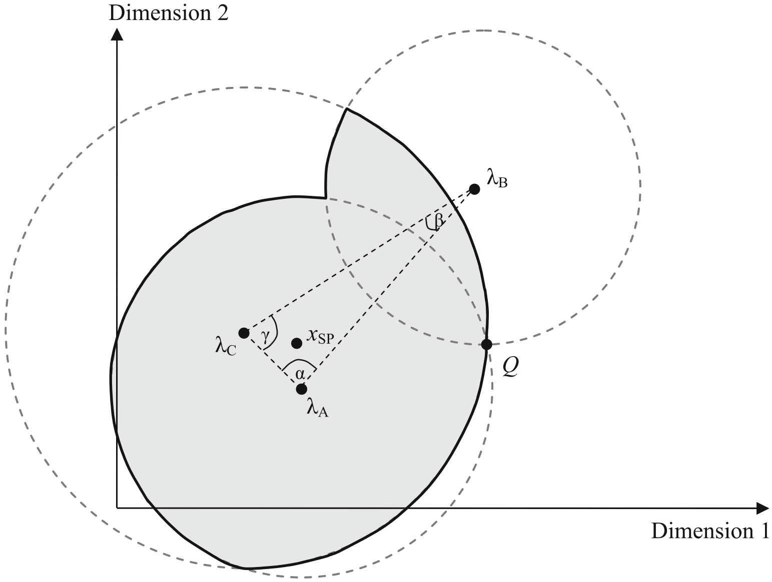

Fig. 1.4 Agenda setting in two-dimensional policy space. The shaded area is the win set of Q

outcome of the decision-making process can be expected to be located at or very close to the strong point (see Grofman et al., 1987). As mentioned earlier, the strong point x_{SP} is a convex combination of players' ideal points, and in particular, it holds that $x_{SP} = \sum_{i=1}^{n} \phi_i^{\lambda} \lambda_i$ (Owen and Shapley, 1989). If $x_g^*(\omega) = x_{SP}$, it follows that player i's ex post power $\partial x_g^*(\omega)/\partial \lambda_i$ is given by the modified power index $\phi^{\lambda} = (\alpha/\pi,\ \beta/\pi,\ \gamma/\pi)$ (see Napel and Widgrén, 2004, p. 532).

The real-world collective decision-making is modeled as a game to which some solution concept needs to be applied in order to derive the collective outcome x_g^* from ω. Yet, while to some models only one 'solution' may lend itself, others allow the application of several solutions or equilibrium concepts each of which explores a different hypothesis about players' rationality and behavior. For example, if the core of a majority voting game in a spatial context is empty, both the center of the *yolk* (McKelvey, 1986), and the strong point (see Grofman et al., 1987) could be considered as reasonable point predictors for the policy outcome.[52] Similarly, many solution concepts do not yield a unique or one-point prediction – even if refinements are used to reduce multiplicity. In a priori analysis, multiple equilibria can be dealt with by making explicit assumptions about their probability (Napel and Widgrén, 2004, p. 523), although this has a slight ad hoc flavor.

Finally, it should be mentioned that the proposals to measure 'power' in a spatial framework have met with scepticism because they, by definition, include the

[52] The yolk (McKelvey, 1986) is the smallest circle that intersects all median lines or median hyperplanes. Its radius can be seen as a measure of how close the game is to having a core outcome.

preferences of players whose power is to be assessed. A number of authors argue that any dependence of power ascriptions on preferences runs counter to the meaning of 'power', which – as the generic ability to affect an outcome – exclusively resides in the game form (cf. p. 16). (See Miller, 1982, Braham and Holler, 2005a, 2005b; for counter-arguments see Napel and Widgrén, 2005).

1.3 Voting Rules and Payoffs

The power distribution in a decision-making body is arguably most crucial in those situations where all individual decision-makers have conflicting interests. Then, the decision that is eventually reached reflects who can enforce his preferences most comprehensively. For example, empirical evidence strongly suggests that budget allocation in the EU is largely driven by the distribution of voting power in the EU Council of Ministers (see Baldwin et al. 2001; Kauppi and Widgrén, 2007 and the references therein). In its purest form, the clash of interests as it occurs in situations of benefit (or cost) allocation (under an assumption of selfish preferences) can be modeled by constant-sum games.

Up to this point, a voter has been considered powerful insofar as he could change the outcome by changing his vote. While it is not generally clear when power can be equated with the benefits or utilities attached to the outcome, the relationship between the *power* of a player and his *value*, i.e., his prospects from playing the game, seems tightest for purely distributive decisions. A second way to establish a link between voting rule and payoffs arises if one looks at the decision-making body through the eyes of a lobbyist. Albeit not identical, the question 'how powerful is committee member i' is intuitively related to the question 'at whom should a lobbyist address his efforts in order to promote a decision he favors'. This section explores some connections between voting rules and payoffs in both cooperative and non-cooperative games.

The canonical problem in distributive politics is 'divide-the-dollar': one dollar – representing, for example, a fixed-size budget – is to be allocated by collective choice among the n members of the committee. Thus, the set of alternatives in 'divide-the-dollar' is the $(n-1)$-dimensional simplex

$$X_{\mathrm{DD}} = \left\{ (x_1, \ldots, x_n) \in [0,1]^n : \sum_{i \in N} x_i = 1 \right\}. \tag{1.20}$$

Example 1.3.1. Suppose that the simple n-player majority game M_n (cf. (1.3)) is used to choose an element of X, and that for all $i \in N$ and $x \in X_{\mathrm{DD}}$, $u_i(x) = x_i$. A winning coalition $S \in 2^N \backslash \emptyset$ gains possession of the prize of victory (e.g., a lump sum budget, or a certain number of cabinet posts), which it completely distributes among its members.

It is well known that the core of this game is empty.[53] This result should also not be surprising in light of the high-dimensionality of the policy space (1.20) (cf. Sect. 1.1.3). Although core emptiness implies that any allocation must leave at least one coalition short of its potential and thus reflects a lot of scope for social conflict, it is not tantamount to instability of outcomes. Rather, an empty core may only indicate that no prediction of outcomes is made.

In Sect. 1.2.1, the Shapley–Shubik index was introduced as the application of the Shapley value to simple games, lending itself to the a priori measurement of power or control in voting situations. Originally, Shapley (1953) suggested to interpret the Shapley value as a measure of players' benefit of playing the game which attempts to relate the size of the share of the pie that accrues to an individual in a cooperative setting to that individual's contribution to the size of the pie. As mentioned earlier, the Shapley value of player i is the average or expected marginal contribution that i makes to the grand coalition when all orders of forming the latter are equally likely. With respect to TU simple games like the one described in Example 1.3.1, it follows that the Shapley–Shubik index of committee member i would correspond to i's *expected payoff* or share in a fixed purse if i received on average his contributed amount.

One immediate way how this can be effected is by the simple procedure Shapley (1953) used to describe the come about of the Shapley value – let players line up in random order and pay to each his marginal worth to the coalition which consists of those who come before him (see p. 18). But giving the pivotal player full credit for having 'baked the pie', or passed the measure, is hardly a conceivable 'real-world' negotiation process. Still, it is possible to sustain the Shapley value by a variety of sophisticated, albeit not necessarily realistic, non-cooperative bargaining games based on the exchange of proposals and counter-proposals (e.g., Gul, 1989; Hart and Mas-Colell, 1996): The vector of players' equilibrium payoffs converges towards the Shapley value in these games.

Alternatively, an outside authority who has the task of allocating collective benefits or costs to individuals might wish to apply the Shapley value, or the Shapley–Shubik index, as an arbitration scheme on the grounds that it would seem fair to give each player his average contribution.[54] If, by contrast, the allocation is decided upon by (weighted) voting, then the outcome *in any play of the game* as it results from the competitive behavior of coalitions does, in general, not correspond to the Shapley–Shubik index which would require the grand coalition to form. In Example 1.3.1, the Shapley–Shubik index is $\phi(v) = (\frac{1}{n}, \ldots, \frac{1}{n})$ [n elements], which is neither an element of the core (being empty) nor of the *Von Neumann–Morgenstern stable set* (see Owen, 1995, pp. 234ff). These latter concepts appear to

[53] In fact, the core is always empty for superadditive constant-sum games; for a proof see Ordeshook (1986, pp. 350f).

[54] For example, Littlechild and Thompson (1977) apply the Shapley value to the problem of allocating airport landing charges to different types of aircraft. Yet, the 'fairness credentials' of the Shapley value hinge on the axioms (see p. 19) which characterize it.

be more qualified than the Shapley value to capture some of the strategic features of the situation.

If majority voting is used as an allocation mechanism, Riker's (1962, pp. 32f and 247ff) *size principle* provides a hypothesis about coalition formation: It states that in n-person constant-sum games with transferable utility only minimum winning coalitions form, that is, coalitions in which every member is critical (cf. p. 7). In a nutshell, the reasoning behind the size principle is that, first, the prize of winning makes it rationally desirable to be a member of a winning coalition, and second, admitting no more members to the coalition than necessary to ensure winning means that the prize can be divided among fewer players. While Riker (1962) justifies the principle on the basis of abstract cases, Schofield (1980) shows that the formation of minimum winning coalitions can be rigorously derived from the additional assumption of $\mathscr{M}_2$-stability (see Owen, 1995, p. 316) for symmetric, constant-sum games with *decreasing returns to scale*.[55] The game in Example 1.3.1 and its five-player-version, Example 1.1.2, fall into this class.

Another link between the voting rule and players' payoffs can be established via vote-selling games. For example, the Penrose–Banzhaf power measure (1.10) allows an interpretation as the amount that a committee member would be paid by a lobbyist buying his or her votes (see Felsenthal and Machover, 1998, p. 45). Consider a lobbyist who will gain one unit of transferable utility if a take-it-or-leave-it committee accepts a certain policy proposal, but who will lose one unit upon rejection of the proposal. How much should the lobbyist be willing to pay for a committee member's vote if, for lack of information on members' attitudes towards the resolution, he deems all coalitions equally likely to form? The answer is, of course, that he ought to be prepared to pay up to the marginal benefit of commanding that vote. Given the voting rule $\mathcal{W}$ of the committee, the a priori chance that the proposal is accepted by the committee without any effort on the part of the lobbyist is $|\mathcal{W}|/2^n$, that is, the number of winning coalitions as a share of all possible bipartitions of the committee. Hence, his expected payoff is given by

$$\mathbf{E}(u) = \frac{|\mathcal{W}|}{2^n}(+1) + \frac{2^n - |\mathcal{W}|}{2^n}(-1) = \frac{|\mathcal{W}|}{2^{n-1}} - 1. \tag{1.21}$$

This is clearly less than the expected payoff in the case that some member i of the committee is *sure* to vote 'yes'. Then, the conditional probability that the proposal gets accepted is $|\mathcal{W}_i|/2^{n-1}$, where $\mathcal{W}_i$ refers to the set of winning coalitions to which i belongs, and 2^{n-1} is the total number of coalitions containing i. This yields the expected payoff

$$\mathbf{E}(u \mid i \text{ votes 'yes'}) = \frac{|\mathcal{W}_i|}{2^{n-1}}(+1) + \frac{2^{n-1} - |\mathcal{W}_i|}{2^{n-1}}(-1) = \frac{2|\mathcal{W}_i|}{2^{n-1}} - 1 \tag{1.22}$$

[55] A game in characteristic function form exhibits decreasing returns to scale (McKelvey and Smith, 1974) if $v(T)/|T| > v(S)/|S|$ for all $T \subset S$, $v(T), v(S) > 0$; for illustration see Example 1.1.2 with Table 1.3.

for the lobbyist. In fact, the difference between the unconditional expected payoff (1.21) and the conditional expected payoff (1.22) is

$$\mathbf{E}\,(u \mid i \text{ votes 'yes'}) - \mathbf{E}(u) = \frac{2|\mathcal{W}_i| - |\mathcal{W}|}{2^{n-1}}$$
$$= \beta_i$$

where the second equality is implied by $2|\mathcal{W}_i| - |\mathcal{W}| = |\{S \subseteq N : i \text{ is critical in } S\}|$, for any $i \in N$, together with definition (1.10).[56] According to this result, the Penrose–Banzhaf measure β_i of committee member i represents the gain which a lobbyist could expect from buying i's vote, and thus, an upper bound to the amount the lobbyist should rationally spent on that vote.

The question 'what amount of transferable utility may players expect from a purely distributional game' can, under certain circumstances, be answered by power indices like the Shapley–Shubik index or the Penrose–Banzhaf measure. Yet, taking a strategic approach to bargaining over the division of one unit of benefits typically leads to conclusions which are not congruent with those of thoroughly cooperative game theory. One important aspect of power, proposal power, is captured by the *legislative ultimatum game*, suggested by Romer and Rosenthal (1978), which has already been considered in the spatial setting (cf. Sect. 1.2.3).

Consider, for example, the interactions between the EU Commission and the Council of Ministers under the majority version of the *consultation procedure*: the Council is presented a proposal by the Commission which it can adopt by qualified majority (or unanimously, depending on the policy area).[57] Treating the Commission as a single actor, and ignoring the possibility that the Commission proposal could be amended by the Council, as well as the fact that interaction is indefinitely repeated, the situation may be illustrated by a simple game with one voting body where all minimum winning coalitions include the Commission and some minimum set of Council members such that the quota is met. In a more non-cooperative vein, if the Commission was in a position to make credible ultimatum offers to Council members, it would obtain the (rational) approval of the latter by making marginal concessions to their interests, assigning (almost) the total prize

[56] Let $\mathcal{W}_{-i}$ denote the set of winning coalitions to which player i does *not* belong. Then, it holds that $|\mathcal{W}| = |\mathcal{W}_i| + |\mathcal{W}_{-i}|$ for any player i. Adding $|\mathcal{W}_i|$ on both sides of this identity and rearranging yields $|\mathcal{W}_i| - |\mathcal{W}_{-i}| = 2|\mathcal{W}_i| - |\mathcal{W}|$. The left hand side indicates the difference between the number of winning coalitions containing i, and the number not containing i, and is easily shown to be equivalent to the number of swings that player i has, i.e., to $|\{S \subseteq N : i \text{ is critical in } S\}|$ (for a proof see Dubey and Shapley, 1979, p. 102).

[57] Under the consultation procedure, the Council can also amend the Commission's proposal by unanimity. Moreover, the European Parliament needs to be consulted, but its opinion is non-binding. Generally, Article 155 grants the Commission the exclusive right to initiate legislation vis-à-vis the Council and the European Parliament, and according to Article 189a (2), it holds the right to modify a proposal at any point of procedure. Nevertheless, how much control the Commission can exercise over the proposal which *ultimately* comes to a vote in the Council probably varies with the EU's different decision procedures (see Garrett and Tsebelis, 1996).

to itself, and leaving (almost) nothing to Council members. Following Napel and Widgrén (2001), all Council members in this example are *inferior players*, i.e., the set of minimum winning coalitions containing any particular Council member is a proper subset of the set of minimum winning coalitions containing the Commission. Arguably, the position of inferior players is somewhat similar to that of dummies (cf. definition (1.2)).[58] Napel and Widgrén (2001) suggest that a player who receives a zero payoff in 'divide-the-dollar' bargaining should be considered powerless. Accordingly, traditional power indices can, for certain applications, be modified by assigning zero power to inferior players. For instance, defining the Penrose–Banzhaf measure (1.10) only with respect to swings by players who are not inferior in the simple voting game gives rise to the *Strict Power Index* (Napel and Widgrén, 2001). There are two points to note. Firstly, while it does not attach any power to players whose payoffs are nil, it is not possible to interpret the *Strict Power Index* in terms of (expected) payoffs. Secondly, the concept of inferior players presupposes that the rules of bargaining are such as to allow one particular player to make credible ultimatum threats to other players.

A prominent approach to distributive decisions in voting committees is the sequential bargaining model by Baron and Ferejohn (1989). In their model, one of the committee members is randomly chosen, in accordance with exogenously given *recognition propobabilities*, to make a proposal – a division of the dollar – which is then put to a vote under 50%-majority rule. All players are assumed to have the same voting weight. Baron and Ferejohn (1989) consider both a closed rule (where the proposer makes a take-it-or-leave-it offer for the current legislative session) and an open rule (where amendments in each session are allowed). Here, we focus on the former. The game ends upon acceptance of the proposal, and the proposal that has the support of a winning coalition materializes. If the proposal $x \in X_{\mathrm{DD}}$ is rejected, the session ends, and the legislature moves to the next session which begins by drawing a proposer again, and so on. From session to session, payoffs are discounted, where a discount factor of 1 indicates maximal patience. A player will vote in favor of a proposal if it gives her at least her continuation value, and against it otherwise. Concentrating on *stationary subgame perfect equilibria* of the game,[59] Baron and Ferejohn (1989) find that, under closed rule, the member recognized first to make a proposal can secure a greater share for herself compared with what she would receive as a mere voter. This proposer advantage stems from the other members' being aware that they might belong to the losing minority in future sessions. It decreases in the discount factor, and increases in the size n of the committee. The predictions from the Baron–Ferejohn model contrast with those that arise from the cooperative analysis of the simple game in Example 1.3.1, the

[58] Indeed, a dummy player is always inferior, and in decisive games (cf. p. 8), the two concepts 'inferior players' and 'dummy' fully coincide (Napel and Widgrén, 2001, p. 213).

[59] A stationary subgame perfect equilibrium is one in which (a) a proposer proposes the same distribution every time he is recognized, and (b) voting members vote only on the basis of the current proposal (and their expectations about future proposals which are time-invariant because of (a)), i.e., players' actions are independent of the history of the game.

main reason being that the non-cooperative model considers both voting power and proposal power.[60]

An extension of the Baron–Ferejohn model of distributive political decisions to weighted majority voting is provided by Snyder et al. (2005). Assuming that no discounting of payoffs occurs, they investigate two different cases concerning the probability of a committee member to be chosen as proposer. First, recognition probabilities may be proportional to voting weights. Second, all members may have the same probability irrespective of their voting weight, which should, for example, at least theoretically apply to the EU Council of Ministers. In the first case, the stationary subgame perfect equilibrium of the game implies that each player's expected payoff is equal to his share of total voting weight. If recognition probabilities are uniform, two types of equilibria emerge: Either expected payoffs are again proportional to committee members' weights, or, for heavily skewed distributions of weights, 'small' players enjoy expected payoffs in excess of their voting weight, deriving from their proposal power. In contrast to power indices which rest on the idea that all coalitions (or minimum winning coalitions, or permutations) are equally likely to form, the non-cooperative model by Snyder et al. (2005) explicitly considers the price of different coalitions. From a cost-minimizing proposer's point of view, one voter with, say, a voting weight of three and a continuation value of four is less desirable to include in the coalition than three voters who have one vote each and a continuation value of one. As a consequence of this 'substitution logic', one committee member with k votes basically receives the same share of the total purse as k members with one vote each. Again, the discrepancy between these results and, for example, the Shapley–Shubik index (which is non-linear in voting weights) can be traced back to the fact that, unlike cooperative solution concepts, the bargaining model accounts for the proposal making *and* the voting process.

[60] Holler and Schein (1979) provide a different model of 'pie' division by majority voting in which the shares of the 'pie' that players can secure for themselves are determined by their proposal rights and the sequence of their proposals.

Chapter 2
Committees as Representative Institutions

A voting game can be looked at from the angle of the individual players, or from that of the designer of the voting game. In the former perspective, it is natural to ask what the 'value' of the game to each of the players is. As stressed in Sects. 1.2 and 1.3, 'value' can refer to a player's ability to change the outcome of a voting game, or to the payoffs he may reasonably expect, but in both cases the game enters the analysis as input. From the designer's perspective, by contrast, the input is a desired value for the individual players, and the problem is to construct a voting game which induces it. This idea, that human interaction can be subjected to deliberate design, possibly most warrants the label 'political'. Assessing the rules of existing political bodies and devising new rules is the foremost area of application for power indices (see Sect. 1.2.1).

This chapter is devoted to one instance of such an *institutional design* problem: finding a weighted voting rule which implements the principle of 'one person, one vote' in a two-tiered government system. Weighted voting is used in many important political bodies such as the EU Council of Ministers, the US Electoral College, and the International Monetary Fund, as well as in some cartels, such as the International Coffee Council. The choice of weights is often a source of considerable dissent in these bodies. With respect to the European Union, the Single European Act of 1986 comprised provisions that weighted voting under a qualified majority rule should be applied for most decisions in the Council of Ministers concerning the Single Market. Like the earlier Treaties of Maastricht and Amsterdam, the Treaty of Nice, which was settled by the governments of the EU member states in December 2000, extended the use of qualified majority rule to new policy domains. Here, as well as at earlier occasions, e.g., the 1995 enlargement to Austria, Finland, and Sweden, the voting weights and quota proved to be a bone of contention. The controversy regained its momentum at the Council of the European Union (the 'EU Summit') in June 2007 due to Poland's lobbying for a square-root allocation of weights in the Council of Ministers.

Section 2.1 provides a brief overview of criteria that the designer of a committee voting game might wish to apply if he is interested in giving fair representation to individual citizens. It will be argued that equality of individual influence on the political outcome should be given priority when the committee is meant to represent some public *in a democratic sense*. So far, this ideal has only been investigated in

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the context of take-it-or-leave-it committees operating under a binary agenda with an exogenously given proposal, giving rise to Penrose's (1946) *square root rule*. In Sect. 2.2, the decision-making is modeled as an idealized two-stage process where alternatives are elements of a one-dimensional policy space. The model thus covers committee decisions that are non-binary and possibly involve strategic behavior. The 'one person, one vote' principle is interpreted as calling for a priori equal indirect influence of citizens on final decisions. In Sect. 2.3, a departure is made towards an analytical solution of the model. Heuristic arguments strongly suggest that, under 50%-majority rule, weights proportional to the square root of population sizes are an approximate solution to the equal representation problem. This finding is corroborated by extensive Monte-Carlo simulations, described in Sect. 2.4. Section 2.5 concludes with a discussion of issues which are left unobserved in this chapter, some of which are taken up in the following one. Except for Sect. 2.1, the presentation draws heavily on Maaser and Napel (2007). Section 2.3 is based on recent work with Stefan Napel.

2.1 Criteria for Representative Committees

A basic characteristic of today's idea of democracy is the use of political representatives who make decisions on behalf of the citizens. The participation of the latter is largely indirect as they only elect their representatives. With respect to analysis and modeling, the fact that all modern democracies are representative implies that a lot of democratic processes have to be conceived as two-tiered. First, the votes of the population determine the composition of a representative decision-making body. Second, this body makes decisions according to some set of rules.

In many two-tier systems, representatives are elected in separate districts and then participate in a governing body at the union level, where they cast a block vote for their district. Most often, voting districts are – for geographical, ethnic, or historical reasons – not equally sized, so that representatives' voting weights have to somehow reflect their constituency's population size. The most prominent example for such two-tier systems is the EU Council of Ministers. The members states of the EU differ widely in population figures, with Germany's 82 million inhabitants at the one extreme, and Malta's 400,000 at the other. A natural question that arises in this context is which rule should be applied to define districts' weight at the union level. Equivalently, thinking of the design of mechanisms as a maximization problem where some criterion – the designer's utility – is maximized under some assumptions about players' behavior, one can ask what the utility function of the designer should be.

Obviously, the answer depends on which properties of the decision-making process itself or its outcomes are regarded as desirable. Concerning the evaluation of alternative decision rules for a representative political body, two classes of normative criteria are particularly relevant: instrumental or outcome-oriented and intrinsic or procedure-oriented. On an instrumental account, one decision rule or another is advocated on the grounds that it is conducive to certain (desirable) substantial outcomes. For instance, one interpretation of 'representation' holds that it means acting in the best interest of those represented (see Pitkin, 1967, pp. 155ff). In this *welfarist* view, representation deals with the relationship between preferences and outcomes.[1] A utilitarian approach to the 'best interest' is to seek efficient outcomes, that is, to maximize total (expected) utility. The design of voting rules that satisfy this ideal is studied in the context of yes–no decision-making by Beisbart et al. (2005), Barberà and Jackson (2006), and Beisbart and Bovens (2007). Another instrumental norm in the welfarist vein is equality of (expected) utility throughout society, which might be seen as flowing from some notion of outcome fairness. This objective is considered by Beisbart and Hartmann (2006) and again Beisbart and Bovens (2007).

Both these ideals are generally incompatible with norms that ascribe intrinsic value to the procedure rather than the outcome. From this viewpoint, the problem is that voting rules which implement instrumental norms may treat individual citizens asymmetrically, that is they fail to guarantee *anonymity*. A voting rule is anonymous if and only if the outcome of collective decision-making is invariant under a permutation of the assignment of preferences to individuals. This property ensures that the collective decision only depends on how many votes an alternative gets, not on whose votes these are. To put the same point differently: a non-anonymous rule – intentionally or accidentally – biases the decision in favor of some individuals. Following Dahl (1956, p. 37), anonymity defines – together with popular sovereignty – the *minimum requirement* for decision-making procedures to be called democratic. Popular sovereignty means that citizens ultimately control political decision-making, and anonymity embodies *political equality*, or equivalently, the principle of 'one person, one vote'. The significance of this principle is intrinsic rather than instrumental since it conveys no orientation towards certain substantial outcomes. Why would the formal or numerical concept of political equality be so important as to take precedence over, for example, the above welfare considerations?[2]

It is illusionary and possibly not even desirable,[3] that collective decisions on public policies, and especially on distributive matters, should be reached by unanimous agreement.[4] Rather, the collective decision will most often infringe upon the will and the interests of some individuals. In as much as the decision bars options that

[1] Manin et al. (1999, p. 9) distinguish representativeness from responsiveness which they take to mean that a government adopts the policies that it understands to be preferred by the citizens. If the government is better informed about the state of the world, responsiveness and representativeness in the above sense can fall apart.

[2] See the debate between Wall (2006) and Christiano (2006) on this issue.

[3] Rae (1975) demonstrates that a decision rule which gives every individual the right of consent is not robust, by which he means that it must return *some* outcome. Acknowledging that it is de facto not possible *not* to make a decision, disagreement in society implies that the outcome is imposed upon at least one individual.

[4] Yet, unanimous consent as a precondition for action is deeply linked with liberalism (see Rae, 1975): It is the source of the mythical social contract in the works of John Locke and Thomas Hobbes, and it is the guiding principle of (supposedly coercion-free) market exchange.

were available to these individuals before, it can be said to interfere with their liberty (in the sense of freedom from coercion). So, if it is true that general consensus is fictitious, and liberty is the proper or initial condition of citizens, how can the use of coercion, implied by the collective decision, be justified?

As unanimous consent cannot reasonably be expected in the realm of political decisions, attention naturally shifts to the procedure by which the conflict of wills is decided. The egalitarian theory of democracy holds that it follows from the basic liberal idea of the fundamental moral equality of all persons that every person must be given equal regard in collective decision-making. The wielding of political power, and the use of coercion following a collective decision, is only legitimate or morally justifiable, if all have an 'equal say' on the decision (see Buchanan, 2002, pp. 710ff). The 'one person, one vote' principle provides this legitimation by assigning to everybody whom the final decision will bind equal formal means to influence the political decision. Of course, it can be contested what it means that citizens have an 'equal say': First, in representative democracy, citizens are confined to electing representatives, which, at best, limits their 'say' to choosing the ends or the direction of policy-making, whereas the choice of the means is usually left to the representatives. Second, some citizens such as parliamentarians or judges have powers to influence collective outcomes that ordinary citizens do not have. Third, wealthy citizens may, by their access to organizational or propaganda resources, wield privileged influence on political decisions. To the first two points, one may answer that functional asymmetries are still compatible with political equality if citizens are the arbiter over those who wield political power and enjoy equal participation in the control of the latter. As regards the third point, it seems that the degree to which political equality is feasible in a society depends on the distribution of income and wealth.

Despite these qualifications, political decision-rules that express a commitment to equality still seem the best possibility to reconcile the use of coercion with liberal individualism (see Buchanan, 2002, pp. 711f). The legitimizing effect of procedural equality is generally absent in instrumental decision rules, e.g., rules that are motivated by welfarist ideals. The standard liberal tenets that every individual is worthy of equal regard and that every individual knows his or her own interest best imply that all individuals must be assigned the same formal weight in making the rules for their lives.

The brief discussion of the egalitarian norm of democracy suggests that other criteria which might govern the choice of the decision procedure, such as utility-maximization, or procedural transparency, should only be taken into account in so far as they do not compromise the more significant 'one person, one vote' principle. It is true that many other properties besides political equality may be deemed desirable for a democratic system, e.g., a high turnout of voters or impartial media coverage of election campaigns. But as these qualities, which could be summarized as the 'deliberative dimension of democracy' (Christiano, 1996, pp. 91ff), are not easily amenable to anticipatory design, it seems reasonable to concentrate on what can be legislated, that is, on procedures which implement numerical equality.

Returning to two-tiered decision systems, democratic rules ought to guarantee equality of the means to influence collective decision-making with respect to the aggregate process (see Christiano, 1996, pp. 232ff). Claims that political equality is established if 'one person, one vote' is satisfied at the electoral stage, where representatives are elected, and that the decision-rule at the legislative stage could follow other goals (e.g., Beitz, 1989), are inconsistent with the view that the first stage is only a means to an end. The election of representatives is not an end in itself.

How can the basic democratic value of political equality be operationalized in practice in two-tiered systems that involve multiple constituencies of different population size? Given non-integer 'ideal shares', which might for example be the result of a general election in a multi-party system with proportional rule, the problem arises how to apportion an integer number of seats for these shares (see Balinski and Young, 2001). If, by contrast, there is one representative from every constituency in the higher-level assembly who casts a block vote on behalf of his constituents (as in the US Electoral College or the EU Council of Ministers), the problem is to determine what representatives' shares or weights should ideally be. The 'one person, one vote' principle can then be interpreted to call for a weighted voting scheme in the representative committee which gives each individual voter in any constituency an equal chance to determine the policy outcome of the two-tier process.

Although it seems straightforward to allocate weights proportional to population sizes, this ignores the combinatorial properties of weighted voting, which often imply stark discrepancies between *voting weight* and actual *voting power*: In an assembly with simple majority rule and three representatives having weight 47, 43, and 10, all three possess exactly the same number of possibilities to form a winning coalition, and hence the same a priori power.

The most well-known solution to this problem is the one first suggested by Penrose (1946). Starting from the ideal world in which only constituency membership[5] distinguishes voters, Penrose found that if members of any constituency are to have the same a priori chance to indirectly determine the outcome of top-tier decisions, then constituencies' voting weights need to be such that their power at the top-tier as measured by the *Penrose–Banzhaf index* (Penrose, 1946; Banzhaf, 1965) is proportional to the square root of the respective constituency's population size (also see Felsenthal and Machover, 1998, Sect. 3.4). This *square root rule* has recently become the benchmark for numerous studies of the EU Council of Ministers (see, e.g., Felsenthal and Machover 2001, 2004, Leech, 2002) and it is at least a reference point for investigations concerning the US (see, e.g., Gelman et al. 2004).

Applying the square root rule has, unfortunately, two weaknesses: First, Penrose's theorem critically depends on equiprobable 'yes' and 'no'-decisions by all voters, or at least a 'yes'-probability which is random and distributed independently across voters with mean exactly 0.5. As shown in Sect. 1.2.1, this assumption leads to the Penrose–Banzhaf index. If the 'yes'-probability is slightly lower or higher, or if it

[5] The constituency configuration is assumed to be given exogenously. See, e.g., Epstein and O'Halloran (1999) on constructing majority–minority voting districts along ethnic, religious, or social lines.

exhibits even minor dependence across voters – say, they are influenced by the same newspapers – then the square root rule may result in highly unequal representation (see Good and Mayer, 1975; Chamberlain and Rothschild, 1981). Related empirical studies in fact have failed to confirm the predictions for average closeness of two-party elections which lie behind the square root rule (see Gelman et al. 2002, 2004).

Second, rigorous justifications for using the square root rule as the benchmark have so far concerned only *preference-free binary voting*.[6] But real decisions are rarely binary, e.g., about *either* introducing a tax, building a road, accepting a candidate, introducing affirmative action, etc. *or* not. At least at intermediate levels there is a preference-driven compromise that involves *many* alternative tax levels, road attributes, suitable candidates, degrees of affirmative action, etc. Even issues that appear to be intrinsically binary, such as declaring war, usually entail decisions of a gradual nature, e.g., how many forces to deploy, or which aims to achieve before stopping aggression.

The first criticism has been addressed in the literature, at least in abstract normative terms. Namely, one can argue that constitutional design should be carried out behind a thick 'veil of ignorance' in which no particular type of dependence or modification of equiprobability (which follows from the 'Principle of Insufficient Reason', cf. p. 22) is justified.[7] Regarding the second issue, the next section presents, to the author's knowledge, the first model which allows to investigate the 'one person, one vote' principle for non-binary decisions that possibly involve strategic behavior.

2.2 The Model

In line with the arguments advanced in the preceding section, the principle of 'one person, one vote' is considered a worthwhile objective in the design of a decision-making rule for a committee of representatives. In the following, it will be equated with the following egalitarian norm: *Each voter in any constituency should have an equal chance to determine the policy implemented by the committee of representatives.* Instead of yes-no-decisions the model considers policy alternatives which are elements of a bounded interval. Two key assumptions are imposed: First, the policy advocated by the top-tier representative of any given constituency coincides with the ideal point of the respective constituency's *median voter* (or the

[6] For rigorous, very comprehensive treatments of the binary or *simple game* world see Felsenthal and Machover (1998) or Taylor and Zwicker (1999).

[7] The term 'veil of ignorance' was coined by John Rawls to characterize a hypothetical 'original position' in which decision-makers agree on principles of justice under uncertainty about the distribution of benefits and burdens that will result from a decision. In the constitutional choice literature, the 'veil of ignorance' is conceived of as a mechanism that makes the hypothetical designers of the constitution choose constitutional rules in an impartial way.

constituency's *core*). Second, the decision taken at the top tier is the position of the *pivotal representative* (or the assembly's *core*), with pivotality determined by the weights assigned to constituencies and a 50%-decision quota. The respective core is meant to capture the result of strategic interaction. As long as this is a reasonable approximation, the actual systems determining collective choices are undetermined and could even differ across constituencies.

In order to produce a recommendation appropriate to the circumstances of 'constitutional choice', it is necessary to be as unspecific as possible about the agenda which will confront the committee, or the ways committee members will evaluate policy alternatives, or the factional structure of the committee. This requirement seems to be met best by assuming that voters choose their most preferred policy independently of each other. In this benchmark case, a given individual's chance to be pivotal at the bottom tier is inversely proportional to the respective constituency's population size. This makes it necessary and sufficient for equal representation of voters that the probability of any given constituency being pivotal at the top tier is proportional to its size.[8]

The population size of a constituency affects the distribution of its median. A given voter's chance to be doubly pivotal thus becomes a rather complex function of (the order statistics of) *differently distributed* independent random variables. This makes a neat analytical statement similar to Penrose's rule exceptionally hard and likely impossible, except for limit situations.

This sketch of the model now needs to be set out more formally. Consider a large population of *voters* partitioned into *m constituencies* $\mathcal{C}_1, \ldots, \mathcal{C}_m$ with $n_j = |\mathcal{C}_j| > 0$ members each. Voters' preferences are single-peaked with *ideal point* λ^i_j (for $i = 1, \ldots, n_j$ and $j = 1, \ldots, m$) in a bounded convex one-dimensional *policy space* $X \subset \mathbb{R}$. Assume for simplicity that all n_j are odd numbers.

For any random policy issue, let $\cdot : n_j$ denote the permutation of voter numbers in constituency $\mathcal{C}_j$ such that

$$\lambda_j^{1:n_j} \leq \ldots \leq \lambda_j^{n_j:n_j}$$

holds. In other words, $k : n_j$ denotes the k-th leftmost voter in $\mathcal{C}_j$ and $\lambda_j^{k:n_j}$ denotes the k-th leftmost ideal point (i.e., $\lambda_j^{k:n_j}$ is the *k-th order statistic* of $\lambda^1_j, \ldots, \lambda^{n_j}_j$).

A policy $x \in X$ is decided on by an *committee of representatives* $\mathcal{R}$ consisting of one representative from each constituency. Without going into details, we assume that the representative of $\mathcal{C}_j$, denoted by j, adopts the ideal point of his constituency's *median voter*,[9] denoted by

[8] If voters' utility is linear in distance, the criterion also guarantees equal expected utility, i.e., a priori *power* and expected *success* are then perfectly aligned. See Laruelle et al. (2006) for a conceptual discussion of the latter.

[9] We are aware of this not being appropriate in all contexts. – The possibility that two ideal points exactly coincide, in which case the median voter (in contrast to the median policy) is not well-defined, is ignored. This is innocuous for any continuous ideal point distribution.

$$\lambda_j \equiv \lambda_j^{(n_j+1)/2:n_j}.$$

Let $\lambda_{k:m}$ denote the k-th leftmost ideal point amongst all the representatives (i.e., the k-th order statistic of $\lambda_1, \dots, \lambda_m$).

In the top-tier assembly or committee of representatives $\mathcal{R}$, each constituency $\mathcal{C}_j$ has *voting weight* $w_j \geq 0$. Any subset $S \subseteq \{1, \dots, m\}$ of representatives which achieves a combined weight $\sum_{j \in S} w_j$ above $q \equiv 0.5 \sum_{j=1}^{m} w_j$, i.e., a *simple majority* of total weight, can implement a policy $x \in X$.

Consider the random variable P defined by

$$P \equiv \min\Big\{l \in \{1, \dots, m\} \colon \sum_{k=1}^{l} w_{k:m} > q\Big\}.$$

Player $P:m$'s ideal point, $\lambda_{P:m}$, is the unique policy that beats any alternative $x \in X$ in a pairwise majority vote, i.e., constitutes the *core* of the voting game defined by weights and quota.[10] Without detailed equilibrium analysis of any decision procedure that may be applied in $\mathcal{R}$ (see Banks and Duggan, 2000 for sophisticated non-cooperative support of policy outcomes inside or close to the core), we assume that the policy agreed by $\mathcal{R}$ is in the core, i.e., it equals the ideal point of the *pivotal representative* $P:m$.

In this setting, the egalitarian norm above can be stated formally as follows: there should exist a constant $c > 0$ such that

$$\forall j \in \{1, \dots, m\} \colon \forall i \in \mathcal{C}_j \colon \Pr\big(j = P:m \wedge i = (n_j + 1)/2:n_j\big) \equiv c. \quad (2.1)$$

We would like to answer the following question: which allocation of weights $w_1, \dots, w_m$ satisfies this norm (at least approximately) for an arbitrary given partition of an electorate into m constituencies? In other words we search for an analogue of Penrose's (1946) rule, which calls for proportionality of a constituency's Penrose–Banzhaf index (1.10) and square root of population.[11]

The probability of a voter's double pivotality in (2.1) depends on the distribution of all voters' ideal points. Though in practice ideal points in different constituencies may come from different distributions on X and may exhibit various dependencies, it is appealing from a normative constitutional-design point of view to presume that the ideal points of all voters in all constituencies are *independently and identically distributed* (i. i. d.).

[10] The policy $\lambda_{P:m}$ – Things are more complicated if $q > 0.5 \sum_{j=1}^{m} w_j$ is assumed. Then, the complement of a losing coalition need no longer be winning. In this case there may not exist *any* policy $x \in X$ which beats all alternatives $x' \neq x$ despite unidimensionality of X and single-peakedness of preferences.

[11] Conditions for when a player's Penrose–Banzhaf index is approximately his/her voting weight are given by Lindner and Machover (2004). – In Chap. 3, the exact version of Penrose's rule is investigated with respect to the present model.

Given that voters' ideal points in constituency $\mathcal{C}_j$ are i. i. d., each voter $i \in \mathcal{C}_j$ has the same probability to be its median. Hence,

$$\forall j \in \{1, \ldots, m\}\colon \forall i \in \mathcal{C}_j\colon \ \Pr\big(i = (n_j + 1)/2{:}n_j\big) = \frac{1}{n_j}.$$

Because the events $\{i = (n_j + 1)/2{:}n_j\}$ and $\{j = P{:}m\}$ are independent, one can thus write (2.1) as

$$\forall j \in \{1, \ldots, m\}\colon \ \frac{\Pr(j = P{:}m)}{n_j} \equiv c. \tag{2.2}$$

So if constituency $\mathcal{C}_j$ is twice as large as constituency $\mathcal{C}_k$, representative j must have twice the chances to be pivotal than representative k in order to equalize individual voters' chances to be pivotal.

Representatives' ideal points $\lambda_1, \ldots, \lambda_m$ are independently but (except in the trivial case $n_1 = \ldots = n_m$) *not* identically distributed. If all voter ideal points come from the (arbitrary) identical distribution F with density f, then $\mathcal{C}_j$'s median position is asymptotically normally distributed (see e.g., Arnold et al., 1992) with mean

$$\mu_j = F^{-1}(0.5) \tag{2.3}$$

and standard deviation

$$\sigma_j = \frac{1}{2\, f(F^{-1}(0.5))\sqrt{n_j}}. \tag{2.4}$$

So, the larger a constituency $\mathcal{C}_j$ is, the more concentrated is the distribution of its median voter's ideal point, λ_j, on the median of the underlying ideal point distribution (assumed to be identical for all λ_j^i). This makes the representative of a larger constituency on average more central in the committee of representatives and more likely to be pivotal in it for a given weight allocation.

It is important to observe that the assumption of the respective *collective preferences* having an identical a priori distribution is inconsistent with the assumption that all *individual preferences* are a priori identically distributed. The intuitively appealing linear rule of giving twice the weight to a constituency double the size (or, in view of the combinatorial aspects of weighted voting, choosing weights such that the respective Shapley–Shubik index[12] is twice as large) violates the 'one person, one vote' principle if one makes the latter assumption. We find it considerably more fitting and will assume i. i. d. ideal points for all bottom-tier voters throughout this paper. Weights and Shapley–Shubik index of constituencies hence need to be increasing in population size but less than linearly.

Probability $\Pr(j = P{:}m)$ in (2.2) depends both on the different distributions of representatives' ideal points (essentially the standard deviations σ_j determined by

[12] Note that the Shapley–Shubik index corresponds to the probability of top-tier pivotality in case that representatives' preferences are i. i. d. (cf. Sect. 1.2.3).

constituency sizes n_j) and the voting weight assignment. This makes computation of the probability of a given constituency $\mathcal{C}_j$ being pivotal a complex numerical task even for the most simple case of *uniform weights*, in which the representative of $\mathcal{C}_j$ with *median* top-tier ideal point is always pivotal, i.e., $P \equiv (m+1)/2$ for odd m. Define $M^{-j} \equiv \{1, \ldots, j-1, j+1, \ldots, m\}$ as the index set of all constituencies except $\mathcal{C}_j$. Then, the probability of constituency $\mathcal{C}_j$ being pivotal is

$$\begin{aligned}\Pr\left(j = \tfrac{m+1}{2}\!:\!m\right) &= \Pr\left(\text{exactly } \tfrac{m-1}{2} \text{ of the } \lambda^k, k \neq j, \text{ satisfy } \lambda_k < \lambda_j\right) \\ &= \int \sum_{\substack{S \subset M^{-j},\\ |S|=(m-1)/2}} \prod_{k \in S} F_k(x) \cdot \prod_{k \in M^{-j} \setminus S} (1 - F_k(x)) \cdot f_j(x)\ dx,\end{aligned} \tag{2.5}$$

where f_j and F_j denote the density and cumulative density functions of λ_j ($j = 1, \ldots, m$). Proceeding to the case of weighted voting ($P \not\equiv (m+1)/2$), even success in more generally approximating

$$\Pr(j = p\!:\!m) = \int \sum_{S \subset M^{-j},\ |S|=p-1} \prod_{k \in S} F_k(x) \cdot \prod_{k \in M^{-j} \setminus S} (1 - F_k(x)) \cdot f_j(x)\, dx.$$

for any *given* realization p of random variable P would be of little help because events $\{P = p\}$ and $\{j = p\!:\!m\}$ are no longer independent.[13] So, typically,

$$\Pr(j = P\!:\!m) \neq \sum_{p=1}^{m} \Pr(P = p) \cdot \Pr(j = p\!:\!m).$$

2.3 Analytic Arguments

For the reasons stated above, it seems unrealistic to aim for a general analytical solution to the equal representation problem (2.1), or equivalently, (2.2) for arbitrary finite configurations $(n_1, \ldots, n_m)$. But is there a way of making progress for a particularly clear layout? The following heuristic arguments suggest that this might indeed be possible.

Assume that representatives' ideal points λ_j are normally distributed with mean $\mu_j = 0$ and standard deviation

$$\sigma_j = \frac{\vartheta\sqrt{2\pi}}{2\sqrt{n_j}} > 0$$

[13] To see this, consider the artificial case of representative j having weight $w_j > 0.5\sum_{j=1}^{m} w_j$ even though all constituencies are of equal size, so that ideal points λ_k ($k = 1, \ldots, m$) are i. i. d. Since j is a dictator, $\Pr(j = P\!:\!m) = 1$. But $\Pr(P = p) = 1/m$ and $\Pr(j = p\!:\!m) = 1/m$ for all p.

[cf. (2.4)], where $\vartheta > 0$ is a constant. Denote the cumulative density function of λ_j by F_{λ_j}, and the density of λ_j by f_{λ_j}. The latter is given by

$$f_{\lambda_j}(x) = \frac{1}{\sigma_j\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_j{}^2}} = \frac{\sqrt{n_j}}{\vartheta\pi} e^{-\frac{x^2 n_j}{\vartheta^2\pi}}. \tag{2.6}$$

Let Ω denote the set of vectors of median ideal points. Finally, to facilitate notation, let $\pi_j \equiv \Pr(j = P:m)$. A representative k with ideal point λ_k is pivotal in the committee of representatives if, for a given realization $\omega \in \Omega$ of median voters' ideal points, the total weight of the representatives who have ideal points to the left of λ_k is greater than or equal to $q - w_k$, but less than q.

Given weights $w_1, \ldots, w_m$, let $\tilde{w}_j(x)$, $x \in X$, be the random variable defined by

$$\tilde{w}_j(x)(\omega) = \begin{cases} w_j & \text{if } \lambda_j(\omega) \le x \\ 0 & \text{if } \lambda_j(\omega) > x. \end{cases}$$

where $\omega \in \Omega$ refers to a particular ideal point realization. The random variable $\tilde{w}_j(x)$ is the contribution of constituency $\mathcal{C}_j$ to the total weight of constituencies which have ideal points weakly to the left of x. Denote the weight accumulated up to x by constituencies other than any fixed constituency $\mathcal{C}_k$ by

$$\tilde{W}_{-k}(x)(\omega) = \sum_{j \neq k} \tilde{w}_j(x)(\omega).$$

Consider any ideal point realization ω such that $\lambda_k(\omega) = x$. Constituency $\mathcal{C}_k$ is pivotal in the committee of representatives iff

$$\tilde{W}_{-k}(x)(\omega) \le q < \tilde{W}_{-k}(x)(\omega) + w_k$$

or

$$q - w_k < \tilde{W}_{-k}(x)(\omega) \le q.$$

The expected value of the probability of this event with respect to the probability density function $f_{\lambda_k}(x)$ yields k's overall power $\Pr(k = P:m)$,

$$\pi_k = \int_{-\infty}^{\infty} \Pr(q - w_k < \tilde{W}_{-k}(x) \le q)\, f_{\lambda_k}(x)\, dx. \tag{2.7}$$

So, π_k is the probability that representative k's median is located between the positions $\tilde{W}_{-k}^{-1}(q - w_k)$ and $\tilde{W}_{-k}^{-1}(q)$ in X at which constituencies $j \neq k$ have accumulated weight $q - w_k$ and q, respectively.[14] This is illustrated in Figs. 2.1 and 2.2.

[14] Note that $\tilde{W}_{-k}(x)$ is a step function. Thus, $\tilde{W}_{-k}^{-1}(\cdot)$ is the *quasi-inverse* of $\tilde{W}_{-k}(x)$, i.e., $\tilde{W}_{-k}^{-1}(y) = \inf\{x \in X \mid y \le \tilde{W}_{-k}(x)\}$.

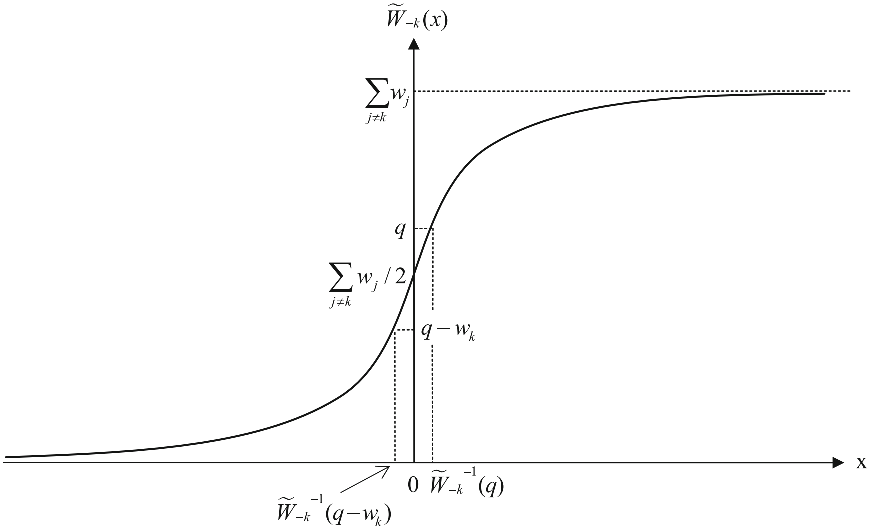

Fig. 2.1 Accumulated weight of constituencies other than k and determination of $\tilde{W}_{-k}^{-1}(q - w_k)$ and $\tilde{W}_{-k}^{-1}(q)$

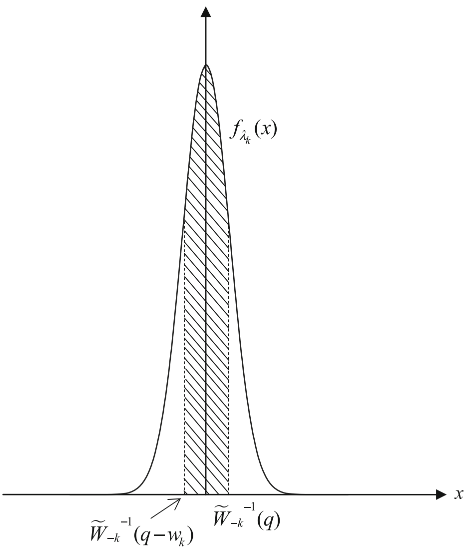

Fig. 2.2 Density of λ_k. The shaded area corresponds to π_k, which is the expectation of the event that λ_k is situated between $\tilde{W}_{-k}^{-1}(q - w_k)$ and $\tilde{W}_{-k}^{-1}(q)$

As mentioned already, the explicit computation of $\tilde{W}_{-k}(x)$'s distribution, and hence that of π_k, is very involved: for any $\overline{W} \in (q - w_k; q]$, one needs to account for all combinatorial possibilities to reach the aggregate weight $\overline{W}$ without $\mathcal{C}_k$. This would amount to the enumeration of all coalitions S not containing representative k

with weight $\sum_{j\in S} w_j = \overline{W}$, followed by the summation of the respective formation probabilities $\prod_{j\in S} F_{\lambda_j}(x) \prod_{j\notin S}(1 - F_{\lambda_j}(x))$.

An approximation which ignores these combinatorial complications is of little use for estimating power for any particular weight distribution, but helps in identifying the general behavior of power, as weight and population size is varied.

The key observation is that, for a large number m of constituencies, $\tilde{W}_{-k}(x)$ – as the sum of $m - 1$ independent random variables – is approximately normally distributed,[15] with mean

$$\mathbf{E}\tilde{W}_{-k}(x) = \sum_{j\neq k} \mathbf{E}\tilde{w}_j(x) = \sum_{j\neq k} w_j F_{\lambda_j}(x). \tag{2.8}$$

As m goes to infinity, the variance of the random variable $\tilde{W}_{-k}(x)$ approaches zero. Hence, the error of replacing $\tilde{W}_{-k}(x)$ in (2.7) by its expected value $\mathbf{E}\tilde{W}_{-k}(x)$ is small when we look at a large number of constituencies. In particular, we can then approximate (2.7) by

$$\begin{aligned}\pi_k \approx \hat{\pi}_k &= \int_{-\infty}^{\infty} \mathbf{1}_{\{x:\, q-w_k<\mathbf{E}\tilde{W}_{-k}(x)\leq q\}}(x)\, f_{\lambda_k}(x)\, dx \\ &= \int_{-\infty}^{\infty} \mathbf{1}_{\{x:\, \mathbf{E}\tilde{W}_{-k}^{-1}(q-w_k)<x\leq\mathbf{E}\tilde{W}_{-k}^{-1}(q)\}}(x)\, f_{\lambda_k}(x)\, dx \\ &= \int_{\mathbf{E}\tilde{W}_{-k}^{-1}(q-w_k)}^{\mathbf{E}\tilde{W}_{-k}^{-1}(q)} f_{\lambda_k}(x)\, dx \end{aligned} \tag{2.9}$$

where $\mathbf{1}_X$ denotes the indicator function of set X.

From the point symmetry of the normal cumulative density function, $F_{\lambda_j}(x) = 1-F_{\lambda_j}(-x)$, together with (2.8), it follows that $\mathbf{E}\tilde{W}_{-k}(x)$ is point symmetric in relation to the point $(0; \sum_{j\neq k} w_j/2)$ (cf. Fig. 2.1). Consider the case of simple majority rule, $q = \sum_j w_j/2$. The quota can be rewritten as $q = \sum_{j\neq k} w_j/2 + w_k/2$, whilst $q-w_k = \sum_{j\neq k} w_j/2 - w_k/2$. Thus, if $\mathbf{E}\tilde{W}_{-k}(z) = q$ for some $z \in \mathbb{R}$, then it holds that $\mathbf{E}\tilde{W}_{-k}(-z) = q - w_k$.

Using (2.6), approximation (2.9) becomes

$$\hat{\pi}_k = \int_{-z}^{z} f_{\lambda_k}(x)\, dx = \frac{\sqrt{n_k}}{\vartheta\pi} \int_{-z}^{z} e^{-\frac{x^2 n_k}{\vartheta^2\pi}}\, dx = 2\frac{\sqrt{n_k}}{\vartheta\pi} \int_{0}^{z} e^{-\frac{x^2 n_k}{\vartheta^2\pi}}\, dx \tag{2.10}$$

where z is implicitly defined by $\mathbf{E}\tilde{W}_{-k}(z) = q$. This integral can be written as the Taylor series

[15] As the random variables $\tilde{w}_j(x)$ are independently, but not identically distributed with finite variance, *Lyapunov's central limit theorem* applies.

$$\hat{\pi}_k = 2\frac{\sqrt{n_k}}{\vartheta\pi}\int_0^z \big(1 - \frac{\eta x^2}{1!} + \frac{\eta^2 x^4}{2!} - \frac{\eta^3 x^6}{3!} + \ldots\big)\,dx$$
$$= 2\frac{\sqrt{n_k}}{\vartheta\pi}\big(z - \frac{\eta z^3}{3\cdot 1!} + \frac{\eta^2 z^5}{5\cdot 2!} - \frac{\eta^3 z^7}{7\cdot 3!} + \ldots\big) \tag{2.11}$$

with $\eta \equiv \frac{n_k}{\vartheta^2\pi}$.

If constituency k were included in the aggregation, the quota $q = \sum_j w_j/2$ would in expectation be accumulated exactly at $x = 0$. If constituency $\mathcal{C}_k$'s weight w_k is 'small' relative to $\sum_j w_j$, then q will in expectation be accumulated slightly to the right of zero, i.e., z is close to zero. For this case, terms in (2.11) with degree greater than one have only a second-order effect. So $\hat{\pi}_k$ can rather well be approximated by

$$\hat{\pi}_k = \frac{2z\sqrt{n_k}}{\vartheta\pi}. \tag{2.12}$$

In the neighborhood of $x = 0$, $F_{\lambda_j}(x)$ can be approximated by its Taylor polynomial of degree 1, i.e.,

$$\hat{F}_{\lambda_j}(x) = F_{\lambda_j}(0) + x f_{\lambda_j}(0) = \frac{1}{2} + x f_{\lambda_j}(0).$$

Then, solving

$$\sum_{j\neq k} w_j/2 + w_k/2 = q = \mathbf{E}\tilde{W}_{-k}(z) \approx \sum_{j\neq k} w_j \hat{F}_{\lambda_j}(z) = \sum_{j\neq k} w_j/2 + \sum_{j\neq k} w_j z f_{\lambda_j}(0),$$

the location z is obtained approximately as

$$z \approx \frac{w_k}{2\sum_{j\neq k} w_j\ f_{\lambda_j}(0)}.$$

This, together with (2.12), leads to the conclusion that

$$\pi_k \approx \frac{w_k\ \sqrt{n_k}}{\vartheta\pi \sum_{j\neq i} w_j\ f_{\lambda_j}(0)}. \tag{2.13}$$

According to (2.13), the probability of constituency $\mathcal{C}_k$ to be pivotal at the top tier is approximately proportional to its weight w_k and to the square root of its population n_k. The square root of population, which first showed up in the density (2.6), describing the distribution of representative j's position, reappears in (2.13). Returning to the equal representation condition (2.2), it follows from (2.13) that (2.2) can be approximately satisfied by choosing weights w_j^* such that

$$w_j^* \propto \sqrt{n_j},$$

where the notation $\propto$ refers to (direct) proportionality between w_j^* and the square root of n_j for all $j = 1, \ldots, m$.

In order to obtain this heuristic result, two major approximations were made: first, the effect of the combinatorial features of a particular weight distribution on power is ignored [leading to formula (2.9)]. Second, the 'lumpiness' of player k's weight which implies that z is actually larger than 0 is not taken into account [leading to (2.12) and (2.13)]. Nevertheless, (2.13) allows a prediction about the equitable weight allocation for large representative committees and exposes the reason why one would expect a square root rule to eventually emerge from the 'double pivot' model introduced in Sect. 2.2.[16]

While the approximative weight allocation rule $w_j^* \propto \sqrt{n_j}$ may be expected to work well under 'limit conditions', it is of limited use when the number of constituencies is 'small'. The following section for this reason uses Monte-Carlo simulation in order to approximate the probability of any constituency $\mathcal{C}_j$ being pivotal for a given partition of an electorate, or *configuration* $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$, and a fixed weight vector $(w_1, \ldots, w_m)$. Based on this, the simulation tries to find weights $(w_1^*, \ldots, w_m^*)$ which approximately satisfy the two equivalent equal representation conditions (2.1) and (2.2).

2.4 Simulation Results

The problem of finding probability $\pi_j \equiv \Pr(j = P\!:\!m)$ is similar to that of evaluating the odds of rolling a '6' with a funny-shaped die: while one may conceivably solve a complex dynamic model with several partial differential equations, it is equally reliable to simply roll the die many times and keep track of '6's. In particular, π_j can be viewed as the *expected value* of the random variable $K_j \equiv g_j^w(\lambda_1, \ldots, \lambda_m)$ which equals 1 if $j = P\!:\!m$ holds for given weight vector w and realized median ideal points $\lambda_1, \ldots, \lambda_m$, and 0 otherwise. The *Monte-Carlo method* (Metropolis and Ulam, 1949) then exploits the fact that the empirical average of s independent draws of K_j,

$$\bar{k}_j^s = \frac{1}{s} \sum_{l=1}^{s} k_j^l,$$

converges to K_j's theoretical expectation

$$\mathbf{E}(K_j) = \pi_j$$

[16] Note that Penrose's square root rule also includes an approximation: In its derivation, *Stirling's formula* is used to approximate the probability that an individual voter is decisive at the lower-tier referendum (or a general two-candidate election).

by the law of large numbers. The speed of convergence in s can be assessed by the sample variance of $\bar{k}_j^s$. Using the central limit theorem, it is then possible to obtain estimates of π_j with a desired precision (e.g., a 95%-confidence interval) if one generates and analyzes a sufficiently large number of realizations.

To obtain a realization k_j^l of K_j, first m random numbers $\lambda_1, \ldots, \lambda_m$ are drawn from distributions $F_1, \ldots, F_m$.[17] Throughout the analysis, F_j is taken to be a *beta distribution* with parameters $\big((n_j + 1)/2, (n_j + 1)/2\big)$. This corresponds to the median of n_j independently [0, 1]-*uniformly distributed* voter ideal points, i.e., all individual voter positions are assumed to be distributed uniformly.[18] Second, the realized constituency positions are sorted and the pivotal position p is determined. Constituency $\mathcal{C}_{p:m}$ is thus identified as the pivotal player of $\mathcal{R}$. It follows that $k_j^l = 1$ for $j = p{:}m$, and 0 for all other constituencies.

The goal is to identify a simple rule for assigning voting weights to constituencies which – if it exists – approximately satisfies equal representation conditions (2.1) or (2.2) for various numbers of constituencies m and population configurations $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$. A natural focus is the investigation of *power laws*

$$w_j = n_j{}^{\alpha} \tag{2.14}$$

with $\alpha \in [0, 1]$. For big m this approximately includes Penrose's square root rule as the special case $\alpha = 0.5$ (see Lindner and Machover, 2004; Chang et al., 2006).

For any given m and population configuration $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$ under consideration, a fixed α is considered and then π_j is approximated by its empirical average $\hat{\pi}_j$ in a run of 10 million iterations. This is repeated for different values of α, ranging from 0 to 1 with a step size of 0.1 or 0.01, in order to find the exponent α which comes 'closest' to implying equal representation for the given configuration.

Two different criteria come into question for evaluating distance between the (estimated) probability vector $\hat{\pi} \equiv (\hat{\pi}_1, \ldots, \hat{\pi}_m)$ realized by weights w and the ideal egalitarian vector $\pi^* \equiv (\sum_{k=1}^m n_k)^{-1} \cdot (n_1, \ldots, n_m)$. A first straightforward criterion is $\hat{\pi}_j$'s cumulative quadratic deviation from π_j^*,

$$\sum_{j=1}^{m} (\hat{\pi}_j - \pi_j^*)^2, \tag{2.15}$$

which is equivalent to considering Euclidean distance between $\hat{\pi}$ and π^* in $\mathbb{R}^m$. The above, a priori, treats deviations from π_j^* equally for all j, i.e., looks at deviations for constituencies as such rather than for *individuals*.

---

[17] A *Java* computer program is used. The source code is available upon request. Directly drawing the constituency medians λ_j provides a huge computational advantage. Unfortunately, it prevents statements about the population median and, e.g., its average distance to the policy outcome.

[18] The mentioned asymptotic results for order statistics imply that only F's median position and density at the median matter when constituency sizes are large. So below findings are *not* specific to the assumption of uniform distributions at the bottom tier.

It seems, however, desirable in an egalitarian context to focus on the latter. So the second criterion considers cumulative quadratic deviations between the realized and the ideal chances of an individual. Any voter in any constituency $\mathcal{C}_j$ would ideally determine the outcome with the same probability $1/\sum_{k=1}^{m} n_k$, but vector $\hat{\pi}$ actually gives him or her the probability $\hat{\pi}_j/n_j$ of doing so. Treating all n_j voters in any constituency $\mathcal{C}_j$ equally then amounts to looking at

$$\sum_{j=1}^{m} n_j \cdot \left(\frac{1}{\sum_{k=1}^{m} n_k} - \frac{\hat{\pi}_j}{n_j} \right)^2 . \tag{2.16}$$

Minimization of (2.16) seems more relevant than that of (2.15). In any case optimal values of α are virtually unaffected by a switch between the two criteria. They are also almost unaffected by a switch from respective quadratic deviations to absolute deviations. So, with little loss of information, only results for measure (2.16) will be presented, referring to it as *cumulative individual quadratic deviation* below. Section 2.4.1 first investigates computer-generated random environments with constituency numbers between 10 and 100; several population configurations for each m are investigated to check the robustness of an optimal α. Sections 2.4.2 and 2.4.3 then briefly look at the EU Council of Ministers and the US Electoral College.

2.4.1 Randomly Generated Configurations

Table 2.1 reports the optimal values of α that were obtained for four sets of configurations $\{\mathcal{C}_1, \ldots, \mathcal{C}_m\}$.[19] For $m \in \{10, 15, 20, 25, 30, 40, 50\}$, constituency sizes $n_1, \ldots, n_m$ were independently drawn from a uniform distribution over $[0.5 \cdot 10^6,\ 99.5 \cdot 10^6]$. Numbers in column (I) are the optimal $\alpha \in \{0, 0.1, \ldots, 0.9, 1\} \subset [0, 1]$, where probabilities $\hat{\pi}_j$ were estimated by a simulation with 10 mio. iterations. Cumulative individual quadratic deviations for optimal α's are shown in brackets. Column (II) reports the respective values obtained for an independent second set of constituency configurations; columns (III) and (IV) do likewise but based on the finer grid $\{0, 0.01, 0.02, \ldots, 0.99, 1\}$ that contains α.[20]

While results for $m = 10$ are still inconclusive, $\alpha \approx 0.5$ emerges as the very robust ideal exponent for larger number of constituencies. The reported cumulative individual quadratic deviations are so small that even if the power laws assumed in (2.14) do not contain the theoretically best rule for equal representation in our median-voter context (because possibly constituencies' sizes are not the right

[19] The configuration draws are independent across different values of m. Thus, the table actually reports optimal values obtained for 28 *independent* configurations.

[20] Hence columns (III) and (IV) each report on $101 \cdot 7$ simulation runs (with 10 mio. iterations each).

Table 2.1 Optimal value of α for uniformly distributed constituency sizes (cumulative individual squared deviations from ideal probabilities in parentheses)

# const	(I)	(II)	(III)	(IV)
10	0.5	0.6	0.39	0.00
	(1.22×10^{-11})	(1.04×10^{-11})	(2.20×10^{-12})	(2.39×10^{-11})
15	0.5	0.5	0.49	0.48
	(1.43×10^{-11})	(1.45×10^{-13})	(2.79×10^{-14})	(8.84×10^{-14})
20	0.5	0.5	0.49	0.49
	(4.80×10^{-14})	(8.59×10^{-14})	(5.66×10^{-15})	(6.91×10^{-15})
25	0.5	0.5	0.49	0.49
	(9.25×10^{-15})	(1.28×10^{-14})	(5.37×10^{-15})	(7.69×10^{-15})
30	0.5	0.5	0.49	0.49
	(1.11×10^{-15})	(5.12×10^{-15})	(7.36×10^{-15})	(2.38×10^{-15})
40	0.5	0.5	0.49	0.49
	(3.38×10^{-15})	(5.11×10^{-15})	(3.69×10^{-15})	(7.02×10^{-15})
50	0.5	0.5	0.50	0.50
	(3.06×10^{-15})	(4.70×10^{-15})	(3.10×10^{-15})	(3.30×10^{-15})

Table 2.2 Optimal value of α for normally distributed constituency sizes ($\mu = 1$ mio., $\sigma =$ 200,000; truncated below 0)

# const	(I)	(II)	(III)	(IV)
10	0.0	0.0	0.0	0.0
	(1.22×10^{-9})	(1.65×10^{-9})	(9.21×10^{-9})	(1.83×10^{-9})
20	0.6	0.0	0.6	0.0
	(2.19×10^{-10})	(2.93×10^{-10})	(2.82×10^{-10})	(3.83×10^{-10})
30	0.1	0.2	0.4	0.5
	(1.07×10^{-10})	(1.07×10^{-10})	(6.94×10^{-11})	(6.76×10^{-11})
40	0.3	0.4	0.4	0.5
	(1.72×10^{-11})	(2.08×10^{-11})	(2.32×10^{-11})	(2.81×10^{-13})
50	0.4	0.2	0.3	0.3
	(1.60×10^{-11})	(7.39×10^{-12})	(3.56×10^{-11})	(4.72×10^{-11})
100	0.5	0.5	0.5	0.5
	(1.01×10^{-13})	(2.30×10^{-12})	(1.99×10^{-13})	(3.44×10^{-13})

reference point, but rather something like their Penrose–Banzhaf or Shapley–Shubik index), they allow a sufficiently good approximation for most practical purposes.

Results in Table 2.1 are strongly suggesting that (an approximation of) Penrose's square root rule holds also in the context of median voter-based policy decisions in $X \subset \mathbb{R}$. But optimality of $\alpha \approx 0.5$ could be an artifact of considering uniformly distributed constituency sizes $n_1, \ldots, n_m$, which perhaps unrealistically makes small constituencies as likely as large ones. Therefore similar investigations using other distributional assumptions need to be conducted.

Constituency sizes seem usually a matter of history, geography, or deliberate design. In the latter case, one might expect them to be clustered around some 'ideal' intermediate level. This makes a (truncated) normal distribution around some value $\bar{n}$ a focal assumption for constituency configurations. Table 2.2 indicates that, in

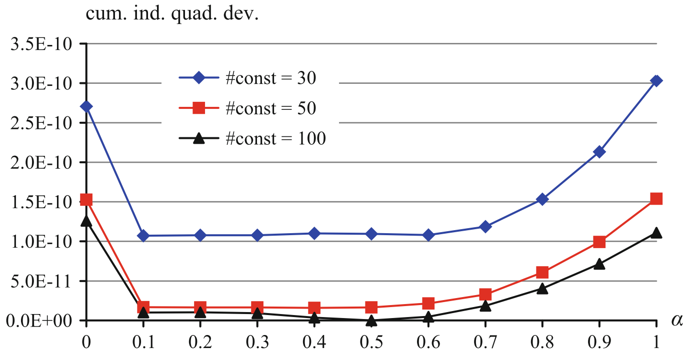

Fig. 2.3 Cumulative individual quadratic deviation in normal-distribution runs (I) for different numbers of constituencies

this case, $\alpha = 0.5$ is no longer the general clear winner from the considered set of parameters $\{0, 0.1, \ldots, 0.9, 1\}$. This is neither very surprising nor – from a square-root-rule point of view – very disturbing: Moderately many and more or less equally sized constituencies give rather little scope for discrimination between constituencies. Assigning slightly larger constituencies substantially more weight risks overshooting the mark, but assigning them only slightly more weight may not translate into an increased number of pivot positions at all. So, first, the optimal α can be expected to be rather sensitive to the precise constituency configuration at hand, especially when a small number of constituencies creates relatively few distinct opportunities to achieve a majority. And, second, in the wide range where extra weight to an above-the-average constituency translates into no or few extra winning coalitions, the objective function is very flat. This is nicely illustrated by Fig. 2.3. Its minimization via Monte Carlo techniques is then particularly sensitive to remaining estimation errors. But note that the importance of these issues decreases as m gets large. This indicates that applicability of the square root rule rests on enough flexibility regarding the formation of distinct winning coalitions.

When historical or geographical boundaries determine a population partition, a yet more natural distributional benchmark for n_j is a power law such as *Zipf's law* (or *zeta distribution*), which has big empirical support in a variety of contexts.[21] As an example, we consider the Pareto distribution with density function

$$g(x \mid \kappa, \underline{x}) = \kappa \, \frac{\underline{x}^{\kappa}}{x^{\kappa+1}} \tag{2.17}$$

[21] Examples for which (approximative) power-law behavior has been observed include sizes of human settlements (Gabaix, 1999; Reed, 2002), the value of oil reserves in oil fields, the size of meteor impacts on the moon, or even frequencies of words in long sequences of text. Explanations for this widespread regularity are based on ideas such as self-organized criticality and highly optimized tolerance (see, e.g., Newman, 2000).

Table 2.3 Optimal value of α for constituency sizes from Pareto distribution on $[0.1;\infty)$

	κ			
# const	1.0	1.8	3.4	5.0
10	0.5	0.5	0.0	0.0
	(1.32×10^{-9})	(3.25×10^{-9})	(3.72×10^{-9})	(1.80×10^{-8})
20	0.5	0.5	0.5	0.0
	(6.99×10^{-11})	(4.78×10^{-11})	(5.64×10^{-11})	(3.61×10^{-9})
30	0.5	0.5	0.5	0.1
	(1.32×10^{-11})	(2.41×10^{-11})	(2.41×10^{-11})	(1.03×10^{-10})
40	0.5	0.5	0.5	0.15
	(1.87×10^{-11})	(2.25×10^{-11})	(3.27×10^{-12})	(2.85×10^{-11})
50	0.5	0.5	0.5	0.1
	(1.31×10^{-10})	(1.86×10^{-11})	(2.67×10^{-12})	(1.91×10^{-10})
100	0.5	0.5	0.5	0.5
	(3.79×10^{-12})	(1.04×10^{-12})	(8.88×10^{-13})	(7.54×10^{-13})

on $[\underline{x}, \infty)$. Parameter $\underline{x}$ provides a lower bound on n_j and parameter κ determines how quickly the probability of drawing a large (rather than small or medium-sized) constituency approaches 0.

Table 2.3 reports simulations with constituency sizes drawn from a Pareto distribution with $\underline{x} = 0.1$ and $\kappa \in \{1, 1.8, 3.4, 5\}$, where numbers refer to million inhabitants. As long as the distribution is only moderately skewed (small κ), findings correspond nicely to those for the uniform distribution: $w_j = \sqrt{n_j}$ performs best and gets close to ensuring equal representation provided that the number of constituencies is sufficiently large. The former is no longer the case for a heavily skewed distribution of constituency sizes, i.e., when there are mostly small constituencies and only one or perhaps two large constituencies (reminiscent of the major players in an otherwise oceanic game). Giving all constituencies equal weight does reasonably well. A coefficient α greater, but still not far from zero, improves on this by creating additional pivot positions for the large constituency. But for a moderate number of constituencies, increasing α after the initial introduction of asymmetry produces quite little effect (again, the objective function is rather flat over a big range as indicated by Fig. 2.4) and then suddenly overshoots, resulting in too much power for the large constituency. For the same combinatorial reasons as in the normal-distribution case, this problem gets less severe, the greater is the total number of constituencies: For $m = 100$ or larger, $\alpha = 0.5$ turns out to be clearly optimal even for high skewness ($\kappa = 5$).

In summary, the above analysis of many different population configurations reveals three things. First, as Table 2.1 and Figs. 2.3 and 2.4 show, $\alpha = 0.5$ results in representation close to being as equal as possible for the given partition of the electorate. Second, for a moderately large number m of constituencies $\alpha \approx 0.5$ is optimal in the considered class of power laws unless all constituency sizes are very similar (e.g., n_j normally distributed with small variance) or rather similar with one or two outliers (corresponding to a heavily skewed distribution). Third, even in these extreme cases the optimal α converges to 0.5 as m gets large. We now turn to two prominent real-world two-tier voting systems.

Fig. 2.4 Cumulative individual quadratic deviation for $m = 30$ and different Pareto distributions

2.4.2 EU Council of Ministers

Together with Commission and Parliament, the Council of Ministers is one of the European Union's chief legislative bodies. It is widely held to be the most influential amongst the three and most voting power analysis concentrates on it.[22] It consists of a national government representative from each of the EU member states, endowed with voting weight that is (weakly) increasing in share of total population.[23]

Figure 2.5 illustrates the probabilities that representatives from differently sized member states are pivotal in the Council assuming a 50% decision quota and assigning voting weight based on populations size via $w_j = n_j^{\alpha}$.[24] In line with above findings for randomly generated two-tier voting systems, $\alpha = 0.5$ performs best amongst all coefficients in $\{0, 0.1, \ldots, 1\}$ (cf. Fig. 2.6). Figure 2.5 shows how close the implied probability of country j being pivotal comes to the respective ideal value, which would implement a priori perfectly equal representation. Only the most populous country, Germany, would be visibly misrepresented (here: over-represented).

With the exceptions of Germany, Spain, and Poland, the current Council weights agreed in the Treaty of Nice correspond roughly to the square root of populations. It follows that *if* a single quota of 50% were used in the Council of Ministers, probabilities $\hat{\pi}_j$ would be close to their egalitarian values (with the mentioned exceptions).

[22] See Felsenthal and Machover (2004), Baldwin and Widgrén (2004), and Leech (2002) for examples. Napel and Widgrén (2006) argue formally that the Commission's and Parliament's positions are nearly irrelevant in the EU25's most common *codecision procedure*.

[23] The current voting rule (based on the Treaty of Nice) is actually quite complex. In addition to standard weighted voting it involves the requirement that the majority weight supporting a policy represents a simple majority of member states and 62% of population.

[24] These and the following numbers are Monte-Carlo estimates obtained from six runs with 10 million iterations each. In case of qualified majority voting, the pivot is identified by assuming a status quo at or to the left of the leftmost representative's ideal point.

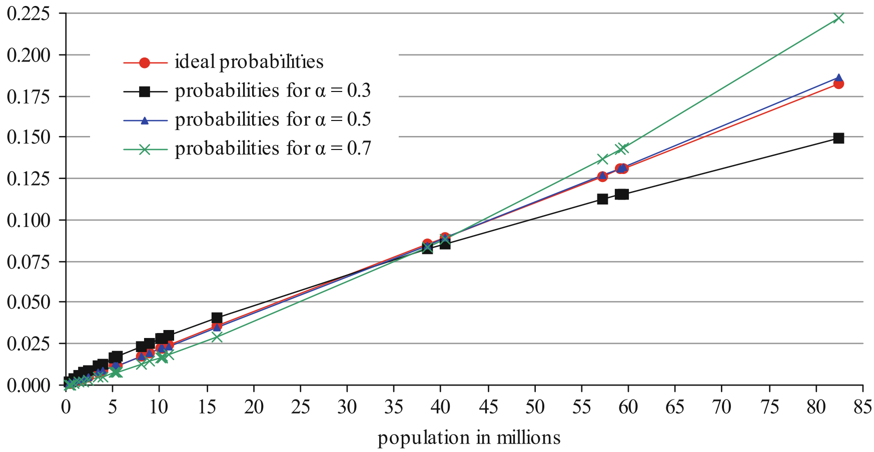

Fig. 2.5 Power law weights for EU25. Probabilities generated by weights of the form $w_j = n_j{}^{\alpha}$ are compared to ideal probabilities

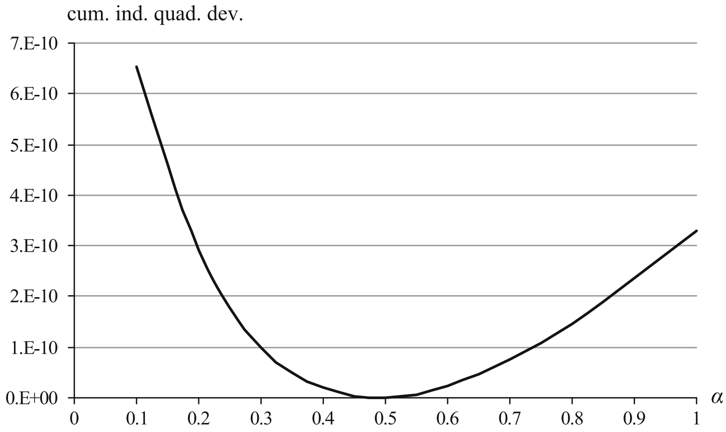

Fig. 2.6 Cumulative individual quadratic deviation for EU Council of Ministers

However, the Council uses a qualified majority of 72.2% of the weight plus additional population and number-of-supporters requirements. The latter have little effect (see Felsenthal and Machover, 2001) but the former makes a real difference.

Comparison of panels (a) and (b) in Fig. 2.7 illustrates this. With a quota significantly above 50%, a priori greater centrality of median opinion in large countries such as Germany or France no longer provides greater chances of being pivotal in the Council. It actually reduces them. So under the qualified majority rule, representation is not only even more biased against German voters, but now also French, British, and Italian representatives are less often pivotal than would be necessary to give all voters in the EU equal representation in the Council.

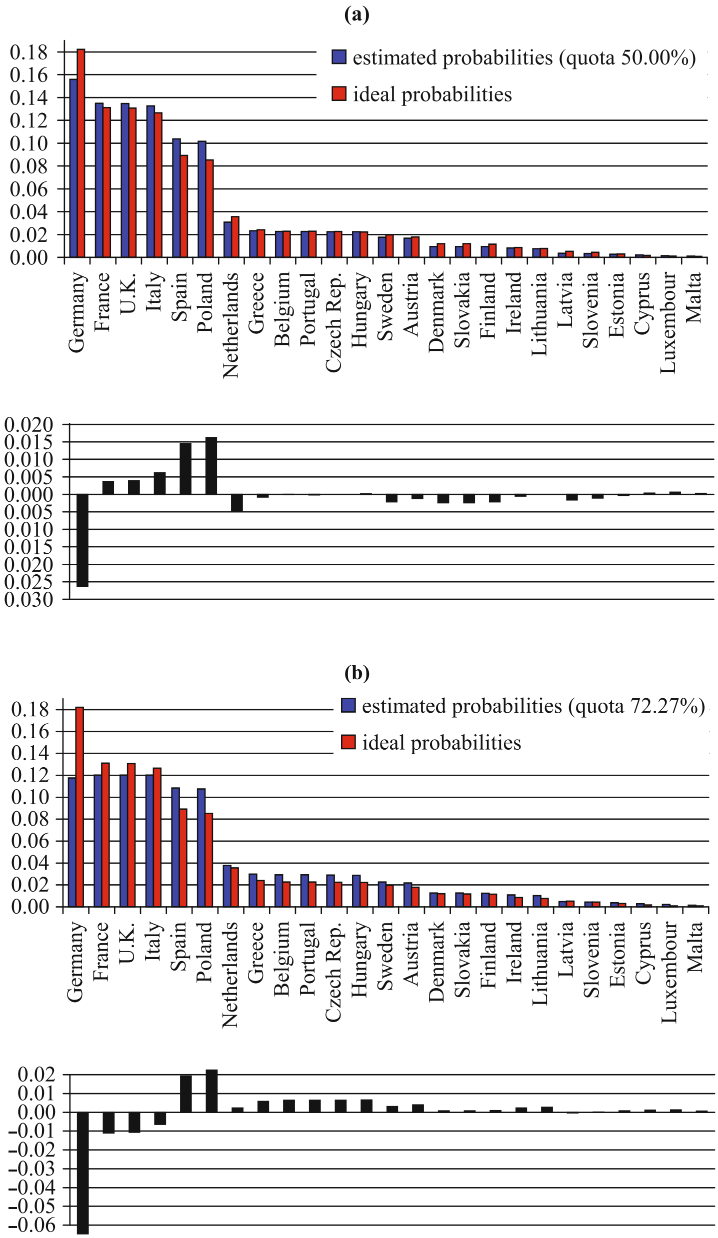

Fig. 2.7 Nice weights. Panel (**a**): EU25 with nice weights under 50% quota; Panel (**b**): EU25 with nice weights under 72.2% single quota

Note that this analysis not only puts historical voting patterns and preference similarities between some members behind a veil of ignorance but also, as do the mentioned applied studies, it disregards differences between the bottom-tier voting procedures which determine national governments. For example, the UK uses plurality rule or a 'first-past-the-post' system, whilst Germany uses a roughly proportional system.[25] This difference might have a systematic effect on the respective accuracy of our median voter assumption at the constituency level. To the extent that it does not, our findings are robust.

2.4.3 US Electoral College

US citizens elect their president via an Electoral College. The 50 states and Washington DC each send representatives to it. Their number is weakly increasing in the represented share of total population. Although most Electors are not legally bound to vote in any particular way, virtually all state representatives cast their vote for the presidential candidate who secured a plurality of the respective state's popular vote with only minor exceptions. The US Electoral College is therefore commonly treated as a weighted voting system. It actually inspired the important development of the *generating function approach* (see Mann and Shapley, 1962 and recently Algaba et al., 2003), which is the main computational technique for evaluating power under weighted voting in binary settings. Large numbers of players could hitherto only be tackled by the Monte Carlo method (see Mann and Shapley, 1960).

Decisions in the Electoral College have in the recent past been essentially binary. The pivotal player amongst the states' median voters might, however, feature prominently in a more sophisticated model of how the two main contestants are selected. In any case, consideration of strategic policy choices in a convex space provides a useful benchmark for the preference-free dichotomous model considered by Penrose (1946) and, specifically addressing the Electoral College, Banzhaf (1968).[26]

Figure 2.8 illustrates the result of determining (hypothetical) weights for state representatives based on current US state population data. Substantiating the findings of Penrose and Banzhaf, the square root rule corresponding to $\alpha = 0.5$ is again extremely successful in ensuring equal representation. Moreover, as shown by Fig. 2.9, it is clearly the best amongst all considered rules.

[25] Germany's system is actually complex: Some members of parliament are directly elected in a first-past-the-post manner, others get seats in proportion to their party's vote. Stratmann and Baur (2002) use this distinction amongst German parliamentarians to show that different electoral procedures indeed translate into different policies.

[26] Early weighted voting analysis of US presidential elections also includes Brams (1978, Chap. 3).

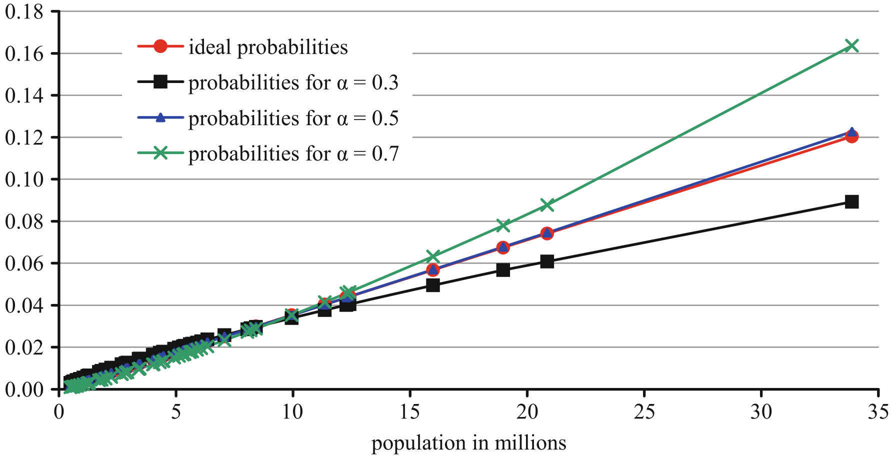

Fig. 2.8 Power law weights for US Electoral College. Probabilities generated by weights of the form $w_j = n_j{}^{\alpha}$ are compared to ideal probabilities

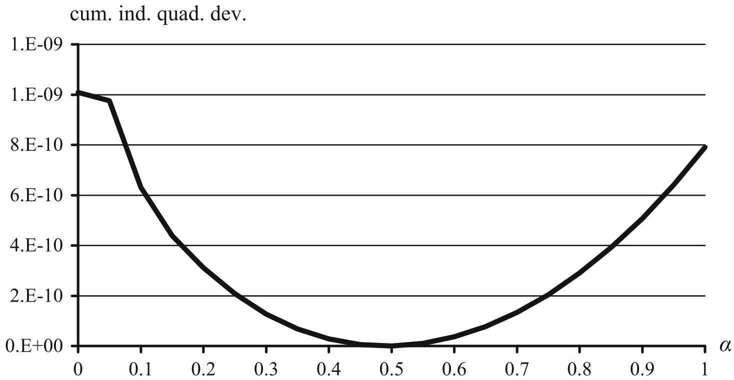

Fig. 2.9 Cumulative individual quadratic deviation for US Electoral College

2.5 Discussion

The analytical arguments in Sect. 2.3 suggest that the probability of constituency $\mathcal{C}_j$ being pivotal in the committee of representatives is approximately proportional to its weight w_j and the square root of its population size, $\sqrt{n_j}$, provided that the number of constituencies is large. This implies that approximately equal representation can be achieved by a simple square root rule. The simulation study in Sect. 2.4 corroborates this finding. Considering a vast number of randomly generated population configurations as well as recent data for the EU and the US, top-tier weights proportional to the square root of population turn out optimal for most practically relevant population configurations. Even for extreme artificial cases, the rule yields good results and becomes optimal if the number of constituencies m gets large.

Since, for large m, the simple square root rule and Penrose's rule practically coincide,[27] these results may be seen as evidence that Penrose's square root rule has a wider area of application than previously thought: in a way, its confinement to take-it-or-leave committees seems overcome.

This result is surprising because apart from the 'veil of ignorance' perspective with a priori identical but independent voters, the setting considered here is very remote from the preference-free binary model considered by Penrose (1946), Banzhaf (1965, 1968) and others. It is relevant because the binary model, from which Penrose's square root rule is derived traditionally, suffers from two serious weaknesses: first, its prediction concerning the average closeness of binary elections is rejected in empirical tests (see Gelman et al., 2002, 2004). Second, it only applies to pure take-it-or-leave-it voting situations which appear to be rare in reality. From a practical point of view, the finding of (near) optimality of a simple square root rule in the 'double pivot' model provides a better foundation for applied studies which use Penrose's square root rule as a benchmark. In contrast to the binary voting model, which considers the intra-institutional power distribution in a take-it-or-leave committee, but ignores proposal and amendment rules, the 'double pivot' model allows for strategic interaction among committee members who have proposal power, provided that the result of this interaction corresponds to the core of the voting game $[q; w_1, \ldots, w_m]$. The square root rule thus seems to be less of an artifact of a particular objective function or setting.

Square root rules have been demonstrated to be optimal under criteria other than the equality of influence. If, in a binary voting model, weights of the delegates, rather than their powers, are made proportional to the square root of their constituency's population size, the *second square root rule* (Felsenthal and Machover, 1998, pp. 72ff) results. This rule minimizes the expected *majority deficit*, which corresponds to the mean deviation of the indirect two-tier decision-making rule from a 'direct democracy' simple majority rule. Beisbart and Hartmann (2006) and Beisbart and Bovens (2007) arrive at the square root rule in a welfarist binary framework starting from the norm that expected utility should be equalized for all countries. Basically the same result is obtained by Barberà and Jackson (2006), who study the design of voting rules that maximize the expected utility in a 'fixed-size-of-blocks model', i.e., for large constituency populations which contain variably many small blocks.[28]

With respect to the EU Council of Ministers, it seems desirable to base the design of decision-making rules on clear principles rather than ad hoc settlements as this would prevent the haggling for influence to start anew each time new members are admitted or constitutional reform stands for debate. Nevertheless, at this point, it would be imprudent to recommend Penrose's square root rule or its simpler cousin

[27] In fact, Penrose (1946) seems to have deliberately blurred the distinction between *voting weight* and *voting power* in his discussion of equal representation in a world assembly. Penrose was aware, however, that approximate proportionality of weight and power generally holds only for sufficiently many constituencies.

[28] Also see Beisbart et al. (2005) for a related utilitarian investigation.

as a standard for constitutional design in the real world.[29] For one thing, the search for the optimal decision rule was confined to 50%-majority rule. Figure 2.7, however, indicates that the Treaty of Nice weights which approximately follow a square root rule produce unequal representation when applying the EU25 decision threshold of 72.2%. For another, the result could critically depend upon the assumption that the preferences of all voters are identically distributed, i.e., that they have no relation with constituency membership. Both points are investigated in Chap. 3.

Regarding the obviously strong assumptions such as one-dimensionality of the policy space or single-peaked preferences, it must be kept in mind that the binary voting model with its assumption of equiprobable yes–no decisions is even narrower than the one used here. Yet, two other potential criticisms of our model are worth noting here. First, it is debatable whether the representatives will be fully responsive to their respective constituency's median voter on every issue. This assumes that they face competition for (re-)election, and that they "formulate policies in order to win elections" (Downs, 1957, p. 28). In reality, a representative may take positions that differ significantly from his district's median when voter preferences within that district are sufficiently heterogeneous (see Gerber and Lewis, 2004 for empirical evidence). Second, the policy space is assumed to be the same for both the general election situation at the bottom tier and the committee decision at the top. In representative democracies, however, citizens have only indirect access to the policy space by choosing representatives for the legislative period. Even if citizens' preferences do not change substantially over that period, representatives do possibly not reflect them equally well at all times, depending, for example, on whether the constituency is at the end or the beginning of the election cycle.

[29] The square root rule already played a significant role in the public discussion of a possible EU Constitution. See, for example, the open letter by Bilbao et al. (2004) to the EU members' governments with repercussions in various national news outlets.

Chapter 3
Robust Equal Representation

This chapter investigates the robustness of square root rules for equal representation in two-tiered voting systems. When policy alternatives are non-binary and decisions are made by simple majority rule, Chap. 2 demonstrated that weight proportional to the square root of population size is approximately optimal, which may be interpreted as extending the scope of Penrose's square root rule beyond the narrow limits of binary decision-making. However, in light of the normative character of this result, the simplifications used in the modeling of a complex real situation, such as, e.g., decision-making in the EU Council of Ministers, require special scrutiny.

Specifically, the aim of this chapter is to conduct a 'sensitivity analysis' regarding the square root rule, addressing the following questions:

- How does a 'simple' voting rule that derives directly from constituency sizes perform compared to more sophisticated rules that use standard power indices as reference points?
- What is the fair voting rule under supermajority rules at the top tier?
- How does the fair voting rule react to heterogeneity across constituencies?

The first point is explored in Sect. 3.1. The implementation of sophisticated rules, and namely of Penrose's square root rule, necessitates finding weights which induce a desired power distribution. As no analytic solution to this *inverse problem* is known, the evaluation involves nonlinear optimization techniques. The reported observations suggest that standard sophisticated rules capture some, but not all combinatorial aspects of weighted voting in the model which was presented in Sect. 2.2. The difference between simple and sophisticated rules turns out to be relevant only for 'small' committees.

Section 3.2 considers the effects of supermajority requirements in the delegates' committee on the representation of individual citizens. A brief review of the literature shows that 'representation' has so far been neglected as a criterion for the choice of the quota. Yet, a high decision threshold not only affects the balance of 'external costs' and 'decision-making costs' (Buchanan and Tullock, 1962) or challenges the so-called 'efficiency' of a decision-making body, but also impairs the equality of representation significantly.

Penrose's square root rule refers to a voting model where voters only differ in the constituency they live in. This implies that the voting behavior of citizens from

N.F. Maaser, *Decision-Making in Committees*, Lecture Notes in Economics and Mathematical Systems 635, DOI 10.1007/978-3-642-04153-2_3,

the same constituency is not more highly correlated than the voting behavior of citizens from distinct constituencies. Although the model introduced in Sect. 2.2 is quite different from Penrose's, it shares with the latter the assumption that all individual citizens are a priori identical. In Sect. 3.3, this premise is relaxed slightly, but without abandoning the 'veil of ignorance' perspective which is appropriate to constitutional design. Whereas citizens' preferences are still identical a priori, issue-specific attitudes are assumed to vary with constituency affiliation. The investigation of this setting gives rise to an entirely new recommendation: that representatives' weights should follow a linear rule based on the Shapley–Shubik index in order to ensure equal representation.

Section 3.4 concludes by a general discussion of the robustness of square root rules in the binary and the spatial voting model.

3.1 Simple and Sophisticated Square Root Rules

In the 'double pivot' model of Chap. 2, the simple square root rule, $w_j \propto \sqrt{n_j}$, has been found to ensure equal representation to an almost optimal extent when the number of constituencies is large. With few constituencies (and representatives), however, it becomes more important to have an index that, at least approximately, captures the power distribution generated by voting weights at the top tier. Standard power indices can be ruled out as candidates for the 'theoretically correct' index because they are based on *identical* stochastic behavior of top-tier voters, which is generally inconsistent with identical stochastic behavior of bottom-tier voters. Still, as a second-best solution, sophisticated rules that are based upon the Shapley–Shubik index (see equation (1.9)) or the Penrose–Banzhaf index (see equation (1.10)) might be expected to do better than the simple rule: the latter ignores all combinatorial aspects of weighted voting, while the former capture them at least for identical top-tier behavior. The latter is not too far off when constituencies have similar sizes.

Implementing such rules requires a solution to the *inverse problem* of finding weights which induce a desired power distribution (see, e.g., Leech, 2003; Leech and Machover, 2003). For a finite number n of committee members, the number of different voting rules is also finite, albeit increasing very quickly in n. Therefore, the set of reachable power vectors is discrete, as illustrated in Fig. 3.1. The problem of enumerating all simple games with n players could be solved by determining all *antichains* on 2^N.[1] This (unsolved) problem is known as *Dedekind's problem*, and the corresponding numbers are called Dedekind numbers. The number of games in the important subclass of non-dictatorial, weighted majority games

[1] A subset of a partially ordered set (or poset) $(P, <_P)$ – where P is a set, and $<_P$ is a partial order relation – is an antichain if any two elements of the subset are incomparable under $<_P$. Applied to simple voting games, the power set 2^N is partially ordered in respect to the inclusion $\subseteq$, and each set of minimum winning coalitions, characterizing a game, corresponds to an antichain.

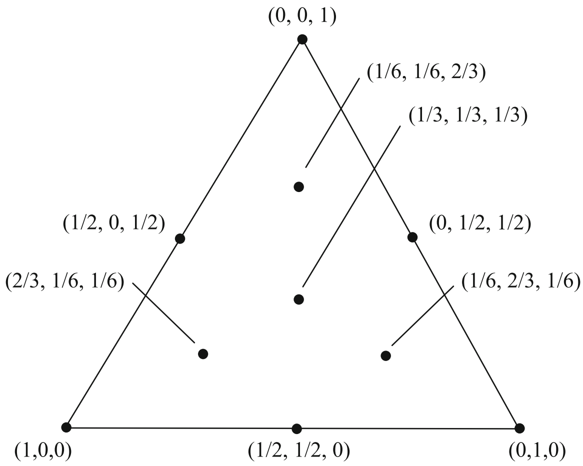

Fig. 3.1 Illustration of the nature of the inverse problem. Numbers are Shapley–Shubik indices for all proper three-player weighted majority games

with a quota of half the total weight is considerably smaller: for example, counting permutations, there are four such games with three players,[2] whereas the Dedekind number (excluding the empty antichain which contains no subsets and the antichain consisting of only the empty set) is 18.

It is worthwhile to compare the performance of the simple square root rule to that of sophisticated rules in the double median setting introduced in Sect. 2.2. For the comparison, we use 30 randomly generated configurations of 15 constituencies each.[3] Experience suggests that at this value the distribution of power is not entirely governed any more by the combinatorial particularities of the configuration at hand, but asymptotic properties only begin to operate (see Chap. 2, or, for example, Chang et al., 2006). For larger numbers of constituencies, it becomes increasingly difficult to make meaningful comparisons between weight-based and index-based rules, as the power ratio (measured by the Penrose–Banzhaf or the Shapley–Shubik index) between any two representatives typically approaches the ratio of their voting weights. This convergence is asserted by Penrose's 1952 Limit Theorem, which has been proved to hold under certain conditions (Lindner and Machover, 2004), and seems to apply whenever the weight distribution is not too skewed.[4]

[2] The minimum integer weight representations of these four games are $(3; 2, 1, 1)$, $(3; 1, 2, 1)$, $(3; 1, 1, 2)$, and $(2; 1, 1, 1)$. In Fig. 3.1, these games correspond to the four points in the interior of the simplex.

[3] In 15 of the configurations, population sizes were drawn from a uniform distribution, and in the other 15 from a Pareto distribution with $\kappa = 1.0$.

[4] For special classes of weighted voting games, Lindner and Machover (2004) prove Penrose's 1952 Limit Theorem with respect to the Penrose–Banzhaf index for $q = 0.5$ and with respect to

Let w_β and w_ϕ denote the weight vectors that are solutions to the inverse problems *"choose weights such that*

(I) $\beta_j(w,q) \propto \sqrt{n_j}$*, and*
(II) $\phi_j(w,q) \propto \sqrt{n_j}$*, for each constituency j "*,

where $\beta(\cdot)$ and $\phi(\cdot)$ refer to the Penrose–Banzhaf measure (1.10) and the Shapley–Shubik index (1.9), respectively. The comparison involves three different weight allocations: simple square root weights, w_β, and w_ϕ. A reference point to evaluate the capacity of these rules to achieve equal representation is provided by the 'best egalitarian weights', as resulting from an unconstrained search for the minimizer of the objective function

$$\sum_{j=1}^{m} n_j \cdot \left(\frac{1}{\sum_{k=1}^{m} n_k} - \frac{\hat{\pi}_j}{n_j} \right)^2 . \tag{3.1}$$

(see Sect. 2.4). These, as well as the inverse weights w_β and w_ϕ, are obtained numerically by the *Nelder–Mead simplex method* (see, for example, Avriel, 1976, Chap. 9).[5] The deviation from ideal probabilities that is associated with the best unconstrained weights can be considered as inevitable. Owing to the discrete nature of the set of possible power allocations, the discrepancy can, in general, not be eliminated completely.

Figure 3.2 suggests systematic differences in the performance of cumulative deviation (3.1) under the four sets of weights. The graphic impression is corroborated by a comparison (including all 30 configurations) of cumulative individual quadratic deviations for (1a) w_β, and (1b) w_ϕ, versus simple square root weights, (2a) w_β, and (2b) w_ϕ, versus best unconstrained weights, and (3) w_β versus w_ϕ, using the *Wilcoxon signed rank test* (see, e.g., Hollander and Wolfe, 1999, Chap. 3). In tests (1a) and (1b), the null hypothesis that the median difference between pairs of observations is zero could be rejected at the 99% significance level, indicating that both the inverse weights w_β and w_ϕ perform significantly better than simple square root weights.[6] Similarly, the null hypothesis in tests (2a) and (2b) was rejected at the 99% significance level, which suggests that both w_β and w_ϕ are none the less not the correct or first-best weights in the double median setting. In test (3), the null

the Shapley–Shubik index for $q \in (0,1)$. Their conjecture that the Theorem holds 'almost always' under rather general conditions is corroborated in a simulation study by Chang et al. (2006).

[5] The Nelder–Mead algorithm does not rely on numerical or analytic gradients, which makes it particularly suitable to non-linear optimization problems like the present. In each step of the search, the probabilities $\pi_j \equiv \Pr(j = P:m)$ of representative j being pivotal in the top-tier committee are approximated by their empirical average over 10 million iterations. A *MATLAB* computer program is used for the computations. The source code is available upon request.

[6] Generally, $\alpha = 0.5$ is not exactly the best exponent among all power laws. Obviously, the best power law weights $w_j = n_j^{\alpha^*}$ for a given configuration result in a lower deviation from egalitarian representation than simple square root weights, but they turn out to perform still worse than w_β and w_ϕ.

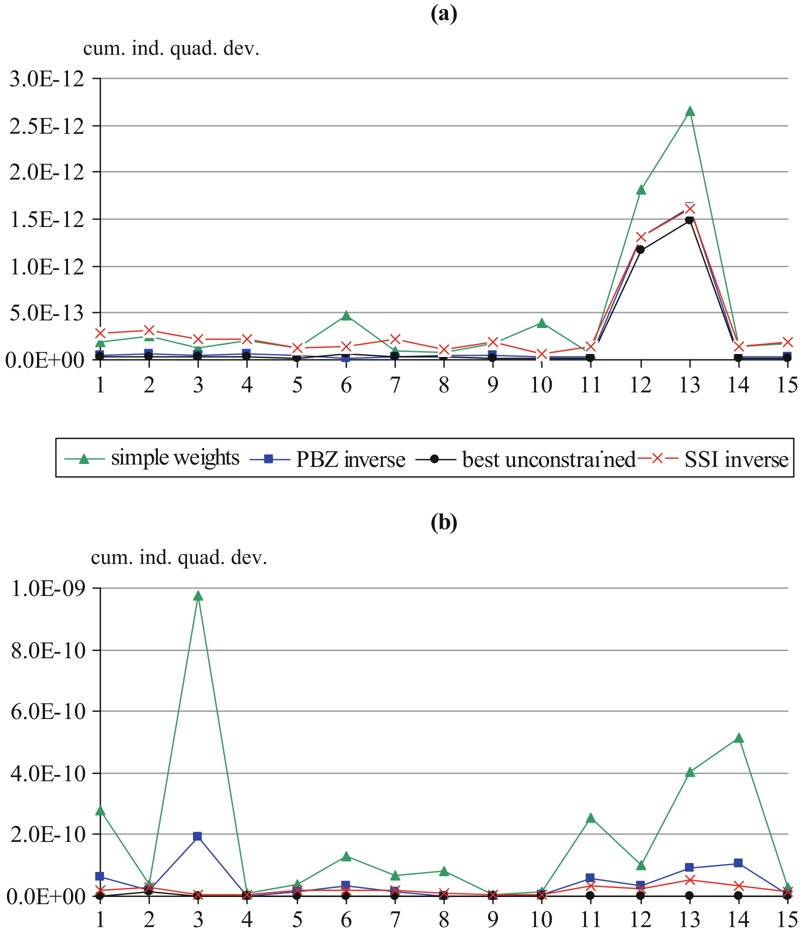

Fig. 3.2 Cumulative individual quadratic deviation under simple weights, w_β, w_ϕ, and best unconstrained weights. Panel (**a**): 15 configurations with uniformly distributed constituency sizes; Panel (**b**): 15 configurations with Pareto distributed constituency sizes

hypothesis could not be rejected, that is, no significant difference between w_β and w_ϕ was detected.

In order to find out whether these differences fade away for larger numbers of constituencies, an additional test including 12 configurations with 30 constituencies each is conducted.[7] For these, cumulative deviations (3.1) under weights w_ϕ are first compared with those under simple square root weights. The null hypothesis that the median difference between pairs of observations is zero could *not* be rejected at the

[7] The 12 configurations consist of 3×4 configurations with population sizes drawn from a uniform, a normal, and a Pareto ($\kappa = 1.0$) distribution, respectively.

95% significance level (it can be rejected at the 90% level). Second, w_ϕ is checked against the best power law weights $w_j = n_j^{\alpha^*}$ with the result that no significant difference in deviations for these two weight allocation rules could be established.

3.2 Quota Variation

A voting game is defined by the non-negative weight vector $w = (w_1, \ldots, w_m)$ and the (relative) quota $q \in (0, 1]$ which specifies the proportion of total weight required to pass a (positive) decision. Decision-thresholds of more than 0.5 are referred to as super (or 'special' or 'qualified') majority rules. Many possible criteria for choosing an 'optimal' quota exist, and we discuss some of them briefly, before we investigate the consequences of supermajority requirements in the spatial setting.

3.2.1 The 'Optimal' Quota

Dating back to Rousseau, economists and political theorists like Wicksell (1969, pp. 110ff) and Buchanan and Tullock (1962) have suggested that collective decisions should ideally be made unanimously. This recommendation is based on the unique property of unanimity rule to effect Pareto-preferred outcomes with certainty as it protects individuals against being coerced by other members of the society.[8] On the other hand, unanimity rule exemplifies the evident fact that supermajority rules favor status quo-loving minorities – or, in the extreme case, individual veto players – over change-loving majorities (see Rae, 1975).

Reaching a unanimous collective decision is also often regarded as costly in terms of decision time, and strategic behavior on the part of individuals may well inflate these costs further. Buchanan and Tullock (1962) therefore portray the optimal majority as emerging from the trade-off between the 'decision-making costs' of a rule and its 'external costs', i.e., the utility loss suffered by outvoted individuals as compared to the utility they would have secured under unanimity. In this line of thought, supermajority rules are sometimes regarded as watered-down substitutes for the ideal of unanimity. Their widespread use for grave and important questions such as amendments to the constitution suggests that the 'external costs' of these decisions may be particularly high, yet the specific quota is usually chosen on no explicit grounds as both 'decision-making' and 'external' costs defy quantification.

Majorities larger than 50% seem harder to be brought about, implying a bias in favor of the status quo. Changes to the status quo are the more difficult to achieve, the closer the quota is to 100%. A voting rule that treats each pair of issues

[8] One criticism advanced by Rae (1975) is that Pareto-optimality is compatible with a possibly outrageous distributional situation which is locked in under unanimity rule.

alike, or symmetrically, irrespective of their content is neutral as defined in May (1952). May's (1952) theorem states that simple majority rule is the only positively responsive decisive voting rule that satisfies anonymity and neutrality. Obviously, supermajority rules violate neutrality as it takes e.g., two thirds of the votes to change the status quo, but only one third to leave it in place. Under the precondition that the voting rule is anonymous, Rae (1969) shows that if, for all individuals, the costs of seeing an unwanted policy adopted are as high as the costs of not getting a policy that they support, then 50% majority rule is optimal on utilitarian grounds. The feeling that supermajority rules are less democratic than simple majority is due to their small responsiveness to individual preferences. Instead, they tend to impose a "decision based on precedent" (McGann, 2004, p. 63). As Rae (1975) argues, the privilege of the status quo over other alternatives which is implied by a high quota may well translate into a privilege of those voter groups who thrive under the status quo, e.g., the prior generation who defined the decision rules as to best serve their preferences. To the extent that they bias institutions – intentionally or accidentally – in favor of some alterative or group of voters, supermajority requirements thus also indirectly violate anonymity, which Sect. 2.1 identified as the most fundamental characteristic of democratic rule.

Moreover, a high quota can result in the inability of the body to reach any (positive) decision at all, i.e., in a lack of 'efficiency' (operationalized by Felsenthal and Machover, 2001; Baldwin et al., 2001 as the probability that a random proposal is passed in a 0-1-setting). Note that retaining the status quo because of a high quota is disadvantageous or 'inefficient' only if the decision-making body would otherwise pass a good decision.

However, the conventional wisdom that high quotas result in gridlock may not hold if side-payments are available. As suggested in Harstad (2005), the quota is irrelevant for passing a decision when opposing veto players can be compensated to accept it. For European Union politics, there is some anecdotal evidence of such transfer payments, and in fact studies of EU decision-making before and after a supposedly efficiency-reducing enlargement have failed so far to find any of the predicted effects.

Another often cited reason for using supermajorities is that they produce 'stability': They are less prone than simple majority rule to cycling across outcomes, or intransitivity (cf. Sect. 1.1.3).[9] Without any restrictions on preferences, electoral cycles can arise under all rules except unanimity, but they are less probable under 'high' quotas. For decisions where some amount of social consensus exists, Caplin and Nalebuff (1988) demonstrate that the possibility of cycles vanishes under 64%-majority rule.[10]

[9] On the other hand, supermajorities could serve to mitigate time inconsistencies if they make it more difficult to revise a policy.

[10] More precisely, the 'social consensus' which Caplin and Nalebuff (1988) presuppose amounts to the restriction that the density of voters' ideal points is a logarithmically concave function, e.g., a uniform density over a convex set. Then, points exist which cannot be defeated by any other alternative under a majority requirement larger than or equal to $1 - 1/e \approx 0.64$.

It is, however, doubtful whether the existence of a consistent choice by the collectivity should be considered a desideratum from a democratic point of view. Following Miller's (1983) and McGann's (2004) argument, cycles could well be beneficial because, by preventing that some individuals are perpetually outvoted, they enhance systemic stability. McGann (2004) suggests that, due to the instability from cycling and the relative ease with which winning coalitions can be formed, majority rule protects minorities better from exploitation than supermajoritarian prerequisites.[11]

A different branch of the literature on decision procedures focuses on their epistemic characteristics, i.e., on their capacities to aggregate information and track the truth (see, e.g., Nitzan and Paroush, 1984). When it is, at least in principle, possible to assess objectively what the correct decision is, as in a jury's verdict, supermajority rules may be preferable to simple majority rule as they could increase the probability that the correct choice is made. As it is hard to argue that an independent standard of correctness for political decisions exists, the procedural perspective seems, however, far more relevant here than the epistemic.

The quota can also be used to make each player's power (as measured by some index) proportional to his or her voting weight. For example, Dubey and Shapley (1979, Theorem 6) prove that expected power as measured by the normalized Penrose–Banzhaf index is equal to relative weight if the quota is chosen randomly (for a discussion of such randomized decision rules see Holler, 1985). The objective of *strictly proportional power* may be motivated by a concern for the transparency or, thinking of the allocation of parliamentary seats, even the stability of the political system. Specifically with regard to voting in the EU Council of Ministers, Słomczyński and Życzkowski (2007) suggest a voting system which is called the "Jagiellonian Compromise": It consists of a square root weight allocation and a voting threshold chosen to achieve maximal proportionality between weights and power.

3.2.2 Supermajorities and Representation

In view of the important role that supermajority rules play in theory, and their widespread use in real-world decision-making, it is worthwhile to investigate the 'double pivot' model (cf. Sect. 2.2) with respect to the effect of using a quota $q \gg 0.5$ in the top-tier assembly. Of course, even approximately equal representation is impossible under unanimity rule (keeping the bottom-tier role of the median). For $0.5 \ll q \ll 1$, optimal assignments can be expected to give large constituencies greater weight than implied by $\alpha = 0.5$.

[11] The claim that supermajorities are minority-protecting rests on the assumption, uncovered by McGann (2004), that the status quo is more benign for the minority than government action to change it.

In contrast to simple majority rule, the voting game under supermajority rule is not decisive. This means that possibly no policy $x \in X$ exists which defeats all alternatives $x' \neq x$ in a pairwise comparison. The probability that the outcome of collective decision-making is merely a confirmation of the status quo is a measure of the *institutional inertia* created by the decision threshold. The following analysis, however, concentrates on creative power rather than representatives' abilities to preserve the status quo.

To this end, the status-quo, Q, is fixed to a point equal to or left of the leftmost representative's ideal point, which implies that it will always be displaced in favor of some policy to its right by a winning coalition. This assumes that committee members agree about the direction of policy change. Suppose, for illustrative purposes, that $X = [0, 1]$ and $Q = 0$. Moreover, let a continuum of representatives have equal weights and their policy positions be distributed uniformly on X. Then, for a given value of $q \in (0.5, 1)$, all policies $x \in (0,\ 2(1-q))$ are preferred to the status quo by a majority of at least q. However, any policy $x < 1-q$ could still be improved upon by a share of representatives greater than q. A continuous process of 'displacement' of the status quo in the top-tier committee can be expected to come to a halt at $x = 1-q$. A further movement to the right will be blocked by at least the representative whose ideal point is equal to $1-q$, and who is strictly necessary to form a winning coalition. For example, the policy outcome under a quota of 0.75 would be the first quartile point of the distribution of representatives' ideal points.

Under weighted voting, and for discrete representatives' ideal points, the above reasoning suggests that the policy adopted in the committee of representatives coincides with the ideal point $\lambda_{P:m}$ of representative $P:m$ who is pivotal 'from the right'. The random variable P is defined by

$$P \equiv \min \left\{ l \in \{1, \ldots, m\} \colon \sum_{k=1}^{l+1} w_{k:m} > (1-q) \sum_{j=1}^{m} w_j \right\}.$$

In principle, it seems feasible, for 'limit' situations, to extend the analytical arguments put forward in Sect. 2.3 concerning the probability of top-tier pivotality under simple majority rule to the case of supermajorities. This being beyond the scope of the present work, we resort to Monte-Carlo simulation to evaluate rules of the form $w_j = n_j^{\alpha}$ (cf. Sect. 2.4) and search for the optimal α given alternative values of the quota. Again, the extent to which the considered rule falls short of the egalitarian norm (2.1) or (2.2) will be measured by cumulative quadratic deviation at the individual level as given by (3.1). First, randomly generated configurations will be investigated, then, we briefly look at the EU Council of Ministers.

3.2.2.1 Randomly Generated Configurations

Under different assumptions about the distribution of constituency sizes, each of the Tables 3.1–3.3 reports optimal values of α for four configurations with $m = 30$ constituencies. As mentioned in Sect. 3.1, the difference between simple

Table 3.1 Optimal value of α for constituency sizes from Uniform distributions $\mathbf{U}(a,b)$ (cumulative individual quadratic deviations from ideal probabilities in parentheses)

q (%)	Distribution of constituency sizes			
	(I) $\mathbf{U}(0, 10^8)$	(II) $\mathbf{U}(0, 10^8)$	(III) $\mathbf{U}(3 \times 10^6, 10^7)$	(IV) $\mathbf{U}(3 \cdot 10^6, 10^7)$
50	0.49	0.49	0.46	0.42
	(4.85×10^{-15})	(4.28×10^{-15})	(3.46×10^{-14})	(9.93×10^{-13})
55	0.50	0.50	0.52	0.54
	(2.05×10^{-15})	(1.58×10^{-15})	(2.33×10^{-14})	(3.23×10^{-14})
60	0.52	0.52	0.50	0.46
	(9.00×10^{-15})	(9.52×10^{-15})	(4.84×10^{-14})	(4.41×10^{-13})
65	0.56	0.56	0.58	0.60
	(5.40×10^{-14})	(3.48×10^{-14})	(1.36×10^{-13})	(4.12×10^{-14})
70	0.62	0.62	0.62	0.58
	(2.16×10^{-13})	(1.50×10^{-13})	(2.60×10^{-13})	(8.56×10^{-14})
75	0.70	0.70	0.70	0.72
	(5.79×10^{-13})	(4.10×10^{-13})	(6.95×10^{-13})	(2.52×10^{-13})
80	0.80	0.80	0.82	0.82
	(1.32×10^{-12})	(9.36×10^{-13})	(1.72×10^{-12})	(5.68×10^{-13})

Table 3.2 Optimal value of α for constituency sizes from Normal distributions $\mathbf{N}(\mu,\sigma)$ (cumulative individual quadratic deviations from ideal probabilities in parentheses)

q (%)	Distribution of constituency sizes			
	(I) $\mathbf{N}(10^7, 4 \times 10^6)$	(II) $\mathbf{N}(10^7, 4 \times 10^6)$	(III) $\mathbf{N}(10^7, 2 \times 10^6)$	(IV) $\mathbf{N}(10^7, 2 \times 10^6)$
50	0.50	0.50	0.40	0.50
	(2.38×10^{-14})	(9.33×10^{-14})	(8.78×10^{-12})	(1.20×10^{-11})
55	0.50	0.50	0.64	0.65
	(1.73×10^{-14})	(8.27×10^{-14})	(3.47×10^{-14})	(1.87×10^{-13})
60	0.52	0.52	0.50	0.50
	(3.75×10^{-14})	(7.64×10^{-14})	(7.72×10^{-12})	(1.08×10^{-11})
65	0.55	0.55	0.68	0.72
	(1.29×10^{-13})	(7.10×10^{-14})	(4.95×10^{-14})	(7.00×10^{-14})
70	0.60	0.62	0.50	0.56
	(4.47×10^{-13})	(3.40×10^{-13})	(4.37×10^{-12})	(7.00×10^{-12})
75	0.68	0.70	0.80	0.84
	(1.12×10^{-12})	(5.00×10^{-13})	(1.57×10^{-13})	(1.72×10^{-13})
80	0.80	0.80	0.66	0.72
	(2.64×10^{-12})	(1.78×10^{-12})	(7.87×10^{-13})	(2.18×10^{-12})

and sophisticated rules becomes insignificant for this number of players.[12] We therefore concentrate on findings regarding optimal rules of the type $w_j = n_j^{\alpha}$. The values of α run from 0 to 1 in 0.01-intervals, and probabilities π_j were estimated by simulations with 10 mio. iterations. Values in parentheses are the deviations (3.1) associated with the optimal α.

[12] The tests for 30-constituency unions reported at the end of Sect. 3.1 concern precisely the configurations which are used in the present simulations on quota variation.

Table 3.3 Optimal value of α for constituency sizes from Pareto distributions $\mathbf{P}(\kappa, x)$ (cumulative individual quadratic deviations from ideal probabilities in parentheses)

q (%)	Distribution of constituency sizes			
	(I) $\mathbf{P}(1.0, 500000)$	(II) $\mathbf{P}(1.0, 500000)$	(III) $\mathbf{P}(1.8, 500000)$	(IV) $\mathbf{P}(1.8, 500000)$
50	0.48	0.46	0.48	0.46
	(1.96×10^{-12})	(7.46×10^{-12})	(3.37×10^{-13})	(1.86×10^{-11})
55	0.50	0.50	0.50	0.48
	(2.59×10^{-13})	(2.44×10^{-12})	(4.34×10^{-13})	(2.55×10^{-12})
60	0.56	0.56	0.52	0.52
	(1.44×10^{-11})	(2.94×10^{-11})	(3.99×10^{-13})	(4.59×10^{-11})
65	0.66	0.68	0.56	0.56
	(1.22×10^{-10})	(2.56×10^{-10})	(1.95×10^{-12})	(2.67×10^{-10})
70	0.78	0.80	0.60	0.62
	(5.22×10^{-10})	(9.72×10^{-10})	(1.50×10^{-11})	(5.73×10^{-10})
75	0.80	0.80	0.70	0.72
	(1.62×10^{-9})	(2.74×10^{-9})	(7.69×10^{-11})	(7.71×10^{-10})
80	0.90	0.90	0.84	0.84
	(2.99×10^{-9})	(4.31×10^{-9})	(2.30×10^{-10})	(8.53×10^{-10})

Three observations apply irrespectively of the distributional assumption. First, α^* increases in the quota. This is due to the fact that the median voter of large constituencies is more central, which lowers the chances of the constituencies to be pivotal when a considerable supermajoritarian rule is used. To compensate this effect, the weight of populous countries has to rise. Second, the deviation from ideal egalitarian probabilities generally also increases. From $q = 55\%$ to $q = 80\%$, the quality of representation deteriorates by up to a factor of 1,000. This decline indicates that any power law either gives not enough or too much pivot probability to large constituencies. Third, while one might have expected cumulative individual quadratic deviations to be lowest under simple majority, they reach their minimum at a quota of 55% (among all quotas considered here). It seems that the slightly higher threshold impairs to the right degree the chances of large constituencies who tend to be a little over-represented with simple square root weights (cf. Figs. 2.5 and 2.8), and thus makes representation more egalitarian.

Table 3.1 relates results for uniformly distributed constituency sizes $n_1, \ldots, n_{30}$. Populations in configurations (I) and (II) come from a uniform distribution over $[0, 10^8]$, and those in (III) and (IV) come from a uniform distribution over $[3 \times 10^6, 10^7]$. It will be readily noticed that the deviations in columns (I) and (II) are smaller than deviations in (III) and (IV) except for the highest quotas where no systematic difference is apparent. Moreover, the optimal α exhibits greater stability from configuration (I) to (II) than between (III) and (IV). As the variance of $\mathbf{U}(0, 10^8)$ is by a factor of 200 greater than that of $\mathbf{U}(3 \times 10^6, 10^7)$, the data suggest a positive relationship between the variance of the population numbers and the accuracy of the equal representation rule. These findings are corroborated by the data for normally distributed populations contained in Table 3.2, where again the two left columns pertain to more variable population configurations than the two right columns. The likely explanation for these patterns is that a great amount of variance in population numbers translates *ceteris paribus* into a great variety of weights in the power law allocation $w_j = n_j^{\alpha}$. This implies, for a given quota, that more distinct winning coalitions exist, enabling a closer match between achievable and ideal probability vectors.

It is worth noting that the variances of $\mathbf{U}(3\times 10^6, 10^7)$ and the normal distribution $\mathbf{N}(10^7, 2 \times 10^6)$ are roughly the same, but a comparison of columns (III) and (IV) in Table 3.1 with corresponding columns in Table 3.2 reveals that estimated probabilities are, in most cases, closer to their ideal values with uniformly distributed constituency sizes. Thus, if one considers configurations of the same distribution type, the more variable distribution can be expected to allow more egalitarian representation, but across different types, variance is a less reliable indicator. Under a normal distribution, many constituencies are of similar size, and the minor differences between them cannot easily be reflected adequately by pivot probabilities. Then, the precise value of α^* and the quality of representation may depend heavily on the particular constituency configuration at hand, as is the case with configuration (III) in Table 3.2.

Table 3.3 shows results for population sizes drawn from a Pareto distribution $\mathbf{P}(\kappa, \underline{x})$ (cf. definition (2.17)). The parameter $\kappa > 0$ determines the shape or skewness of the distribution, and $\underline{x} > 0$ is the minimum possible value.[13] Here, only a single or very few large constituencies exist, which are particularly disadvantaged by their central position when supermajority rules apply. A high value of α would give them a power monopoly, but a moderate α gives them insufficient pivot probabilities. This logic drives the rather low values of α under simple majority rule as well as the comparatively high values for the most demanding quotas.

3.2.2.2 EU Council of Ministers

The EU Council of Ministers decides the largest part of issues by qualified majority voting. A proposal is adopted if, first, 255 out of 345 votes (73.9%) are cast in its favor. The number of votes allocated to each member state roughly reflect the square root of population size. Additionally, the majority weight supporting a proposal must represent a simple majority of member states. Finally, any member state may ask for confirmation that the approving votes represent at least 62% of the EU's total population. The latter two requirements are, however, insignificant as they are in the great majority of cases fulfilled whenever the qualified majority is met (see Felsenthal and Machover, 2001). With regard to EU decision-making the assumption of a status quo fixed to the left of the leftmost representative's ideal point is

[13] It is not possible to compare the columns in Table 3.3 with respect to the variance of constituency sizes because the variance of $\mathbf{P}(\kappa, \underline{x})$ is infinity for $\kappa \leq 2$.

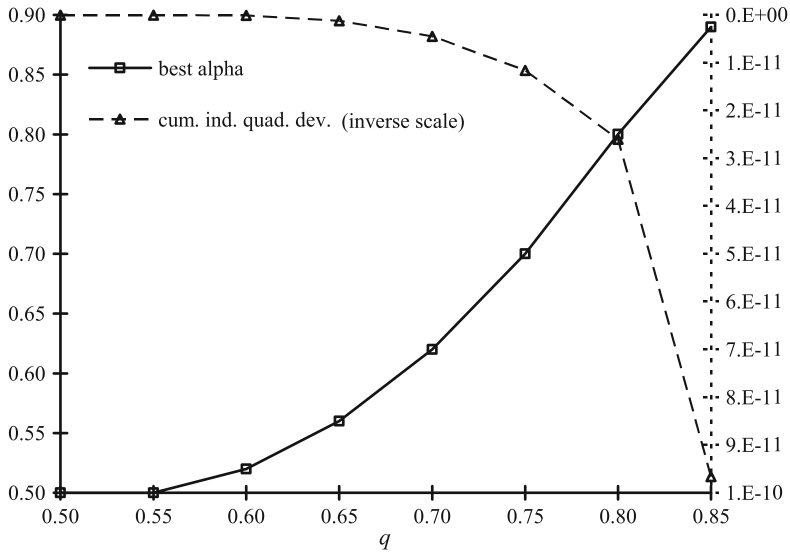

Fig. 3.3 Effect of quota variation for EU27

particularly interesting, because often not the direction of new legislation, but only the extent of change is subject to the process of legislative bargaining.

Figure 3.3 shows, for EU27 population data, the effect of a quota $q > 0.5$ on representation. The respective best value of $\alpha \in \{0, 0.02, \ldots, 0.98, 1\}$ is represented by the solid graph which is measured on the left vertical axis. The figure suggests that the optimal α is approximately a quadratic function of q. The right vertical axis measures the corresponding cumulative individual quadratic deviation. As it has its zero point in the upper right corner, the dashed graph can be interpreted as indicating the closeness between ideal egalitarian probabilities and the (estimated) pivot probabilities under the optimal α-rule. The drop in closeness (or rise in deviation) means that representation becomes increasingly unequal as the quota is increased. The following numbers may help to get an idea of the deterioration of fit: The ideal probabilities of Malta and Germany are 0.08% and 16.62%, respectively. Already for $q = 0.7$, however, their (estimated) pivot probabilities under the best weight assignment rule $w_j = n_j^{0.62}$ are 0.06% and 15.45%, which is 25% and 7% short of the ideal values.

3.3 Heterogeneity Across Constituencies

The presumably most questionable assumption in our previous analysis (and that of Penrose, too) is that, for each issue, all voters' opinions are assumed to be identically distributed. They have no relation to constituency membership whatsoever.

In contrast to the model considered so far, with i. i. d. ideal points of all individual voters throughout, this section explores the idea that voters within a country are somewhat similar.

Similarities in the attitudes of citizens from the same constituency could be the result of a sorting process à la Tiebout (1956), i.e., voting with one's feet. In Alesina and Spolaore (2003), preference homogeneity within a country is argued to develop over time due to geographical proximity and national policies fostering cultural uniformity. The fact that citizens of one country usually share historical experience, traditions, language, communication etc. can be expected to induce some kind of common set of values or 'common belief' in them.[14] The existence of diverse 'common belief' systems is just the reason why countries persistently differ in their population sizes and why they can hardly be redistricted so as to produce equal-size constituencies. Under the premise of Penrose's square root rule that the preferences of the individuals within one country are unconnected, there is no justification – apart from the tough practical realization – not to regroup citizens into purely administrative districts with equal numbers of voters. When all voters are i. i. d., what, abstracting from potential epistemic reasons, is the rationale for a committee of representatives instead of a single president-like decision-maker?

The case being argued for here is that the fact that voter preferences within a constituency tend to 'have more in common' than preferences *across* constituencies should be considered as known behind the veil of ignorance, and should be taken into account for normative, or constitutional design, purposes. In Fig. 3.4, which illustrates different degrees of using information in the assessment of voting situations, the assumption is referred to as 'a priori II'. It is in line with Braham and Steffen (2002) who take the term 'a priori' to describe a perspective which incorporates the 'structure' (as opposed to particular preferences) that conditions voters' behavior.[15] The analysis based on 'a priori II' is to be distinguished from studies that model, especially with regard to the EU, similarities or dissimilarities between countries based on economic or social dimensions (see Widgrén, 1995), size or geographical position (see Beisbart and Hartmann, 2006), and that possibly contribute to understand decision-making from an *interim* or *a posteriori* perspective.

Generalizing our earlier model, the 'a priori II' assumption is implemented by introducing constituency-specific distributions of individual ideal points. Given a policy issue, the ideal points λ_i^j of voters in constituency $\mathcal{C}_j$ come from an arbitrary identical distribution F_j with density f_j and *distribution median* $\tilde{\Delta}_j$.[16] It is assumed that, rather than being identical, the $\tilde{\Delta}_j$ are random variables with an identical distribution $H_j = H$ for all $j = 1, \ldots, m$. The expected value of $\tilde{\Delta}_j$ is assumed to

[14] A 'common belief' is also represented by Straffin's (1977) *homogeneity assumption* under which the probability of a voter 'affecting the outcome' coincides with the Shapley–Shubik index.

[15] Concerning the probabilistic interpretation of power measures (cf. Sect. 1.2.1), Braham and Steffen (2002) argue that the whole range of partial homogeneity assumptions is no less a priori than its two borderline cases, i.e., the Banzhaf index and the Shapley–Shubik index.

[16] If the distributions of individual ideal points are symmetric, $\tilde{\Delta}_j$ could also refer to the mean of the distribution f_j.

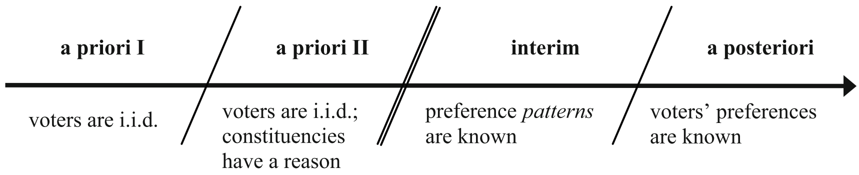

Fig. 3.4 Increasing degree of information usage in the assessment of decision-making situations

be zero, and the standard deviation is given by $\sigma_H > 0$, measuring the degree of heterogeneity across constituencies. Let h denote the density function of H. Chapter 2 dealt with the special case without heterogeneity across constituencies, i.e., $\sigma_H = 0$. Generally, a realized distribution F_j is specific to constituency $\mathcal{C}_j$, thus expressing the 'common belief' of that constituency, whilst the ideal points of voters from different constituencies are independent. It is worthwhile to emphasize that, in expectation, all voter ideal points still have an identical a priori distribution, but group membership now makes a difference interim, and this is acknowledged in our a priori analysis.

As noted before, in the case of i. i. d. voters' ideal points, the representative of a larger constituency is on average more central in the electoral college, and given simple majority rule, more likely to be pivotal in it for a given weight allocation. In the light of the standard deviation of the (approximate) distribution of the population median λ_j as given by

$$\sigma_j = \frac{1}{2\, f_j(F_j^{-1}(0.5))\sqrt{n_j}} \tag{3.2}$$

(cf. Sect. 2.2), it is possible that slight differences in the countries' ideal point distributions suffice to make representatives' ideal points virtually identically distributed. The extent of the necessary perturbation depends on the population sizes involved. For example, according to (3.2), the largest standard deviation of the median ideal point in the EU27 (belonging to Malta as the smallest member state) is $\sigma_{\max} = 7.8 \times 10^{-4}$. Any amount of heterogeneity greater than that, say $\sigma_H = 0.001$, practically removes the greater centrality which would otherwise be implied by a larger population.

Let us make this intuition more precise. Given the distribution F_j of individual voters in constituency $\mathcal{C}_j$, the representative's (or median voter) ideal point λ_j is asymptotically normally distributed with mean $\mu_j = F_j^{-1}(0.5) = \tilde{\Delta}_j$, and the standard deviation given by (3.2). For a specific realization Δ of the median of all voters' ideal points in constituency $\mathcal{C}_j$, $\tilde{\Delta}_j$, let $f_{\lambda_j\,|\,\Delta}$ be the conditional density of λ_j. For any Δ, this density is a shifted version of $f_{\lambda_j\,|\,0}$. In particular, it holds that $f_{\lambda_j\,|\,\Delta}(x + \Delta) = f_{\lambda_j\,|\,0}(x)$ for all x. Now recall that Δ is a specific realization of the random variable $\tilde{\Delta}_j$ with distribution H and density h. The unconditional density of the median voter's position in $\mathcal{C}_j$ (and representative j's ideal point) λ_j is hence given by

$$f_{\lambda_j}(x) = \int_{-\infty}^{+\infty} f_{\lambda_j\,|\,\Delta}(x)h(\Delta)d\Delta$$
$$= \int_{-\infty}^{+\infty} f_{\lambda_j\,|\,0}(x-\Delta)h(\Delta)d\Delta. \quad (3.3)$$

Consider first the case that $h(\Delta)$ is a uniform density on the interval $[-a,+a]$, $a > 0$. Then, (3.3) becomes

$$f_{\lambda_j}(x) = \frac{1}{2a}\int_{-a}^{+a} f_{\lambda_j\,|\,0}(x-\Delta)d\Delta,$$

so

$$f_{\lambda_j}(x) = \frac{1}{2a}\left[-F_{\lambda_j\,|\,0}(x-\Delta)\right]_{-a}^{+a}$$
$$= \frac{1}{2a}\left[F_{\lambda_j\,|\,0}(x+a) - F_{\lambda_j\,|\,0}(x-a)\right].$$

As the standard deviation of $F_{\lambda_j\,|\,0}$ is small for 'large' constituency size n_j, we have $F_{\lambda_j\,|\,0}(x+a) \approx 1$ and $F_{\lambda_j\,|\,0}(x-a) \approx 0$ if $a \gg 0$, and therefore $f_{\lambda_j}(x) \approx 1/(2a)$ in the 'center' of the interval $[-a,+a]$, irrespective of which constituency $\mathcal{C}_j$ one considers. For x 'close' to the boundaries, $f_{\lambda_j}(x)$ depends on the constituency-specific $F_{\lambda_j\,|\,0}$, and thus differs across constituencies. Figure 3.5 illustrates the above reasoning.

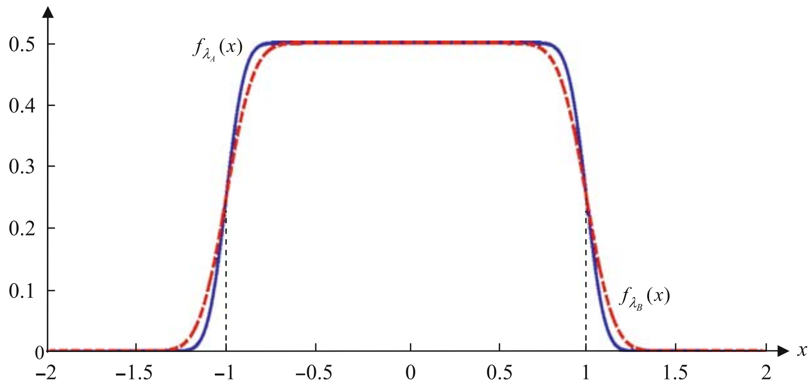

Fig. 3.5 Densities $f_{\lambda_j}(x)$ of median voter positions λ_j, $j = A, B$. The underlying $f_{\lambda_j\,|\,0}$ are normal distributions with standard deviations $\sigma_A = 0.08$ (*solid graph*) and $\sigma_B = 0.12$ (*dashed graph*). The heterogeneity function $h(\Delta)$ is a uniform density over $[-1,+1]$

If, in (3.3), $h(\Delta)$ is a normal distribution with zero mean and standard deviation σ_H, then one obtains

$$f_{\lambda_j}(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_j} \exp\left(-\frac{1}{2}\frac{(x-\Delta)^2}{\sigma_j^2}\right) \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_H} \exp\left(-\frac{1}{2}\frac{\Delta^2}{\sigma_H^2}\right) d\Delta$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma_j^2+\sigma_H^2}} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma_j^2+\sigma_H^2}\right) \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\sigma_j^2+\sigma_H^2}}{\sigma_j\sigma_H} \exp\left(-\frac{1}{2}\frac{(\Delta - \frac{x\sigma_H^2}{\sigma_j^2+\sigma_H^2})^2}{\frac{\sigma_j^2\sigma_H^2}{\sigma_j^2+\sigma_H^2}}\right) d\Delta}_{=1}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma_j^2+\sigma_H^2}} \exp\left(-\frac{1}{2}\frac{x^2}{\sigma_j^2+\sigma_H^2}\right).$$

This establishes that the ideal point λ_j of the representative from constituency $\mathcal{C}_j$ is normally distributed with mean zero and standard deviation $\sqrt{\sigma_j^2+\sigma_H^2}$. For two constituencies $\mathcal{C}_j$ and $\mathcal{C}_k$ with large but different populations sizes, it holds that

$$\lambda_j \sim \mathbf{N}(0, \sqrt{\sigma_j^2+\sigma_H^2}) \;\approx\; \mathbf{N}(0, \sqrt{\sigma_k^2+\sigma_H^2}) \sim \lambda_k$$

under the condition that σ_j and σ_k are small compared to σ_H.

Figure 3.6 shows, for two constituencies A and B of different size, sample median distributions for seven realizations of $\tilde{\Delta}_j$ $(j = A, B)$, respectively. Due

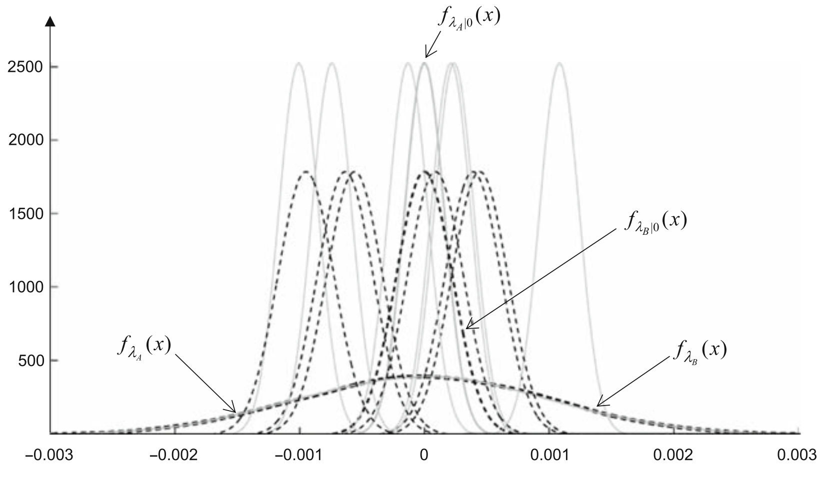

Fig. 3.6 Density functions of median voter positions of median voter positions in large constituency A (*solid gray curves*) and small constituency B (*dashed black curves*) for varying median/mean voter positions from a normal heterogeneity function. Uncertainty about the median/mean voter position results into flattened ex ante median densities $f_{\lambda_j}(x)$, $j = A, B$

to A's larger population size, the $f_{\lambda_A \mid \Delta}$ distributions are much more concentrated around the realizations of $\tilde{\Delta}_A$ than is the case for the $f_{\lambda_B \mid \Delta}$ distributions. Ex ante, however, considering the random variables $\tilde{\Delta}_j$ rather than realizations of them, the density functions $f_{\lambda_A}(x)$ and $f_{\lambda_B}(x)$ practically coincide.

More generally, note that (3.3) is the definition of the *convolution* $(h * f_{\lambda_j \mid 0})(x)$. For two independent random variables Y_1, Y_2 with $Y_1 \sim H$ and $Y_2 \sim F_{\lambda_j \mid 0}$, the convolution of their individual density functions gives the probability density of the sum $Y_1 + Y_2$. If the variance of Y_2 is small compared to that of Y_1 (as is the case if $F_{\lambda_j \mid 0}$ is the distribution of the median of a large population), then the random variable Y_2 can be viewed as almost constant. Hence, the variance of $Y_1 + Y_2$ is practically determined only by the variance of Y_1.

The above arguments demonstrate that representatives' ideal points are virtually i. i. d. under the assumption of some heterogeneity between constituencies. This implies that all $m!$ orderings of representatives are equiprobable. The chances π_j of any representative j to be the pivot at the top tier are then captured by the Shapley–Shubik index $\phi_j(w, q)$. Hence a simple rule ensuring equal representation emerges:

Shapley–Shubik linear rule (SSLR): With any amount of heterogeneity $\sigma_H \gg \max_j\{\sigma_j\}$ and for a given decision quota q, the weights w^* which satisfy the egalitarian norm (2.1) or, equivalently, (2.2) are defined implicitly as solutions to the inverse problem

$$\phi_j(w^*, q) \propto n_j, \qquad j = 1, \ldots, m. \tag{3.4}$$

The finding that in the presence of heterogeneity weights should be chosen such that the Shapley–Shubik index is proportional to population size for each constituency is illustrated by Fig. 3.7 for the EU Council of Ministers. Note that the proportional Shapley–Shubik rule holds for *any* quota used at the top tier, but the remarks concerning the inverse problem under high quotas still apply: implementing the above rule requires a solution to the *inverse problem* of finding weights that yield the desired values. In general, only approximative solutions to this problem exist because the number of distinct voting games on the set of players $\{1, \ldots, m\}$ is finite, whereas the number of combinations of desired values is infinite. This technical problem is perceivable in Fig. 3.7: When a 50%-majority rule is used, the Shapley–Shubik indices associated with best unconstrained weights are located nicely on the 45°-line, but under the 73.9%-quota, they rather meander around that line.

In line with the above discussion, Fig. 3.8 demonstrates that the transition from square root rule to a near-linear rule takes place very quickly. The square root rule survives for $\sigma_H = 0.00001$, but already for $\sigma_H = 0.00005$ we get $\alpha^* = 0.58$. Given the small variation in the setting – in Chap. 2 mean ideal points came from the degenerate normal distribution $\mathbf{N}(0, \sigma_H = 0)$ – the result differs strikingly from our previous finding $\alpha^* \approx 0.5$. The square root rule ceases to apply already for small degrees of heterogeneity.

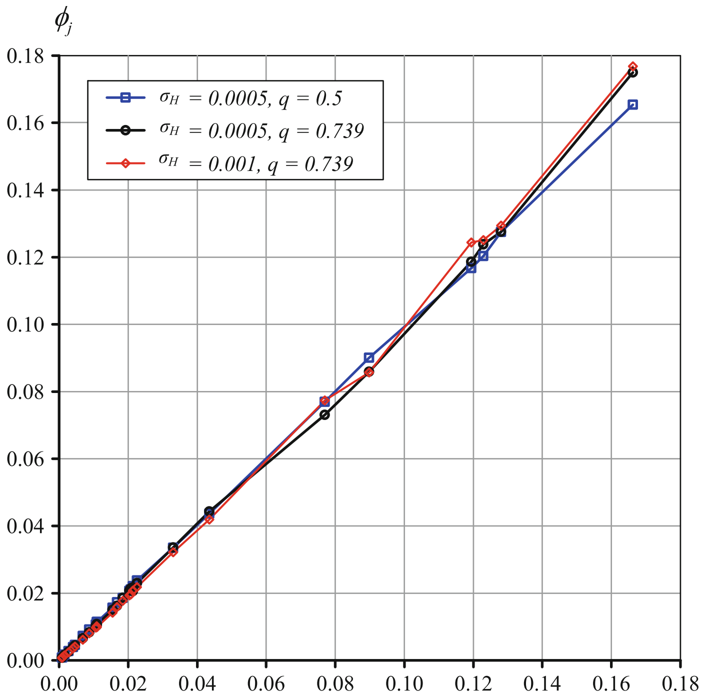

Fig. 3.7 Shapley–Shubik linear rule: Shapley–Shubik indices for best unconstrained weights in EU27

Figure 3.9 plots the objective criterion (3.1) versus α.[17] The mean ideal points μ_j of the 27 countries are drawn from the normal distribution $\mathbf{N}(0, \sigma_H = 0.001)$ and voters' ideal points in constituency j are uniformly distributed on $[\mu_j - 0.5, \mu_j + 0.5]$. Amongst all coefficients in $\{0, 0.01, \ldots, 1\}$, $\alpha = 0.94$ performs best with a cumulative individual quadratic deviation of 7.02×10^{-13}. For these best simple weights and SSLR weights, Fig. 3.10 shows the absolute deviations of estimated from ideal egalitarian probabilities, applying a hypothetical quota of $q = 0.5$. Apart from Germany's being over-represented under the simple rule, the overall difference between the two weight allocations is small, which is in line with our remarks at the end of Sect. 3.1 (albeit concerning square root rules).

A second form of interim heterogeneity occurs if constituencies differ in their degree of 'preference cohesion'. In particular, it is conceivable that the strength of the 'common belief' decreases in the size of the society.[18] The variance σ_j^2 of the ideal point distribution F_j $(j = 1, \ldots, m)$ in constituency $\mathcal{C}_j$ can then be written as

[17] Eurostat population numbers for EU27 countries as of 01/01/2007 are used as simulation input.

[18] The assumption that preferences are more heterogeneous in large populations is also made in Alesina and Spolaore (2003), and the trade-off between the costs of differences and the economies of scope in large jurisdictions determines nation size in their framework.

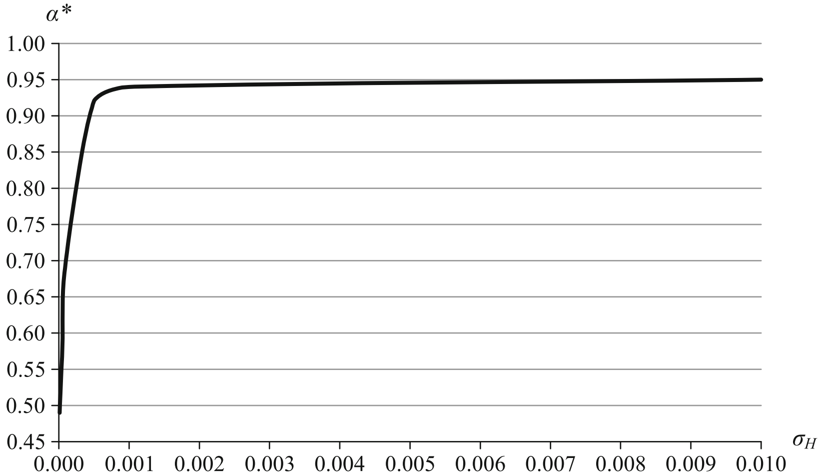

Fig. 3.8 Rapid transition from square root rule to near-linear rule for EU27 ($q = 0.5$)

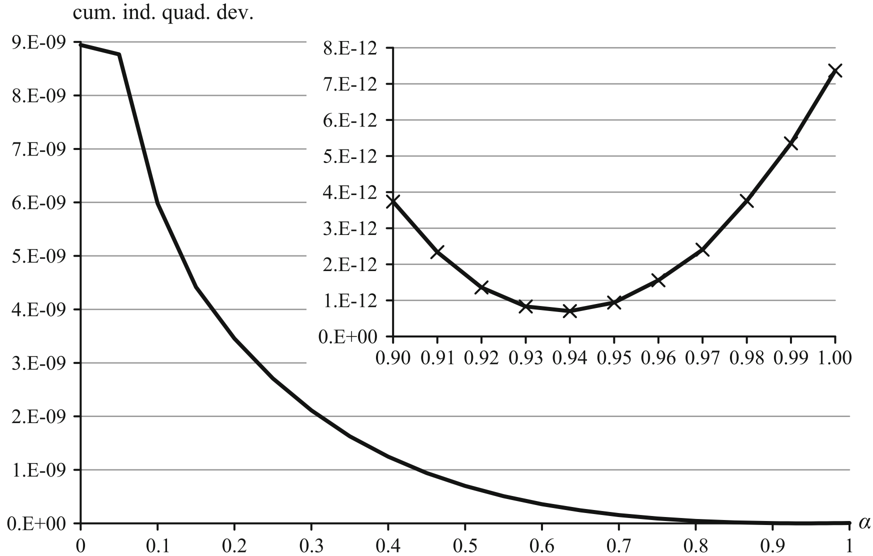

Fig. 3.9 Cumulative individual quadratic deviation for EU27 ($\sigma_H = 0.001$, $q = 0.5$)

$$\sigma_j^2 = g(n_j)$$

where $g(\cdot)$ is a monotonically increasing function.

Consider the simple case of proportionality, i.e., $g(n_j) = a \cdot n_j$ where $a > 0$ is a constant, and assume that, in each constituency j, voters' ideal points are uniformly distributed on the interval $[x_{1j}, x_{2j}]$ with common mean $\mu_j = \mu$ for all for $j = 1, \ldots, m$. The density in constituency j is

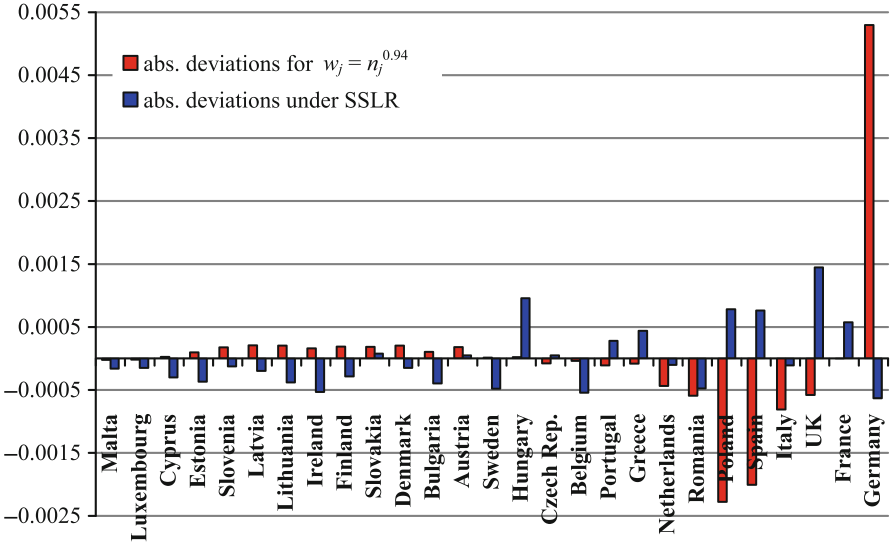

Fig. 3.10 Absolute deviations of estimated from ideal probabilities for EU27 ($\sigma_H = 0.001$, $q = 0.5$)

$$f_j(x) = \begin{cases} \frac{1}{x_{2j}-x_{1j}} & x_{1j} \leq x \leq x_{2j} \\ 0 & \text{elsewhere,} \end{cases}$$

and the variance of the uniform distribution on $[x_{1j}, x_{2j}]$ is $\sigma_j^2 = (x_{2j} - x_{1j})^2/12$. Then, the length of the interval $[x_{1j}, x_{2j}]$ for constituency j with population n_j is proportional to $\sqrt{n_j}$. As the density $f_j(x)$ appears in (3.2), the standard deviation of the median ideal point is equalized for all constituencies and for all values of a. When the variance of voters' uniformly distributed ideal points increases proportionately in population size, the representatives' ideal points are i. i. d. For the reasons stated before, we arrive again at the recommendation to allocate weights such that each representative's Shapley–Shubik power is proportional to his constituency size.

Though the assumption that preferences are more widely spread in large constituencies could seem plausible at first glance, it is less convincing if one thinks about the policy space X as the carrier of individual preferences. Rather, preferences in a small society can be as varied as in a large society. Therefore we deem the assumption that 'preference heterogeneity' is independent of population size, i.e., identical variances σ_j^2 of the λ_i^j for all constituencies, more plausible.

It is interesting to note that SSLR coincides with the 'neutral' voting rule that Laruelle and Valenciano (2008a) obtain in the context of a bargaining committee (cf. Sect. 1.2.2). A bargaining committee consists of a voting rule $\mathcal{W}$ specifying the winning coalitions, and a m-person Nash bargaining problem $B = (U, d)$, where

$U \subseteq \mathbb{R}^m$ is the set of feasible payoff vectors and d is the vector of status quo or disagreement payoffs. Unlike classical bargaining which is thought of as a unanimous decision process, an agreement in a bargaining committee only needs the support of a winning coalition to be implemented. Under the condition that the Shapley–Shubik index ϕ is accepted as a valid measure of bargaining power, Laruelle and Valenciano (2007) axiomatically derive a solution $F(B, \mathcal{W})$ to the bargaining committee problem $(B, \mathcal{W})$:

$$F(B, \mathcal{W}) = \text{Nash}^{\phi(\mathcal{W})}(B) = \arg\max_{\substack{u \in U, \\ u \geq d}} \prod_{j=1}^{m} (u_j - d_j)^{\phi_j(\mathcal{W})}, \qquad (3.5)$$

that is, an asymmetric Nash solution with weights given by the Shapley–Shubik indices of the committee members under the voting rule. Solution (3.5) can be regarded as a reasonable expectation of the utility levels when a *general* agreement is achieved in the bargaining committee.

A voting rule is called ‘neutral’ if every citizen in every constituency $\mathcal{C}_j$ is indifferent between bargaining for himself with all other citizens in the union or leaving the bargaining to a representative who bargains on behalf of $\mathcal{C}_j$ in the bargaining committee. As shown by Laruelle and Valenciano (2008a), a rule with this property exists if citizens’ preferences are ‘partition-symmetric’. This property requires that all citizens in constituency $\mathcal{C}_j$ have a common status quo payoff d_j, and that the set of payoff vectors which are attainable for them must be symmetric for any fixed distribution of payoffs among non-members of $\mathcal{C}_j$. Under partition-symmetry, citizens within the same constituency have the same bargaining characteristics. This makes it possible to condense the bargaining problem that would be faced by the ‘committee of the whole’, i.e., by all citizens bargaining directly, into the m-person bargaining problem $B = (U, d)$. Let u_i^{dNB} denote the payoff to citizen i under direct (and unweighted) Nash bargaining. Now suppose that i is a member of constituency $\mathcal{C}_j$. It is quite obvious that i would get a payoff equal to u_i^{dNB} if, for all j, the weight of representative j in an asymmetric Nash bargaining solution is proportional to the number of citizens in $\mathcal{C}_j$. In view of (3.5), this implies that a voting rule $\mathcal{W}$ is neutral if the Shapley–Shubik index of representative j under $\mathcal{W}$ is proportional to his constituency’s population number n_j.

3.4 Discussion

The aim of this chapter has been to examine the effects of two particularly relevant modifications to the ‘double pivot’ model of two-tier voting systems. First, top-tier decision thresholds of more than 50% have been studied by means of simulations. Second, the analysis has been made more comprehensive by portraying ‘constituencies’ not as purely administrative entities, but as reflecting heterogeneities between voters.

In the model without heterogeneity, supermajority requirements were demonstrated to drastically reduce the extent to which the egalitarian ideal can be satisfied. The conclusion from these findings would be that the quota applied in a committee of representatives has important implications for the equity and hence the legitimacy of decision-making. This contrasts with the yes-no-setting, where the mathematical validity of Penrose's square root rule is not limited to any specific value of the quota. The latter can therefore, in principle, be adjusted to pursue objectives other than equal representation. This ignores, however, that Penrose's square root rule requires a solution to the inverse problem (cf. Sect. 3.1). As the set of different weighted voting games shrinks when q increases, the inverse problem cannot be solved anymore to a satisfying degree of approximation. Practically, a high threshold impedes the appropriate implementation of Penrose's rule, and thus impairs the equality of representation in the binary model.

As suggested, for example, by Harstad (2008), the decision threshold in the representative committee may have important implications for strategic delegation. In particular, if the majority requirement is large, then it could be beneficial for the median voter to strategically delegate decision-making, or bargaining, at the top tier to a representative who is more conservative, i.e., status quo biased, than himself. The 'double pivot' model could possibly be extended by modeling explicitly the legislative bargaining in the committee of representatives, e.g., appoint at random one representative as agenda setter and let all representatives vote under closed rule for or against the proposal. Then, the identity between the median voter and the representative in each constituency could be relaxed allowing for strategic delegation, or similarly, for a distinction between the policies a representative advocates and those he votes for.

As a theoretical contribution, the present chapter has established that, in the presence of almost arbitrarily small heterogeneity among constituencies, weights for which each constituency's Shapley–Shubik index becomes (approximately) proportional to its population number ensure equal representation of individual citizens, interpreted as identical (and positive) indirect expected influence on final outcomes. The existence of such heterogeneities is regarded as 'a priori II' information, available behind an appropriately thinned veil of ignorance: individual voters are still assumed to be independently and – *in expectation* – identically distributed. Penrose's square root rule (or its simpler limit version) turns out not to be robust in light of even slight heterogeneity across constituencies. Figure 3.11 summarizes the results.

Square root rules have been found to lack robustness in other contexts before. However, the literature heretofore has considered only the binary voting model and other forms of correlation among voters' preferences. As highlighted by Good and Mayer (1975), Chamberlain and Rothschild (1981), and Kaniovski (2008), even minor changes regarding the equiprobability assumption at the individual or collective decision level in the binary model lead to substantially different swing probabilities, and thus to recommendations for operationalizing the one-person, one-vote principle that disagree with Penrose's square root rule. Also considering the binary voting model, Kirsch (2007) finds that square root weights minimize the

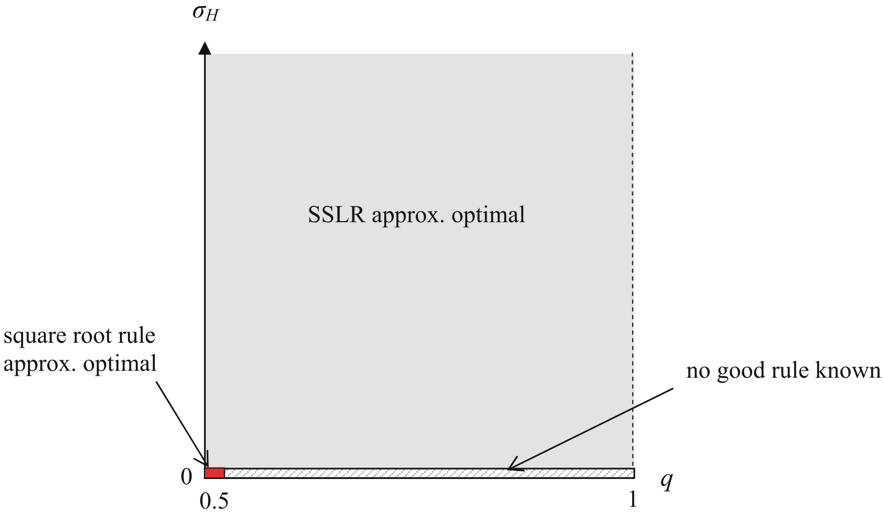

Fig. 3.11 Summary of findings of the sensitivity analysis

difference between the margin of representatives accepting or rejecting a proposal and the size of the popular margin.[19] Yet, if each constituency exhibits a 'collective bias' whose strength is independent of constituency size, the optimal weights with respect to that minimization problem turn out to be proportional to population numbers rather than to the square roots of the latter. Investigating the ideals of maximizing and equalizing expected utility, respectively, Beisbart and Bovens (2007) come to basically the same conclusion: With i.i.d. voters and simple majority rule, both ideals are met by simple square root weights. But if correlations of individual utilities within each constituency are introduced, then optimality of the square root rule breaks down quickly.

Proportional rules have emerged as normative recommendations from various models. Yet so far, they have not been shown to produce equal representation as it is understood here. Barberà and Jackson (2006) show in their 'fixed-number-of-blocks model' that, if each constituency consists of a given number of blocks of identical voters, setting weights proportional to population sizes maximizes expected utility irrespective of the voting threshold. Similarly, considering the case that the interests of people within the same constituency are perfectly correlated (but independent across constituencies), Beisbart and Bovens (2007) find that the ideal of maximizing overall expected utility requires proportional weights and a threshold of 50%.

Whereas the above findings pertain to binary decision-making, Laruelle and Valenciano (2008a) study a committee whose members bargain over a convex space of alternatives. They demonstrate that a voting rule ensures 'neutral representation'

[19] The difference between these two quantities is very similar to the *mean majority deficit* which is also minimized under square root weights (see Felsenthal and Machover, 1998, pp. 72ff).

in the sense that all individual citizens are indifferent between bargaining themselves or putting bargaining in the hands of a representative if each representative is provided with bargaining power, measured by the Shapley–Shubik index, proportional to his constituency's population size.

Neutrality of the voting rule can be interpreted in terms of equivalence between direct and indirect democracy. In the 'double pivot' model, it requires that the outcome of the two-tiered decision-making process should equal the outcome of bargaining among all individual voters in the union. Under single-peakedness and one-dimensionality, the latter corresponds to the median ideal point of all citizens. Generally, the direct and the indirect outcome differ, but it seems worthwhile to search for a weight allocation rule that is 'most neutral' in the sense of minimizing the discrepancy.

Chapter 4
Committees and Lobby Coalition Formation

This chapter finally descends from the heights of constitutional design to the domain of 'ordinary' politics: It analyzes a situation where lobbyists seek to influence decision-making in a legislature (or legislative committee) by offering payments to its members. While Chap. 1 has asked "who gets what" with respect to committee members themselves, Chaps. 2 and 3 have studied individual citizens' 'derivative' influence on decisions in a committee of representatives. The present chapter considers the question how much clout lobbyists have with a legislative committee.

Lobbyists are assumed to share common interests with respect to the political outcome, whereby lobbying efforts become a public good for them, and incentives to free-ride arise. Each lobbyist prefers someone else to contribute so that he may enjoy the benefits from a more favorable political decision without incurring the costs of bringing it about. Although increasing returns to sharing these costs exist, which would imply the formation of a 'grand coalition' of lobbyists, "The Logic of Collective Action" (Olson, 1965) suggests that, in the absence of enforceable agreements, (rational) individual lobbyists cannot be expected to act on group interest, that is, to provide an efficient level of lobbying efforts. At the same time, in the real world, lobbyists often seem able to organize themselves in order to obtain changes in legislation or regulation.

The impact of the 'legislative environment' – the decision-making procedure used by the legislature together with legislators' policy preferences – on the lobbyists' *collective action problem*, i.e., their incentives to form coalitions, has, to our best knowledge, so far not been examined by the theoretical literature. To study this topic, we propose a game-theoretic model and determine the equilibrium lobby coalition structure. The model helps to understand the lobbying of supranational legislative institutions like the EU Commission and the EU Council of Ministers.

Section 4.1 introduces our framework and clarifies the intuition behind the basic results. Section 4.2 formally presents a simple model of lobby coalition formation in anticipation of the subsequent legislative process. In Sect. 4.3, the game is solved by backward induction: the analysis reveals a link between the legislature's status quo bias and the coalition formation of the lobbyists. Section 4.4 discusses the results and their robustness, considering various modifications of the model.

N.F. Maaser, *Decision-Making in Committees*, Lecture Notes in Economics and Mathematical Systems 635, DOI 10.1007/978-3-642-04153-2_4,

4.1 Motivation

A difficult problem in political theory is to assess to which extent lobbyists can influence the outcomes of a legislative process, and which factors determine lobby success. The large empirical literature concerning these questions has yielded mixed results, both with respect to the legislative institutions of the US and those of the EU (see, e.g., Wright, 1990; Fordham and McKeown, 2003; Bernhagen and Bräuninger (2005); Mahoney, 2007; for an overview of the literature concerning the EU also see Dür, 2005). Formal work on the interaction between lobbying and legislation is still scarce. Denzau and Munger (1986) propose a model in which the legislator's decision either serves the interests of organized groups or the interests of unrepresented constituents, so as to maximize votes. Snyder (1991) analyzes lobby contributions and influence in a spatial voting model where legislators have different ideal points. Helpman and Persson (2001) investigate the implications of alternative legislative bargaining processes (US congressional and European parliamentary system) for policy outcomes and lobbying contributions.

The effect of the legislative environment on the incentives of lobbyists to form coalitions has, to our best knowledge, so far not been examined by the theoretical literature.[1] In the following, this issue is addressed with particular regard to the EU.

Empirical findings on lobbying at the EU level cast doubt on the ability of so-called Eurogroups to organize themselves, and promote the interests of their members effectively (see, for example, Jordan and McLaughlin, 1993; McLaughlin et al. 1993). Eurogroups are umbrella associations or 'associations of associations', that are officially recognized by the European Commission. Examples include the European Chemical Industry Council (CEFIC) with a membership of chemical associations from 22 countries, or the European Environmental Bureau (EEB), a federation of more than 140 environmental organizations. The perceived weakness of many Eurogroups is surprising as one might expect that the progressive extension of the EU's policy-making competencies at the expense of local and national governments (van Schendelen, 2005, p. 66f) would in fact boost the supranational organization of lobbying activities.[2] Moreover, the formation of Eurogroups is encouraged by a preference on the part of the EU institutions to consult such federations (Mazey and Richardson, 1993).[3] The accommodativeness of EU institutions to Eurogroups and other interest groups is generally attributed to the enormous information problems that legislators face with regard to the consequences and the evaluation of policies in the 27 member states.

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[1] Coalition formation among organized interests in the US has, however, been studied empirically by Hojnacki (1997). With respect to the EU, Pijnenburg (1998) presents a case study on the formation of *ad hoc* lobby coalitions of companies.

[2] The reasons commonly cited to explain their perceived ineffectiveness are internal heterogeneity and difficulties in forming a common position (Pijnenburg, 1998; Michalowitz, 2004, p. 124).

[3] The institutionalized consultation of group interests is seen as related to corporatist ideas prevailing in EU policy-making (see Michalowitz, 2004, p. 27). Yet, associations do not sit at the table when decisions are actually made, and inclusion of their suggestions is only rarely obligatory.

As a framework for studying the decision of lobbyists, which could be (national) special interest groups or individual companies,[4] to participate in a lobby coalition, this chapter provides a three-stage game-theoretic model. At the coalition-formation stage, modeled as the unanimity game Γ proposed by Von Neumann and Morgenstern (1944, p. 243f) and Hart and Kurz (1983, 1984), lobbyists simultaneously decide upon which coalition they wish to belong to. Lobby coalitions act to maximize the sum of their members' utilities. Ruling out side-payments within coalitions allows us to use the *valuation approach* pioneered by Shenoy (1979) and Hart and Kurz (1983, 1984).[5] The outcome of this stage is a partition of the lobbyists into disjoint coalitions, called a *coalition structure*.

In the second stage, according to the coalition structure established, lobbyists make contributions to members of the legislature in order to change the outcome of legislation. As in the standard *public good game* (Ray and Vohra, 1997), it turns out that only the largest coalition (in terms of members) will engage in lobbying. Lobby contributions may be thought of as pieces of information that help legislators to draft legislation or to repel criticism, or that confront them with arguments countervailing their current position. They could also be monetary or other material benefits. In the context of EU lobbying, contributions are also known to include valuable technical information or proposals on wordings of regulations. The modeling of the contribution stage builds upon the model of vote-buying with price-discrimination suggested by Snyder (1991). Unlike that model, we explicitly consider finitely many lobbyists, and include an 'independent' agenda-setter in the legislature.[6]

Finally, in the third stage, the legislature chooses a policy outcome from a one-dimensional convex policy space. The legislature's decision affects N constituencies, each of which is represented by one legislator. The model of the legislative process follows Romer and Rosenthal (1978) where an agenda-setter chooses a position that maximizes his utility subject to the constraint that a majority prefers the position to the status quo. The proposal is executed if supported by the required majority.

Lobbyists' preferences over policy outcomes are assumed to be aligned, which provides a natural potential for the formation of a grand or all-encompassing coalition. But lobbying efforts are prone to free-riding (Olson, 1965): The provision of resources to the political decision-makers is a contribution to a public good for the lobbyists. In order to concentrate on the collective action problem of the lobbyists, we assume that their policy goals are not only aligned, but coincide fully. Concerning the possibilities to influence the decision process, we impose two

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[4] Of course, collective action problems *inside* these groups or companies may exist. These are assumed to already have been resolved in the present analysis.

[5] A valuation is a mapping which associates to each coalition structure π a vector in $\mathbb{R}^n$, representing individual payoffs. A player's valuation of a certain coalition thus depends on the entire coalition structure. Hart and Kurz (1983, 1984) refer to the evaluation of players' prospects for any coalition structure as the *coalition structure value*.

[6] In Snyder's (1991) model, a single lobbyist (or a unitary lobby group) has full agenda-setting power, and there is a continuum of legislators.

assumptions: First, all lobbyists may approach the agenda-setter and offer contributions to him. Second, as in Helpman and Persson (2001), we assume a one-to-one relationship between legislators and lobbyists, i.e., an individual lobbyist can lobby exclusively one legislator, 'his own'.

The objective of the present chapter is to explore, in this setting, the relationship between the distribution of preferences in the legislature and the 'severity' of the lobbyists' collective action problem. Our main finding is that the *status quo bias* of legislators plays a key role in the endogenous formation of lobby coalitions. More specifically, there is a dichotomy between situations where the bias for the status quo on a given issue is 'small', that is, decision-makers exhibit a pronounced disposition to *change* current policy, and situations where the status quo bias is 'large'. In the former case, lobbyists can focus their efforts on the agenda-setter. In the coalition-formation game, lobbyists' *symmetry* with respect to accessing the agenda-setter leads to the formation of the grand coalition of all lobbyists, and the policy outcome under lobbying tends to be close to the lobbyists' ideal point.[7] By contrast, a large status quo bias means that the overall willingness to alter the status quo is low. We demonstrate that, in this case, recruiting the votes of some legislators is necessary, and the identity of the lobbyists who form the lobby coalition matters. The fact that the votes of a certain subset of legislators are needed to have the desired policy accepted creates *asymmetry* among the lobbyists. In equilibrium, the lobbying coalition may then be smaller than the grand coalition, which is inefficient from the lobbyists' point of view as it fails to maximize the sum of their utilities. The lobby coalition's success is small in the sense that it is, in general, very costly to achieve policies that are far away from those that would have occurred without lobby interference, and the coalition may therefore content itself with rather meek 'modifications' to the outcome. Concerning the question of *who* lobbies, the analysis suggests that only those lobbyists who are associated with legislators that are pivotal regarding the desired policy outcome, participate in the lobby group, while all others free-ride. In a twist of Axelrod's (1970) famous prediction for parliamentary democracies that ideologically connected minimal winning coalitions will form, the lobby coalition is made up of lobbyists who, rather than having themselves adjacent ideal points, are linked to ideologically connected legislators.

The major application that we have in mind for the model above is lobbying of the European institutions, namely the EU Commission and the EU Council of Ministers. The interaction between these bodies has been found to bear some resemblance to the legislative bargaining game used here, and has been modeled accordingly before (e.g., Steunenberg et al., 1999). The Commission is regularly considered as the agenda-setter *par excellence*, because it virtually has a monopoly to initiate EU legislation (Vaubel, 1997, p. 444). Thus, the considered legislative ultimatum game, with the Commission (and its bureaucracy) making a take-it-or-leave-it proposal, is quite realistic. Although the Council of Ministers does usually not draft policy

[7] How close exactly depends on how easily the agenda-setter is influenced, and how many members the legislature (and thus the lobby coalition) has.

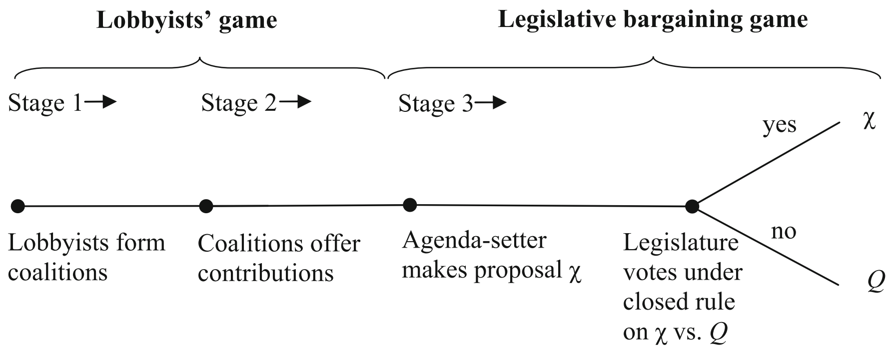

Fig. 4.1 Time structure of the game

proposals itself, it is considered the most powerful decision-making body in the EU (e.g., Mazey and Richardson, 1993, p. 14). Its influence has been substantiated in theoretical work by Napel and Widgrén (2006), with special regard to the co-decision procedure of the EU. The Council of Ministers includes one minister (or other government representative) from each member state. The assumption of a one-to-one-relationship between legislators and lobbyists hence translates into the idea that government officials are much more easily accessed by lobbyists from the same country.

4.2 The Model

The relationship between the preference distribution in the legislature and coalition formation among lobbyists is studied using a three-stage game. In the first stage, lobbyists simultaneously decide on which coalitions they wish to form in order to coordinate their lobbying efforts. In the second stage, the resulting coalitions simultaneously offer contributions in order to influence the legislature's policy choice. In the third stage, the legislature decides. The sequential structure of interaction is illustrated in Fig. 4.1. We start by describing the third stage of the game, the legislative policy-making, in Sect. 4.2.1. The second and first stage, the lobbyists' game, follow in Sect. 4.2.2.

4.2.1 The Legislature

Consider a legislature consisting of an agenda-setter A and a set $N = \{1, \ldots, n\}$ of legislators (without agenda-setting authority), where $n \geq 3$ is, for simplicity, an odd number. Each legislator represents one constituency and has one vote. For the time being, we assume that decisions are taken by simple majority rule.

The set of possible policies is described by a compact convex one-dimensional Euclidean space $X \subset \mathbb{R}$, with each point corresponding to a different policy, such as, e.g., the amount of money to be spent on a certain public good, a tax rate, or the level of carbon dioxide emission constraints for cars. Each member of the legislative body has single-peaked preferences on X, and can thus be identified with his *ideal point*. Let λ_A denote the agenda-setter's ideal point, and $\Lambda = (\lambda_1, \ldots, \lambda_n)$ the profile of legislators' ideal points. Without loss of generality, assume that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ for a given random policy issue. Then, $\lambda_M \equiv \lambda_{(n+1)/2}$ is the ideal point of the *median legislator*.

The legislature operates under a *closed rule*, i.e., no amendments can be made, with a legislative session consisting of the following sequence of events:

1. The agenda-setter confronts the legislators with a take-it-or-leave-it proposal $\chi \in X$, where the agenda-setter is assumed to have perfect information about the ideal points of the legislators.
2. The legislators vote on the proposal. The outcome of the game is χ if a simple majority approve of it. If the proposal is rejected, the status quo $Q \in X$ (or a default outcome) prevails. The status quo is known to all players.

This simple model of legislative bargaining is basically a spatial version of the legislative 'ultimatum' game proposed by Romer and Rosenthal (1978).[8]

To keep things simple, preferences are assumed to be given by utility functions of the linear absolute deviation form:

$$u_A(x, \lambda_A) = -a|x - \lambda_A| + R_A \tag{4.1}$$

for the agenda-setter, and

$$u_j(x, \lambda_j) = -b|x - \lambda_j| + R_j, \tag{4.2}$$

for legislator j, where $x \in X$ denotes a policy, $a, b > 0$ are parameters reflecting the intensity of the agenda-setter's and the legislators' preferences, and R_A and R_j are the amounts of politically valuable resources that they receive from lobbyists. Resources might take several forms. One could imagine that the lobbyists compile (possibly biased) information about policy consequences, or that they have expenditures for surveys, research, or commissioned studies. Resources might also be campaign contributions or outright bribes.

Legislators care about how they vote *per se*, just as the agenda-setter cares about which proposal he makes. In particular, they do not wish to depart too much from their ideal point which could be interpreted as the prevailing opinion of their constituency on the issue at stake, but could also reflect the policy-makers' personal preferences.[9] Lobbyists make their contributions to the members of the legislature

[8] Napel and Widgrén (2004) use the above legislative bargaining model in order to propose a new index of decision power which takes players' spatial preferences into account (cf. Sect. 1.2.3).

[9] The 'costs' that members of the legislature incur when deviating from their ideal point could, e.g., be moral qualms, or reduced chances of reelection if constituents can monitor voting records, and hold legislators accountable for their votes.

contingent on *individual actions*: Offers to the agenda-setter (to legislator j) take the form "I will provide to you an amount R_A (R_j) if you propose (vote in favor of) policy x".[10] Though, in reality, the extent to which legislators are susceptible to lobbyists certainly differs, it is natural to start on the assumption that they can be influenced in an equal fashion.

Legislator j votes for a policy x tied to lobbying rent R_j, whenever he prefers it to the status quo Q, i.e., whenever his individual rationality constraint

$$-b|x-\lambda_j|+R_j \geq -b|Q-\lambda_j| \tag{4.3}$$

is satisfied. Legislators who are indifferent between passing and rejecting are assumed to vote in favor of the proposal. Thus, given a proposal χ, the coalition $N_{\chi \succeq Q} \equiv \{j \in N: -b|\chi-\lambda_j|+R_j \geq -b|Q-\lambda_j|\} \subseteq N$, will form. This implies that, if a lobbyist makes an offer R_A (R_j) such that the agenda-setter (legislator j) is indifferent between accepting and rejecting, the agenda-setter (legislator j) always accepts. Thus, in the bargaining between lobbyist and legislator, the lobbyist has all bargaining power as he will only provide an amount of resources equal to the reservation value of the decision-maker, making (4.3) hold with equality.

4.2.2 The Lobbyists' Game

The game includes as many lobbyists as legislators. Concerning the feasible actions of the lobbyists, we impose two assumptions: First, *all* lobbyists have access – and can offer resources – to the agenda-setter. Second, each lobbyist is associated with exactly *one* legislator, 'his' legislator, to whom he can make contributions. To emphasize the one-to-one relationship between legislators and lobbyists, the set $N = \{1, \ldots, n\}$ will refer to the latter as well, and lobbyist j may be thought of as being based in constituency j. The fixed link could then arise because the lobbyist comes from the same geographical location as the legislator, or because of ideological ties. Empirical evidence suggests that indeed contributions to a legislator come in large part from interests within his own electoral district (see, e.g., Wright, 1989).

The lobbyists are assumed to all have the same ideal point L towards which they seek to shift the outcome of the legislative process. Their preferences are also equally intense. A particular policy outcome x is evaluated according to the quadratic utility function

$$v_j(x, C_j) = -(x-L)^2 - C_j, \tag{4.4}$$

[10] A different kind of offers are 'pivotal contracts', that are contingent on the *collective decision*. Here, a legislator receives a bribe if and only if his vote turns out to be pivotal to the collective decision that the lobbyist seeks to bring about. As discussed by Dal Bó (2007), pivotal contracts allow the outside party to manipulate committee decisions at no cost.

where C_j are the costs lobbyist j incurs.[11] These costs arise from the provision of politically valuable resources to the agenda-setter and the legislators. Throughout the analysis we assume parameters a, b, and L to be such that, for every lobbyist j, the utility from the outcome under lobby influence is greater than the utility associated with the no-lobbying outcome, i.e., the participation constraint never binds. If lobbyist j acts on his own, his costs C_j equal the sum of the resources which he provides to the agenda-setter, denoted by $(R_A)_j$, and the resources R_j which he provides to his legislator.

In the first stage of the lobbyists' game, the players may form coalitions in order to coordinate their lobbying efforts. The result of this phase is a *coalition structure* $\pi = \{S_1, \ldots, S_m\}$, i.e., a partition of the set of lobbyists $N = \{1, \ldots, n\}$, such that $S_i \cap S_k = \emptyset$ for all $i \neq k$ and $\bigcup_{i=1}^{m} S_i = N$. In the second stage, the coalitions that emerged from the previous stage engage non-cooperatively in manipulating the legislature.

Coalition formation among the lobbyists is modeled as a *simultaneous* game, i.e., all lobbyists announce their decision at the same time. Specifically, we model coalition formation as the *unanimity game* originally proposed by Von Neumann and Morgenstern (1944, p. 243f), and later by Hart and Kurz (1983, 1984) as their game Γ.[12] Each player $j \in N$ chooses his strategy from the set $\Sigma_j = \{S \subseteq N : j \in S\}$, i.e., the set of coalitions to which j belongs. A strategy $\sigma_j \in \Sigma_j$ for player j amounts to a message announcing which coalition S player j wishes to commit to. Only if all of its members have chosen it, the coalition is formed. The outcome function that maps a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ into a partition of the set of lobbyists is given by

$$\pi(\sigma) = \{S_1, \ldots, S_m\}$$

where

$$S_i(\sigma) = \begin{cases} S_i & \text{if and only if } \sigma_j = S_i \quad \forall\ j \in S_i, \\ \{j\} & \text{otherwise.} \end{cases}$$

This coalition formation rule states that the formation of a particular coalition S_i requires that all of its prospective members choose S_i. In the case that there is no unanimous consent on the formation of S_i among the $j \in S_i$, those players j who selected the strategy $\sigma_j = S_i$ remain singletons.[13] This implies that a coalition breaks down if a deviation occurs, which seems plausible if one views the coalition as the result of unanimous consent that has then been withdrawn.

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[11] One might also consider a more general utility function $v_j(x, C_j) = -\theta_j(x - L)^2 - C_j$, where $\theta_j > 0$ varies across lobbyists. The assumptions of quadratic utility for the lobbyists and linear utility for the members of the legislative body simplify the analysis, but they are not essential to the qualitative results.

[12] The game Γ, as well as the game Δ in Hart and Kurz (1983, 1984), are also called *exclusive membership* games of coalition formation (Bloch, 1997). Membership is exclusive because players are not free to join an existing coalition without the consent of its members.

[13] By contrast, in the related game Δ, also studied by Hart and Kurz (1983, 1984), a coalition is formed of all members who announce a given coalition, irrespective of whether all members of that coalition announce it.

Once a coalition structure has formed, payoffs to the lobbyists are given by the utilities obtained in the second stage. Players belonging to the same coalition are assumed to jointly maximize their aggregate payoff, and the costs of lobbying are shared equally among coalition members. One could think about the costs C_j of each member j of a lobby coalition S as the (uniform) membership fee j pays to participate in S, and the sum of membership fees exactly covers the resources provided by S: $\sum_{j \in S} C_j = (R_A)_S + \sum_{j \in S} R_j$.

Equal division of costs allows us to represent gains from cooperation by a *valuation*, which assigns to each coalition-structure a vector of individual payoffs (rather than coalitional values to be distributed among coalition members).[14] Thus, the normal form game Γ corresponds to a game *without side payments*.

4.3 Analysis of the Model

The objective of this section is to solve the game for the equilibrium strategies by backward induction. The equilibrium concept is subgame perfect equilibrium, and as the set of Nash equilibria of the coalition formation game may be quite large, *strong Nash equilibrium* (Aumann, 1959) is used as refinement. First, we analyze the legislative game in the absence of lobbying activities, and then proceed to derive the lobbyists' optimal policy choice and coalitional strategies in Sects. 4.3.2, 4.3.3, and 4.3.4.

4.3.1 Legislative Decision-Making Without Lobbyists

The agenda setter and the lobbyists are, for presentational purposes, assumed to have most preferred policies such that $L > \lambda_A \geq Q$ and $\lambda_M \geq Q$ hold.

Which proposal will the agenda-setter choose if no lobbying occurs? As a proposal can be passed by majority, the agenda-setter will not forgo any more distance than is inevitable to secure that majority. The agenda-setter's problem then is to choose the most favorable proposal for himself among those policies which the median player (marginally) prefers to the status quo.

The subgame perfect equilibrium proposal for any particular realization (λ_A, Λ) of ideal points is

$$\chi^*(\lambda_A, \Lambda) = \chi^*(\lambda_A, \lambda_M) = \begin{cases} \lambda_A, & \text{if } \lambda_M \geq x_d \\ 2\lambda_M - Q, & \text{if } \lambda_M \in (Q, x_d) \\ Q, & \text{if } \lambda_M \leq Q \end{cases} \tag{4.5}$$

[14] To allow for a valuation representation, it is necessary that the second-stage game has a unique equilibrium. As shown in Sect. 4.3.3, this is indeed the case.

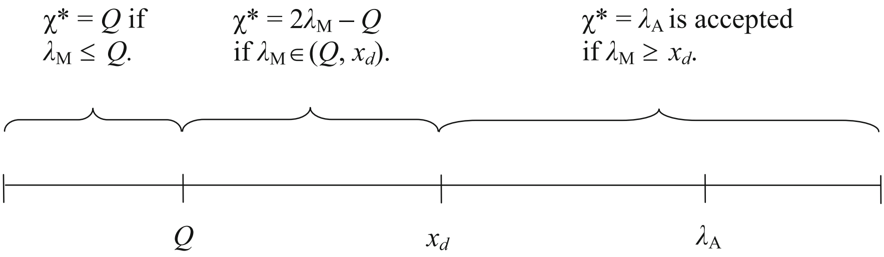

Fig. 4.2 The Nash equilibrium proposal χ^* of the legislative bargaining game

where $x_d = (\lambda_A + Q)/2$ is the *dividing point* that separates the area in which the agenda-setter gets his own ideal point accepted from the area in which he has to compromise with the pivotal player. The equilibrium proposal $\chi^*(\lambda_A, \lambda_M)$ is accepted by at least $(n + 1)/2$ voters, and hence it becomes the outcome of the collective decision-making. Figure 4.2 illustrates the equilibrium of the legislative bargaining game.

Obviously, the legislative bargaining game endows the agenda-setter with considerable power over the outcome. However, this power does not go unchecked. It varies considerably with the position of both the status quo and the median legislator.

For a given profile Λ of legislators' ideal points, the *status quo bias*, or *inertia*, γ of the legislature can be measured by

$$\gamma(\Lambda) = 1 - |\lambda_M - Q|.$$

It is a measure of the legislators' 'aggregate' inclination to *keep* the status quo, ranging from 0 to 1. A large value of γ indicates a large bias in favor of the status quo, as associated either with a situation where opinions in the legislature are divided, or a situation where many legislators (including the median) are comfortable with the status quo. This measure derives only from legislators' preferences, and is to be distinguished from bias – in favor of or against the status quo – created by institutional features such as supermajority requirements.

For illustration purposes consider a continuum of legislators with ideal points distributed uniformly over the interval $X \equiv [-0.5, +0.5]$. Then, the median ideal point is $\lambda_M = 0$. If $Q = -0.5$, we have $\gamma = 0.5$, and the agenda-setter is effective for all positions he might wish to take. If, by contrast, $Q = 0$, i.e., the status quo coincides with the median ideal point, γ equals 1, and the agenda-setter cannot achieve any position other than 0.

4.3.2 Lobbyists' Contributions

As in Snyder's (1991) model of bribes and voting in a majoritarian legislature, the lobbyists will in equilibrium only provide resources to a minimum winning coalition of decision-makers. By doing so, the lobbyists can, in principle, capture the policy-making process. Given the status quo and the ideal points of the members of the legislature, the resources which are necessary to obtain a certain policy outcome x *in the least costly way* are given by

$$\rho(x) = \begin{cases} a|x - \lambda_A| & \text{if } \lambda_M \geq x_d, \\ a|x - \lambda_A| + \sum_{j=M}^{r} 2b\left(\frac{x+Q}{2} - \lambda_j\right) & \text{if } \lambda_M < x_d \end{cases} \tag{4.6}$$

with $x_d = (x+Q)/2$, and where $r \in N$ is defined by the condition $\{\lambda_r \in [\lambda_M, x_d) \wedge \lambda_{r+1} \geq x_d\}$. In the case that $\lambda_M \geq x_d$, it is sufficient to induce the agenda-setter to propose x by providing him his reservation value $a|x - \lambda_A|$. But if $\lambda_M < x_d$, resources must additionally be offered to the set $\{M, \ldots, r\}$ of pivotal legislators, i.e., those who have ideal points in the interval $[\lambda_M, x_d)$, so as to make them indifferent between accepting x and rejecting it. The index number r refers to the legislator with the rightmost ideal point who the lobbyists still have to bribe in order to get a given policy x accepted.

For all the lack of consensus in the lobbying literature whether lobbyists will focus on those sympathetic to their position, or on the rather hostile, the claim made here, that lobbyists wish to pick *pivotal* members of the legislature, appears to be uncontroversial.

Note that (4.6) involves price-discrimination across legislators: The legislators at whom lobbying efforts are aimed are those marginally opposed to the lobbyists' policy, and within this pivotal set the amounts of resources received are higher the more opposed a legislator is to x. In the context of a small legislature or decision-making committee, price-discrimination seems a plausible prediction.

4.3.3 Cartel Formation and Small Status Quo Bias

Consider first the case $\lambda_M \geq (L + Q)/2$, i.e., the policy choice sought by the lobbyists is closer to the median legislator than the latter is to the status quo. We shall use lower case letters to denote the cardinality of coalitions, e.g., s refers to the number of players in coalition S. Viewed in isolation, a coalition of size s solves

$$\max_x \; -s(x - L)^2 - \rho(x) \tag{4.7}$$

yielding a policy choice

$$x^*(s) = L - \frac{a}{2s} \tag{4.8}$$

and resource provision

$$\rho(x^*(s)) = a|L - \frac{a}{2s} - \lambda_A|. \tag{4.9}$$

The larger S, the closer to L is the policy outcome that S can obtain, and the greater the amount of resources it spends on lobbying efforts. Under the equal sharing rule, any member j in coalition S incurs costs $C_j = \rho(x^*(s))/s$, and receives a per-capita utility

$$v_j(s) = \frac{a^2}{4s^2} - \frac{a(L-\lambda_A)}{s} \qquad \forall\ j \in S. \tag{4.10}$$

The function $v_j(s)$ is strictly increasing in s, thus exhibiting increasing returns to size in lobbying activities.

In coalition structure π, coalition S will contribute to lobbying efforts as long as, given the other contributions, the policy choice is less than $x^*(s)$. Suppose the complement $N \setminus S$ provides an amount of resources $\rho(y)$, then the best response of S is to contribute max $[\rho(x^*(s)) - \rho(y), 0]$. In equilibrium, as greater coalitions provide more resources, only the coalition (or coalitions) in π with the largest number of members engages in lobbying, while all other coalitions free-ride. Let $\ell \geq 1$ be the number of such maximal lobby coalitions in coalition structure π. In the case that π includes a unique lobbying coalition S, i.e., $\ell = 1$, utilities under π are fully determined by the size s of S. The utility of the members of S is given by (4.10). If several largest coalitions have formed with s members each, any division of the resources $\rho(x^*(s))$ among these cartels is possible in equilibrium, while non-maximal coalitions still free-ride. In view of the symmetry among the ℓ cartels, it seems plausible to require that each of them bears the share $\rho(x^*(s))/\ell$ of total resource provision. Thus, a unique vector of per-capita utilities is associated to each π. In a coalition structure π with ℓ maximal coalitions S_i of size s, free-riding players obtain utilities

$$v_j(\pi) = v_j(s) = -\frac{a^2}{4s^2} \qquad \forall j \in N \setminus \{\cup_{i=1}^{\ell} S_i\}, \tag{4.11}$$

which is clearly greater than (4.10). Moreover, non-members of the S_i benefit from an increase in the size of the lobbying coalition(s). Formally speaking, in the lobbyists' game, the following conditions defined by Yi (1997) hold that express *positive spillovers* and *negative interrelation between coalition size and payoff*, respectively.

Property 4.3.1. $\forall j \in T,\ T \in \pi, \pi'\colon\ v_j(\pi) < v_j(\pi')$ if $\pi' \setminus T$ can be derived from $\pi \setminus T$ by merging coalitions in $\pi \setminus T$.

Property 4.3.2. For any coalition structure π and any two coalitions $S, T \in \pi\colon\ \forall i \in S, j \in T\colon\ v_i(\pi) < v_j(\pi)$ if and only if $s > t$.

The public good character of lobbying efforts makes the grand coalition the efficient coalition structure from the point of view of the lobbyists: it maximizes the sum of lobbyists' utilities. Whether it will actually be formed is determined in the coalition-formation game to which we turn now.

First, it is useful to observe that the lobbyists are *symmetric*: The utility of a player in a particular coalition only depends on the number of individuals in the largest coalition in the partition, but not on their identities. A strategy σ_j for player

j is therefore equivalent to an announcement of the size s of the coalition j wishes to be a member of. Thus, all players have the same strategy set. Formally, player symmetry is captured by the properties (a) $\Sigma_i = \Sigma_j$ for all $i, j \in N$, and (b) $v_i(\pi) = v_j(\pi'), i, j \in N$, and $v_k(\pi) = v_k(\pi'), k \neq i, j$, where π' is the coalition structure that results from π when player i's and j's strategy choices σ_i and σ_j are switched.[15]

Second, note that the game Γ allows players to commit to staying alone. If all players decide to remain singletons, the game could, in principle, become a war of attrition where each player hopes that one of the others will provide contributions in the expectation of free-riding on him. However, as the no-lobbying outcome is not desirable to any of the lobbyists, each of them is willing to engage in lobbying the agenda-setter, effectuating the outcome $x^*(1)$ according to (4.8). From the above assumption that resource provision is shared equally among all maximal coalitions, it follows that, in the *atomic* coalition structure $\hat{\pi} = \{\{1\}, \{2\}, \ldots, \{n\}\}$, each lobbyist incurs costs of $C_j = \rho(x^*(1))/n$. The payoff of each player j in $\hat{\pi}$ is

$$v_j(\hat{\pi}) = \frac{a^2(2-n)}{4n} - \frac{a(L-\lambda_A)}{n} \qquad \forall\, j \in N, \tag{4.12}$$

and one can easily check that it is always less than the grand coalition per-capita payoff $v_j(N)$.

A strong Nash equilibrium is immune to any coalitional deviation, i.e., it is a strategy profile σ for which there does not exist a coalition $T \subset N$ with strategy profile σ'_T for players in T such that $v_j\left(\pi(\sigma'_T, \sigma_{N \setminus T})\right) > v_j\left(\pi(\sigma)\right)\ \forall j \in T$.

The game Γ generally has multiple strong Nash equilibria. If, for example, some player j committed to stay on his own, then it would be a best response for the remaining players to form the coalition $N \setminus \{j\}$, and vice versa. Since it does not pay in the game Γ to leave any lobbying coalition S once it has formed as the remainder then dissolves into singletons, the coalition structure would be stable iff $v_j(n-1) > v_j(n)$, i.e., iff the single player does not have an incentive to merge with the grand coalition. Yet, taking the idea of simultaneous decisions on coalition announcements σ_j seriously, players have no means to *coordinate* on an asymmetric equilibrium. We therefore focus on *symmetric* strong equilibria as the only plausible results in the case of a small status quo bias γ. Let x_N^* denote the optimal policy, as defined by (4.8), which is available to the grand coalition N. When γ is smaller than a threshold given by $1 - |x_N^* - \lambda_M|$, the lobbyists' game is symmetric.

Proposition 4.3.1. *Suppose that* $\gamma(\Lambda) \leq 1 - |x_N^* - \lambda_M|$*. Then, in the simultaneous coalition-formation game* Γ*, the strategy profile* σ^* *with* $\sigma_j^* = \{N\}$ *for every* $j = 1, \ldots, n$ *is the unique symmetric strong equilibrium, and the grand coalition* N *is formed.*

Proof. The idea behind Proposition 4.3.1 is simple. If bias $\gamma(\Lambda) \leq 1 - |x_N^* - \lambda_M|$, the distance between λ_M and Q exceeds the distance between x_N^* and λ_M. It is

[15] Put differently, a symmetric game is invariant under a permutation of players.

therefore sufficient for the lobbyists to influence the agenda-setter's proposal in their favor. Because of symmetry, a strategy σ_j for player j reduces to an announcement of the size of the coalition he wishes to form. Equation (4.11) shows that all players who do not belong to the maximal coalition S in π receive the same payoff regardless of how the player set $N \setminus S$ is organized. Also, by (4.10), a player in a greater lobbying coalition receives a larger payoff than a player in a smaller lobbying coalition. Hence, no player chooses a strategy $\sigma_j = S$ where $S \subset N, j \in S$, is a coalition of size s with $1 < s < n$. As the grand coalition is strictly preferred to the atomic coalition structure [equation (4.12)] by all players, it follows that $\sigma_j = N$ is the best strategy for all $j \in N$, leading to the formation of the lobbying coalition N. □

4.3.4 Cartel Formation and Large Status Quo Bias

We next analyze the more complex case in which the lobbyists' optimal policy choices require provision of resources to both the agenda-setter and to some legislators in order to recruit the votes that are necessary to pass the proposal. First note that the number of coalitions that will form can safely be limited to 'one or two', as still only the largest coalition, which is in a position to lobby, will do so, and the organization structure of non-lobbying players does not matter. Thus, either the grand coalition N, or the coalition structure $\pi = \{S_L, S_{FR}\}$ will be established, where S_L is the lobbying coalition, and $S_{FR} = N \setminus S_L$ comprises all free-riding players.[16]

Lobbyists differ in their preferences for these two potential coalitions. If, for example, lobbyist M, i.e., the player associated with the median legislator, joined S_{FR}, no successful lobbying would be possible.[17]

Given x, let $r \in N$ again be the index number that is (uniquely) determined by the condition $\{\lambda_r \in [\lambda_M, x_d) \wedge \lambda_{r+1} \geq x_d\}$. Then, the resource schedule (4.6) for the case $\lambda_M < x_d = (x + Q)/2$ can be rewritten as

$$\rho(x) = a|x - \lambda_A| + kb(x + Q) - 2b \sum_{j=M}^{M+k-1} \lambda_j \tag{4.13}$$

where $k \in \{1, 2, \ldots, r + 1 - M\}$ is such that $x \in (2\lambda_{M+k-1} - Q, 2\lambda_{M+k} - Q]$.

[16] Belleflamme (2000) imposes the restriction that maximally two coalitions can form in order to make progress on the analysis of cartel formation among asymmetric firms.

[17] The general possibility that some lobbyist with an index number smaller than M could step into the breach and bring his legislator to vote in favor of the proposal is ignored in the analysis that follows. The focus is on least-cost lobbying strategies as described by (4.6).

In the case $r+1 > n$, let $2\lambda_{M+k} - Q := 0.5$, i.e., a policy $x > 2\lambda_n - Q$ is obtained by providing resources to the legislator set $\{M, M+1, \ldots, n\}$. The piecewise linear schedule (4.13) is continuous throughout its domain $(2\lambda_M - Q, 2\lambda_{r+1} - Q]$.

Let us first consider the policy that would be enacted by the grand coalition. Obviously, the grand coalition is in a position to approach all legislators that it wants. In order to achieve a given policy $x \in (2\lambda_{M+k-1} - Q, 2\lambda_{M+k} - Q]$ for some $k \in \{1, 2, \ldots, r+1-M\}$, contributions must be made to k legislators. At points $x = 2\lambda_{M+k-1} - Q$, $k = 1, 2, \ldots, r+1-M$, the cardinality of the set of pivotal legislators jumps from $k-1$ to k.[18]

Using (4.13), one finds that the policy x^* solving (4.7), i.e., maximizing the utility of the grand coalition, is defined by the fixed point problem

$$x^* \in \tau(x^*) \tag{4.14}$$

where $\tau : X \rightrightarrows [0, L]$ is defined by

$$\tau(x) = \begin{cases} \left\{L - \frac{a+bk}{2n}\right\}, \text{ if } x \in \left(2\lambda_{M+k-1} - Q, 2\lambda_{M+k} - Q\right) \text{ for some } k = 1, 2, \ldots, r+1-M \\ \left[L - \frac{a+b(k-1)}{2n}, L - \frac{a+bk}{2n}\right], \text{ if } x = 2\lambda_{M+k-1} - Q \text{ for some } k = 1, 2, \ldots, r+1-M. \end{cases} \tag{4.15}$$

The assumption that the participation constraint does not bind implies that the range of τ has zero as lower bound. As k monotonically increases in x, the graph of τ is monotonically decreasing in x. It is readily checked that the correspondence τ fulfills the criteria of Kakutani's Fixed Point Theorem.[19] Thus, the existence of a fixed point is guaranteed. Moreover, $\tau(x)$ has a unique fixed point. This follows from the fact that τ is monotonically decreasing. Hence, a unique outcome exists for every coalition structure. Problem (4.14) is illustrated in Fig. 4.3.

Comparing the outcome defined by (4.14) together with (4.15) to (4.8), the corresponding expression under a small status quo bias, it is straightforward to see that the policy outcome is now strictly further to the left of L than in the case $\lambda_M \geq (L+Q)/2$. As more resources are required to achieve a particular outcome x, the lobbyists settle for a less extreme policy. *Ceteris paribus*, the policy is further to the right, the smaller a and b are; that is, the more amenable to lobbying efforts the legislature is.

How does the policy enacted by a lobbying coalition $S_L \neq N$ differ from the one defined in (4.14)? The policy that S_L aspires to is in principle also given by

[18] Thus, k as a function of x depends on the particular profile of legislators' ideal points. In the limiting case where legislators form a continuum with uniformly distributed ideal points on X, the policy that balances at the margin the gains in terms of policy and the costs of lobbying can be calculated explicitly and is given by $x^* = (4nL + 2b\lambda_M - bQ - 2a)/(4n + b)$.

[19] Kakutani's fixed point theorem states the existence of a fixed point of $f : A \rightrightarrows A$ under the conditions that (a) $A \subset \mathbb{R}^n$ is a nonempty, compact, convex set, and (b) f is an upper hemicontinuous correspondence from A into itself with the property that the set $f(x) \subset A$ is nonempty and convex for every $x \in A$.

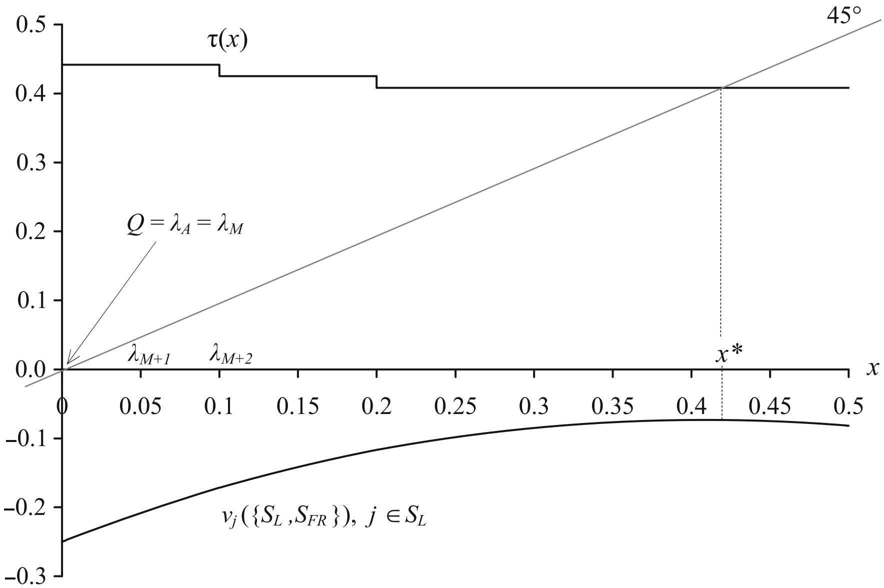

Fig. 4.3 The policy choice x^*, maximizing the utility of members j of the lobbying coalition S_L, is obtained as fixed point of correspondence τ

(4.14), the only difference being that n must be replaced by the cardinality of S_L. Yet, S_L may not include the lobbyists necessary to establish this outcome. For some $k \in \mathbb{N}^*$, $k \leq r + 1 - M$, define S_L as

$$S_L(k,t) = \{M, M+1, \ldots, M+k-1\} \cup T,$$

where $T \subseteq N \setminus \{M, M+1, \ldots, M+k-1\}$ is a (potentially empty) coalition of size t. Hence, the number of players in S_L is $k + t$.

In the case that $k = r+1-M$, S_L does include the lobbyist who is associated to the legislator with ideal point λ_r, and any policy that is desirable to the coalition is principally within its reach. If $k < r+1-M$, the most extreme policy achievable is given by $2\lambda_{M+k} - Q$. Let $\hat{\tau}(x)$ denote the correspondence that results from replacing n with $(k+t)$ in (4.15), and denote the unique fixed point of $\hat{\tau}(x)$ by $\hat{x}$. Then, the optimal policy choice of a coalition S_L is given by the following expression:

$$x^*(k,t) = \begin{cases} \hat{x}, & \text{if } \hat{x} \leq 2\lambda_{M+k} - Q \\ 2\lambda_{M+k} - Q, & \text{if } \hat{x} > 2\lambda_{M+k} - Q. \end{cases} \tag{4.16}$$

$x^*(k,t)$ weakly increases in both k and t until it eventually reaches the policy (4.14) enacted by the grand coalition.

Building on these findings, we can formulate the following result.

Proposition 4.3.2. *Suppose that $\gamma(\Lambda) > 1 - |x_N^* - \lambda_M|$. Then, there exist preference profiles such that the strong Nash equilibrium of the lobbying game involves a coalition structure $\pi = \{S_L, S_{FR}\}$ with $S_L, S_{FR} \neq \emptyset$.*

The proposition states that the lobbying coalition may be smaller than the grand coalition, and all remaining players free-ride. The condition $\gamma(\Lambda) > 1-|x_N^*-\lambda_M|$ is equivalent to saying that λ_M is closer to Q than to the outcome x_N^* which would be available to the grand coalition of lobbyists by influencing the agenda-setter alone. As a corollary, one immediately obtains that the Nash equilibrium structure may be inefficient in the sense that it does not maximize the sum of lobbyists' payoffs, and the policy outcome under lobbying in this case is less 'extreme' than the policy that the grand coalition would obtain.

Before proving the proposition by providing an example, let us briefly review the conditions for a strong equilibrium. In a Nash equilibrium coalition structure $\pi^* = \{S_L, S_{FR}\}$, no player wishes to switch coalitions unilaterally. First, as the participation constraint is assumed to be never binding, the player associated to the median legislator engages in lobbying efforts. For some $k \in \{2, \ldots, r\}$ and $t \in \{0, 1, \ldots, n-1\}$, the incentive of lobbyist $j = M + k - 1$ who is associated to the 'next' potentially pivotal legislator, to join a coalition $S_L(k-1, t)$ is given by

$$I_j(k-1, t) = v_j(k, t) - v_j(k-1, t).$$

Similarly, the incentive of any lobbyist $l \in T = N \setminus \{M, \ldots, M + k - 1\}$ outside the pivotal set to join $S_L(k, t-1)$ is

$$I_l(k, t-1) = v_l(k, t) - v_l(k, t-1).$$

For coalition structure $\{S_L(k,t), S_{FR}(k,t)\}$ to be an equilibrium structure, it must hold that (a) $I_j(k-1,t) \geq 0$ for $j = M+k-1$, but $I_j(k,t) < 0$ for $j = M+k$, and (b) $I_l(k,t-1) \geq 0$ for $l \in N \setminus S_L(k,t-1)$, but $I_l(k,t) < 0$ for all $l \in N \setminus S_L(k,t)$.

As a coalition left by some member disintegrates into singletons, no player can benefit from leaving the lobby coalition $S_L(k,t)$. For an equilibrium structure $\pi^* = \{S_L(k,t), S_{FR}(k,t)\}$ to be a strong Nash equilibrium of the coalition formation game, it is therefore sufficient that members of $S_{FR}(k,t)$ do not have an incentive to merge with $S_L(k,t)$, i.e., to form the grand coalition. This is the case if

$$v_j(k,t) - v_j(N) \geq 0 \quad \text{for all } j \in S_{FR}(k,t).$$

We now construct an example to prove Proposition 4.3.2.[20]

[20] As the equilibrium coalition structure depends on the entire profile of ideal points, the location of the status quo, and parameters a and b, it is difficult to characterize in a general way. Conditions on the parameters could possibly be derived under some restrictions, such as, e.g., equidistant ideal points of all legislators.

Proof of Proposition 2. Consider the following legislature with $N = 5$ legislators. Let $\Lambda = (\lambda_1, \lambda_2, 0, 0.05, 0.1)$ with $\lambda_1 \leq \lambda_2 < 0$ be the vector of their ideal points, and $\lambda_A = 0$ the agenda-setter's ideal point. The parameters in the agenda-setter's and the legislators' utility functions are $a = 1/4$ and $b = 1/10$. The status quo and the lobbyists' ideal point are given by $Q = 0$ and $L = 0.5$, respectively.

If no lobbying takes place, the outcome of the legislative game will be $\chi^* = 0$ (regardless of the position λ_A of the agenda-setter). To be *effective*, a coalition must at least include lobbyist 3 who is associated to the median legislator. Table 4.1 lists all effective lobby coalitions S_L and the valuations of players in S_L and in the free-riding coalition S_{FR}. It can be seen from Table 4.1 that $\pi^* = \{S_L, S_{FR}\} = \{\{345\}, \{12\}\}$ is the unique strong Nash equilibrium structure: start with the situation where only lobbyist 3 makes a lobbying effort, which results in a payoff of −0.1600 for the free-riding players. Yet, lobbyist 4 has an incentive to join {3}, since this increases his payoff from −0.1600 to −0.1300. Similarly, lobbyist 5 benefits from entering the coalition {34}. Given the optimal lobbying efforts of the coalition {345}, the remaining players 1 and 2 do not wish to participate as their free-riding payoff (−0.0084) is higher than what they would receive if only one of them joined the lobbying coalition (−0.0565), or if both joined (−0.0460). Thus, in the coalition-formation game Γ with the number of coalitions that can be established limited to a maximum of two, the equilibrium strategies are $\sigma_j^* = \{345\}$ for lobbyists $j = 3, 4, 5$, and $\sigma_j^* = \{12\}$ for lobbyists $j = 1, 2$.

The policy outcome under lobbying is $x^*(3, 0) = 0.4083$, whereas the grand coalition policy would be 0.445. This is the situation depicted in Fig. 4.3. As indicated by the sum of valuations over all players in the last column in Table 4.1, the lobbyists' joint payoff is *not* maximal in equilibrium. □

The above example demonstrates that, in the case of a large status quo bias, the identity of coalition members becomes important. This asymmetry allows those lobbyists, whose legislators are not necessary to achieve a desired policy outcome, to give free rein to their free-riding instincts *without having to fear* that the atomic coalition structure, or no lobbying effort at all, would be the result. To be effective,

Table 4.1 Lobby coalitions and payoffs

S_L	k	t	$x^*(k,t)$	v_j ($j \in S_L$)	v_j ($j \in S_{FR}$)	$\sum_j v_j$
{3}	**1**	**0**	**0.1**	**−0.1950**	**−0.1600**	**−0.8350**
{31} or {32} or {35}	1	1	0.1	−0.1775	−0.1600	−0.8350
{312} or {315} or {325}	1	2	0.1	−0.1717	−0.1600	−0.8350
{3125}	1	3	0.1	−0.1688	−0.1600	−0.8350
{34}	**2**	**0**	**0.2**	**−0.1300**	**−0.0900**	**−0.5300**
{341} or {342}	2	1	0.2	−0.1167	−0.0900	−0.5300
{3412}	2	2	0.2	−0.1100	−0.0900	−0.5300
{345}	**3**	**0**	**0.4083**	**−0.0733**	**−0.0084**	**−0.2366**
{3451} or {3452}	3	1	0.4313	−0.0565	−0.0047	−0.2308
{12345}	**3**	**2**	**0.4450**	**−0.0460**	–	**−0.2299**

the lobbying coalition must consist of, or at least contain, lobbyists whose legislators have adjacent ideal points. This result is reminiscent of Axelrod's (1970) theory of coalition formation which predicts that coalitions will be ideologically 'connected' along a policy dimension. The conclusion that the grand coalition may not form with heterogeneous players is related to results in Belleflamme (2000) who studies the formation of Cournot oligopolies in an open membership game among firms with different cost functions.

4.4 Discussion

This section discusses the robustness of our findings and modifications of the setting, and then looks at the limitations of the model. To start with, let us reconsider the coalition-formation process.

The game described above is not immune to the common critique that outcomes depend to a great extent on the rules of coalition formation (see Yi, 1997). In principle, the lobbyists' agreement to form the grand coalition, as any agreement concerned with the production of public goods, is subject to the free-rider problem as characterized by the following observation in Stigler (1950, p. 25f) about mergers:

> "The major difficulty in forming a merger is that it is more profitable to be outside a merger than to be a participant. [...] Hence, the promoter of merger is likely to receive much encouragement from each firm – almost every encouragement, in fact, except participation."

In the game Γ, however, the incentive for a lobbyist to stay out of the lobbying coalition, hoping that others will join it nevertheless, vanishes, and the free-riding problem can be resolved. Application of the coalition unanimity game Γ seems appropriate in our context as the number of lobbyists concerned about a certain (regulative) issue is relatively small. This makes the implication of Γ, that the formation of a coalition requires unanimous consent of its members, more plausible, and it seems more likely than in the case of a great number of players that a coalition breaks up after the defection of one of its members. Of course, more complex games of coalition formation exist, namely sequential games with fixed distribution of payoffs as in Bloch (1996) and the equilibrium binding agreements reached by far-sighted players with transferable utility proposed by Ray and Vohra (1997).

Usually, in games of public good provision, cooperation can only be sustained if some kind of information imperfection prevails. This is also true here: the simultaneity of the coalition-formation decision creates informational imperfection. The grand coalition might not emerge in a sequential game with perfect information.

As a normative benchmark, observing the policy that would be chosen by a benevolent social planner is worthwhile. So far, no assumption was made about the origin of the agenda-setter's ideal point. It might derive from the ideal points of the legislators, or from the preferences of citizens. Define the socially efficient policy as the maximizer of the sum of utilities of individual citizens in all constituencies.

Under the additional assumption that individuals' utility is linear in distance and equally intense, the policy corresponding to the *median citizen*'s ideal point would maximize overall welfare. Generally, the median citizen's and the *median legislator*'s ideal points will fall apart, even if each legislator's position corresponds to the median of his constituency and constituencies are equally sized. But λ_M could then be expected to be a good proxy for the socially optimal policy choice.

Suppose the agenda-setter is benevolent in the sense that he would propose λ_M in the absence of any lobbying efforts, so that λ_M would then be the outcome of the legislative process. A small status quo bias allows lobbyists to exercise more influence in the sense that they can achieve outcomes that differ to a great extent from λ_M. The shift of the state of affairs in the lobbyists' favor turns out to be much smaller when they have to engage in recruiting the votes of legislators than in the case where they only have to influence the agenda-setter. The reasons are that, first, assessing both agenda-setter and legislators is more expensive, and, second, lobbyists may not be able to overcome the collective action problem fully.

From the optimal policy choices of the lobbyists for the case of a small and a large status quo bias as given by (4.8) and (4.14), respectively, it can be seen that, the larger the lobbying coalition, the more 'extreme' is the legislative outcome under lobbying. A *large* legislature, implying an equally large number of lobbyists, allows *ceteris paribus* for outcomes that are more congenial to the lobbyists' interest. This is a rather disturbing conclusion given the fact that the number of interests active in EU lobbying has multiplied with the enlargement of the Union.

Given the assumption of equal voting weights, the legislator holding the median position is always pivotal. He is the only other player apart from the agenda-setter who may have an impact on the outcome in the sense that a change in his preferences might alter the outcome of the collective decision-making. Under weighted voting, as used in the EU Council of Ministers, however, the pivotal position is generally different from the median.[21] In the definition of the status quo bias (see Sect. 4.3.1) of the legislature, λ_M would then have to be replaced by the ideal point of the player who is pivotal from the right. Generally speaking, weighted voting can be expected to increase the asymmetry among lobbyists, and thus to intensify their free-riding problem.

The Council of Ministers not only decides by weighted voting, but the decision threshold is also much higher than simple majority. Currently 255 out of 345 votes (73.9%) are needed to pass a proposal. One can check that such a supermajority requirement narrows the set of policies for which the agenda-setter is fully effective. It follows from the present analysis that the potential for the formation of the grand coalition of lobbyists is smaller the higher the quota used in the legislature.

In order to single out the effect of status quo bias in the legislature, the most preferred policy was assumed to be the same for all lobbyists. While this assumption

[21] 'A priori', i.e., without taking preferences into account, the pivotal position would be a random variable. Under the condition that the ideal points $\lambda_1, \ldots, \lambda_n$ are independently and identically distributed, the a priori probability that a legislator is pivotal is given by the respective Shapley–Shubik index (cf. Sect. 1.2).

seems to be close to reality for many classes of lobbyists – one could, e.g., imagine that all cigarette producers uniformly prefer the laxest possible anti-smoking regulation – one can also conceive of situations where different national lobby associations have antagonistic preferences. For example, the car industry of one country could be greatly advanced in producing cars with low CO_2 emission levels whilst the car makers of another country lag behind.

With lobbyist ideal points $L_1, \ldots, L_n$, the formation of a lobby coalition in the case that the agenda-setter is potentially effective for all L generally necessitates side-payments among coalition members. Otherwise some members might prefer to lobby the agenda-setter on their own. Provided that all lobbyists in the coalition have the same preference intensity over policy outcomes (see Footnote 11), the sum of squares $\sum_{j \in S} -(x - L_j)^2$ is minimal for $x = \bar{L} = 1/n \sum_{j \in S} L_j$, i.e., if the outcome of the legislative bargaining is the mean of the lobbyists' ideal points. Therefore, $\bar{L}$ is the ideal point of the lobby coalition. When the issue at stake has a different importance to lobbyists from different constituencies, $\bar{L}$ has to be replaced with a weighted mean. It seems intuitively plausible that a lobby coalition might not be effective if the preferences of its members differ very much.[22] If the lobbyists' positions are scattered on both sides of the agenda-setter's ideal point, it seems most natural to assume that two cartels opposing each other would form.[23]

In summary, the grand coalition of lobbyists is more likely to form when the bias among legislators for the status quo is small, lobbyists' preferences over policy outcomes are similar, the quota is close to simple majority, and when legislators do not differ much in voting weights.

There are, however, several limitations to drawing too wide conclusions from our analysis: First, the one-dimensionality of the policy space is obviously a strong assumption. It might be justified, however, for the application to Eurogroups whose domains of concern are mostly narrow and often 'low-brow' issues such as, e.g., technical standards. Also, the analysis has been confined to a particular simple model of legislative bargaining. It is well known from bargaining theory that modest variations in the institutional details can result in very different outcomes. It would, for example, probably be more realistic for EU decision-making to assume multi-round bargaining, with lobbyists making contributions in each bargaining round. Second, contributions are part of a quid-pro-quo exchange. Lobbyists are able to persuade legislators to vote differently by providing resources to them, but the mechanism through which they exercise their influence is a black box. One key resource that interest groups 'exchange' for influence is information, and an influential branch of the lobbying literature (see, e.g., Austen-Smith and Wright, 1992; Lohmann, 1995) has emphasized the role of interest groups as strategic providers of

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[22] To the extent that a positive relationship exists between the degree of collusion in an industry and the similarity of firms' preferences over policy outcomes, this result is in line with Damania and Fredriksson (2000) who show formally that more collusive industries with higher profits from collusion have a greater incentive to form industry lobby groups, and more easily overcome the free-rider problems involved.

[23] The situation of two competing lobbyists is studied by Groseclose and Snyder (1996).

information which is relevant to policy-decisions. Third, it is important that lobbyist j is unable to influence legislators other than 'his own'. The situation of n lobbyists allocating their resources non-cooperatively to n legislators would constitute a Colonel Blotto game (see Shubik, 1982, p. 226f and 322ff) which is very difficult to analyze. In the case of a common ideal point of the lobbyists, it does not seem too restrictive to exclude this kind of competitive behavior, but if lobbyists' preferences differ, the assumption is more limiting.

Finally, the assumption of a fixed sharing rule might seem a severe restriction on the cooperation possibilities that can be captured. But note that optimal policy choices of the lobbyists would be left unchanged in the transferable utility case. Coalition S_L makes contributions such as to maximize joint payoff (4.7). Since the utility functions of the lobbyists are additively separable, it actually does not matter how contributions are divided. Instead of being financed through membership fees levied uniformly on members, they could be split in line with some efficient cooperative bargaining solution as the result of a bargaining process among the lobbyists. Players within one coalition could hence carry out transfers of utility without affecting the strategic environment for other coalitions. The scope for cooperation could possibly be enhanced if the allocation of the coalitional surplus among cartelists was not determined in advance. But, as argued by Bloch (1996, p. 91f), valuations seem rather natural because the prospective coalition member, when contemplating his coalition decision, must apply some fixed rule to evaluate the payoffs he receives in different coalition structures.[24] In the context of a transnational lobby group, it is quite plausible to assume that the decision to join the group entails bearing a predefined share of the costs. For symmetric games, equal division of coalition payoffs is derived endogenously in a bargaining game by Ray and Vohra (1999).

This chapter has studied the endogenous formation of lobby coalitions and the resulting policy outcome in a highly stylized model of a supranational decision-making process. The novel element in the present chapter lies in characterizing the relationship between the status quo bias of the legislature, which is a source of asymmetry among lobbyists, and lobbyists' ability to overcome a collective action problem. If the status quo bias, or legislative inertia, is large, then the equilibrium coalition structure can feature a lobbying coalition which is smaller than the grand coalition, and thus is inefficient from the lobbyists' point of view.

The empirical literature on EU lobbying has identified *ad hoc coalitions* of lobbyists, especially large companies, as an important type of actor in European lobbying. Moreover, depending on the issue at stake, this form of 'spontaneous' organization often seems to be more attractive to individual lobbyists than representation by a Eurogroup (Pijnenburg, 1998). The analysis provides some indication why, so far, lobbying in the European Union has not been dominated by all-European groups or grand coalitions, but could rather be characterized by ephemeral ad hoc coalitions targeted on specific issues. Likely reasons are the large supermajority requirement

[24] Different players might, however, use different rules, leading to incompatible expectations when it comes to bargaining over the actual allocation.

in the Council as the chief decision-making body, and possibly, prevalent preferences for keeping the status quo, or divergence between Council members on the future direction of policy. For example, opinions on the role of nuclear energy do not only greatly vary in detail among European decision-makers, there is not even a consensus as whether to expand or reduce reliance on this energy source.

One hypothesis generated by the model is that formation of ad hoc coalitions between lobbyists should be expected among lobbyists *whose legislators form an ideologically compact set*. This is a testable prediction, suggesting further empirical work.

There are several other directions that should be taken up by future research. First, it is desirable to characterize more systematically under which parameter constellations the grand coalition will not form, and, ideally, to describe all equilibria that might arise in the game. Second, the problem of lobbyists with divergent interests deserves further attention. While, for some matters, there may only be one dominant 'type' of lobbyists that has a particularly high stake in the issue, other issues provoke activity by different 'types' of lobbyists who oppose each other to varying degrees (e.g., the bio engineering industry and consumer food associations). The effects of heterogeneity in lobbyists' preferences, up to the point of fully opposite interests, on the formation of lobby cartels seem a very challenging topic.

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